# PROBLEMS IN AFFINE ALGEBRAIC GEOMETRY: ON TRIVIALITY AND EMBEDDING OF LINEAR HYPERPLANES AND ON RIGIDITY OF PHAM-BRIESKORN SURFACES

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# Declaration

The work presented in this thesis has been carried out by me under the guidance of Professor Neena Gupta of Indian Statistical Institute, Kolkata. I have properly acknowledged all my co-authors who helped me in developing different part of this thesis.

The work reported in this thesis is original and has not been submitted in part or in full for any degree or diploma or fellowship to any other University or Institute.

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Date : 17/04/2025

# Abstract

My thesis consists of two topics from Affine Algebraic Geometry: one of the topics is on Linear varieties (varieties defined by polynomials which are linear in one variable) and the second topic explores when Pham-Brieskorn surfaces do not admit non-trivial $\mathbb{G}_a$-actions.

Linear varieties over a field $k$ have been playing a central role in the study of some of the challenging problems on affine spaces like the Zariski Cancellation Problem and the Linearization Problem. Breakthroughs on such problems have occurred by examining two questions on linear polynomials of the form

$H := \alpha(X_1, \ldots, X_m)Y - F(X_1, \ldots, X_m, Z, T) \in D := k[X_1, \ldots, X_m, Y, Z, T]$ :

(i) Whether $H$ defines a hyperplane i.e., the affine variety $\mathbb{V} \in \mathbb{A}_k^{m+3}$ defined by $H$ is isomorphic to the affine space $\mathbb{A}_k^{m+2}$.

(ii) If $\mathbb{V}$ is isomorphic to an affine space, then whether $H$ is a coordinate in $D$.

Question (i) connects to the Characterization Problem of identifying affine spaces among affine varieties; Question (ii) is a special case of the formidable Embedding Problem for affine spaces.

In Chapter 3 of the thesis, using $K$-theory and $\mathbb{G}_a$-actions, we address these questions under certain conditions on $\alpha$ and $F$. For instance, we show that when the characteristic of $k$ is zero, $F \in k[Z, T]$ and $H$ defines a hyperplane, then $H$ is a coordinate in $D$ along with $X_1, X_2, \ldots, X_m$.

Our results yield certain family of higher-dimensional hyperplanes satisfying the Abhyankar–Sathaye Conjecture on the Epimorphism Problem and an infinite family of higher dimensional non-isomorphic varieties which are counterexamples to the Zariski Cancellation Problem in positive characteristic and $\mathbb{A}^2$-fibration Problem in positive characteristic.

We have also discussed the above two questions by replacing the field $k$ with a Noetherian integral domain $R$.

In Chapter 4 of the thesis, we have discussed the rigidity of Pham-Brieskorn rings. Over any field $k$, for $n \in \mathbb{Z}_{\geqslant 3}$ and $a_1, \ldots, a_n \in \mathbb{Z}_{\geqslant 1}$, Pham-Brieskorn rings are denoted by $B_{(a_1,\ldots,a_n)}$ and defined by $B_{(a_1,\ldots,a_n)} := k[X_1, \ldots, X_n]/(X_1^{a_1} + \cdots + X_n^{a_n})$.

We showed that every non-domain Pham-Brieskorn ring, for $n \in \mathbb{Z}_{\geqslant 3}$ is non-rigid. For any three integers $a, b, c \geqslant 1$, we give some sufficient conditions on $(a, b, c)$ for which Pham-Brieskorn domain $B_{(a,b,c)}$ is rigid.

This gives an alternative approach to show that over a field $k$ of characteristic $p > 0$, there does not exist any non-trivial exponential map on $k[X, Y, Z, T]/(X^m Y + T^{p^r q} + Z^{p^e}) = k[x, y, z, t]$, for $m, q > 1$, $p \nmid mq$ and $e > r \geqslant 1$, which fixes $y$, a crucial result used in "*On the cancellation problem for the affine space* $\mathbb{A}^3$ *in characteristic* $p$, Invent. Math. **195**" by Neena Gupta to show that the Zariski Cancellation Problem does not hold for the affine 3-space.

We also provide a sufficient condition for $B_{(a,b,c)}$ to be stably rigid.

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# Notation

| | | |
|---|---|---|
| $\mathbb{Z}$ | : | Ring of Integers. |
| $\mathbb{Z}_{\geqslant n}$ | : | Set of integers greater than or equal to $n \in \mathbb{Z}$. |
| $\mathbb{Q}$ | : | Field of Rational numbers. |
| $\mathbb{R}$ | : | Field of Real numbers. |
| $\mathbb{C}$ | : | Field of Complex numbers. |
| DVR | : | Discrete Valuation Ring. |
| PID | : | Principal Ideal Domain. |
| UFD | : | Unique Factorization Domain. |

$k$ : a field.

For a $k$-algebra $B$,

EXP$(B)$ : Set of all exponential maps on $B$.

For any field $L$,

$\overline{L}$ : An algebraic closure of $L$

and for any finite number of elements $\nu_1, \ldots, \nu_n \in \overline{L}$,

$L(\nu_1, \ldots, \nu_n)$ : Smallest subfield of $\overline{L}$ containing $\nu_1, \ldots, \nu_n$ and $L$.

For a commutative ring $R$, a prime ideal $p$ of $R$, an $R$-module $M$, and $R$-algebras $A$, $B$ the following notation will be used:

| | | |
|---|---|---|
| $R^*$ | : | Group of units of $R$. |
| Cl$(R)$ | : | Ideal class group of $R$. |
| $R^{[n]}$ | : | Polynomial ring in $n$ indeterminates over $R$ ($R^{[0]} := R$). |
| Spec$(R)$ | : | The set of all prime ideals of $R$. |
| Max$(R)$ | : | The set of all maximal ideals of $R$. |
| $k(p)$ | : | Residue field $R_p/pR_p$. |
| $A_p$ | : | $S^{-1}A$ where $S = R \setminus p$; also identified with $A \otimes_R R_p$. |
| Aut$_R(M)$ | : | The group of all $R$-linear automorphisms of $M$ over $R$. |
| $1_M$ | : | The identity morphism of $M$ |
| $A \ncong_R B$ | : | There does not exists any $R$-algebra isomorphism between $A$ and $B$. |

For integral domains $R \subseteq A$,

| | | |
|---|---|---|
| Frac$(R)$ | : | The field of fractions of $R$. |
| tr.deg$_R(A)$ | : | Transcendence degree of Frac$(A)$ over that of Frac$(R)$. |

All rings considered in this thesis are commutative with unity and any ring homomorphism fixes unity. Capital letters like $X, Y, Z, T, U, V, X_1, \ldots, X_n$ etc., will designate indeterminates over respective ground rings and fields.

If $B$ is a subset of $A$, we shall use the notation $B \subseteq A$ when $B = A$ is a possibility and the notation $B \subsetneqq A$ for a proper subset when we want to emphasise that $B \neq A$.

For two rings $A, B$, we will use the notation $B \hookrightarrow A$ to mean that $B$ can be realised as a subring of $A$ via the inclusion map.

# Contents

# Chapter 1

# Introduction

## Aim

The main aim of the thesis is to address the following problems, over a field $k$ of arbitrary characteristic:

(1) To investigate under what condition a polynomial $H \in k[X_1, \ldots, X_m, Y, Z, T]$, linear in the indeterminate $Y$ and of the following form

$$H := \alpha(X_1, \ldots, X_m)Y - F(X_1, \ldots, X_m, Z, T),$$

with $\alpha \notin k$ and $F(0, \ldots, 0, Z, T) \neq 0$

(a) Defines a linear hyperplane over $k$ in $k^{[m+3]}$ i.e.,

$$\frac{k[X_1, \ldots, X_m, Y, Z, T]}{(H)} = k^{[m+2]} \ ?$$

(b) Is a coordinate in $k[X_1, \ldots, X_m, Y, Z, T]$ i.e.,

$$k[X_1, \ldots, X_m, Y, Z, T] = k[H]^{[m+2]} \ ?$$

(c) Is a coordinate along with $X_1, \ldots, X_m$ in $k[X_1, \ldots, X_m, Y, Z, T]$ i.e.,

$$k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]} \ ?$$

(2) To classify rings of the form $\frac{k[X,Y,Z]}{(X^a+Y^b+Z^c)}$ (viz. Pham-Brieskorn surfaces), for $(a, b, c) \in \mathbb{Z}^3_{\geqslant 1}$, in terms of rigidity.

The first question has been discussed in Chapter 3 of the thesis entitled "On triviality and embedding of Linear Hyperplanes" while the second question has been taken up in Chapter 4 under the heading "On rigidity of Pham-Brieskorn surfaces".

Now we are going to present a preview of the above two problems and state the main results obtained in Chapters 3 and 4 consecutively.

## 1.1 On triviality and embedding of Linear Hyperplanes: Main results

First we recall some standard terms about elements in polynomial rings.

**Definitions:**

1. A polynomial $f \in k[X_1, \ldots, X_n](= k^{[n]})$ is called a *hypersurface in* $k[X_1, \ldots, X_n]$ if $(f)$ is a proper ideal of $k[X_1, \ldots, X_n]$.

2. A hypersurface $f \in k[X_1, \ldots, X_n]$ is said to be a *hyperplane over $k$ in* $k^{[n]}$ if $\frac{k[X_1, \ldots, X_n]}{(f)}$ is a polynomial ring over $k$.

3. If $f \in k[X_1, \ldots, X_n]$ is a linear polynomial in one of the indeterminates and is also a hyperplane over $k$ in $k^{[n]}$, then $f$ is said to be a *linear hyperplane over $k$*.

4. A polynomial $f \in k[X_1, \ldots, X_n]$ is said to be a *coordinate in* $k^{[n]}$ if $k[X_1, \ldots, X_n] = k[f]^{[n-1]}$.

5. A set of polynomials $\{F_1, \ldots, F_n\}$ in the polynomial ring $k[X_1, \ldots, X_n]$ is called a *system of coordinates* if $k[X_1, \ldots, X_n] = k[F_1, \ldots, F_n]$.

6. For a ring $R$, a polynomial $f \in R[X, Y](= R^{[2]})$ is called a *residual coordinate* if for every prime ideal $\mathfrak{p}$ of $R$, $R[X, Y] \otimes_R k(\mathfrak{p}) = (R[f] \otimes_R k(\mathfrak{p}))^{[1]}$.

7. A polynomial $f \in k[X, Y](= k^{[2]})$ is said to be a *line over $k$ in* $k^{[2]}$ if $\frac{k[X,Y]}{(f)} = k^{[1]}$.

8. A line $f$ over $k$ in $k^{[2]}$ is said to be a *non-trivial line* if $f$ is not a coordinate in $k^{[2]}$.

Next we recall A. Sathaye's definition of an affine fibration over a ring.

**Definition 1.1.1.** For $n \in \mathbb{Z}_{\geqslant 1}$, a finitely generated flat $R$-algebra $B$ over a ring $R$ is said to be an $\mathbb{A}^n$-*fibration over* $R$ if $B \otimes_R k(p) = k(p)^{[n]}$ for every prime ideal $p$ of $R$.

One of the leading problems concerning affine fibration is the *Affine Fibration Problem* due to Dolgachev and Weisfeiler ([17]). It asks the following:
**Question**: Let $R$ be a regular ring and let $n \in \mathbb{Z}_{\geqslant 1}$. Is every $\mathbb{A}^n$-fibration $B$ over $R$ locally trivial i.e., $B_p = R_p^{[n]}$, for all $p \in \operatorname{Spec}(R)$?

$\mathbb{A}^1$-fibrations over $k^{[n]}$, for any $n \in \mathbb{Z}_{\geqslant 0}$ are known to be trivial ([38]). In this thesis we encounter $\mathbb{A}^2$-fibrations over a polynomial ring. A. Sathaye in 1983 ([50]) has shown that any $\mathbb{A}^2$-fibration over a PID $R$, containing $\mathbb{Q}$, is isomorphic to $R^{[2]}$. However, T. Asanuma in 1987 has given the following example of a non-trivial $\mathbb{A}^2$-fibration over a PID not containing $\mathbb{Q}$ ([3, Theorem 5.1]):

**Example 1.1.2.** Let $D$ be a DVR with maximal ideal $\pi D$ and suppose $k := D/\pi D$ is of characteristic $p > 0$. Then

$$A := \frac{D[Y, Z, T]}{(\pi^r Y + Z^{p^e} + T + T^{sp})} \quad \text{where} \quad r, e, s \in \mathbb{Z}_{\geqslant 1},\ p^e \nmid sp \text{ and } sp \nmid p^e$$

is a non-trivial $\mathbb{A}^2$-fibration over $D$.

In [39], H. Kraft listed eight fundamental problems on affine spaces, including the *Embedding Problem* (or *Epimorphism Problem*), the *Zariski Cancellation Problem* (ZCP) and the *Linearisation Problem.* The ZCP asks the following:
**Question** : For $n \in \mathbb{Z}_{\geqslant 1}$, if $B$ is an affine domain such that $B^{[1]} = k^{[n+1]}$, then does this imply that $B = k^{[n]}$? i.e., is $k^{[n]}$ cancellative?

We are now going to describe the Linearisation Problem. For any $r \in \mathbb{Z}_{\geqslant 1}$, let us consider an arbitrary $\mathbb{Z}^r$-grading on the polynomial ring $B = k^{[n]}$. Any $\mathbb{Z}^r$-grading on the polynomial ring $B$ induces an algebraic action of the multiplicative group $(k^*)^r$ on $B$ (also called the action of $(k^*)^r$ on $B$), i.e., the $\mathbb{Z}^r$-grading induces a group homomorphism from $(k^*)^r$ to $\operatorname{Aut}_k(B)$.

**Definition 1.1.3.** This action is said to be *linearisable* if there exists a system of homogeneous coordinates $X_1, \ldots, X_n$ of $B$ i.e., $B = k[X_1, \ldots, X_n]$ and $X_1, \ldots, X_n$ are homogeneous with respect to the $\mathbb{Z}^r$-grading.

The Linearisation Problem asks the following:

**Question** : For any $r, n \in \mathbb{Z}_{\geqslant 1}$, is every algebraic action of the group $(k^*)^r$ on $k^{[n]}$ linearisable?

The ring theoretic formulation of the *Embedding Problem* in codimension-one asks the following question, known as the *Epimorphism Problem.*

**Question 1**: Let $m, n$ be two positive integers and let $\phi : k[X_1, \dots, X_n] \rightarrow k[Y_1, \dots, Y_m]$, be a $k$-algebra epimorphism. Does it follow that there exists a system of coordinates $\{F_1, \dots, F_n\}$ of $k[X_1, \dots, X_n]$ such that ker $\phi = (F_1, \dots, F_{n-m})$?

In particular, when $n - m = 1$, i.e., when $\ker(\phi)$ is a principal ideal $(H)$, we have the following version of the Epimorphism Problem:

**Question 2.** For $n \in \mathbb{Z}_{\geqslant 2}$, let $H \in k[X_1, \dots, X_n]$ be such that $\frac{k[X_1,\dots,X_n]}{(H)} = k^{[n-1]}$. Does it follow that $k[X_1, \dots, X_n] = k[H]^{[n-1]}$?

When $k$ is a field of positive characteristic $p > 0$, there are counterexamples to Question 2 given by B. Segre and M. Nagata ([52], [45]). We mention such an example below:

**Example 1.1.4.** Let $k$ be a field of characteristic $p > 0$. Then

$$f := Z^{p^e} + T + T^{sp} \quad \text{where} \quad e, s \in \mathbb{Z}_{\geqslant 2},\ p \nmid s$$

is a non-trivial line in $k[Z, T]$.

We can extend the above example and get counterexample to Question 2 over a field $k$ of positive characteristic $p > 0$, for every $n \geqslant 3$. Since, by Theorem 2.4.4 (stated later), it follows that for any $m \in \mathbb{Z}_{\geqslant 1}$ and for the same $f \in k[Z, T]$ as Example 1.1.4 we have $\frac{k[X_1,\dots,X_m,Z,T]}{(f)} = k^{[m+1]}$ and $k[X_1, \dots, X_m, Z, T] \ncong_k k[f]^{[m+1]}$.

When $k$ is a field of characteristic zero and $n = 2$, Abhyankar-Moh ([2]) and Suzuki ([55]) gave an affirmative answer to Question 2 — this is popularly known as the "Epimorphism Theorem" (Theorem 2.4.1). However, when $k$ is a field of characteristic zero and $n \geqslant 3$, a complete solution to Question 2 remains open. The celebrated Epimorphism Theorem has inspired several researchers over the years, to make partial contributions to Question 2. The famous *Abhyankar-Sathaye Conjecture* asserts an affirmative answer to Question 2 when the characteristic of $k$ is zero. Note that when the characteristic of $k$ is positive, though Question 2 has a negative answer in general, the question may still be asked for specific forms of $H$. Indeed, many partial affirmative results for Question 2 have been proved even when $k$ is of arbitrary characteristic (cf. [49], [47], [56], [48], [15], [28], [24]). A general survey on the Epimorphism Problem has been made in [19].

For $n = 3$, an affirmative solution to Question 2 was obtained when $H \in k[X_1, X_2, X_3]$ is a linear plane over $k$ in $k^{[3]}$ (i.e., linear in one of the three coordinates $X_1, X_2$ or $X_3$ and $k[X_1, X_2, X_3]/(H) = k^{[2]}$) first by A. Sathaye in characteristic zero ([49]) and later by P. Russell in arbitrary characteristic ([47]). Let $B = k[X_1, X_2, X_3] = A[Y]$ where $A = k^{[2]}$. Sathaye and Russell proved that if the linear plane $H \in A[Y]$ over $k$ in $k^{[3]}$ is of the form $aY + b$ for some $a, b \in A$ with $a \neq 0$, then the coordinates $X, Z$ of $A$ can be so chosen that $a \in k[X]$ and $k[X_1, X_2, X_3] = k[X, H]^{[1]}$; i.e., linear planes over $k$ in $k^{[3]}$ were shown to be of the form $a(X)Y + b(X, Z)$ and a coordinate along with $X$.

In the last few decades, some of the central problems in affine spaces involved the following questions on certain linear polynomials: (i) whether a specified linear polynomial $H \in k[X_1, \ldots, X_n]$ is a hyperplane over $k$ in $k^{[n]}$ and (ii) whether linear hyperplanes over $k$ of a certain form are coordinates – a special case of Question 2 which seeks a possible generalisation of the above Sathaye-Russell Theorem on linear planes.

For example, a crucial step in settling the Linearisation Problem for $\mathbb{C}^*$-actions on $\mathbb{C}^3$ involved deciding whether certain linear polynomials in $\mathbb{C}^{[4]}$ defined by M. Koras and P. Russell, like the Russell cubic

$$H(X, Y, Z, T) = X^2Y + X + Z^2 + T^3,$$

were hyperplanes over $k$ in $k[X, Y, Z, T]$. The nontriviality of the Russell cubic $H$ above was first shown by L. Makar-Limanov in 1996 over a field of characteristic zero ([40]) and later A. Crachiola in 2005 extended it over arbitrary fields ([11]). The full class of Koras-Russell threefolds were shown to be non-trivial by S. Kaliman and L. Makar-Limanov in 1997 ([35]).

Another powerful example of a linear affine variety is the "Asanuma threefold" over a field $k$ of positive characteristic $p > 0$, defined below:

$$R = \frac{k[X, Y, Z, T]}{(X^rY + Z^{p^e} + T + T^{sp})} \quad \text{where} \quad r, e, s \in \mathbb{Z}_{\geqslant 1},\ p^e \nmid sp \text{ and } sp \nmid p^e,$$

a version of Example 1.1.2 that T. Asanuma had constructed as an example of a non-trivial $\mathbb{A}^2$-fibration over a DVR not containing $\mathbb{Q}$ ([3]). In 1994, Asanuma used the above ring $R$ to construct a non-linearizable $k^*$ action on $\mathbb{A}^n_k$, for all $n \geqslant 4$, over an infinite field $k$ of positive characteristic ([4]). He also showed that $R^{[1]} = k^{[4]}$.

In 2014, N. Gupta showed that $R \ncong_k k^{[3]}$ when $r \geqslant 2$, thus providing a negative solution to the ZCP for the affine space $\mathbb{A}^3_k$ in positive characteristic ([27]). Subsequently, following a question of Russell, in the same year, she studied the more general family of linear threefolds

$$R_1 = \frac{k[X, Y, Z, T]}{(X^r Y + F(X, Z, T))},$$

where $F(X, Z, T) \in k[X, Z, T]$ is such that

$$\frac{k[Z, T]}{(F(0, Z, T))} = k^{[1]} \quad \text{and} \quad k[Z, T] \ncong_k k[F(0, Z, T)]^{[1]}$$

over an arbitrary field $k$ in [28]. N. Gupta in 2014 ([29]) and later P. Ghosh and N. Gupta in 2023 ([24]), studied affine domains over an arbitrary field $k$ of the form

$$R_m = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z, T) - X_1 \cdots X_m g(X_1, \ldots, X_m, Z, T))} \tag{1.1.1}$$

for $r_i > 1$, $1 \leqslant i \leqslant m$ and established results connecting the Epimorphism Problem, the ZCP and the Affine Fibration Problem.

In view of the wide applications of linear polynomials to central problems in Affine Algebraic Geometry, we consider a general family of linear polynomials in $k[X_1, \ldots, X_m, Y, Z, T]$, over an arbitrary field $k$, of the following form:

$$H := \alpha(X_1, \ldots, X_m)Y - f(Z, T) - h(X_1, \ldots, X_m, Z, T) \tag{1.1.2}$$

such that $\alpha \notin k$ and $f \neq 0$. In most of our results, we shall also assume that

$$\text{every prime factor of } \alpha \text{ divides } h \text{ in } k[X_1, \ldots, X_m, Z, T], \tag{1.1.3}$$

a generalisation of the earlier conditions $\alpha = X_1^{r_1} \cdots X_m^{r_m}$ and $X_1 \cdots X_m \mid h$ in $R_m$. Let $A$ be an affine $k$-domain defined as follows:

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(H)}, \tag{1.1.4}$$

where $H$ is as in (1.1.2). We investigate the following questions under the hypothesis (1.1.3):

**Question 3**: (i) Under what condition $A = k^{[m+2]}$?

(ii) Does $A = k^{[m+2]} \implies k[X_1, \ldots, X_m, Y, Z, T] = k[H]^{[m+2]}$?

(iii) If so, is $H$ necessarily a coordinate in $k[X_1, \ldots, X_m, Y, Z, T]$ along with $X_1, \ldots, X_m$, i.e., $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$?

An affirmative answer to Question 3(ii) or Question 3(iii) would yield a higher-dimensional generalisation of the Sathaye-Russell Theorem on linear planes.

When $\alpha(X_1, \ldots, X_m) = X_1^{r_1} \cdots X_m^{r_m}$, $r_i > 1$ for all $i \in \{1, \ldots, m\}$ and $X_1 \cdots X_m \mid h$, P. Ghosh and N. Gupta have provided thirteen equivalent conditions for $A = k^{[m+2]}$ including the condition "$f$ is a coordinate in $k[Z, T]$" and the condition "$H$ is a coordinate in $k[X_1, \ldots, X_m, Y, Z, T]$" (cf. [24, Theorem 3.10] and [25, Theorem 4.5]). When $m = 1$, S. Kaliman, S. Vénéreau, M. Zaidenberg ([37]) and S. Maubach ([42]) have answered Question 3 over $\mathbb{C}$ and M. E. Kahoui, N. Essamaoui and M. Ouali over a field of characteristic zero ([34]).

In this direction we first study the ring $A$ (as in 1.1.4) over a field $k$ of arbitrary characteristic and for any $m \in \mathbb{Z}_{\geqslant 1}$ and prove the following result (Theorems 3.2.3 and 3.2.6).

**Theorem A.** Let $k$ be an algebraically closed field, $H$ be a polynomial as in (1.1.2) satisfying condition (1.1.3) and $A$ be as in (1.1.4). Suppose that $A^{[l]} = k^{[l+m+2]}$ for some $l \geqslant 0$ and $\frac{k[Z,T]}{(f)}$ is a regular domain. Then the following statements hold:

(i) $\frac{k[Z,T]}{(f)} = k^{[1]}$.

(ii) Moreover, if ch.$k = 0$, then $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$.

Using the above result we have proved the following theorem which establishes the Abhyankar-Sathaye Conjecture for a family of hyperplanes in $k^{[m+3]}$ over a field $k$ of characteristic zero (Theorem 3.2.7). Note that for a polynomial $g \in k[X_1, \ldots, X_m]$, $(g)_{X_i}$ denotes $\frac{\partial g}{\partial X_i}$, $1 \leqslant i \leqslant m$.

**Theorem B.** Let $k$ be a field of characteristic zero and let $A$ be an affine $k$-domain as in (1.1.4) such that $H$ is a polynomial as in (1.1.2) satisfying condition (1.1.3). Let $\alpha = \prod_{i=1}^{n} p_i^{s_i}$ be a prime factorization of $\alpha$ in $k[X_1, \ldots, X_m]$. Suppose that one of the following conditions is satisfied:

(I) $s_i = 1$ for some $i$.

(II) $s_i > 1$ for every $i$ and at least one of the following holds:

(a) $p_j^2 \mid h$ for some $j$.

(b) $(p_j, (p_j)_{X_1}, \ldots, (p_j)_{X_m})k[X_1, \ldots, X_m]$ is a proper ideal for some $j$.

(c) $(p_l, p_j)k[X_1, \ldots, X_m]$ is a proper ideal for some distinct $l, j \in \{1, \ldots, n\}$.

Let $x_1, \ldots, x_m$ be the images of $X_1, \ldots, X_m$ in $A$ respectively. Then the following statements are equivalent:

(i) $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$.

(ii) $k[X_1, \ldots, X_m, Y, Z, T] = k[H]^{[m+2]}$.

(iii) $A = k[x_1, \ldots, x_m]^{[2]}$.

(iv) $A = k^{[m+2]}$.

(v) $k[Z, T] = k[f(Z, T)]^{[1]}$.

(vi) $A$ is an $\mathbb{A}^2$-fibration over $k[x_1, \ldots, x_m]$.

(vii) $A^{[l]} = k^{[m+l+2]}$ for some $l \geqslant 0$.

In the above theorem the conditions mentioned as (I) and (II) ensure the regularity of the ring $\frac{k[Z,T]}{(f)}$ when $A$ is a regular ring. Note that over a field $k$ of characteristic zero, the polynomials of the type

$$\alpha(X_1, \ldots, X_m)Y - f(Z, T) \in k[X_1, \ldots, X_m, Y, Z, T] \quad \text{with} \quad \alpha \notin k \quad \text{and} \quad f \neq 0$$

are contained in the family of hypersurfaces mentioned in Theorem B.

Next, we investigate Question 3 for hypersurfaces $H$ over a field $k$ of arbitrary characteristic, where $H$ is as in (1.1.2) and $\alpha$ (in $H$) is of the following form:

$$\begin{aligned} \alpha \;=\; & X_1^{r_1}(X_1\beta_1(X_1, \ldots, X_m) + \cdots + X_{m-1}^{r_{m-1}}(X_{m-1}\beta_{m-1}(X_{m-1}, X_m) \\ & + X_m^{r_m}(X_m\beta_m(X_m) + \alpha_{m+1})) \ldots ) \end{aligned} \tag{1.1.5}$$

in a system of coordinates $\{X_1, \ldots, X_m\}$ of $k^{[m]}$, for some $\beta_i \in k[X_i, \ldots, X_m]$, $1 \leqslant i \leqslant m$, $\alpha_{m+1} \in k^*$ and some $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 1}$, i.e., $X_1^{r_1} \mid \alpha(X_1, \ldots, X_m)$ and for $2 \leqslant i \leqslant m$, $X_i \mid \alpha_{i-1}(0, X_i, \ldots, X_m)$, where

$$\begin{aligned} \alpha_1 &= \alpha(X_1, \ldots, X_m)/X_1^{r_1} \\ \alpha_i &= \alpha_{i-1}(0, X_i, \ldots, X_m)/X_i^{r_i}, \; 2 \leqslant i \leqslant m. \end{aligned} \tag{1.1.6}$$

In Chapter 3, we make a precise definition (Definition 3.4.1) and call the polynomials of the form (1.1.5) "**r**-divisible polynomials". In this connection, we first establish the following result (follows from Theorems 3.4.11 and 3.4.12):

**Theorem C.** Let $k$ be an infinite field. Let

$$A = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - F(X_1, \ldots, X_m, Z, T))}$$

be a domain such that

(a) $f(Z, T) := F(0, \ldots, 0, Z, T) \neq 0$.

(b) For $\mathbf{r} = (r_1, \ldots, r_m)$, $\alpha$ is **r**-divisible with respect to $\{X_1, \ldots, X_m\}$ in $k^{[m]}$ i.e., $\alpha$ is as in (1.1.5) and $\alpha_i$, $1 \leqslant i \leqslant m$ are as in (1.1.6).

(c) $r_i > 1$, for each $i \in \{1, \ldots, m\}$.

(d) $\gcd(\alpha_i(0, X_{i+1}, \ldots, X_m), F(0, \ldots, 0, X_{i+1}, \ldots, X_m, Z, T)) = 1$, for all $i \in \{1, \ldots, m\}$.

(e) Either $\mathrm{ML}(A) = k$ or $\mathrm{DK}(A) = A$ (for the definition of ML and DK please refer to Section 2.2).

Then there exist a system of coordinates $\{Z_1, T_1\}$ of $k[Z, T]$ and $a_0, a_1 \in k^{[1]}$, such that $f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1$. Furthermore, if $f$ is a line in $k[Z, T]$ i.e., $k[Z, T]/(f) = k^{[1]}$, then $k[Z, T] = k[f]^{[1]}$.

Observe that Theorems 5.22 and 5.23 of [26] follows from the above theorem. Combined the statement is as follows:

**Corollary.** *Let $k$ be an infinite field. Let $H$ be as in* (1.1.2) *such that $X_1 \mid h$, $\alpha$ be as in* (1.1.5) *with respect to $\{X_1, \ldots, X_m\}$ in $k^{[m]}$, where $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 2}$ and $A$ be as in* (1.1.4). *Suppose* $\mathrm{ML}(A) = k$ *or* $\mathrm{DK}(A) = A$. *Then there exist a system of coordinates $\{Z_1, T_1\}$ of $k[Z, T]$ and $a_0, a_1 \in k^{[1]}$, such that $f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1$. Furthermore, if $k[Z, T]/(f) = k^{[1]}$, then $k[Z, T] = k[f]^{[1]}$.*

Theorem C will enable one to readily recognise that a large family of certain affine varieties (cf. Corollary 3.4.13 and 3.4.14) are not affine spaces; for instance, the varieties defined by the polynomial $H$ below are not affine spaces (cf. Examples 3.4.3 and 3.4.4 for **r**-divisibility of coefficients of $Y$)

$$H = X^2(X+1)^2Y-(Z^2+T^3)-Xh_1(X,Z,T) \in k[X,Y,Z,T], \text{ for any } h_1 \in k^{[3]}$$

or

$$H = X_1X_2^2(X_1+X_2^2)Y-(Z^2+T^3)-h_2(X_1,X_2) \in k[X_1,X_2,Y,Z,T],\ h_2 \in k^{[2]}$$

or

$$H = X_1X_2^2(X_1+X_2^2)Y-(Z^2+T^3)-h_3(X_1,X_2,Z,T),\ h_3 \in k^{[4]},\ h_3(0,0,Z,T) = 0.$$

Using Theorem C, we have proved the following theorem (Theorem 3.5.3) which addresses Question 3.

**Theorem D.** Let $H$ be a polynomial as in (1.1.2) satisfying (1.1.3) and $A$ be an affine $k$-domain as in (1.1.4). For $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 2}$, let $\alpha$ be a **r**-divisible polynomial in the system of coordinates $\{X_1 - \lambda_1, \ldots, X_m - \lambda_m\}$, for some $\lambda_i \in \overline{k}$, $1 \leqslant i \leqslant m$ such that each $\lambda_i$ is separable over $k$. Let $x_1, \ldots, x_m$ be the images of $X_1, \ldots, X_m$ in $A$ respectively. Then the following statements are equivalent:

(i) $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$.

(ii) $k[X_1, \ldots, X_m, Y, Z, T] = k[H]^{[m+2]}$.

(iii) $A = k[x_1, \ldots, x_m]^{[2]}$.

(iv) $A = k^{[m+2]}$.

(v) $k[Z,T] = k[f(Z,T)]^{[1]}$.

In fact, we have obtained equivalence of nine more statements involving stable isomorphisms, affine fibrations and two invariants — the Makar-Limanov invariant and the Derksen invariant. Observe that the family of polynomials given by

$$(X_1^{r_1+1} + X_1^{r_1}X_2^{r_2+1} + \cdots + X_1^{r_1} \cdots X_{m-1}^{r_{m-1}}X_m^{r_m+1})Y - f(Z,T),$$

for $f \neq 0$, $r_i \geqslant 2$, $1 \leqslant i \leqslant m$ and the family of polynomials given by

$$a_1(X_1)\cdots a_m(X_m)Y - f(Z,T) - h(X_1,\ldots,X_m,Z,T),\ f \neq 0,$$

where every prime divisor of $a_1(X_1)\cdots a_m(X_m)$ in $k[X_1,\ldots,X_m]$ divides $h$, and every $a_i(X_i)$ has a separable multiple root $\lambda_i$ over $k$, are included in the family of hypersurfaces mentioned in Theorem D.

Note that Question 3(i) is addressed by the equivalence (iv) $\Leftrightarrow$ (v); Question 3(ii) by the equivalence of (iv) $\Leftrightarrow$ (ii) and Question 3(iii) by (i) $\Leftrightarrow$ (ii) in Theorem B (for ch.$k = 0$) and Theorem D. In particular, the Abhyankar–Sathaye Conjecture holds affirmatively for the hypersurfaces $H$ satisfying the assumptions in Theorems B and D.

Theorem D also yields an infinite family of non-isomorphic varieties which are counterexamples to the Zariski Cancellation Problem in positive characteristic (Corollary 3.5.4 and Remark 3.5.5).

We also prove partial generalisations of Theorems B and D over any Noetherian seminormal domain or a Noetherian domain containing $\mathbb{Q}$ (Theorems 3.6.1 and 3.6.2).

J. K. Verma, looking at $R_m$ (as in (1.1.1)), once raised the question whether results analogous to Ghosh-Gupta ([24]) can be obtained when the monomial "$X_1^{r_1}\cdots X_m^{r_m}$, $r_i > 1$, $1 \leqslant i \leqslant m$", the coefficient of $Y$, is replaced by a binomial. A particular case of Verma's question has also been answered by Theorem D. The next result displays it.

**Corollary.** *Let $H$ be a polynomial as in* (1.1.2) *satisfying* (1.1.3) *and $A$ be an affine $k$-domain as in* (1.1.4)*, such that $\alpha$ is a binomial. By change of coordinate, if necessary*

$$\alpha = X_1^{r_1}\cdots X_{i-1}^{r_{i-1}}(\lambda X_i^{r_i}\cdots X_m^{r_m} + \mu X_i^{s_i}\cdots X_m^{s_m}),$$

*for some $\lambda, \mu \in k^*$, $i \in \{1,\ldots,m\}$ and there exists $j \in \{i,\ldots,m\}$ such that $r_l < s_l$, for all $i \leqslant l \leqslant j$ and $r_l > s_l$, for all $j < l \leqslant m$. Suppose $(r_1,\ldots,r_{i-1}) \in \mathbb{Z}_{\geqslant 2}^{i-1}$. If either $r_l \geqslant 2$, for all $i \leqslant l \leqslant j$ or $s_l \geqslant 2$, for all $j < l \leqslant m$ then all the conclusions of Theorem D holds.*

In the above corollary $\alpha$ is $\mathbf{r_1} :=(r_1,\ldots,r_m)$ divisible in the system of coordinates $\{X_1,\ldots,X_m\}$ and $\mathbf{r_2} := (r_1,\ldots,r_{i-1},s_{j+1},\ldots,s_m,s_i,\ldots,s_j)$ divisible in the system of coordinates $\{X_1,\ldots,X_{i-1},X_{j+1},\ldots,X_m,X_i,\ldots,X_j\}$ in $k^{[m]}$. Thus either $\mathbf{r_1} \in \mathbb{Z}_{\geqslant 2}^m$ or $\mathbf{r_2} \in \mathbb{Z}_{\geqslant 2}^m$ helps satisfy the hypothesis of Theorem D.

We now give a layout of Chapter 3. In Section 3.1 we study a few properties, like factoriality, of the affine domain $A$. We will prove Theorems A and B in Section 3.2; Theorem C in Section 3.4 and Theorem D in Section 3.5. Generalisation of Theorems B and D will be discussed in Section 3.6.

## 1.2 On rigidity of Pham-Brieskorn surfaces: Main results

Now we recall some definitions involving existence of non-trivial exponential maps on a $k$-algebra (for definition of non-trivial exponential map, see Section 2.2).

**Definitions:**

1. A $k$-algebra $A$ with no non-trivial exponential map is called *rigid*.
2. A $k$-algebra $A$ with at least one non-trivial exponential map is called a *non-rigid ring*.
3. A $k$-algebra $A$ is said to be *stably rigid* if for every $n \in \mathbb{Z}_{\geqslant 0}$ and every non-trivial exponential map $\phi$ on $A[X_1, ....., X_n](= A^{[n]})$, $A \subseteq A[X_1, ...., X_n]^{\phi}$. Note that stable rigidity implies rigidity.

For $n \in \mathbb{Z}_{\geqslant 3}$ and $\underline{a} := (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1}$, let

$$B_{\underline{a}} = B_{(a_1,a_2,\ldots,a_n)} = \frac{k[X_1, X_2, \ldots, X_n]}{(X_1^{a_1} + X_2^{a_2} + \cdots + X_n^{a_n})}.$$

When $k$ is a field of characteristic zero, for $n \in \mathbb{Z}_{\geqslant 3}$ and an arbitrary $n$-tuple $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1}$, the integral domains of the form $B_{(a_1,a_2,\ldots,a_n)}$ are known as Pham-Brieskorn rings. We shall call the rings of the form $B_{(a_1,a_2,\ldots,a_n)}$ *Pham-Brieskorn rings (or domains)* even over a field $k$ of arbitrary characteristic $p$ ($\geqslant 0$).

In the 1960s, the Pham-Brieskorn rings over $\mathbb{C}$ were first introduced by Egbert Brieskorn and studied by Frédéric Pham as an interesting example of a manifold with one singular point at $(0, \ldots, 0)$. The topology of these manifolds for $n \geqslant 3$ has also been studied by Milnor and others. Detailed surveys are given in [43] and [51].

The rigidity of Pham-Brieskorn rings has been studied by several experts including M. Zaidenberg, S. Kaliman, G. Freudenburg and D. Daigle when

$k$ is an algebraically closed field of characteristic zero ([36], [23], [22, Section 9.2], [14], [10]) to name a few.

We already know that in [27], N. Gupta has shown that, over a field $k$ of positive characteristic $p > 0$, the Asanuma threefolds of the form

$$\frac{k[X,Y,Z,T]}{(X^rY + Z^{p^e} + T + T^{sp})} \quad \text{where} \quad r \in \mathbb{Z}_{\geqslant 2}, e, s \in \mathbb{Z}_{\geqslant 1} \quad \text{with} \quad p^e \nmid sp,\ sp \nmid p^e,$$

are counterexamples to the Zariski Cancellation Problem. A crucial step in the proof was to show that on any ring of the form

$$B := \frac{k[X,Y,Z,T]}{(X^rY + T^{sp^m} + Z^{p^e})} \quad \text{where} \quad s, m, e \in \mathbb{Z}_{\geqslant 1}, r \in \mathbb{Z}_{\geqslant 2} \text{ with } sp^m \nmid p^e,\ p^e \nmid sp^m,$$

there does not exist any non-trivial exponential map fixing $y$ (the image of $Y$ in $B$) and hence the $k(y)$-algebra

$$B \otimes_{k[y]} k(y) = \frac{k(y)[X,Z,T]}{(yX^r + T^{sp^m} + Z^{p^e})},$$

where $s, m, e \in \mathbb{Z}_{\geqslant 1}, r \in \mathbb{Z}_{\geqslant 2}$ with $sp^m \nmid p^e$, $p^e \nmid sp^m$; is rigid.

In view of the importance of the rigidity of rings of the form $k[X,Y,Z] \,/\, (X^a + Y^b + Z^c)$ for $(a,b,c) \in \mathbb{Z}^3_{\geqslant 1}$, over a field of arbitrary characteristic $k$, we undertake classification of such surfaces, in terms of rigidity.

With the help of Eisenstein's criterion for irreducibility, it follows that, over a field $k$ of characteristic $p \geqslant 0$, for $n \in \mathbb{Z}_{\geqslant 3}$ and $a_1, a_2, \ldots, a_n \in \mathbb{Z}_{\geqslant 1}$, the polynomial $X_1^{a_1} + X_2^{a_2} + \cdots + X_n^{a_n} \in k[X_1, X_2, \ldots, X_n]\ (= k^{[n]})$ is irreducible if and only if $p \nmid gcd(a_1, a_2, \ldots, a_n)$. For a field $k$ of characteristic $p \geqslant 0$ and for any $n \in \mathbb{Z}_{\geqslant 3}$, we consider the sets

$F_n := \{(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1} \mid p \nmid gcd(a_1, a_2, \ldots, a_n)\}$,
$T_n := \{(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1} \mid a_i = 1 \text{ for some } i, \text{ or } \exists\, i, j \in \{1, 2, \ldots, n\}, i \neq j \text{ and } a_i = a_j = 2\}$
and
$R_n := \{(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1} \mid a_i = 1 \text{ for some } i, \text{ or } \exists\, i, j \in \{1, 2, \ldots, n\}, i \neq j \text{ and } a_i = p^r, a_j = sp^e \text{ for some } r, s, e \in \mathbb{Z}_{\geqslant 1} \text{ with } r \leqslant e\}$.

By [22, Section 9.2] and [23, Theorem 7.1], it is known that over an algebraically closed field of characteristic zero, $B_{(a,b,c)}$ is rigid if and only if $(a,b,c) \in \mathbb{Z}^3_{\geqslant 1} \setminus T_3$. In this thesis we will establish the following results on stable rigidity (Theorem 4.1.4), non-rigidity (Section 4.2) and rigidity (Theorem 4.2.12) of Pham-Brieskorn rings.

**Theorem E.** Over any field $k$ of characteristic $p \geqslant 0$, the Pham-Brieskorn domain $B_{(a,s_2p^r,s_3p^e)}$ is stably rigid when $a, s_2, s_3 \in \mathbb{Z}_{\geqslant 1}$, $r, e \in \mathbb{Z}_{\geqslant 0}$ and $p \nmid as_2s_3$ with $\frac{1}{a} + \frac{1}{s_2} + \frac{1}{s_3} \leqslant 1$.

**Theorem F.** For any field $k$ of characteristic $p \geqslant 0$ and for $n \in \mathbb{Z}_{\geqslant 3}$,

(i) $B_{(a_1,a_2,\ldots,a_n)}$ is non-rigid if $(a_1, a_2, \ldots, a_n) \in R_n$.

(ii) $B_{(a_1,a_2,\ldots,a_n)}$ is non-rigid when $(a_1, a_2, \ldots, a_n) \in T_n$ and $k$ contains a square root of $-1$.

(iii) $B_{(a_1,a_2,\ldots,a_n)}$ is non-rigid when $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1} \setminus F_n$.

(iv) $B_{(a,b,c)}$ is rigid when $(a, b, c) \in F_3 \setminus (R_3 \cup T_3 \cup S_3)$, where $S_3 := \{(2, 2m, 2p^e) \mid m \in \mathbb{Z}_{\geqslant 2},\ e \in \mathbb{Z}_{\geqslant 1} \text{ and } p \nmid 2m\}$.

In Section 4.3, we prove rigidity of certain surfaces (Theorem 4.3.4).

**Theorem G.** Let $a, b, c \in \mathbb{Z}_{\geqslant 2}$ be such that $gcd(a, b, c) = 1$. Then for any $F(Y) \in k[Y]$, $k[X, Y, Z]/(X^a + Y^bZ^c + F(Y))$ is a rigid domain.

In Section 4.4, we present a few applications of our results.

The results obtained in Chapter 3 are based on the joint work with Parnashree Ghosh and my supervisor Neena Gupta ([26]) and the results in Chapter 4 are obtained in the joint work with my supervisor ([32]).

# Chapter 2

# Preliminaries

In this chapter we recall some necessary concepts (Grothendieck group, Whitehead group, exponential maps, admissible $\mathbb{Z}$-filtration etc.) in the first three sections and state some well-known results in Section 2.4. We begin by recalling some relevant results from $K$-theory.

## 2.1 Some preliminary results on $K$-theory

In this section we consider a Noetherian ring $R$ and state some $K$-theoretic aspects of $R$ (cf. [6], [9]). Let $\mathscr{M}(R)$ denote the category of all finitely generated $R$-modules and let $\mathscr{P}(R)$ denote the category of all finitely generated projective $R$-modules. Let $G_0(R)$ and $G_1(R)$, respectively denote the *Grothendieck group* and the *Whitehead group* of the category $\mathscr{M}(R)$. Let $K_0(R)$ and $K_1(R)$, respectively denote the *Grothendieck group* and the *Whitehead group* of the category $\mathscr{P}(R)$.

We now recall the definition of $G_0(R)$ and $G_1(R)$.

For $M \in \mathscr{M}(R)$, let $[M]$ denote the isomorphism class of $M$. Now $G_0(R)$ is the Abelian group presented by generators $[M]$, for $M \in \mathscr{M}(R)$, subject to the relations $[M] = [M'] + [M'']$ whenever $M'$, $M$, $M''$ form an exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{M}(R)$.

Let $\mathscr{M}^{\mathbb{Z}}(R)$ denote the category whose objects are pairs $(M, \sigma)$, where $M \in \mathscr{M}(R)$ and $\sigma$ is an $R$-linear automorphism of $M$ i.e., $\sigma \in \mathrm{Aut}_R(M)$. A morphism $\phi : (M, \sigma) \to (M', \sigma')$ in $\mathscr{M}^{\mathbb{Z}}(R)$ is defined to be an $R$-linear map $\phi : M \to M'$ such that $\sigma'\phi = \phi\sigma$. An isomorphism $\phi : (M, \sigma) \to (M', \sigma')$ in $\mathscr{M}^{\mathbb{Z}}(R)$ is a morphism in $\mathscr{M}^{\mathbb{Z}}(R)$ such that $\phi : M \to M'$ is an isomorphism in $\mathscr{M}(R)$. For $(M,\sigma) \in \mathscr{M}^{\mathbb{Z}}(R)$, let $[M,\sigma]$ denote the isomorphism class of $(M,\sigma)$. Now $G_1(R)$ is the Abelian group presented by generators $[M,\sigma]$, where $(M,\sigma) \in \mathscr{M}^{\mathbb{Z}}(R)$, subject to the relations $[M,\sigma] = [M',\sigma'] + [M'',\sigma'']$ whenever $(M',\sigma'),(M,\sigma),(M'',\sigma'')$ form an exact sequence

$$0 \to (M',\sigma') \to (M,\sigma) \to (M'',\sigma'') \to 0$$

in $\mathscr{M}^{\mathbb{Z}}(R)$ and $[M,\sigma\eta] = [M,\sigma] + [M,\eta]$ whenever $M \in \mathscr{M}(R)$ and $\sigma,\eta \in \mathrm{Aut}_R(M)$.

Recall that for any $M \in \mathscr{M}(R)$ and $\sigma \in \mathrm{Aut}_R(M)$, $\sigma$ is said to be an *unipotent automorphism* if $\sigma - 1_M$ is nilpotent.

The next result from [6, APPENDIX (4.10)] describe when the images of two elements from $\mathscr{M}(R)$ are equal in $G_0(R)$.

**Lemma 2.1.1.** *For $M,N \in \mathscr{M}(R)$, the following statements are equivalent:*

(i) $[M] = [N]$ *in* $G_0(R)$.

(ii) *There exist exact sequences*

$$0 \to W' \to T_1 \to W'' \to 0 \quad \textit{and} \quad 0 \to W' \to T_2 \to W'' \to 0$$

*in $\mathscr{M}(R)$ such that $M \oplus T_1 \cong N \oplus T_2$.*

Adapting the proof of Lemma 2.1.1 and with the help of Whitehead's Lemma one can sketch a similar condition describing when two elements from $\mathscr{M}^{\mathbb{Z}}(R)$ are equal in $G_1(R)$.

**Lemma 2.1.2.** *For $(M,\sigma_M),(N,\sigma_N) \in \mathscr{M}^{\mathbb{Z}}(R)$, the following statements are equivalent:*

(i) $[M,\sigma_M] = [N,\sigma_N]$ *in* $G_1(R)$.

(ii) *There exist exact sequences*

$$0 \to (W',\sigma_{W'}) \to (T_1,\sigma_{T_1}) \to (W'',\sigma_{W''}) \to 0$$

*and*

$$0 \to (W',\epsilon\sigma_{W'}) \to (T_2,\sigma_{T_2}) \to (W'',\sigma_{W''}) \to 0$$

*in $\mathscr{M}^{\mathbb{Z}}(R)$, where $\epsilon$ is a composition of unipotent automorphisms of $W'$, such that $(M \oplus T_1, \sigma_M \oplus \sigma_{T_1}) \cong (N \oplus T_2, \sigma_N \oplus \sigma_{T_2})$.*

*Proof.* **(i) ⇒ (ii)** : Since $[M,\sigma_M]=[N,\sigma_N]$ in $G_1(R)$, there exist $m\in\mathbb{Z}_{\geqslant 1}$ and $W_i\in\mathscr{M}(R)$, $\sigma_i,\eta_i\in \mathrm{Aut}_R(W_i)$, $1\leqslant i\leqslant m$ such that

$$[M,\sigma_M]-[N,\sigma_N]=\sum_{i=1}^{m}([W_i,\sigma_i\eta_i]-[W_i,\sigma_i]-[W_i,\eta_i]) \text{ in } G_0(\mathscr{M}^{\mathbb{Z}}(R)).$$

Therefore, by Lemma 2.1.1, there exist exact sequences

$$0\to(U',\sigma_{U'})\to(X,\sigma_X)\to(U'',\sigma_{U''})\to 0$$

and

$$0\to(U',\sigma_{U'})\to(Y,\sigma_Y)\to(U'',\sigma_{U''})\to 0$$

in $\mathscr{M}^{\mathbb{Z}}(R)$, such that $(M,\sigma_M)\oplus(\sum_{i=1}^{m}((W_i,\sigma_i)\oplus(W_i,\eta_i)))\oplus(X,\sigma_X)\cong(N,\sigma_N)\oplus(\sum_{i=1}^{m}(W_i,\sigma_i\eta_i))\oplus(Y,\sigma_Y)$. Now Whitehead's lemma states that, for any $P\in\mathscr{M}(R)$ and $\sigma\in\mathrm{Aut}_R(P)$, there exist $\epsilon'\in\mathrm{Aut}_R(P\oplus P)$, a composition of unipotent automorphisms, such that $(P\oplus P,\sigma\oplus\sigma^{-1})=(P\oplus P,\epsilon')$. Hence it follows that

$$\sum_{i=1}^{m}(W_i\oplus W_i,1_{W_i}\oplus\sigma_i\eta_i)=(\sum_{i=1}^{m}(W_i\oplus W_i),\widetilde{\epsilon}\sum_{i=1}^{m}(\sigma_i\oplus\eta_i)),$$

where $\widetilde{\epsilon}$ is a composition of unipotent automorphisms of $\oplus_{i=1}^{2m}W_i$. Thus, by taking

$$(W',\sigma_{W'})=(U',\sigma_{U'})\oplus((\oplus_{i=1}^{m}W_i)\oplus(\oplus_{i=1}^{m}W_i),(\oplus_{i=1}^{m}\sigma_i)\oplus(\oplus_{i=1}^{m}\eta_i)),$$

$$(T_1,\sigma_{T_1})=(X,\sigma_X)\oplus((\oplus_{i=1}^{m}W_i)\oplus(\oplus_{i=1}^{m}W_i),(\oplus_{i=1}^{m}\sigma_i)\oplus(\oplus_{i=1}^{m}\eta_i)),$$

$$(T_2,\sigma_{T_2})=(Y,\sigma_Y)\oplus((\oplus_{i=1}^{m}W_i)\oplus(\oplus_{i=1}^{m}W_i),\widetilde{\epsilon}((\oplus_{i=1}^{m}\sigma_i)\oplus(\oplus_{i=1}^{m}\eta_i)))$$

and

$$(W'',\sigma_{W''})=(U'',\sigma_{U''})$$

we get our result.

**(ii) ⇒ (i)** : It follows from the fact that for any $M\in\mathscr{M}(R)$ and an unipotent automorphism $\sigma\in\mathrm{Aut}_R(M)$, $[M,\sigma]=0$ in $G_1(R)$ (cf. [6, Corollary (4.9)]). □

The definitions of $G_i(R)$ and $K_i(R)$, for $i\in\mathbb{Z}_{\geqslant 2}$ can be found in ([53], Chapters 4 and 5). Let $C$ be a Noetherian ring and $\phi:R\to C$ be a flat ring homomorphism. Then for any $i \geqslant 0$, $G_i(\phi) : G_i(R) \to G_i(C)$ is induced by the functor $\otimes_R C : \mathscr{M}(R) \to \mathscr{M}(C)$ which is defined by $M \to M \otimes_R C$ (cf. [53, 5.8]).

Next we recall a few theorems regarding $G_i$. The following two results can be found in [53], Proposition 5.16 and Theorem 5.2 respectively.

**Theorem 2.1.3.** *Let $C$ be a Noetherian ring and $\phi : R \to C$ be a flat ring homomorphism. Let $t$ be a regular element of $R$ and $u := \phi(t)$. Then the natural maps $\bar{\phi} : \frac{R}{tR} \to \frac{C}{uC}$ and $t^{-1}\phi : R[t^{-1}] \to C[u^{-1}]$ induce the following commutative diagram of long exact sequences of groups for all $i \in \mathbb{Z}_{\geqslant 1}$ :*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & G_i(\frac{R}{tR}) & \longrightarrow & G_i(R) & \longrightarrow & G_i(R[t^{-1}]) & \longrightarrow & G_{i-1}(\frac{R}{tR}) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle G_i(\bar{\phi})} & & \downarrow{\scriptstyle G_i(\phi)} & & \downarrow{\scriptstyle G_i(t^{-1}\phi)} & & \downarrow{\scriptstyle G_{i-1}(\bar{\phi})} & \\
\cdots \longrightarrow & G_i(\frac{C}{uC}) & \longrightarrow & G_i(C) & \longrightarrow & G_i(C[u^{-1}]) & \longrightarrow & G_{i-1}(\frac{C}{uC}) & \longrightarrow \cdots .
\end{array}$$

**Theorem 2.1.4.** *For an indeterminate $T$ over $R$, the map $G_i(R) \to G_i(R[T])$, induced by the inclusion $R \hookrightarrow R[T]$, is an isomorphism, for all $i \in \mathbb{Z}_{\geqslant 0}$. Hence $G_i(k[X_1, \dots, X_m]) = G_i(k)$, for all $i \in \mathbb{Z}_{\geqslant 0}$ and for all $m \in \mathbb{Z}_{\geqslant 1}$.*

Now we recall some well-known facts about $G_0(R)$ and $G_1(R)$.

**Remark 2.1.5.** (i) For a regular ring $R$, $G_i(R) = K_i(R)$, for every $i \in \mathbb{Z}_{\geqslant 0}$. In particular,

(a) $G_0(k[X_1, \dots, X_m]) = G_0(k) = K_0(k) = \mathbb{Z}$

(b) $G_1(k[X_1, \dots, X_m]) = G_1(k) = K_1(k) = k^*$

(ii) There is a canonical group homomorphism $\theta : R^* \hookrightarrow G_1(R)$ defined by $\theta(u) = [R, \sigma_u]$, for $u \in R^*$, where $\sigma_u : R \to R$ is the $R$-linear automorphism defined by $\sigma_u(r) = ru$ for all $r \in R$.

We now recall the concept of an exponential map on a $k$-algebra and some related concepts.

## 2.2 Exponential maps and related invariants

**Definition 2.2.1.** Let $A$ be a $k$-algebra and let $\phi_U : A \to A[U]$ be a $k$-algebra homomorphism. We say that $\phi = \phi_U$ is an *exponential map on $A$*, if $\phi$ satisfies the following two properties:

(i) $\varepsilon_0\phi_U$ is identity on $A$, where $\varepsilon_0 : A[U] \to A$ is the evaluation map at $U = 0$.

(ii) $\phi_V\phi_U = \phi_{V+U}$; where $\phi_V : A \to A[V]$ is extended to a homomorphism $\phi_V : A[U] \to A[U, V]$ by defining $\phi_V(U) = U$.

The above definition can also be rewritten in terms of the following five properties:

For each $a \in A$, we can write $\phi(a) = \sum_{i=0}^{\infty} \phi^{(i)}(a)U^i$ in $A[U]$.

(i) The sequence of maps $\{\phi^{(i)}\}_{i=0}^{\infty}$ is a sequence of linear maps on $A$.

(ii) For each $a \in A$, the sequence $\{\phi^{(i)}(a)\}_{i=0}^{\infty}$ has only finitely many non-zero terms.

(iii) $\phi^{(0)}$ is the identity map on $A$.

(iv) (Leibniz Rule) For all $n \in \mathbb{Z}_{\geqslant 0}$ and for $a, b \in A$, $\phi^{(n)}(ab) = \sum\limits_{i+j=n} \phi^{(i)}(a)\phi^{(j)}(b)$.

(v) For all $i, j \in \mathbb{Z}_{\geqslant 0}$, $\phi^{(i)}\phi^{(j)} = \binom{i+j}{i}\phi^{(i+j)}$.

Note that when characteristic of $k$ is zero, then for any $n \in \mathbb{Z}_{\geqslant 1}$, $\phi^{(n)} = \frac{1}{n!}\phi^{(1)}$. Thus when $k$ is a field of characteristic zero, exponential maps on a $k$-algebra $A$ are equivalent to locally nilpotent derivations (LNDs) on $A$. We now write down the definition of an LND on a $k$-algebra $A$.

**Definition 2.2.2.** A $k$-linear map $D : A \to A$ is said to be a *locally nilpotent derivation on* $A$ if it satisfies the following two properties:

(i) (Leibniz Rule) For $a, b \in A$, $D(ab) = aD(b) + D(a)b$.

(ii) For each $a \in A$, there exists $n \in \mathbb{Z}_{\geqslant 0}$ such that $D^n(a) = 0$.

When $k$ is an algebraically closed field of characteristic zero and $A$ is an affine $k$-domain, then there is a bijective correspondence between the set of all LNDs on $A$ and the set of all algebraic $\mathbb{G}_a$-actions ($(k,+)$ actions) on Max$(A)$ (cf. [22]). Therefore, in the same setting, there is a bijective correspondence between EXP$(A)$ and the set of all algebraic $\mathbb{G}_a$-actions on Max$(A)$.

Given an exponential map $\phi = \phi_U$ on a $k$-algebra $A$, we define the $\phi$-degree of an element $a \in A \setminus \{0\}$ by $\deg_\phi(a) = \deg_U(\phi(a))$ and $\deg_\phi(0) := -\infty$.

Moreover, if $A$ is an integral domain then the function $\deg_\phi$ is a degree function (Definition 2.3.2) on $A$.

The ring of invariants of $\phi = \phi_U$ is a subring of $A$ given by

$$\begin{aligned} A^\phi &= \{a \in A \mid \phi(a) = a\} = \{a \in A \mid \deg_\phi(a) \leqslant 0\} \\ &= \{a \in A \mid \deg_U(\phi(a)) = 0\} \cup \{0\}. \end{aligned}$$

An exponential map $\phi$ is said to be *non-trivial* if $A^\phi \neq A$.

The *Derksen invariant* of $A$ is a subring of $A$ defined by

$$\mathrm{DK}(A) = k[A^\phi \mid \phi \text{ is a non-trivial exponential map on } A]$$

and the *Makar-Limanov invariant* of $A$ is a subring of $A$ defined by

$$\mathrm{ML}(A) = \bigcap_{\phi \in \mathrm{EXP}(A)} A^\phi.$$

Let $\mathrm{ML}_0(A) = A$ and for any $n \in \mathbb{Z}_{\geqslant 1}$, $\mathrm{ML}_n(A) := \mathrm{ML}(\mathrm{ML}_{n-1} A)$. Then the *rigid core* of $A$, denoted by $\mathcal{R}(A)$, is defined by

$$\mathcal{R}(A) = \bigcap_{n \geqslant 1} \mathrm{ML}_n(A).$$

We list below some useful properties of exponential maps (cf. [44], [11], [27] and [31]).

**Lemma 2.2.3.** *Let $A$ be an affine $k$-domain and $\phi$ be a non-trivial exponential map on $A$. Then the following holds:*

(i) *$A^\phi$ is factorially closed in $A$ i.e., for any $a, b \in A \setminus \{0\}$, if $ab \in A^\phi$ then $a, b \in A^\phi$. Hence, $A^\phi$ is also algebraically closed in $A$.*

(ii) $\mathrm{tr.deg}_k(A^\phi) = \mathrm{tr.deg}_k(A) - 1$.

(iii) $\deg_\phi(\phi^{(i)}(a)) \leqslant \deg_\phi(a) - i$, *for all $a \in A$ and $i \in \mathbb{Z}_{\geqslant 0}$. If $a \neq 0$ then $\phi^{(\deg_\phi a)}(a) \in A^\phi$.*

(iv) *Suppose $x \in A$ has the minimal positive $\phi$-degree $n$ and $c := \phi^{(n)}(x)$. Then $c \in A^\phi$ and $A[c^{-1}] = A^\phi[c^{-1}][x] = A^\phi[c^{-1}]^{[1]}$.*

(v) *If $\mathrm{tr.deg}_k(A) = 1$, then $A = \widetilde{k}^{[1]}$, where $\widetilde{k}$ is the algebraic closure of $k$ in $A$ and $A^\phi = \widetilde{k}$.*

(vi) *Let $S$ be a multiplicative closed subset of $A^{\phi} \setminus \{0\}$. Then $\phi$ extends to a non-trivial exponential map $S^{-1}\phi$ on $S^{-1}A$ defined by $S^{-1}\phi(a/s) = \phi(a)/s$, for all $a \in A$ and $s \in S$. Moreover, the ring of invariants of $S^{-1}\phi$ is $S^{-1}(A^{\phi})$.*

**Lemma 2.2.4.** *Let $A = k^{[n]}$ then $\mathrm{ML}(A) = k$ and $\mathrm{DK}(A) = A$ for all $n \geqslant 2$.*

**Remark 2.2.5.** (i) Whenever for a $k$-domain $A$, $A \otimes_k \overline{k}$ is a rigid ring (for definition of rigid rings check Section 1.2) then $A$ is also rigid, since an exponential map $\phi$ on $A$ can be extended to an exponential map $\phi \otimes id$ on $A \otimes_k \overline{k}$.

(ii) Let $A$ be a $k$-algebra such that $A \otimes_k \overline{k}$ is an integral domain and $\operatorname{tr.deg}_k A = 1$. Then $k$ is algebraically closed in $A$ and hence by Lemma 2.2.3(v), $A$ admits a non-trivial exponential map if and only if $A = k^{[1]}$.

## 2.3 Proper and admissible $\mathbb{Z}$-filtration

For this section $A$ denotes an affine $k$-domain. We define below a proper and admissible $\mathbb{Z}$-filtration on $A$.

**Definition 2.3.1.** A collection $\{A_n \mid n \in \mathbb{Z}\}$ of $k$-linear subspaces of $A$ is said to be a *proper $\mathbb{Z}$-filtration on $A$* if the following properties hold:

(i) $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{Z}$.

(ii) $A = \cup_n A_n$.

(iii) $\cap_n A_n = \{0\}$.

(iv) $(A_n \setminus A_{n-1})(A_m \setminus A_{m-1}) \subseteq A_{m+n} \setminus A_{m+n-1}$ for all $m, n \in \mathbb{Z}$.

Next we define related concepts of degree function and semi-degree function on $A$.

**Definition 2.3.2.** A *degree function on $A$* is a function $\deg : A \to \mathbb{Z} \cup \{-\infty\}$ satisfying the following properties:

(i) $\deg(a) = -\infty$ if and only if $a = 0$.

(ii) $\deg(ab) = \deg(a) + \deg(b)$.

(iii) $\deg(a+b) \leqslant max\{\deg(a), \deg(b)\}$ for all $a, b \in A$.

When condition (ii) of the above definition is replaced by a weaker condition "$\deg(ab) \leqslant \deg(a) + \deg(b)$ for all $a, b \in A$" then we call the function deg to be a *semi-degree function on A*.

**Remark 2.3.3.** Any proper $\mathbb{Z}$-filtration $\{A_n\}_{n\in\mathbb{Z}}$ on $A$ defines a degree function deg on $A$, where $\deg(0) := -\infty$ and $\deg(a) := \min\{d \in \mathbb{Z} \mid a \in A_d\}$ for all $a \in A \setminus \{0\}$. Conversely, any degree function, deg on $A$ determines a proper filtration on $A$ given by $A_n = \{a \in A \mid \deg(a) \leqslant n\}$, for all $n \in \mathbb{Z}$.

Any proper $\mathbb{Z}$-filtration on $A$ determines an associated $\mathbb{Z}$-graded integral domain

$$\mathrm{gr}(A) := \bigoplus_{i\in\mathbb{Z}} \frac{A_i}{A_{i-1}}.$$

There exists a natural map $\rho : A \to \mathrm{gr}(A)$ defined by $\rho(a) = a + A_{n-1}$ if $a \in A_n \setminus A_{n-1}$.

**Definition 2.3.4.** A proper $\mathbb{Z}$-filtration $\{A_n\}_{n\in\mathbb{Z}}$ on $A$ is said to be *admissible* if there exists a finite generating set $\Gamma$ of $A$ such that, for any $n \in \mathbb{Z}$ and $a \in A_n$, $a$ can be written as a finite sum of monomials in elements of $\Gamma$ and each of them is a monomial in $A_n$ also.

**Remark 2.3.5.** Suppose that $A$ has a proper $\mathbb{Z}$-filtration and a finite generating set $\Gamma$ which makes the filtration admissible. Then $\mathrm{gr}(A)$ is generated by $\rho(\Gamma)$ (cf. [27, Remark 2.2]).

**Remark 2.3.6.** Let $A$ be a $\mathbb{Z}$-graded algebra say, $A = \bigoplus_{i\in\mathbb{Z}} C_i$. Then there exists a proper $\mathbb{Z}$-filtration $\{A_n\}_{n\in\mathbb{Z}}$ on $A$ defined by $A_n = \bigoplus_{i\leqslant n} C_i$ and $\mathrm{gr}(A) = \bigoplus_{n\in\mathbb{Z}} A_n/A_{n-1} \cong \bigoplus_{n\in\mathbb{Z}} C_n = A$. Moreover, for any element $a \in A$, $\rho(a)$ is the highest degree homogeneous summand of $a$ in $A$. Thus the above filtration on $A$ is admissible.

We now define a homogeneous exponential map over a $\mathbb{Z}$-graded $k$-algebra.

**Definition 2.3.7.** Let $\phi : A \to A[U]$ be an exponential map on a $\mathbb{Z}$-graded $k$-algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$. For any $x \in A$, let $\phi(x) := \sum_{i=0}^{\infty} \phi^{(i)}(x)U^i$ in $A[U]$. Then $\phi$ is said to be a *homogeneous exponential map* if there exists some $d \in \mathbb{Q}$ such that for every homogeneous element $a$ of degree $r$, $\phi^{(i)}(a)$ is a homogeneous element of degree $r + id$ i.e., $\phi^{(i)}(A_r) \subseteq A_{r+id}$, for all $i \in \mathbb{Z}_{\geqslant 0}$ and $r \in \mathbb{Z}$.

Next we record a theorem on homogenization of exponential maps by H. Derksen, O. Hadas and L. Makar-Limanov ([16]) as presented in ([11, Theorem 2.6]).

**Theorem 2.3.8.** *Let $A$ be an affine $k$-domain with an admissible $\mathbb{Z}$-filtration. Suppose $\phi$ is a non-trivial exponential map on $A$. Then $\phi$ induces a non-trivial homogeneous exponential map $\widehat{\phi}$ on the associated graded domain* $\operatorname{gr}(A)$ *such that $\rho(A^{\phi}) \subseteq (\operatorname{gr}(A))^{\widehat{\phi}}$, where $\rho$ is the natural map $A \to \operatorname{gr}(A)$.*

## 2.4 Some known results

For convenience of readers we quote here some known results. First we state a formulation of the well-known Epimorphism Theorem due to S.S. Abhyankar and T.T. Moh ([2, Theorem 1.1]).

**Theorem 2.4.1.** *Let $\varphi : k[X, Y] \to k[T]$ be a $k$-algebra epimorphism. Suppose* $\deg_T(\varphi(X)) = n \geqslant 1$ *and* $\deg_T(\varphi(Y)) = m \geqslant 1$ *and $ch.k \nmid gcd(m, n)$. Then either $m \mid n$ or $n \mid m$.*

*In particular, it states that over a field $k$ of characteristic zero, any line $f$ over $k$ in $k^{[2]}$ is a coordinate i.e., if $\frac{k[Z,T]}{(f)} = k^{[1]}$ then $k[Z, T] = k[f]^{[1]}$.*

M. Suzuki had independently shown that any line over $\mathbb{C}$ in $\mathbb{C}^{[2]}$ is a coordinate ([55]).

Earlier Segre and Nagata have constructed examples, to show that the above theorem does not hold over a field of positive characteristic ([52], [45]). Example 1.1.4 is one such example.

We now state the cancellative property of $k^{[1]}$ ([1, 2.8]).

**Theorem 2.4.2.** *Let $B$ be $k$-domain such that $B^{[n]} = k^{[n+1]}$ for some $n \in \mathbb{Z}_{\geqslant 1}$. Then $B = k^{[1]}$.*

The above result was later extended by E. Hamann over any ring containing $\mathbb{Q}$ or a seminormal domain ([33]). For convenience we recall the definition of a seminormal domain here.

**Definition 2.4.3.** An integral domain $R$ is called a *seminormal domain* if for any $a \in K$(the fraction field of $R$) with $a^2, a^3 \in R$ we have $a \in R$.

**Theorem 2.4.4.** *Let $R$ be a ring containing $\mathbb{Q}$ or let $R$ be a seminormal domain. Let $B$ be a $R$-algebra such that $B^{[n]} = R^{[n+1]}$ for some $n \in \mathbb{Z}_{\geqslant 1}$. Then $B = R^{[1]}$.*

Next we give a characterisation of $k^{[1]}$ for an algebraically closed field $k$ (cf. [22, Lemma 2.9]).

**Lemma 2.4.5.** *Let $k$ be an algebraically closed field and let $B$ be a finitely generated $k$-algebra. Suppose that $B$ is a PID and $B^* = k^*$. Then $B = k^{[1]}$.*

We now state the celebrated Quillen-Suslin Theorem on projective modules ([46], [54]).

**Theorem 2.4.6.** *Every finitely generated projective module over $k^{[n]}$ is free.*

Next we state a version of the Russell-Sathaye criterion for a ring to be a polynomial ring in one indeterminate over a given subring [48, Theorem 2.3.1], as presented in [8, Theorem 2.6].

**Theorem 2.4.7.** *Let $C \subseteq D$ be integral domains such that $D$ is a finitely generated $C$-algebra. Let $S$ be a multiplicatively closed subset of $C \setminus \{0\}$ generated by some prime elements of $C$ which remain prime in $D$.*

*Suppose $S^{-1}D = (S^{-1}C)^{[1]}$ and, for every prime element $p \in S$, we have $pC = pD \cap C$ and $\frac{C}{(p)}$ is algebraically closed in $\frac{D}{(p)}$.*

*Then $D = C^{[1]}$.*

We now recall a fundamental result on residual coordinates ([7, Theorem 3.2]).

**Theorem 2.4.8.** *Let $R$ be a Noetherian domain such that either $\mathbb{Q} \subseteq R$ or $R$ is seminormal* (Definition 2.4.3). *Then the following statements are equivalent:*

(i) *$h \in R[X,Y]$ is a residual coordinate i.e., $R[X,Y] \otimes_R k(p) = (R[h] \otimes_R k(p))^{[1]}$ for all prime ideal $p$ of $R$.*

(ii) *$h \in R[X,Y]$ is a coordinate i.e., $R[X,Y] = R[h]^{[1]}$.*

Next we quote a theorem of A. K. Dutta [18, Theorem 7].

**Theorem 2.4.9.** *Let $L$ be a separable field extension of $k$, $A$ a $k$-algebra and $B$ an $A$-algebra such that $L \otimes_k B \cong (L \otimes_k A)^{[1]}$ as $L \otimes_k A$-algebras. Then $B$ is isomorphic to the symmetric algebra of a finitely generated rank one projective module over $A$.*

The next result on triviality of separable $\mathbb{A}^1$-forms over $k^{[1]}$ is a special case of the above theorem.

**Lemma 2.4.10.** *Let $f \in k[Z,T]$ be such that $L[Z,T] = L[f]^{[1]}$ for some separable field extension $L$ of $k$. Then $k[Z,T] = k[f]^{[1]}$.*

The next lemma from [27, Lemma 3.3] provides an important criterion for the existence of non-trivial exponential map on certain affine domains.

**Lemma 2.4.11.** *Let $A$ be an affine $k$-domain over an infinite field $k$. Let $f \in A$ be such that $f - \lambda$ is a prime element of $A$ for infinitely many $\lambda$ in $k$. Let $\phi : A \to A[U]$ be a non-trivial exponential map on $A$ such that $f \in A^{\phi}$. Then there exists a $\beta \in k$ such that $f - \beta$ is a prime element of $A$ and $\phi$ induces a non-trivial exponential map on $A/(f-\beta)A$.*

Next we recall two lemmas from [13], Lemma 3.3 and 3.4 respectively.

**Lemma 2.4.12.** *Let $k$ be a field of characteristic $p \geqslant 0$ and let $\phi$ be an exponential map on a $k$-domain $A$. Let $f, g \in A$ be such that there exist $n, m \in \mathbb{Z}_{\geqslant 2}$ and $c_1, c_2 \in A^{\phi} \setminus \{0\}$ for which $c_1 f^n + c_2 g^m \in A^{\phi} \setminus \{0\}$ and neither $n$ nor $m$ is a power of $p$. Then $f, g \in A^{\phi}$.*

**Lemma 2.4.13.** *Let $k$ be a field of characteristic $p > 0$ and let $\phi$ be an exponential map on a $k$-domain $A$. Let $f, g \in A$ be such that $f$ is prime in $A$ and there exist $n \in \mathbb{Z}_{\geqslant 2}$ with $p \nmid n$ and $l \in \mathbb{Z}_{\geqslant 1}$, $c_1, c_2 \in A^{\phi} \setminus \{0\}$ such that $c_1 f^n + c_2 g^{p^l} \in A^{\phi} \setminus \{0\}$. Then $f, g \in A^{\phi}$.*

We now state a known criterion for flatness [41, Corollary of Theorem 22.6].

**Lemma 2.4.14.** *Let $A$ be a Noetherian ring and $B = A[X_1, \ldots, X_m] = A^{[m]}$. Let $g(X_1, \ldots, X_m) \in B$ be such that its coefficient over $A$ generate the unit ideal of $A$. Then $B/gB$ is a flat $A$-algebra.*

We end this section by stating a criterion for a simple birational extension of a UFD to be a UFD ([26, proposition 3.5]). For convenience we include the proof.

We first prove a lemma in this regard.

**Lemma 2.4.15.** *Let $R$ be a UFD, $u, v \in R\setminus\{0\}$ and $C := \frac{R[Y]}{(uY-v)}$ be an integral domain. Let $u := \prod_{i=1}^{n} u_i^{r_i}$ be a prime factorization of $u$ in $R$. Suppose that for every $i, 1 \leqslant i \leqslant n$ whenever, $(u_i, v)R$ is a proper ideal, $\prod_{j \neq i} u_j^{s_j} \notin (u_i, v)R$, for arbitrary $s_j \in \mathbb{Z}_{\geqslant 0}$. Then for each $i \in \{1, \ldots, n\}$ either $u_i$ is irreducible in $C$ or $u_i \in C^*$.*

*Proof.* Note that $R \hookrightarrow C \hookrightarrow R[u_1^{-1}, \ldots, u_n^{-1}]$. Suppose $u_j \notin C^*$ for some $j, 1 \leqslant j \leqslant n$. Now we are going to show that $u_j$ is irreducible. Suppose $u_j = c_1c_2$ for some $c_1, c_2 \in C$. If $c_1, c_2 \in R$, then either $c_1 \in R^*$ or $c_2 \in R^*$, as $u_j$ is irreducible in $R$. Therefore, we can assume that at least one of them is not in $R$. Suppose $c_1 \notin R$. Let $c_1 = \frac{h_1}{u_1^{i_1} \cdots u_n^{i_n}}$ and $c_2 = \frac{h_2}{u_1^{l_1} \cdots u_n^{l_n}}$, for some $h_1, h_2 \in R$ and $i_s, l_s \geqslant 0$, $1 \leqslant s \leqslant n$. Therefore, we have

$$h_1h_2 = u_j(u_1^{i_1+l_1} \cdots u_n^{i_n+l_n}). \tag{2.4.1}$$

As $c_1 \notin R$, using (2.4.1), without loss of generality, we can assume that

$$c_1 = \lambda \frac{\prod_{i \leqslant s} u_i^{p_i}}{\prod_{i=s+1}^{n} u_i^{p_i}}, \quad \text{for some} \quad \lambda \in C^* \quad \text{and} \quad s < n, \tag{2.4.2}$$

where $p_i \geqslant 0$ for $1 \leqslant i \leqslant n$, and $p_i > 0$ for $i \geqslant s+1$.

Now when $n = 1$ or $p_i = 0$ for every $i \leqslant s$, then $c_1 \in C^*$, and we are done. If not, then $n > 1$ and without loss of generality, we assume that $p_1 > 0$.

Therefore, from (2.4.2), we have $u_1^{p_1} \ldots u_s^{p_s} \in u_iC \cap R = (u_i, v)\,R$ for every $i \geqslant s+1$. Hence by the given hypothesis, for every $i \geqslant s+1$, we get that $(u_i, v)R = R$, i.e., $u_i \in C^*$. Thus we get

$$c_1 = \mu \prod_{i \leqslant s} u_i^{p_i}, \tag{2.4.3}$$

for some $\mu \in C^*$ and hence

$$\mu c_2 = \frac{u_j}{\prod_{i \leqslant s} u_i^{p_i}}.$$

If $\prod_{i \leqslant s} u_i^{p_i} \in u_jR$, then $c_2 \in C^*$ and we are done. If not, then $u_j \in u_iC \cap R = (u_i, v)R$ for every $i \leqslant s$ with $p_i > 0$. If for such an $u_i$ with $i \leqslant s$ and $p_i > 0$, $(u_i, v)R$ is a proper ideal then we get a contradiction by the given hypothesis. Therefore, for all such $u_i$, $(u_i, v)R = R$ i.e., $u_i \in C^*$ and hence by (2.4.3), $c_1 \in C^*$ and we are done.

Therefore we obtain that $u_j$ must be an irreducible element in $C$. □

**Proposition 2.4.16.** *Let $R$ be a UFD, $u, v \in R \setminus \{0\}$ and $C = \frac{R[Y]}{(uY-v)}$ be an integral domain. We consider $R$ as a subring of $C$. Let $u := \prod_{i=1}^{n} u_i^{r_i}$ be a prime factorization of $u$ in $R$. Suppose that for every $i \in \{1, \ldots, n\}$ for which $(u_i, v)R$ is a proper ideal, we have $\prod_{j \neq i} u_j^{s_j} \notin (u_i, v)R$, for arbitrary $s_j \in \mathbb{Z}_{\geqslant 0}$.*

*Then the following statements are equivalent:*

(i) *$C$ is a UFD.*

(ii) *For each $i \in \{1, \ldots, n\}$, either $u_i$ is prime in $C$ or $u_i \in C^*$.*

(iii) *For each $i \in \{1, \ldots, n\}$, either $(u_i, v)R \in Spec(R)$ or $(u_i, v)R = R$, i.e., the image of $v$ in $\frac{R}{u_iR}$ is either a prime in $\frac{R}{u_iR}$ or a unit in $\frac{R}{u_iR}$.*

*Proof.* **(ii) ⇔ (iii)** : For every $j, 1 \leqslant j \leqslant n$, we have

$$\frac{C}{u_jC} \cong \left(\frac{R}{(u_j, v)}\right)^{[1]}. \tag{2.4.4}$$

Note that $u_j$ is either a prime element or a unit in $C$ according as $\frac{C}{u_jC}$ is either an integral domain or a zero ring. Hence the equivalence follows from (2.4.4).

**(i) ⇒ (ii)** : By Lemma 2.4.15 we obtain that for each $i \in \{1, \ldots, n\}$ either $u_j \in C^*$ or $u_j$ is an irreducible element in $C$. Since $C$ is a UFD it follows that either $u_j \in C^*$ or $u_j$ is prime in $C$.

**(ii) ⇒ (i)** : Note that $R \hookrightarrow C \hookrightarrow R[u_1^{-1}, \ldots, u_n^{-1}]$. Without loss of generality we assume that $u_1, \ldots, u_{i-1} \in C^*$ and $u_i, \ldots, u_n$ are primes in $C$ for some $i$, $1 \leqslant i \leqslant n$. Since $C[u_1^{-1}, \ldots, u_n^{-1}] = C[u_i^{-1}, \ldots, u_n^{-1}] = R[u_1^{-1}, \ldots, u_n^{-1}]$ is a UFD, by Nagata's criterion for UFD ([41, Theorem 20.2]) we obtain that $C$ is a UFD. □

# Chapter 3

# On triviality and embedding of Linear Hyperplanes

The main aim of this chapter is to prove extended versions of Theorems A, B, C and D mentioned in the Introduction. In Section 3.1 we study a few properties, like factoriality, of the affine domain $A$. We prove Theorems A and B in Section 3.2; Theorem C in Section 3.4 and Theorem D in Section 3.5. After proving the main theorems the last Section 3.6 is devoted to the discussion of partial generalisations of Theorems B and D over any Noetherian domain containing $\mathbb{Q}$ or a Noetherian seminormal domain.

## 3.1 Some properties of the ring $A$

**Throughout this chapter, $\boldsymbol{A}$ will denote the following ring:**

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z,T) - h(X_1, \ldots, X_m, Z, T))} \tag{3.1.1}$$

and $H := \alpha(X_1, \ldots, X_m)Y - f(Z,T) - h(X_1, \ldots, X_m, Z, T)$ is an polynomial in $k[X_1, \ldots, X_m, Y, Z, T](= k^{[m+3]})$ such that $\boldsymbol{f(Z,T) \neq 0}$ and **every prime factor of $\boldsymbol{\alpha}$ in $\boldsymbol{k[X_1, \ldots, X_m]}$ divides $\boldsymbol{h}$ in $\boldsymbol{k[X_1, \ldots, X_m, Z, T]}$**.

So $\boldsymbol{A = k[X_1, \ldots, X_m, Y, Z, T]/(H)}$. Note that

(i) if $\alpha = 0$, then $h = 0$ and hence $A$ is a polynomial ring if and only if $\frac{k[Z,T]}{(f)} = k^{[1]}$ (cf. Theorem 2.4.2).

(ii) if $\alpha \in k^*$, then $A = k^{[m+2]}$.

Henceforth, we assume that $\boldsymbol{\alpha \notin k}$.

(iii) $A$ and $A \otimes_k \overline{k}$ both are integral domains.

Let $x_1, \ldots, x_m, y, z, t$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in $A$ respectively. **$E$ will denote the subring of $A$ generated by $x_1, \ldots, x_m$ over $k$**, i.e.,

(iv) $\boldsymbol{E = k[x_1, \ldots, x_m] = k^{[m]}}$.

(v) $A \hookrightarrow A\left[\frac{1}{\alpha}\right] = E[z,t]\left[\frac{1}{\alpha}\right]$.

Let $\alpha = \prod_{i=1}^{n} p_i^{s_i}$ be a prime factorization of $\alpha$ in $k[X_1, \ldots, X_m]$. Since every prime factor of $\alpha$ divides $h$ in $k[X_1, \ldots, X_m, Z, T]$, we have $h = (\prod_{i=1}^{n} p_i)h_1$ for some $h_1 \in k[X_1, \ldots, X_m, Z, T]$. Hence $\prod_{i=1}^{n} p_i(\prod_{i=1}^{n} p_i^{s_i - 1}y - h_1) = f(z,t)$ in $A$. Therefore,

(vi) if $f(Z,T) \in k^*$, then $\prod_{i=1}^{n} p_i \in A^*$ and so $\alpha \in A^*$ and hence $A = A\left[\frac{1}{\alpha}\right] = E[z,t]\left[\frac{1}{\alpha}\right]$.

**Lemma 3.1.1.** *The ring $A$ is a flat $E$-algebra.*

*Proof.* Let $H_1 = \alpha(x_1, \ldots, x_m)Y - f(Z,T) - h(x_1, \ldots, x_m, Z, T) \in E[Y, Z, T]$. Since every prime factor of $\alpha$ in $E$ divides $h$ in $E[Z,T]$ and $f \neq 0$, the coefficients of $H_1$ over $E$ generate the unit ideal in $E$. Now the result follows from Lemma 2.4.14. □

Next we prove an equivalent condition for $A$ to be an $\mathbb{A}^2$-fibration over $E$.

**Lemma 3.1.2.** *The following statements are equivalent:*

(i) *$A$ is an $\mathbb{A}^2$-fibration over $E$.*

(ii) $\dfrac{\frac{E_p}{pE_p}[Z,T]}{(f(Z,T))} = \left(\dfrac{E_p}{pE_p}\right)^{[1]}$ *for all $p \in Spec(E)$ with $\alpha \in p$.*

*Proof.* **(i) ⇒ (ii)** : Since $A$ is an $\mathbb{A}^2$-fibration over $E$, we have

$$\frac{A_p}{pA_p} = A \otimes_E \left(\frac{E_p}{pE_p}\right) = \left(\frac{E_p}{pE_p}\right)^{[2]} \text{ for every } p \in \operatorname{Spec}(E).$$

Let $p \in \operatorname{Spec}(E)$ be such that $\alpha \in p$. Then

$$\left(\frac{E_p}{pE_p}\right)^{[2]} = \frac{A_p}{pA_p} = \left(\frac{\frac{E_p}{pE_p}[Z,T]}{(f(Z,T))}\right)^{[1]}.$$

Therefore, by Theorem 2.4.2, we have $\dfrac{\frac{E_p}{pE_p}[Z,T]}{(f(Z,T))} = \left(\dfrac{E_p}{pE_p}\right)^{[1]}$.

**(ii) ⇒ (i)** : By Lemma 3.1.1, $A$ is a flat $E$-algebra and therefore, it remains to show that

$$A \otimes_E \left(\frac{E_p}{pE_p}\right) = \frac{A_p}{pA_p} = \left(\frac{E_p}{pE_p}\right)^{[2]} \text{ for every } p \in \operatorname{Spec}(E).$$

Let $p \in \operatorname{Spec}(E)$. We now consider two cases and show that in each case the above criterion is satisfied.

**Case I :** $\alpha \notin p$. Then $A_p = E_p[z,t] = E_p^{[2]}$ and therefore, $\frac{A_p}{pA_p} = \left(\frac{E_p}{pE_p}\right)^{[2]}$.

**Case II :** $\alpha \in p$. Then

$$\frac{A_p}{pA_p} = \frac{\frac{E_p}{pE_p}[Y,Z,T]}{(f(Z,T))} = \left(\frac{\frac{E_p}{pE_p}[Z,T]}{(f(Z,T))}\right)^{[1]} = \left(\frac{E_p}{pE_p}\right)^{[2]}.$$

□

**Remark 3.1.3.** Note that if there exists a $p \in \operatorname{Spec}(E)$ such that $\alpha \in p$ and $\frac{E_p}{pE_p}$ is a separable field extension over $k$, then by Lemma 3.1.2 and the fact that separable $\mathbb{A}^1$-forms over any field is trivial (cf. [18, Lemma 5]), it will follow that $A$ is an $\mathbb{A}^2$-fibration over $E$ if and only if $f(Z,T)$ is a line over $k$ in $k[Z,T]$.

Next we deduce a necessary condition for the affine domain $A$ to be a UFD.

**Lemma 3.1.4.** *Suppose that $A$ is a UFD. Then either $f(Z,T)$ is irreducible in $k[Z,T]$ or $f(Z,T) \in k^*$.*

*Proof.* Let $\alpha(X_1,\dots,X_m) = \prod_{i=1}^{n} p_i(X_1,\dots,X_m)^{l_i}$ be a prime factorization of $\alpha$ in $k[X_1,\dots,X_m]$. Putting $R = k[X_1,\dots,X_m,Z,T]$, $u = \alpha(X_1,\dots,X_m)$, $u_i = p_i$, $1 \leqslant i \leqslant n$ and $v = f(Z,T) + h(X_1,\dots,X_m,Z,T)$ in Proposition 2.4.16, we have $A = \frac{R[Y]}{(uY-v)}$. Let

$$R_i := \frac{R}{u_iR} = \frac{k[X_1,\dots,X_m]}{(p_i)}[Z,T]$$

and $x_{i1},\dots,x_{im},z_i,t_i$ denote the images of $X_1,\dots,X_m,Z,T$ respectively in $R_i$, for $1 \leqslant i \leqslant n$. Note that $vR_i = f(z_i,t_i)R_i$, for all $i$, $1 \leqslant i \leqslant n$. Therefore, for any $s_j \in \mathbb{Z}_{\geqslant 0}$, $\prod_{j\neq i} p_j(x_{i1},\dots,x_{im})^{s_j} \notin vR_i$ whenever $vR_i$ is a proper ideal. Therefore, the assumption in Proposition 2.4.16 is satisfied and hence either $(p_1, f(Z,T))R$ is a prime ideal in $R$ or $(p_1, f(Z,T))R = R$. Thus, either

$f(z_1, t_1)$ is irreducible in $R_1$ or it is a unit in $R_1$. Hence it follows that either $f(Z, T)$ is irreducible in $k[Z, T]$ or $f(Z, T) \in k^*$. □

Next we show that the converse of the above lemma is not true in general with the help of an example.

**Example 3.1.5.** Let $k$ be a non-perfect field of positive characteristic $p > 0$. Then there exists $\lambda \in k$ such that $\lambda = \beta^p$ for some $\beta \in \overline{k} \setminus k$. Let

$$A = \frac{k[X_1, X_2, Y, Z, T]}{((X_1^p + \lambda)(X_2^p + \lambda)Y - (Z^p + \lambda T^p))}$$

and $f(Z, T) := Z^p + \lambda T^p \in k[Z, T]$. Let $x_1, x_2$ be the image of $X_1, X_2$ in $A$ respectively. Note that:

(i) $f(Z, T)$ is irreducible in $k[Z, T]$ and $f(Z, T) = (Z + \beta T)^p$ in $k(\beta)[Z, T]$.

(ii) $\frac{A}{(x_1^p+\lambda)} \cong k(\beta)[X_2, Y, Z, T]/(Z + \beta T)^p$, therefore, $x_1^p + \lambda$ is not a prime element of $A$. Similarly, $x_2^p + \lambda$ is also not a prime element of $A$.

(iii) $x_1^p + \lambda, x_2^p + \lambda \notin A^*$ and all the conditions of Lemma 2.4.15, are satisfied. Hence by Lemma 2.4.15, $x_1^p + \lambda$ and $x_2^p + \lambda$ both are irreducible in $A$.

Therefore, by (ii) and (iii), it follows that $A$ is not a UFD.

However, the converse of Lemma 3.1.4 is true if $f(Z, T)$ is irreducible in $\overline{k}[Z, T]$ or $f(Z, T) \in k^*$.

**Lemma 3.1.6.** *Suppose that either $f(Z, T)$ is irreducible in $\overline{k}[Z, T]$ or $f(Z, T) \in k^*$. Then $A \otimes_k L$ is a UFD, for every algebraic extension $L$ of $k$.*

*Proof.* Let $L$ be an algebraic extension of $k$ and let $A_L := A \otimes_k L$. Suppose $\alpha = \prod_{i=1}^n p_i^{l_i}$ is a prime factorization of $\alpha$ in $L[X_1, \ldots, X_m]$. If $f(Z, T) \in k^*$, then $\prod_{i=1}^n p_i \in A_L^*$ and hence

$$A_L = A_L[p_1^{-1}, \ldots, p_n^{-1}] = L[X_1, \ldots, X_m, Z, T, p_1^{-1}, \ldots, p_n^{-1}]$$

is a UFD. Suppose $f(Z, T) \notin k^*$ then $f(Z, T)$ is irreducible in $\overline{k}[Z, T]$. Since

$$A_L[p_1^{-1}, \ldots, p_n^{-1}] = L[X_1, \ldots, X_m, Z, T, p_1^{-1}, \ldots, p_n^{-1}]$$

is a UFD, by Nagata's criterion for UFD ([41, Theorem 20.2]) it is enough to show that $p_1, \ldots, p_n$ are primes in $A_L$. We show this below.

Fix $i \in \{1, \dots, n\}$. Let $E_i = \frac{L[X_1,\dots,X_m]}{(p_i)}$ and $F_i$ be the field of fractions of $E_i$. Since $f(Z,T)$ is irreducible in $\overline{k}[Z,T]$ it follows that $f$ is irreducible in $F_i[Z,T]$. Therefore, $\frac{F_i[Z,T]}{(f(Z,T))}$ is an integral domain and hence $\frac{E_i[Z,T]}{(f(Z,T))}$ is an integral domain. Thus

$$\frac{A_L}{p_i A_L} = \frac{E_i[Y,Z,T]}{(f(Z,T))} = \left(\frac{E_i[Z,T]}{(f(Z,T))}\right)^{[1]}$$

is an integral domain and hence $p_i$ is a prime in $A_L$. □

We now determine some conditions on $\alpha$ and $h$ such that the regularity of $A$ implies the regularity of the ring $k[Z,T]/(f(Z,T))$.

**Lemma 3.1.7.** *Let $k$ be a perfect field and let $R = k[X_1, \dots, X_m, Y, Z, T]/(G)$ be an affine domain, where $G := vY + h - f(Z,T)$ for some $v \in k[X_1, \dots, X_m] \setminus k$, $h \in k[X_1, \dots, X_m, Z, T]$ and $f(Z,T) \in k[Z,T]$. Let $v = \prod_{i=1}^{n} p_i^{s_i}$ be a prime factorization of $v$ in $k[X_1, \dots, X_m]$. Suppose that one of the following conditions is satisfied:*

(I) *$s_i = 1$ for some $i$ and $p_i \mid h$.*

(II) *$s_i > 1$ for every $i$ and at least one of the following holds.*

- (a) *$p_j^2 \mid h$ for some $j$.*
- (b) *$(p_j, (p_j)_{X_1}, \dots, (p_j)_{X_m})k[X_1, \dots, X_m]$ is a proper ideal for some $j \in \{1, \dots, n\}$ and $p_j \mid h$.*
- (c) *$(p_l, p_j)k[X_1, \dots, X_m]$ is a proper ideal for some distinct $l, j \in \{1, \dots, n\}$ and $p_l p_j \mid h$.*

*Then $k[Z,T]/(f(Z,T))$ is a regular ring whenever $R$ is a regular ring.*

*Proof.* Let $D = k[X_1, \dots, X_m, Y, Z, T]$ and $I = (G, G_Y, G_{X_1}, \dots, G_{X_m}, G_Z, G_T)D$. Since $k$ is a perfect field it follows that $R$ is a regular ring if and only if $I = D$ ([41, Theorem 30.5]). Suppose that $R$ is a regular ring.

**(I)** Without loss of generality, we assume that $s_1 = 1$ and hence $p_1 \mid h$. Let $\beta = \prod_{i \geqslant 2} p_i^{s_i}$ and $\tilde{h} = h/p_1$. Therefore $G = vY + h - f = p_1(\beta Y + \tilde{h}) - f$. Then it can be observed that $I \subseteq (f, f_Z, f_T, \beta Y + \tilde{h}, p_1)D$. Since $R$ is regular we have

$$(f, f_Z, f_T, \beta Y + \tilde{h}, p_1) = D. \tag{3.1.2}$$

Suppose, if possible, that $(f, f_Z, f_T) \subseteq \mathfrak{m}k[Z,T]$ for some maximal ideal $\mathfrak{m}$ of $k[Z,T]$. Then as $p_1 \nmid \beta$ the ideal $(\mathfrak{m}, \beta Y + \tilde{h}, p_1)D$ is a proper ideal of $D$ containing $(f, f_Z, f_T, \beta Y + \tilde{h}, p_1)D$, a contradiction. Thus $k[Z,T]/(f)$ must be a regular ring.

**(II)** Suppose condition (a) is satisfied. Then $I \subseteq (p_j, f, f_Z, f_T)D$ and since $I = D$, we have $(f, f_Z, f_T)k[Z,T] = k[Z,T]$. Now suppose condition (b) or (c) is satisfied. When $p_j \mid h$, then $I \subseteq (p_j, (p_j)_{X_1}, \dots, (p_j)_{X_m}, f, f_Z, f_T)D$ and when $p_j p_l \mid h$ then $I \subseteq (p_l, p_j, f, f_Z, f_T)D$. Since $v = \prod_{i=1}^{n} p_i^{s_i} \in k[X_1, \dots, X_m]$ and $I = D$, we have $(f, f_Z, f_T)k[Z,T] = k[Z,T]$ in either case.

Thus it follows that $k[Z,T]/(f)$ is a regular ring if any one of the condition (a), (b) or (c) is satisfied. □

However, the next result shows that if $k[Z,T]/(f)$ is a regular ring then $A$ is always regular.

**Lemma 3.1.8.** *Let $k$ be a perfect field and let $R = k[X_1, \dots, X_m, Y, Z, T]/(G)$ be an affine domain, where $G = vY + h - f(Z,T)$ for some $v \in k[X_1, \dots, X_m] \setminus k$, $h \in k[X_1, \dots, X_m, Z, T]$ and $f(Z,T) \in k[Z,T]$. Suppose that every prime factor of $v$ divides $h$ in $k[X_1, \dots, X_m, Z, T]$. Then $R$ is a regular ring if $k[Z,T]/(f(Z,T))$ is a regular ring.*

*Proof.* Let $D = k[X_1, \dots, X_m, Y, Z, T]$ and $I = (G, G_Y, G_{X_1}, \dots, G_{X_m}, G_Z, G_T)D$. Since $k$ is a perfect field, $R$ is a regular ring if and only if $I = D$ ([41, Theorem 30.5]). Suppose, if possible, that $R$ is not a regular ring. Then there exists a maximal ideal $M$ of $k[X_1, \dots, X_m, Y, Z, T]$ such that $(G, G_Y, G_Z, G_T) \subseteq M$. Hence, there exists a prime factor $q$ of $v = G_Y$ such that $q \in M$. Now as $q \mid h$, $q \mid h_Z$ and $q \mid h_T$, we have

$$(f, f_Z, f_T) \subseteq M \cap k[Z,T]$$

and hence $k[Z,T]/(f(Z,T))$ is not a regular ring, a contradiction. Hence our result follows. □

**Remark 3.1.9.** Note that the hypersurfaces defined by the polynomials $H = \alpha(X_1, \dots, X_m)Y - f(Z,T) \in k[X_1, \dots, X_m, Y, Z, T]$, where $\alpha \in k[X_1, \dots, X_m] \setminus k$ are contained in the family of hypersurfcaes considered in Lemma 3.1.7.

## 3.2 On Theorems A and B

We begin by proving a small lemma which is crucial in proving the main theorems (viz. Theorems B and D).

**Lemma 3.2.1.** *Let $R$ be an integral domain and*

$$H := \alpha Y - f(Z,T) - h(Z,T) \in R[Y,Z,T]$$

*be an irreducible polynomial such that $\alpha \in R$, $\alpha$ can be expressed as product of prime elements of $R$ and each of the prime factors of $\alpha$ divides $h$ in $R[Y,Z,T]$. Suppose $R[Z,T] = R[f]^{[1]}$. Then $R[Y,Z,T] = R[H]^{[2]}$.*

*Proof.* Since $R[Z,T] = R[f]^{[1]}$, there exists $g \in R[Z,T]$ such that $R[Z,T] = R[f,g]$. Let $C = R[H,g]$ and $D = R[Y,Z,T]$. Note that every prime divisor of $\alpha$ in $R$ remains prime when considered as elements of $C$ and $D$. Let $S$ be the multiplicative closed subset of $C$ generated by the prime divisors of $\alpha$ in $R$. Then $S^{-1}D = (S^{-1}C)^{[1]}$ and $\frac{D}{qD} = \left(\frac{C}{qC}\right)^{[1]}$, for every prime divisor $q$ of $\alpha$ in $R$. Therefore, by Theorem 2.4.7, we have $D = C^{[1]}$. □

Next we prove a lemma which is crucial in proving Theorem A.

**Lemma 3.2.2.** *Let $k$ be an algebraically closed field. Suppose that $C$ is a regular affine $k$-domain, $R$ is a reduced affine $k$-algebra and the map $R \hookrightarrow R \otimes_k C$ induces surjective maps $G_i(R) \to G_i(R \otimes_k C)$ for $i = 0, 1$. Then the canonical inclusion $\tau : k \hookrightarrow C$ induces isomorphisms of $K_i$-groups for $i = 0, 1$ and hence $K_0(C) = \mathbb{Z}$ and $K_1(C) = k^*$.*

*Proof.* Observe that the inclusion map $\tau : k \hookrightarrow C$ induces inclusions $\iota : K_0(k) = \mathbb{Z} \hookrightarrow K_0(C)$ and $\eta : K_1(k) = k^* \hookrightarrow K_1(C)$. We will show that both $\iota$ and $\eta$ are isomorphism of groups.

We first show that $\iota$ is a surjective map.

Let $P_C$ be a finitely generated projective $C$-module. Since $G_0(R) \to G_0(R \otimes_k C)$ is surjective, there exist finitely generated $R$-modules $M_R$ and $N_R$ such that

$$[R \otimes_k P_C] = [M_R \otimes_k C] - [N_R \otimes_k C].$$

Therefore, by Lemma 2.1.1, there exist exact sequences of $R \otimes_k C$-modules

$$0 \to W' \to T_j \to W'' \to 0, \text{ for } j = 1, 2,$$

such that

$$(R \otimes_k P_C) \oplus (N_R \otimes_k C) \oplus T_1 \cong (M_R \otimes_k C) \oplus T_2. \tag{3.2.1}$$

Let $S$ be the set of all non-zero divisors of $R$. Since $R$ is a reduced affine $k$-algebra, $S^{-1}R = \prod_{i=1}^{n} L_i$, where $L_i$ is a finitely generated field extension of $k$ for each $i$, $1 \leqslant i \leqslant n$. Therefore, from (3.2.1), we have

$$(S^{-1}R \otimes_k P_C) \oplus (S^{-1}N_R \otimes_k C) \oplus S^{-1}T_1 \cong (S^{-1}M_R \otimes_k C) \oplus S^{-1}T_2. \tag{3.2.2}$$

Then from (3.2.2), we have,

$$(L_1 \otimes_k P_C) \oplus (L_1 \otimes_R N_R \otimes_k C) \oplus (L_1 \otimes_R T_1) \cong (L_1 \otimes_R M_R \otimes_k C) \oplus (L_1 \otimes_R T_2).$$

Let $\widetilde{C} = L_1 \otimes_k C$. Further it follows that for $j = 1, 2$,

$$0 \to L_1 \otimes_R W' \to L_1 \otimes_R T_j \to L_1 \otimes_R W'' \to 0$$

are exact sequences of $\widetilde{C}$-modules. Note that $L_1 \otimes_R M_R \cong L_1^{r_1}$ and $L_1 \otimes_R N_R \cong L_1^{r_2}$ as $L_1$-vector spaces for some $r_1, r_2 \in \mathbb{Z}_{\geqslant 1}$. Now, by Lemma 2.1.1, it follows that

$$[L_1 \otimes_k P_C] = [L_1 \otimes_R M_R \otimes_k C] - [L_1 \otimes_R N_R \otimes_k C] = [\widetilde{C}^{r_1}] - [\widetilde{C}^{r_2}] \text{ in } G_0(\widetilde{C}).$$

Note that $\widetilde{C}$ is a regular ring as $k$ is algebraically closed. Therefore, $G_0(\widetilde{C}) = K_0(\widetilde{C})$. Hence, there exists $s \in \mathbb{Z}_{\geqslant 0}$ such that

$$(L_1 \otimes_k P_C) \oplus \widetilde{C}^{r_2+s} \cong \widetilde{C}^{r_1+s}. \tag{3.2.3}$$

Since $P_C$ is a finitely generated projective $C$-module, there exists a finitely generated $k$-algebra $\widehat{L}$ such that

$$(\widehat{L} \otimes_k P_C) \oplus (\widehat{L} \otimes_k C)^{r_2+s} \cong (\widehat{L} \otimes_k C)^{r_1+s}. \tag{3.2.4}$$

Now let $m$ be a maximal ideal of $\widehat{L}$. Then $\frac{\widehat{L}}{m} = k$, as $k$ is an algebraically closed field. Therefore, as $(\widehat{L} \otimes_k C)/m(\widehat{L} \otimes_k C) \cong C$ and $(\widehat{L} \otimes_k P_C)/m(\widehat{L} \otimes_k P_C) \cong P_C$, from (3.2.4), we have

$$[P_C] = [C^{r_1}] - [C^{r_2}] \text{ in } K_0(C).$$

Therefore, the map $\iota$ is surjective, and thus $K_0(C) = \mathbb{Z}$.

Similarly, we now show that $\eta : K_1(k) = k^* \hookrightarrow K_1(C)$ is a surjective map. We consider an element $[Q_C, \sigma_C] \in K_1(C)$. Since $G_1(R) \to G_1(R \otimes_k C)$ is surjective, we have finitely generated $R$-modules $\widetilde{M}_R$ and $\widetilde{N}_R$ with $\gamma_R \in \mathrm{Aut}_R(\widetilde{M}_R)$ and $\delta_R \in \mathrm{Aut}_R(\widetilde{N}_R)$ such that

$$[R \otimes_k Q_C, 1_R \otimes \sigma_C] = [\widetilde{M}_R \otimes_k C, \gamma_R \otimes 1_C] - [\widetilde{N}_R \otimes_k C, \delta_R \otimes 1_C]$$

in $G_1(R \otimes_k C)$. Therefore, by Lemma 2.1.2, there exist exact sequences in the category $\mathscr{M}^{\mathbb{Z}}(R \otimes_k C)$ (defined as in Section 2.1)

$$0 \to (U', \sigma_{U'}) \to (T_1', \sigma_{T_1'}) \to (U'', \sigma_{U''}) \to 0$$

and

$$0 \to (U', \epsilon\sigma_{U'}) \to (T_2', \sigma_{T_2'}) \to (U'', \sigma_{U''}) \to 0,$$

where $\epsilon$ is a composition of unipotent automorphisms of $U'$ such that

$$(R\otimes_k Q_C, 1_R\otimes\sigma_C)\oplus(\widetilde{N}_R\otimes_k C, \delta_R\otimes 1_C)\oplus(T_1', \sigma_{T_1'}) \cong (\widetilde{M}_R\otimes_k C, \gamma_R\otimes 1_C)\oplus(T_2', \sigma_{T_2'}). \tag{3.2.5}$$

Henceforth in this proof, to avoid complication of notation, we are going to use the symbol $1_C$ to denote identity morphism of the free $C$-module, $C^l$ for appropriate $l \in \mathbb{Z}_{\geqslant 1}$, understood from the context.

Now, localising equation (3.2.5) by $S$ (the set of all non-zero divisors of $R$), and considering the first component, we have the following

$$(L_1 \otimes_k Q_C, 1_{L_1} \otimes \sigma_C) \oplus (\widetilde{C}^{s_2}, \delta_{L_1} \otimes 1_C) \oplus (L_1 \otimes_R T_1', \sigma_{(L_1 \otimes_R T_1')}) \cong$$

$$(\widetilde{C}^{s_1}, \gamma_{L_1} \otimes 1_C) \oplus (L_1 \otimes_R T_2', \sigma_{(L_1 \otimes_R T_2')}),$$

where $L_1 \otimes_R \widetilde{M}_R \cong L_1^{s_1}$ and $L_1 \otimes_R \widetilde{N}_R \cong L_1^{s_2}$ as $L_1$-vector spaces, for some $s_1, s_2 \in \mathbb{Z}_{\geqslant 1}$ with $\gamma_{L_1} \in \mathrm{Aut}_{L_1}(L_1^{s_1})$ and $\delta_{L_1} \in \mathrm{Aut}_{L_1}(L_1^{s_2})$ along with the following exact sequences in the category $\mathscr{M}^{\mathbb{Z}}(\widetilde{C})$

$$0 \to (L_1 \otimes_R U', \sigma_{L_1 \otimes_R U'}) \to (L_1 \otimes_R T_1', \sigma_{L_1 \otimes_R T_1'}) \to (L_1 \otimes_R U'', \sigma_{L_1 \otimes_R U''}) \to 0,$$

and

$$0 \to (L_1 \otimes_R U', \epsilon'\sigma_{L_1 \otimes_R U'}) \to (L_1 \otimes_R T_2', \sigma_{L_1 \otimes_R T_2'}) \to (L_1 \otimes_R U'', \sigma_{L_1 \otimes_R U''}) \to 0,$$

where $\epsilon'$ is a composition of unipotent automorphisms of $L_1 \otimes_R U'$. Now using

Lemma 2.1.2 we have,

$$[L_1 \otimes_k Q_C, 1_{L_1} \otimes \sigma_C] + [\widetilde{C}^{s_2}, \delta_{L_1} \otimes 1_C] = [\widetilde{C}^{s_1}, \gamma_{L_1} \otimes 1_C] \text{ in } G_1(\widetilde{C}).$$

As before $G_1(\widetilde{C}) = K_1(\widetilde{C})$, and thus there exist $r \in \mathbb{Z}_{\geqslant 1}$ and an automorphism $\widetilde{\sigma}_{\widetilde{C}^r} \in \mathrm{Aut}_{\widetilde{C}}(\widetilde{C}^r)$ such that

$$(L_1 \otimes_k Q_C, 1_{L_1} \otimes \sigma_C) \oplus (\widetilde{C}^{s_2}, \delta_{L_1} \otimes 1_C) \oplus (\widetilde{C}^r, \widetilde{\sigma}_{\widetilde{C}^r}) \cong (\widetilde{C}^{s_1}, \gamma_{L_1} \otimes 1_C) \oplus (\widetilde{C}^r, \widetilde{\sigma}_{\widetilde{C}^r}). \tag{3.2.6}$$

Since $Q_C$ is a finitely generated projective $C$-module, there exists a finitely generated $k$-algebra $\widetilde{L}$ such that

$$(\widetilde{L} \otimes_k Q_C, 1_{\widetilde{L}} \otimes \sigma_C) \oplus ((\widetilde{L} \otimes_k C)^{s_2}, \delta_{\widetilde{L}} \otimes 1_C) \oplus ((\widetilde{L} \otimes_k C)^r, \tilde{\sigma}_{(\widetilde{L} \otimes_k C)^r}) \cong$$

$$((\widetilde{L} \otimes_k C)^{s_1}, \gamma_{\widetilde{L}} \otimes 1_C) \oplus ((\widetilde{L} \otimes_k C)^r, \tilde{\sigma}_{(\widetilde{L} \otimes_k C)^r}), \tag{3.2.7}$$

for some $\delta_{\widetilde{L}} \in \mathrm{Aut}(\widetilde{L}^{s_2})$, $\gamma_{\widetilde{L}} \in \mathrm{Aut}(\widetilde{L}^{s_1})$ and $\tilde{\sigma}_{(\widetilde{L} \otimes_k C)^r} \in \mathrm{Aut}((\widetilde{L} \otimes_k C)^r)$. Since $k$ is an algebraically closed, for any maximal ideal $m$ of $\widetilde{L}$, we have $\frac{\widetilde{L}}{m} = k$. Then from (3.2.7), we have

$$(Q_C, \sigma_C) \oplus (C^{s_2}, \delta_k \otimes 1_C) \oplus (C^r, \widetilde{\sigma}_{C^r}) \cong (C^{s_1}, \gamma_k \otimes 1_C) \oplus (C^r, \widetilde{\sigma}_{C^r}),$$

for some $\delta_k \in \mathrm{Aut}(k^{s_2})$, $\gamma_k \in \mathrm{Aut}(k^{s_1})$ and $\widetilde{\sigma}_{C^r} \in \mathrm{Aut}(C^r)$. Hence, we have

$$[Q_C, \sigma_C] = [C^{s_1}, \gamma_k \otimes 1_C] - [C^{s_2}, \delta_k \otimes 1_C] \text{ in } K_1(C).$$

Thus, the map $\eta$ is surjective, and hence $K_1(C) = k^*$. □

We now prove the first part of Theorem A.

**Theorem 3.2.3.** *Let $k$ be an algebraically closed field and $A$ be an affine $k$-domain as in* (3.1.1). *Suppose that $C := \frac{k[Z,T]}{(f)}$ is a regular domain and $A^{[l]} = k^{[m+l+2]}$ for some $l \geqslant 0$. Then $C = k^{[1]}$.*

*Proof.* Note that $E := k[x_1, \ldots, x_m] \hookrightarrow A$ is flat (cf. Lemma 3.1.1). Consider the inclusion maps $k \xhookrightarrow{\sigma} E \xhookrightarrow{\gamma} A \xhookrightarrow{\delta} A^{[l]}$. By Theorem 2.1.4, $G_i(k) \xrightarrow{G_i(\sigma)} G_i(E)$, $G_i(A) \xrightarrow{G_i(\delta)} G_i(A^{[l]})$ are isomorphisms and since $A^{[l]} = k^{[l+m+2]}$, $G_i(k) \xrightarrow{G_i(\delta\gamma\sigma)} G_i(A^{[l]})$ are also isomorphisms for every $i \geqslant 0$. Therefore, it follows that $G_i(\gamma) : G_i(E) \to G_i(A)$ are isomorphisms for every $i \geqslant 0$. Let $p_1, \ldots, p_n$ be distinct prime factors of $\alpha$ in $E$ and let $u = \prod_{i=1}^n p_i$.

Then $A[u^{-1}] = E[u^{-1}]^{[2]}$, therefore by Theorem 2.1.4, the inclusion map $E[u^{-1}] \hookrightarrow A[u^{-1}]$ induce isomorphism of groups $G_i(E[u^{-1}]) \to G_i(A[u^{-1}])$, for every $i \geqslant 0$. Now by Theorem 2.1.3, we get the following commutative diagram

$$\begin{array}{ccccccccc}
G_j(E) & \longrightarrow & G_j(E[u^{-1}]) & \longrightarrow & G_{j-1}(\frac{E}{uE}) & \longrightarrow & G_{j-1}(E) & \longrightarrow & G_{j-1}(E[u^{-1}]) \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
G_j(A) & \longrightarrow & G_j(A[u^{-1}]) & \longrightarrow & G_{j-1}(\frac{A}{uA}) & \longrightarrow & G_{j-1}(A) & \longrightarrow & G_{j-1}(A[u^{-1}]).
\end{array}$$

By the Five lemma the canonical maps

$$G_i\left(\frac{E}{uE}\right) \to G_i\left(\frac{A}{uA}\right) \tag{3.2.8}$$

are isomorphisms for every $i \geqslant 0$. Let $R = \frac{E}{uE}$. Then $\dfrac{A}{uA} = \dfrac{R[Y,Z,T]}{(f)} = (R \otimes_k C)^{[1]}$. Note that $R$ is a reduced affine $k$-algebra. Thus from (3.2.8) and Theorem 2.1.4, it follows that the inclusion $R \hookrightarrow \frac{R[Z,T]}{(f)}$ induces canonical isomorphisms

$$G_i(R) \to G_i\left(\frac{R[Z,T]}{(f)}\right) = G_i(R \otimes_k C) \quad \text{for every} \quad i \geqslant 0.$$

Therefore, by Lemma 3.2.2, we get that the canonical inclusion $k \hookrightarrow \dfrac{k[Z,T]}{(f)}$ induces isomorphisms $\iota : K_0(k) \to K_0(C)$ and $\eta : K_1(k) \to K_1(C)$. Since $\eta$ maps $k^*$ onto $C^*$, we have $K_1(C) = C^* = k^*$. Now since $K_0(C) = K_0(k) = \mathbb{Z}$, we have the ideal class group of $C$, $\mathrm{Cl}(C) = 0$ (cf. [53, Example 1.2]) and hence $C$ is a PID. Therefore, by Lemma 2.4.5, we have $C = k^{[1]}$. $\square$

**Remark 3.2.4.** The above result helps us to conclude that affine $k$-domains of the form $A$ when $\overline{k}[Z,T]/(f)$ is a regular domain and $f$ is not a line over $\overline{k}$ in $\overline{k}[Z,T]$, over a field $k$ of characteristic zero is never an affine space. For example, let $k$ be a field of characteristic zero and

$$A := \frac{k[X_1, X_2, Y, Z, T]}{(\alpha(X_1, X_2)Y - (ZT + Z + T) - h(X_1, X_2, Z, T))},$$

for some $\alpha \in k^{[2]}$, $h \in k^{[4]}$ be a $k$-domain such that $\alpha \notin k$ and every prime factor of $\alpha$ divides $h$ in $k[X_1, X_2, Z, T]$. By a suitable choice of $\alpha$ and $h$, the ring of the form $A$ can be made to satisfy several properties of polynomial rings. However, by the above theorem it follows that $A$ is not stably isomorphic to a polynomial ring over $k$.

**Corollary 3.2.5.** *Let $A$ and $H$ be as in* (3.1.1). *Suppose that $f(Z,T) = a_0(Z) + a_1(Z)T$, for some $a_0, a_1 \in k^{[1]}$ and $A^{[l]} = k^{[m+l+2]}$ for some $l \geqslant 0$. Then $k[Z,T] = k[f]^{[1]}$ and $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$.*

*Proof.* Since $A^{[l]} = k^{[l+m+2]}$ we have $\overline{A} := A \otimes_k \overline{k}$ is a UFD and $\overline{A}^* = \overline{k}^*$. Therefore by Lemma 3.1.4 it follows that $f$ is irreducible in $\overline{k}[Z,T]$. We now consider two cases:

**Case I :** $a_1(Z) = 0$.
Then $f(Z,T) = a_0(Z)$. Thus $a_0(Z)$ is irreducible in $\overline{k}[Z,T]$ and hence a linear polynomial in $Z$ over $k$.

**Case II :** $a_1(Z) \neq 0$.
We show that $a_1(Z) \in \overline{k}^*$, and hence $a_1(Z) \in k^*$. Since $f$ is irreducible in $\overline{k}[Z,T]$ we have $\gcd_{\overline{k}[Z]}(a_0(Z), a_1(Z)) = 1$. Hence $\frac{\overline{k}[Z,T]}{(f)} = \overline{k}\left[Z, \frac{1}{a_1(Z)}\right]$ which is a regular domain. Therefore by Theorem 3.2.3, $\left(\frac{\overline{k}[Z,T]}{(f)}\right)^* = \overline{k}^*$. Hence $a_1(Z) \in \overline{k}^*$.

Therefore, $f$ is a coordinate in $k[Z,T]$. Hence by Lemma 3.2.1, we have $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$. □

We now prove the remaining part of Theorem A.

**Theorem 3.2.6.** *Let $k$ be a field of characteristic zero with $A$ and $H$ be as in* (3.1.1). *Suppose that $A^{[l]} = k^{[l+m+2]}$ for some $l \geqslant 0$ and $k[Z,T]/(f)$ is a regular ring. Then $k[Z,T] = k[f]^{[1]}$ and $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$.*

*Proof.* Let $\overline{A} = A \otimes_k \overline{k}$. Since $\overline{A}^{[l]} = \overline{k}^{[l+m+2]}$, we have $\overline{A}$ is a UFD and $\overline{A}^* = \overline{k}^*$. Therefore, by Lemma 3.1.4, it follows that $f$ is irreducible in $\overline{k}[Z,T]$. Hence $\overline{k}[Z,T]/(f)$ is a domain. As $k$ is a field of characteristic zero and $k[Z,T]/(f)$ is regular, we have $\overline{k}[Z,T]/(f)$ is a regular domain. Thus, by Theorem 3.2.3 we have $\overline{k}[Z,T]/(f) = \overline{k}^{[1]}$. Since $k$ is a field of characteristic zero, by applying Theorem 2.4.1 and Lemma 2.4.10 consecutively, we have $k[Z,T] = k[f]^{[1]}$. Now, by Lemma 3.2.1, we have $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$. □

Next we prove Theorem B.

**Theorem 3.2.7.** *Let $k$ be a field of characteristic zero and $A$ and $H$ be as in* (3.1.1) *and $A$ be an affine domain. Let $\alpha = \prod_{i=1}^{n} p_i^{s_i}$ be the prime factorization of $\alpha$ in $k[X_1, \ldots, X_m]$. Suppose that one of the following condition is satisfied.*

(I) *$s_i = 1$ for some $i$.*

(II) *$s_i > 1$ for every $i$ and at least one of the following holds.*

  (a) *$p_j^2 \mid h$ for some $j$.*

  (b) *$(p_j, (p_j)_{X_1}, \ldots, (p_j)_{X_m})k[X_1, \ldots, X_m]$ is a proper ideal for some $j \in \{1, \ldots, n\}$.*

  (c) *If $n \geqslant 2$, then $(p_l, p_j)k[X_1, \ldots, X_m]$ is a proper ideal for some distinct $l, j \in \{1, \ldots, n\}$.*

*Then the following statements are equivalent:*

(i) *$k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$.*

(ii) *$k[X_1, \ldots, X_m, Y, Z, T] = k[H]^{[m+2]}$.*

(iii) *$A = k[x_1, \ldots, x_m]^{[2]}$.*

(iv) *$A = k^{[m+2]}$.*

(v) *$k[Z, T] = k[f(Z, T)]^{[1]}$.*

(vi) *$A$ is an $\mathbb{A}^2$-fibration over $k[x_1, \ldots, x_m]$.*

(vii) *$A^{[l]} = k^{[m+l+2]}$ for some $l \geqslant 0$.*

*Proof.* We prove the equivalence of the statements as follows.

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(vii)} \Longrightarrow \text{(v)} \Longrightarrow \text{(i)}$$
$$\text{(i)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(vi)} \Longrightarrow \text{(v)}$$

Note that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (vii) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (vi) follow trivially. (v) $\Rightarrow$ (i) follows from Lemma 3.2.1. Therefore, it is enough to prove (vi) $\Rightarrow$ (v) and (vii) $\Rightarrow$ (v).

**(vi) $\Rightarrow$ (v)** : Let $m$ be a maximal ideal of $E := k[X_1, \ldots, X_m]$ containing $\alpha$. Note that $L := \frac{E}{m}$ is a finite field extension of $k$. By Lemma 3.1.2, we have

$\frac{L[Z,T]}{(f)} = L^{[1]}$. Since ch.$k = 0$, by Theorem 2.4.1, we have $L[Z,T] = L[f]^{[1]}$. Now the assertion follows by Lemma 2.4.10.

**(vii) ⇒ (v)** : Since $A^{[l]} = k^{[m+l+2]}$ we have, $A$ is a UFD and $(A \otimes_k \overline{k})^* = \overline{k}^*$. Hence by Lemma 3.1.4, it follows that $f$ is irreducible in $\overline{k}[Z,T]$. Note that $A$ is also a regular domain, hence by Lemma 3.1.7, $\frac{k[Z,T]}{(f)}$ is a regular domain. Therefore, the result follows from Theorem 3.2.6. □

**Remark 3.2.8.** Let $k$ be a field of characteristic zero and $H$ be as in (3.1.1) with $h = 0$, i.e., $H := \alpha(X_1, \ldots, X_m)Y - f(Z,T)$. Then $H$ satisfies either condition (I) or condition (II)(a) of Theorem 3.2.7. Thus we know that for the above mentioned form of $H$, the Abhyankar-Sathaye Conjecture is satisfied and moreover, $H$ is a hyperplane over $k$ in $k^{[m+3]}$ if and only if $f$ is a coordinate in $k[Z,T]$.

## 3.3 On the admissibility of a $\mathbb{Z}$-filtration

In this section we construct an associated graded ring (Corollary 3.4.10) and use it to prove Theorem C in two parts (Theorems 3.4.11 and 3.4.12) in the next section.

Let $R$ be a UFD and

$$A_R := \frac{R[X,Y,Z,T]}{(X^d\alpha_1(X)Y - F(X,Z,T))} \quad \text{for some} \quad d \geqslant 1 \tag{3.3.1}$$

with $\alpha_1(0) \neq 0$, $F(0,Z,T) \neq 0$ be a domain. Let $x, y, z, t$ denote the images of $X, Y, Z, T$ in $A_R$ respectively. Then $A_R \subseteq R\left[x, \frac{1}{x}, \frac{1}{\alpha_1(x)}, z, t\right]$. Note that every element $r \in R\left[x, \frac{1}{x}, \frac{1}{\alpha_1(x)}, z, t\right]$ is of the form

$$r = x^{i_1}p, \text{ for some } i_1 \in \mathbb{Z},\, p \in R\left[x, \frac{1}{\alpha_1(x)}, z, t\right] \text{ with } x \nmid p.$$

Therefore, the function $w_R$ on $R\left[x, \frac{1}{x}, \frac{1}{\alpha_1(x)}, z, t\right]$ defined by

$$w_R(r) = -i_1, \text{ for an } r \text{ as above,}$$

is a degree function. Thus $w_R$ restricts to a degree function on $A_R$ and by

Remark 2.3.3 it induces a proper $\mathbb{Z}$-filtration $\{A_R^n\}_{n\in\mathbb{Z}}$ on $A_R$ such that

$$A_R^n := \{g \in A_R : w_R(g) \leqslant n\}, n \in \mathbb{Z}.$$

Note that $y = \frac{F(x,z,t)}{x^d\alpha_1(x)} \in A_R$ and $F(0,z,t) \neq 0$, therefore, it follows that $w_R(y) = d$.

In this section, we shall prove the following theorem.

**Theorem 3.3.1.** *Let $R$ be a UFD and a finitely generated $k$-algebra. Let $R$, $A_R$ and $w_R$ be as above with $\gcd_{R[Z,T]}(\alpha_1(0), F(0,Z,T)) = 1$. Let $\{c_1, \dots, c_n\}$ be a generating set of the affine $k$-algebra $R$. Then $w_R$ induces an admissible $\mathbb{Z}$-filtration on $A_R$ with respect to the generating set $\{c_1, \dots, c_n, x, y, z, t\}$ such that*

$$gr(A_R) \cong \frac{R[X,Y,Z,T]}{(\alpha_1(0)X^dY - F(0,Z,T))}.$$

*Proof.* Follows from Proposition 3.3.6 and Lemma 3.3.7 proved below. □

**Remark 3.3.2.** Note that when $R$ is a field $k$, then the condition $\gcd_{k[Z,T]}(\alpha_1(0), F(0,Z,T)) = 1$ becomes redundant as $\alpha_1(0) \in k^*$.

**Throughout the rest of this subsection the notation and hypotheses are assumed as in Theorem** 3.3.1. We are now going to prove Proposition 3.3.6. We first prove three technical lemmas (Lemmas 3.3.3, 3.3.4 and 3.3.5) in this connection.

**Lemma 3.3.3.** *Let $b \in A_R$ be an element such that $w_R(b) < 0$. Suppose that $b = \sum_{i=1}^s m_i$, where $m_i$ is a monomial in $R[x, x^dy, z, t]$ with $w_R(m_i) = 0$ for all $i, 1 \leqslant i \leqslant s$. Then $b = xb_1$ for some $b_1 \in R[x, x^dy, z, t]$.*

*Proof.* Since $b = \sum_{i=1}^s m_i$, with $w_R(m_i) = 0$, it follows that $m_i \in R[x^dy, z, t]$. As $w_R(b) < 0$, it follows that $w_R(\alpha_1(0)^i b) < 0$ for every integer $i > 0$ and further it follows that $\alpha_1(0)^r b \in (\alpha_1(0)x^dy - F(0,z,t))R[x^dy, z, t]$ for some $r > 0$. Now

$$\alpha_1(0)x^dy - F(0,z,t) = -(\alpha_1(x) - \alpha_1(0))x^dy + (F(x,z,t) - F(0,z,t)) \in xR[x, x^dy, z, t]. \tag{3.3.2}$$

Thus, $\alpha_1(0)^r b \in xR[x, x^dy, z, t]$. Note that

$$R[x, x^dy, z, t] \cong \frac{R[X,U,Z,T]}{(\alpha_1(X)U - F(X,Z,T))}.$$

Hence $R[x, x^d y, z, t]/(x) \cong R[U, Z, T]/(\alpha_1(0)U - F(0, Z, T))$. Since $\gcd_{R[Z,T]}(\alpha_1(0), F(0, Z, T)) = 1$, it follows that $\{x, \alpha_1(0)\}$ is a regular sequence in $R[x, x^d y, z, t]$ and hence we get that $b \in xR[x, x^d y, z, t]$. Thus the result follows. □

**Lemma 3.3.4.** *Let $b \in R[x, x^d y, z, t] \subseteq A_R$ be an element such that $w_R(b) = -n$ for some $n \geqslant 0$. Then $b$ can be represented as a sum of monomials $m_i$ in $\{x, x^d y, z, t\}$ with $w(m_i) \leqslant -n$, for every $i$.*

*Proof.* We show that $b = x^n b_n$ for some $b_n \in R[x, x^d y, z, t]$ with $w_R(b_n) = 0$. The lemma will follow since for any monomial $m = \lambda x^\iota (x^d y)^\beta z^l t^j$ with $\lambda \in R \setminus \{0\}$ and $\iota, \beta, l, j \in \mathbb{Z}_{\geqslant 0}$ we have $w_R(m) = -\iota \leqslant 0$.

We prove that $b = x^n b_n$ by induction on $n \geqslant 0$. Now if $n = (-w_R(b)) = 0$, then we are already done. Now consider $n > 0$. Then we have

$$b = \sum_{i=1}^{s} m_i' + \sum_{i=s+1}^{r} m_i' \in R[x, x^d y, z, t],$$

where $m_i'$'s are monomials in $\{x, x^d y, z, t\}$ with $w_R(m_i') < 0$ for $1 \leqslant i \leqslant s$ and $w_R(m_i') = 0$ for $s+1 \leqslant i \leqslant r$. Since $w_R(b) < 0$ and $w_R(\sum_{i=1}^{s} m_i') < 0$, we have $w_R(\sum_{i=s+1}^{r} m_i') < 0$ and hence by Lemma 3.3.3, $\sum_{i=s+1}^{r} m_i' = x\widetilde{b_1}$ for some $\widetilde{b_1} \in R[x, x^d y, z, t]$. Also for every $i, 1 \leqslant i \leqslant s$ we know that $x \mid m_i'$ in $R[x, x^d y, z, t]$. Hence $b = xb_1$ for some $b_1 \in R[x, x^d y, z, t]$ with $w_R(b_1) = -(n-1)$. Hence by induction hypothesis, we get the desired result. □

**Lemma 3.3.5.** *Let $b \in A_R$ be an element such that $w_R(b) = n$. If $b$ is a sum of monomials in $\{x, y, z, t\}$ each of degree $n+m$ for some $m > 0$, then $x \mid b$ and $b$ has another representation as a sum of monomials in $\{x, y, z, t\}$ each of degree less than or equal to $n$.*

*Proof.* Let $b = \sum_{i=1}^{s} m_i$, where $m_i$ is a monomial in $\{x, y, z, t\}$ with $w_R(m_i) = n + m$ for some $m > 0$. We first show that there exist $\iota, \beta \in \mathbb{Z}_{\geqslant 0}$ such that $d\iota - \beta = n + m$ and $m_i = \lambda_i y^\iota x^\beta m_i'$ for each $i$, $1 \leqslant i \leqslant s$ with $\lambda_i \in R \setminus \{0\}$ and $m_i'$ being a monomial in $\{x^d y, z, t\}$ (i.e., $w_R(m_i') = 0$).

If $n + m \leqslant 0$, then we can take $\iota = 0$ and $\beta = -(n+m) \geqslant 0$. If $n+m > 0$, then we can take $\iota$ to be the least integer greater than equal to $(n+m)/d$ and $\beta = d\iota - (n+m)$. Therefore, $b = y^\iota x^\beta \tilde{b}$, where $\tilde{b} \in R[x, x^d y, z, t]$ and $w_R(\tilde{b}) = -m$. Now the result follows by Lemma 3.3.4. □

We now assume that $R$ is also an affine $k$-algebra.

**Proposition 3.3.6.** *Let $\{c_1, \ldots, c_n\}$ be a generating set of the affine $k$-algebra $R$. Then the $\mathbb{Z}$-filtration $\{A_R^n\}_{n\in\mathbb{Z}}$ on $A$ is admissible with respect to the generating set $\{c_1, \ldots, c_n, x, y, z, t\}$.*

*Proof.* Let $b \in A_R$ be such that $w_R(b) = n$. Suppose

$$b = b_1 + \cdots + b_r,$$

such that for every $j, 1 \leqslant j \leqslant r$, $b_j = \sum_{l=1}^{s_j} m_{jl}$, where $m_{jl}$ is a monomial in $\{x, y, z, t\}$ with $w_R(m_{jl}) = n_j$, and $n_1 < \cdots < n_r$.

If $n = n_r$, then we are done. If not, then $v_r := w_R(b_r) < n_r$ as $n < n_r$. By Lemma 3.3.5 there exists a representation of $b_r$ as sum of monomials, $b_r = \sum_{l=1}^{s'} m'_{rl}$ with $w_R(m'_{rl}) \leqslant v_r$. Hence

$$b = \widetilde{b_1} + \cdots + \widetilde{b_{r'}},$$

such that for every $i, 1 \leqslant i \leqslant r'$ we get $\widetilde{b_i}$ to be a sum of monomials of $w_R$-degree $n'_i$, where $n'_1 < \cdots < n'_{r'}$ and $n'_{r'} < n_r$. Hence repeating the above process finitely many times we get a representation of $b$ as a sum of monomials, each of them having $w_R$-degree not more than $n$. Thus, the result follows. □

The next lemma describes the structure of the associated graded ring of $A_R$ with respect to the filtration $\{A_R^n\}_{n\in\mathbb{Z}}$.

**Lemma 3.3.7.** *Let $\tilde{A} := \bigoplus_{n\in\mathbb{Z}} \frac{A_R^n}{A_R^{n-1}}$ be the associated graded ring of $A$ with respect to the filtration $\{A_R^n\}_{n\in\mathbb{Z}}$. Then*

$$\tilde{A} \cong \frac{R[X, Y, Z, T]}{(\alpha_1(0)X^dY - F(0, Z, T))}.$$

*Proof.* Let $\{c_1, \ldots, c_n\}$ be a generating set of the affine $k$-algebra $R$. Since $\{A_R^n\}_{n\in\mathbb{Z}}$ is a proper admissible $\mathbb{Z}$-filtration with respect to the generating set $\{c_1, \ldots, c_n, x, y, z, t\}$ we have $\tilde{A} = R[\overline{x}, \overline{y}, \overline{z}, \overline{t}]$, where $\overline{x}, \overline{y}, \overline{z}, \overline{t}$ denote the images of $x, y, z, t$ in $\mathrm{gr}(A_R)$ respectively, via the natural map $\rho : A_R \to \mathrm{gr}(A_R)$ (cf. Remark 2.3.5).

Now $\alpha_1(0), x^dy, F(0, z, t) \in A_0^R \setminus A_{-1}^R$. Since $x^d\alpha_1(x)y - F(x, z, t) = 0$ in $A_R$ by (3.3.2), $\alpha_1(0)x^dy - F(0, z, t) \in A_{-1}^R$. Therefore, $\alpha_1(0)\overline{x}^d\overline{y} - F(0, \overline{z}, \overline{t}) = 0$ in $\mathrm{gr}(A_R)$. Hence we have the following onto $k$-homomorphism

$$\pi : \frac{R[X, Y, Z, T]}{(\alpha_1(0)X^dY - F(0, Z, T))} \to \mathrm{gr}(A_R).$$

Now both L.H.S and R.H.S are integral domains with $\operatorname{tr.deg}_k \operatorname{gr}(A_R) = \operatorname{tr.deg}_k R + 3 < \infty$, and thus their dimensions are equal. Therefore, $\pi$ is an isomorphism. □

Now we note down a few useful remarks for $A_R$ with $\gcd_{R[Z,T]}(\alpha_1(0), F(0, Z, T)) = 1$.

**Remark 3.3.8.** (i) Let $m$ be a monomial of $A_R = R[x, y, z, t]$. If $w_R(m) < 0$ then $m \in xA_R$. If $w_R(m) > 0$ then $y \mid m$.

(ii) For any $g \in A_R$, if $w_R(g) < 0$ then $g = \sum_{i=1}^{n} m_i$, where $m_i$ is a monomial in $\{x, y, z, t\}$ with $w_R(m_i) \leqslant w_R(g)$. Thus $g \in xA_R$.

(iii) For any $g \in A_R$, if $w_R(g) > 0$ then $\rho(y) \mid \rho(g)$, where $\rho : A_R \to \operatorname{gr}(A_R)$ is the natural map.

(iv) Let $g \in A_R$ be such that $w_R(g) = 0$. Then by Theorem 3.3.1, $g = \sum_{i=1}^{n} m_i$, where $m_i$'s are monomials in $A = R[x, y, z, t]$ with $w_R(m_i) \leqslant w_R(g) = 0$. Hence $g \in R[x, x^d y, z, t]$, as for any monomial $m \in R[x, y, z, t]$ with $w_R(m) \leqslant 0$, $m \in R[x, x^d y, z, t]$. Therefore, there exists $r \geqslant 0$ such that $\alpha_1(0)^r g \in R[x, \alpha_1(0)x^d y, z, t]$. By the equality (3.3.2), $\alpha_1(0)x^d y = F(0, z, t) + xv$ for some $v \in R[x, x^d y, z, t]$. Thus, it follows that

$$\alpha_1(0)^r g = g_1(z, t) + xu(x, x^d y, z, t),$$

for some $g_1 \in R[z, t]$ and $u \in R[x, x^d y, z, t]$ with $w_R(u) \leqslant 0$.

## 3.4 On Theorem C

Theorem C will be proved for the domains of the form

$$A = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - F(X_1, \ldots, X_m, Z, T))} \tag{3.4.1}$$

when $\alpha(X_1, \ldots, X_m)$ satisfies certain divisibility hypothesis in $k[X_1, \ldots, X_m]$. For convenience, we call a polynomial $\alpha$ satisfying the hypothesis a "**r**-divisible polynomial". We now give the precise definition.

**Definition 3.4.1.** Let $R$ be an integral domain. For $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 1}$, we say a non-zero polynomial $\alpha \in R^{[m]}$ is **r**-*divisible* if there exists a system of coordinates $\{X_1, \dots, X_m\}$ of $R^{[m]}$ such that

$$\alpha(X_1, \dots, X_m) = X_1^{r_1}\alpha_1(X_1, \dots, X_m), \text{ for some } \alpha_1 \in R^{[m]} \text{ with } X_1 \nmid \alpha_1;$$

for any $i \in \{2, \dots, m\}$,

$$\alpha_i(X_i, \dots, X_m) := \alpha_{i-1}(0, X_i, \dots, X_m)/X_i^{r_i} \in R[X_i, \dots, X_m] \text{ with } X_i \nmid \alpha_i$$

and $\alpha_{m+1} := \alpha_m(0) \in R \setminus \{0\}$,

i.e., for $i \in \{1, \dots, m\}$, there exist $\beta_i \in R^{[m-i+1]}$ such that

$$\begin{aligned}
\alpha &= X_1^{r_1}\alpha_1(X_1, \dots, X_m) \\
&= X_1^{r_1}(X_1\beta_1(X_1, \dots, X_m) + \alpha_1(0, X_2, \dots, X_m)) \\
&= X_1^{r_1}(X_1\beta_1(X_1, \dots, X_m) + X_2^{r_2}\alpha_2(X_2, \dots, X_m)) \\
&= X_1^{r_1}(X_1\beta_1(X_1, \dots, X_m) + X_2^{r_2}(X_2\beta_2(X_2, \dots, X_m) + X_3^{r_3}\alpha_3(X_3, \dots, X_m))) \\
&= \dots \\
&= X_1^{r_1}(X_1\beta_1(X_1, \dots, X_m) + \dots + X_{m-1}^{r_{m-1}}(X_{m-1}\beta_{m-1}(X_{m-1}, X_m) + \\
&\quad X_m^{r_m}\alpha_m(X_m)) \dots) \\
&= X_1^{r_1}(X_1\beta_1(X_1, \dots, X_m) + \dots + X_{m-1}^{r_{m-1}}(X_{m-1}\beta_{m-1}(X_{m-1}, X_m) + \\
&\quad X_m^{r_m}(X_m\beta_m(X_m) + \alpha_{m+1})) \dots).
\end{aligned}$$

**Remark 3.4.2.** Note that for $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{Z}^m_{\geqslant 1}$ and a system of coordinates $\{X_1, \dots, X_m\}$ in $R^{[m]}$ we can conclude that $\alpha$ is $\mathbf{r}$-divisible if and only if $r_1$ is the highest power of $X_1$ dividing $\alpha(X_1, \dots, X_m)$ and $r_j$ is the highest power of $X_j$ dividing $\alpha_{j-1}(0, X_j, \dots, X_m)$ in $R[X_1, \dots, X_m]$, for all $j \in \{2, \dots, m\}$.

We illustrate the above concept with two examples which will be used later.

**Example 3.4.3.** $X^2(1 + X)^2 \in k[X]$ is $(2)$-divisible in $\{X\}$ in $k^{[1]}$.

**Example 3.4.4.** $X_1X_2^2(X_1 + X_2^2) \in k[X_1, X_2]$ is $(1, 4)$-divisible in $\{X_1, X_2\}$ in $k^{[2]}$ and is $(2, 2)$-divisible in $\{X_2, X_1\}$ in $k^{[2]}$.

Next we recall the definition of Generalised Asanuma varieties from [25, Page 3].

**Definition 3.4.5.** A variety whose coordinate ring is of the following form

$$B := \frac{k[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m}Y - F(X_1, \dots, X_m, Z, T))} \quad \text{for some} \quad r_i > 1, 1 \leqslant i \leqslant m, \tag{3.4.2}$$

with $f(Z,T) := F(0,\ldots,0,Z,T) \neq 0$ is known as a *Generalised Asanuma variety* and we call an affine domain of the form $B$ to be a *Generalised Asanuma domain.*

Let $x_1,\ldots,x_m,y,z,t$ denote the images of $X_1,\ldots,X_m,Y,Z,T$ in $B$ respectively. Note that the coefficient of $Y$ in the defining equation of a Generalised Asanuma variety is "$X_1^{r_1}\cdots X_m^{r_m}$" with $r_i > 1$ for all $i, 1 \leqslant i \leqslant m$ and observe that it is $\mathbf{r}$-divisible in $\{X_1,\ldots,X_m\}$, where $\mathbf{r} = (r_1,\ldots,r_m)$.

We recall below a few results from [28] and [29], [24] on Generalised Asanuma domains. We first state a lemma about its Derksen invariant ([29, Lemma 3.3]).

**Lemma 3.4.6.** *Let $B$ be a Generalised Asanuma domain as in* (3.4.2)*. Then* $k[x_1,\ldots,x_m,z,t] \subseteq \mathrm{DK}(B)$.

Now we quote a technical result obtained by combining results from [28], [29] and [24].

**Proposition 3.4.7.** *Let $B$ be a Generalised Asanuma domain as in* (3.4.2)*. Suppose* $k[x_1,\ldots,x_m,z,t] \subsetneqq \mathrm{DK}(B)$*. Then the following statements hold:*

(i) *If $m = 1$ or if $k$ is an infinite field, then there exist $Z_1, T_1 \in k[Z,T]$ and $a_0, a_1 \in k^{[1]}$ such that $k[Z,T] = k[Z_1,T_1]$ and $f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1$.*

(ii) *If $f$ is a line over $k$ in $k^{[2]}$, i.e., $\frac{k[Z,T]}{(f)} = k^{[1]}$ then $k[Z,T] = k[f]^{[1]}$.*

*Proof.* **(i)** The case $m = 1$ is done in [28, Proposition 3.7]. As observed in [24, Proposition 2.10], the case when $k$ is an infinite field follows from [29, Proposition 3.4(i)].

**(ii)** The result follows from [29, Proposition 3.4] when $k$ is infinite and [24, Proposition 2.12] when $k$ is a finite field. □

Next we record another technical lemma from [24, Lemma 3.13].

**Lemma 3.4.8.** *Let $B$ be a Generalised Asanuma domain in* (3.4.2)*. Suppose $m \in \mathbb{Z}_{\geqslant 2}$ and $k[x_1,\ldots,x_m,z,t] \subsetneqq DK(B)$. Then there exist an integer $l \in \{1,\ldots,m\}$ and an integral domain $B_l$ of the form*

$$B_l = \frac{k(X_l)[X_1,\ldots,X_{l-1},X_{l+1},\ldots,X_m,Y,Z,T]}{(X_1^{r_1}\cdots X_m^{r_m}Y - f(Z,T))}$$

*such that* $\mathrm{DK}(B_l) = B_l$.

The next proposition illustrates that using suitable degree functions and applying Theorem 3.3.1 successively to a ring as in (3.4.1) with $\alpha$ being a $\mathbf{r}$-divisible polynomial, we can obtain a Generalised Asanuma domain as the last associated graded domain.

**Proposition 3.4.9.** *Let*

$$A = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - F(X_1, \ldots, X_m, Z, T))},$$

*be a domain such that*

(a) $f(Z,T) := F(0, \ldots, 0, Z, T) \neq 0.$

(b) *For* $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 1}$ *the polynomial* $\alpha(X_1, \ldots, X_m)$ *is* $\mathbf{r}$*-divisible in the system of coordinates* $\{X_1, \ldots, X_m\}$ *in* $k^{[m]}$*. Let* $\alpha_1 := \alpha(X_1, \ldots, X_m)/X_1^{r_1}$ $\alpha_i := \alpha_{i-1}(0, X_i, \ldots, X_m)/X_i^{r_i}$, $2 \leqslant i \leqslant m$ *and* $\alpha_{m+1} := \alpha_m(0) \in k^*$.

(c) $\gcd(\alpha_i(0, X_{i+1}, \ldots, X_m), F(0, \ldots, 0, X_{i+1}, \ldots, X_m, Z, T)) = 1$, $1 \leqslant i \leqslant m$ *in* $k[X_1, \ldots, X_m, Z, T]$.

*Then we can define a degree function* $w_j$ *on* $A_{j-1}$ *such that* $w_j$ *induces an admissible* $\mathbb{Z}$*-filtration on* $A_{j-1}$*, where* $A_0 := A$ *and* $A_j := \mathrm{gr}(A_{j-1})$ *(the associated* $\mathbb{Z}$*-graded integral domain with respect to* $w_j$*), for each* $j$, $1 \leqslant j \leqslant m$*, such that*

$$A_m \cong \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} X_2^{r_2} \ldots X_m^{r_m} Y - f(Z,T))}.$$

*Proof.* Since $A_0 = A$ is an affine domain and $X_1 \mid \alpha$, we have $F(0, X_2, \ldots, X_m, Z, T) \neq 0$. Now $\alpha_0 := \alpha$ is $\mathbf{r}$-divisible in the system of coordinates $\{X_1, \ldots, X_m\}$ in $k^{[m]}$. Hence, $\alpha_1(0, X_2, \ldots, X_m) = X_2^{r_2}\alpha_2(X_2, \ldots, X_m)$ is $\tilde{\mathbf{r}}$-divisible in the system of coordinates $\{X_2, \ldots, X_m\}$, where $\tilde{\mathbf{r}} = (r_2, \ldots, r_m) \in \mathbb{Z}^{m-1}_{\geqslant 1}$. Let $x_{10}, \ldots, x_{m0}, y_0, z_0, t_0$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in $A_0$ respectively. We first consider the degree function $w_1$ on $A_0$ defined by

$$w_1(x_{10}) = -1,\ w_1(y_0) = r_1,\ w_1(z_0) = w_1(t_0) = w_1(x_{i0}) = 0 \quad \text{for} \quad 2 \leqslant i \leqslant m.$$

Since $\gcd(\alpha_1(0, X_2, \ldots, X_m), F(0, X_2, \ldots, X_m, Z, T)) = 1$, by Theorem 3.3.1 $w_1$ induces an admissible $\mathbb{Z}$-filtration on $A_0$ with respect to the generating set $\{x_{10}, \ldots, x_{m0}, y_0, z_0, t_0\}$ such that the associated graded domain is

$$A_1 = \mathrm{gr}(A_0) = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} X_2^{r_2}\alpha_2(X_2, \ldots, X_m)Y - F(0, X_2, \ldots, X_m, Z, T))}.$$

Let $x_{11}, \ldots, x_{m1}, y_1, z_1, t_1$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in $A_1$ respectively. We consider the function $w_2$ on $A_1$, defined as follows:

$$w_2(x_{21}) = -1,\ w_2(y_1) = r_2,\ w_2(z_1) = w_2(t_1) = w_2(x_{i1}) = 0 \quad \text{for} \quad i \in \{1, 3, \ldots, m\}.$$

Since $\gcd(X_1^{r_1}\alpha_2(0, X_3, \ldots, X_m), F(0, 0, X_3, \ldots, X_m, Z, T)) = 1$ in $k[X_1, \ldots, X_m, Z, T]$ by Theorem 3.3.1, $w_2$ induces an admissible $\mathbb{Z}$-filtration on $A_1$ such that the associated graded domain is

$$A_2 = \operatorname{gr}(A_1) = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1}X_2^{r_2}X_3^{r_3}\alpha_3(X_3, \ldots, X_m)Y - F(0, 0, X_3, \ldots, X_m, Z, T))}.$$

Thus proceeding successively for any $j \in \{2, \ldots, m\}$, we get domains

$$A_{j-1} \quad = \quad \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_j^{r_j}\alpha_j(X_j, \ldots, X_m)Y - F(0, \ldots, 0, X_j, \ldots, X_m, Z, T))}.$$

Let $l = j-1$ and $x_{1l}, \ldots, x_{ml}, y_l, z_l, t_l$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in $A_{j-1}$ respectively. Now we define a degree function $w_j$ on $A_{j-1}$ as follows:

$$w_j(x_{jl}) = -1,\ w_j(y_l) = r_j,\ w_j(z_l) = w_j(t_l) = w_j(x_{il}) = 0,\ i \in \{1, \ldots, m\} \setminus \{j\}.$$

Since $\gcd(X_1^{r_1} \cdots X_l^{r_l}\alpha_j(0, X_{j+1}, \ldots, X_m), F(0, \ldots, 0, X_{j+1}, \ldots, X_m) = 1$ by Theorem 3.3.1, $w_j$ induces an admissible $\mathbb{Z}$-filtration on $A_{j-1}$ for which the associated graded domain is of the form

$$A_j = \operatorname{gr}(A_{j-1}) = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_{j+1}^{r_{j+1}}\alpha_{j+1}(X_{j+1}, \ldots, X_m)Y - F(0, \ldots, 0, X_{j+1}, \ldots, X_m, Z, T))}.$$

Since $\alpha_{m+1} \in k^*$, at the $m$th step we get the domain

$$A_m \cong \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1}X_2^{r_2} \cdots X_m^{r_m}Y - f(Z, T))}, \quad f(Z, T) = F(0, \ldots, 0, Z, T),$$

as an associated graded domain of $A_{m-1}$ with respect to the degree function $w_m$. □

As an illustration of the hypothesis in Proposition 3.4.9 we note the following implication of the above result.

**Corollary 3.4.10.** *Let $A$ be an affine domain as in* (3.1.1) *such that*

(a) *For $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 1}$ the polynomial $\alpha(X_1, \ldots, X_m)$ is $\mathbf{r}$-divisible in* *the system of coordinates* $\{X_1, \ldots, X_m\}$ *in* $k^{[m]}$.

(b) $X_1 \mid h(X_1, \ldots, X_m, Z, T)$ *in* $k[X_1, \ldots, X_m, Z, T]$.

*Then we can define a degree function* $w_j$ *on* $A_{j-1}$ *such that* $w_j$ *induces an admissible* $\mathbb{Z}$*-filtration on* $A_{j-1}$*, where* $A_0 := A$ *and* $A_j := \mathrm{gr}(A_{j-1})$ *for each* $j$*,* $1 \leqslant j \leqslant m$*, such that*

$$A_m \cong \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} X_2^{r_2} \ldots X_m^{r_m} Y - f(Z, T))}.$$

We now establish a more general version of Theorem C. For convenience we split the result in two parts. We first prove Theorem C for the case $\mathrm{ML}(A) = k$. In the statement and proof below we use the notation as in Proposition 3.4.9.

**Theorem 3.4.11.** *Let* $A$ *be as in Proposition* 3.4.9 *with* $\mathbf{r} \in \mathbb{Z}^m_{\geqslant 2}$*. Suppose* $\mathrm{ML}(A) = k$*. Then the following statements hold:*

(i) *There exists an integral domain* $\tilde{A}$ *of the form*

$$\tilde{A} = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} X_2^{r_2} \ldots X_m^{r_m} Y - f(Z, T))} \quad \text{with} \quad f(Z, T) = F(0, \ldots, 0, Z, T)$$

*such that* $\mathrm{DK}(\tilde{A}) = \tilde{A}$.

(ii) *When* $m = 1$ *or* $k$ *is an infinite field then there exist a system of coordinates* $\{Z_1, T_1\}$ *of* $k[Z, T]$ *and* $a_0, a_1 \in k^{[1]}$*, such that* $f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1$.

(iii) *When* $f$ *is a line over* $k$ *in* $k^{[2]}$ *i.e.,* $\frac{k[Z,T]}{(f)} = k^{[1]}$*, then* $k[Z, T] = k[f]^{[1]}$.

*Proof.* Recall that $f(Z, T) \neq 0$ and $x_{10}, \ldots, x_{m0}, y_0, z_0, t_0$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in $A$ $(=A_0)$ respectively. For convenience we rename $x_{i0}$ to be $x_i$, for all $i \in \{1, \ldots, m\}$ and $y_0, z_0, t_0$ to be $y, z, t$ respectively i.e., now $x_1, \ldots, x_m, y, z, t$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in $A(= A_0)$ respectively. Since $\mathrm{ML}(A) = k$, there exists a non-trivial exponential map $\phi$ on $A$ such that $x_1 \notin A^{\phi}$.

Recall that $w_1$ is the degree function on $A$ defined as follows:

$$w_1(x_1) = -1, w_1(y) = r_1, w_1(z) = w_1(t) = w_1(x_i) = 0 \text{ for all } i \in \{2, \ldots, m\}.$$

Since $\alpha$ is $\mathbf{r}$-divisible in the system of coordinates $\{X_1, \ldots, X_m\}$, we have $\alpha = X_1^{r_1}\alpha_1$ for some $\alpha_1 \in k^{[m]}$. Since $\gcd(\alpha_1(0, X_2, \ldots, X_m), F(0, X_2, \ldots, X_m, Z, T)) = 1$, by Theorem 3.3.1 the degree function $w_1$ on $A$ induces an admissible $\mathbb{Z}$-filtration on $A$ with respect to $\{x_1, \ldots, x_m, y, z, t\}$ such that

$$A_1 = \mathrm{gr}(A) \cong \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1}\alpha_1(0, X_2, \ldots, X_m)Y - F(0, X_2, \ldots, X_m, Z, T))}.$$

For $a \in A$, let $\overline{a}$ denote its image in $A_1$. For convenience, we rename $x_{i1}$ to be $\overline{x_i}$ for all $i \in \{1, \ldots, m\}$ and $y_1, z_1, t_1$ to be $\overline{y}, \overline{z}, \overline{t}$ in $A_1$ respectively i.e., now $\overline{x_1}, \ldots, \overline{x_m}, \overline{y}, \overline{z}, \overline{t}$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in $A_1$ respectively. By Theorem 2.3.8, $\phi$ induces a non-trivial exponential map $\phi_1$ on $A_1$. We shall show that $\overline{y} \in A_1^{\phi_1}$.

We first note that $w_1(a) \geqslant 0$ for all $a \in A^{\phi}$. Suppose not. Therefore, there exists $a \in A^{\phi}$ such that $w_1(a) < 0$. Then $x_1 \mid a$ in $A$ (cf. Remark 3.3.8(ii)) and hence $x_1 \in A^{\phi}$ (cf. Lemma 2.2.3(i)) which contradicts the choice of $\phi$.

Suppose, if possible, $w_1(a) = 0$ for all $a \in A^{\phi}$. Since $\mathrm{tr.deg}_k A^{\phi} = m+1$ (cf. Lemma 2.2.3(ii)), there exist $m+1$ algebraically independent elements $f_1, \ldots, f_{m+1} \in A^{\phi}$ with $w_1(f_i) = 0$, for $1 \leqslant i \leqslant m+1$. Set $\beta := \alpha_1(0, x_2, \ldots, x_m)$. By Remark 3.3.8(iv), for each $i \in \{1, \ldots, m+1\}$, there exists $s_i \in \mathbb{Z}_{\geqslant 0}$ such that

$$\beta^{s_i} f_i = p_{i0}(x_2, \ldots, x_m, z, t) + x_1 q_i(x_1, \ldots, x_m, y, z, t), \text{ where } w_1(q_i) \leqslant 0.$$

Then for any $i \in \{1, \ldots, m+1\}$,

$$\overline{\beta}^{s_i}\overline{f_i} = p_{i0}(\overline{x_2}, \ldots, \overline{x_m}, \overline{z}, \overline{t}), \text{ i.e., } \overline{f_i} = \frac{p_{i0}(\overline{x_2}, \ldots, \overline{x_m}, \overline{z}, \overline{t})}{\overline{\beta}^{s_i}} \tag{3.4.3}$$

and $\overline{f_i} \in A_1^{\phi_1}$ for $1 \leqslant i \leqslant m+1$ (cf. Theorem 2.3.8). Thus

$$k[\overline{f_1}, \ldots, \overline{f_{m+1}}] \subseteq D := k\left[\overline{x_2}, \ldots, \overline{x_m}, \overline{z}, \overline{t}, \frac{1}{\overline{\beta}}\right].$$

Suppose, if possible, that $\{\overline{f_1}, \ldots, \overline{f_{m+1}}\}$ is an algebraically independent set. Since

$$k^{[m+1]} = k[\overline{x_2}, \ldots, \overline{x_m}, \overline{z}, \overline{t}] \hookrightarrow A_1 \hookrightarrow k\left[\overline{x_1}, \overline{x_2} \ldots, \overline{x_m}, \overline{z}, \overline{t}, \frac{1}{\overline{x_1}^{r_1}\alpha_1(0, \overline{x_2}, \ldots, \overline{x_m})}\right]$$

and

$$\overline{x_1}^{r_1}\alpha_1(0, \overline{x_2}, \ldots, \overline{x_m})\overline{y} = F(0, \overline{x_2}, \ldots, \overline{x_m}, \overline{z}, \overline{t}),$$

we have, $\operatorname{tr.deg}_k D = m+1$. Therefore, $D$ is algebraic over $k[\overline{f_1}, \dots, \overline{f_{m+1}}]$ and hence $k[\overline{x_2}, \dots, \overline{x_m}, \overline{z}, \overline{t}] \subseteq A_1^{\phi_1}$ (cf. Lemma 2.2.3(i)). Thus $\overline{x_1}, \overline{y} \in A_1^{\phi_1}$, which contradicts the fact that $\phi_1$ is non-trivial. Therefore, there exists a non-zero $Q \in k^{[m+1]}$ such that $Q(\overline{f_1}, \dots, \overline{f_{m+1}}) = 0$. Hence it follows that

$$Q\left(\frac{p_{10}(\overline{x_2}, \dots, \overline{x_m}, \overline{z}, \overline{t})}{\overline{\beta}^{s_1}}, \dots, \frac{p_{(m+1)0}(\overline{x_2}, \dots, \overline{x_m}, \overline{z}, \overline{t})}{\overline{\beta}^{s_{m+1}}}\right) = 0. \tag{3.4.4}$$

Now since $f_i = \frac{p_{i0}(x_2,\dots,x_m,z,t)}{\beta^{s_i}} + \frac{x_1 q_i}{\beta^{s_i}}$, $1 \leqslant i \leqslant m$, using (3.4.4), we have $Q(f_1, \dots, f_{m+1}) = \frac{x_1 q}{\beta^s}$ for some $q \in A$ and $s \in \mathbb{Z}_{\geqslant 0}$, i.e., $x_1 q \in \beta^s A$. Now since

$$\frac{A}{x_1 A} = \frac{k[X_2, \dots, X_m, Y, Z, T]}{(F(0, X_2, \dots, X_m, Z, T))},$$

$\beta$ is a non-zero divisor in $\frac{A}{x_1 A}$. Thus it follows that $Q(f_1, \dots, f_{m+1}) = x_1 \widetilde{q}$ for some $\widetilde{q} \in A$. Since $Q(f_1, \dots, f_{m+1}) \in A^{\phi}$, it implies that $x_1 \in A^{\phi}$ (cf. Lemma 2.2.3(i)), a contradiction to the choice of $\phi$.

Therefore, there exists an $a \in A^{\phi}$ with $w_1(a) > 0$. Hence $\overline{y} \mid \overline{a}$ (cf. Remark 3.3.8(iii)) and thus $\overline{y} \in A_1^{\phi_1}$.

By Proposition 3.4.9, there exists an admissible $\mathbb{Z}$-filtration induced by the degree function $w_j$ on $A_{j-1}$ for all $j, 1 \leqslant j \leqslant m$, where $A_0 = A$ and $A_i = \operatorname{gr}(A_{i-1})$, $1 \leqslant i \leqslant m$ such that

$$\tilde{A} := A_m \cong \frac{k[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} X_2^{r_2} \dots X_m^{r_m} Y - f(Z, T))},$$

with $f(Z,T) = F(0, \dots, 0, Z, T)$. Since $\overline{y} (= y_1) \in A_1^{\phi_1}$, by repeated application of Theorem 2.3.8, we see that $\phi_1$ induces a non-trivial exponential map $\tilde{\phi}$ on $\tilde{A}$ such that $y_m \in \tilde{A}^{\tilde{\phi}}$. Hence by Lemma 3.4.6, $\operatorname{DK}(\tilde{A}) = \tilde{A}$. Thus the first statement is proved. Now by Proposition 3.4.7, the last two statements follow. □

We now prove Theorem C for the case $\operatorname{DK}(A) = A$. In fact we prove a more general statement where the hypothesis $\operatorname{DK}(A) = A$ is replaced by a weaker condition that $\operatorname{DK}(A)$ contains an element of positive degree with respect to a certain degree function. In the statement and proof below we use the notation as in Proposition 3.4.9.

**Theorem 3.4.12.** *Let $A$ be as in Proposition 3.4.9 with $\mathbf{r} \in \mathbb{Z}^m_{\geqslant 2}$. Recall that $w_1$ is the degree function on $A$ defined as follows, $w_1(x_{10}) = -1, w_1(y_0) =$*

$r_1, w_1(x_{i0}) = w_1(z_0) = w_1(t_0) = 0$, $2 \leqslant i \leqslant m$. *Suppose* $\mathrm{DK}(A)$ *contains an element with positive* $w_1$*-degree. Then the following statements hold:*

(i) *There exists an integral domain* $\tilde{A}$ *of the form*

$$\frac{k[X_1,\ldots,X_m,Y,Z,T]}{(X_1^{r_1}X_2^{r_2}\ldots X_m^{r_m}Y - f(Z,T))} \quad \textit{where} \quad f(Z,T) = F(0,\ldots,0,Z,T)$$

*such that* $\mathrm{DK}(\tilde{A}) = \tilde{A}$.

(ii) *When* $m = 1$ *or* $k$ *is an infinite field then there exist a system of coordinates* $\{Z_1, T_1\}$ *of* $k[Z,T]$ *and* $a_0, a_1 \in k^{[1]}$, *such that* $f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1$.

(iii) *When* $f$ *is a line over* $k$ *in* $k^{[2]}$ *i.e.,* $\frac{k[Z,T]}{(f)} = k^{[1]}$ *then* $k[Z,T] = k[f]^{[1]}$.

*Proof.* Recall that $x_{10},\ldots,x_{m0},y_0,z_0,t_0$ denote the images of $X_1,\ldots,X_m,Y,Z,T$ in $A$ ($=A_0$) respectively. For convenience we rename $x_{i0}$ to be $x_i$, for all $i \in \{1,\ldots,m\}$ and $y_0,z_0,t_0$ to be $y,z,t$ respectively i.e., now $x_1,\ldots,x_m,y,z,t$ denote the images of $X_1,\ldots,X_m,Y,Z,T$ in $A$ respectively. As $\alpha$ is $\mathbf{r}$-divisible with respect to $\{X_1,\ldots,X_m\}$, we have $\alpha = X_1^{r_1}\alpha_1$ for some $\alpha_1 \in k^{[m]}$. Since $\gcd(\alpha_1(0,X_2,\ldots,X_m), F(0,X_2,\ldots,X_m,Z,T)) = 1$ in $k[X_1,\ldots,X_m,Z,T]$, by Theorem 3.3.1 the degree function $w_1$ on $A$, defined by

$$w_1(x_1) = -1, w_1(y) = r_1, w_1(z) = w_1(t) = w_1(x_i) = 0 \text{ for all } i \in \{2,\ldots,m\},$$

induces an admissible $\mathbb{Z}$-filtration on $A$ with respect to $\{x_1,\ldots,x_m,y,z,t\}$ such that

$$A_1 := \mathrm{gr}(A) \cong \frac{k[X_1,\ldots,X_m,Y,Z,T]}{(X_1^{r_1}\alpha_1(0,X_2,\ldots,X_m)Y - F(0,X_2,\ldots,X_m,Z,T))}.$$

For $a \in A$, let $\overline{a}$ denote its image in $A_1$. For convenience we rename $x_{i1}$ to be $\overline{x_i}$ for all $i \in \{1,\ldots,m\}$ and $y_1,z_1,t_1$ to be $\overline{y},\overline{z},\overline{t}$ in $A_1$ respectively i.e., now $\overline{x_1},\ldots,\overline{x_m},\overline{y},\overline{z},\overline{t}$ denote the images of $X_1,\ldots,X_m,Y,Z,T$ in $A_1$ respectively. Since $\mathrm{DK}(A)$ contains an element of positive $w_1$-degree, there exist a non-trivial exponential map $\phi$ on $A$ and $a \in A$ with $w_1(a) > 0$ such that $a \in A^{\phi}$. Hence $\overline{y} \mid \overline{a}$ (cf. Remark 3.3.8(iii)). By Theorem 2.3.8, $\phi$ induces a non-trivial exponential map $\phi_1$ on $A_1$ such that $\overline{a} \in A_1^{\phi_1}$. Therefore $\overline{y} \in A_1^{\phi_1}$ (cf. Lemma 2.2.3(i)). By Proposition 3.4.9, there exists an admissible $\mathbb{Z}$-filtration induced by the degree function $w_j$ on $A_{j-1}, 1 \leqslant j \leqslant m$, where $A_0 = A$ and

$A_i = \mathrm{gr}(A_{i-1})$ for all $i, 1 \leqslant i \leqslant m$ such that

$$\tilde{A} := A_m \cong \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} X_2^{r_2} \ldots X_m^{r_m} Y - f(Z,T))} \quad \text{with} \quad r_i \geqslant 2,\, 1 \leqslant i \leqslant m.$$

Now by repeated application of Theorem 2.3.8, $\phi_1$ induces a non-trivial exponential map $\tilde{\phi}$ on $\tilde{A}$ such that $y_m \in \tilde{A}^{\tilde{\phi}}$. Hence by Lemma 3.4.6, $\mathrm{DK}(\tilde{A}) = \tilde{A}$. Thus the first statement is proved. Now by Proposition 3.4.7, the last two statements follow. □

As a consequence we see that if $f$ cannot be made linear by change of coordinates, then the ring $A$ is not an affine space. More precisely:

**Corollary 3.4.13.** *Let $A$ be an affine domain as in Proposition* 3.4.9 *with* $\mathbf{r} \in \mathbb{Z}^m_{\geqslant 2}$. *Suppose $F(X_1, \ldots, X_m, Z, T) = f_1(Z,T) + h(X_1, \ldots, X_m)$ for some $f_1 \in k^{[2]} \setminus \{0\}$ and $h \in k^{[m]}$. Moreover, let $f_1(Z,T) + h(0, \ldots, 0)$ has the property that for any $Z_1, T_1 \in \overline{k}[Z,T]$ with $\overline{k}[Z,T] = \overline{k}[Z_1,T_1]$ there does not exist $a_0, a_1 \in \overline{k}^{[1]}$ for which $f_1(Z,T) + h(0, \ldots, 0) = a_0(Z_1) + a_1(Z_1)T_1$. Then* $\mathrm{ML}(A) \neq k$ *and* $\mathrm{DK}(A) \neq A$. *In particular, $A \ncong_k k^{[m+2]}$.*

*Proof.* Suppose, if possible, that $\mathrm{ML}(A) = k$ or $\mathrm{DK}(A) = A$. Let $\overline{A} = A \otimes_k \overline{k}$. Then $\mathrm{ML}(\overline{A}) = \overline{k}$ or $\mathrm{DK}(\overline{A}) = \overline{A}$. Therefore, by Theorem 3.4.11(ii) or Theorem 3.4.12(ii), we have $f_1 + h(0, \ldots, 0) = a_0(Z_1) + a_1(Z_1)T_1$ for some $a_0, a_1 \in \overline{k}^{[1]}$ and $\overline{k}[Z,T] = \overline{k}[Z_1,T_1]$, a contradiction to the given hypothesis. Thus $\mathrm{ML}(A) \neq k$ and $\mathrm{DK}(A) \neq A$. Hence $A \ncong_k k^{[m+2]}$ (cf. Lemma 2.2.4). □

**Corollary 3.4.14.** *Let $A$ be as in Corollary* 3.4.10 *with* $\mathbf{r} \in \mathbb{Z}^m_{\geqslant 2}$. *Suppose that $f(Z,T)$ has the property that for any $Z_1, T_1 \in \overline{k}[Z,T]$ with $\overline{k}[Z,T] = \overline{k}[Z_1,T_1]$ there does not exist $a_0, a_1 \in \overline{k}^{[1]}$ for which $f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1$. Then* $\mathrm{ML}(A) \neq k$ *and* $\mathrm{DK}(A) \neq A$. *In particular, $A \ncong_k k^{[m+2]}$.*

*Proof.* Follows similarly like Corollary 3.4.13. □

**Remark 3.4.15.** The above results enable one to construct (or recognise) a large class of affine domains of Russell-Koras type (rings which are regular, factorial, contractible and satisfy many properties of polynomial rings, without being a polynomial ring itself). For instance, consider

$$A_1 := \frac{k[X,Y,Z,T]}{(X^2(X+1)^2Y - (Z^2+T^3) - Xh_1(X,Z,T))} \text{ for some } h_1 \in k^{[3]},$$

$$A_2 := \frac{k[X_1, X_2, Y, Z, T]}{(X_1X_2^2(X_1 + X_2^2)Y - (Z^2 + T^3) - h_2(X_1, X_2)} \text{ for some } h_2 \in k^{[2]},$$

and

$$A_3 := \frac{k[X_1, X_2, Y, Z, T]}{(X_1X_2^2(X_1 + X_2^2)Y - (Z^2 + T^3) - X_1h_3(X_1, Z, T) - X_2h_4(X_1, X_2, Z, T))},$$

for some $h_3 \in k^{[3]}$ and $h_4 \in k^{[4]}$. By suitable choices of $h_i$, $1 \leqslant i \leqslant 4$ the rings $A_i$ $(1 \leqslant i \leqslant 3)$ can be made to satisfy several properties of polynomial rings. Observe that $Z^2+T^3+\lambda$, for any $\lambda \in k$ cannot be made linear in any system of coordinates. Hence with the help of Examples 3.4.3 and 3.4.4 and Corollaries 3.4.14 and 3.4.13 we can show that both the Makar-Limanov invariants and the Derksen invariants of $A_i$, for all $i, 1 \leqslant i \leqslant 3$ are different from that of a polynomial ring; in particular, they are not polynomial rings.

## 3.5 On Theorem D

In this section we prove an extended version of Theorem D for an affine domain $A$ defined as follows

$$A = \frac{k[X_1, \dots, X_m, Y, Z, T]}{(\alpha(X_1, \dots, X_m)Y - f(Z, T) - h(X_1, \dots, X_m, Z, T))}, \tag{3.5.1}$$

where $H := \alpha(X_1, \dots, X_m)Y - f(Z, T) - h(X_1, \dots, X_m, Z, T)$ such that

(a) For $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{Z}^m_{\geqslant 2}$ the polynomial $\alpha$ is $\mathbf{r}$-divisible in the system of coordinates $\{X_1 - \lambda_1, \dots, X_m - \lambda_m\}$ for some $\lambda_i \in \overline{k}, 1 \leqslant i \leqslant m$.

(b) Every prime divisor of $\alpha(X_1, \dots, X_m)$ divides $h$ in $k[X_1, \dots, X_m, Z, T]$.

Let $x_1, \dots, x_m, y, z, t$ denote the images of $X_1, \dots, X_m, Y, Z, T$ in $A$ respectively. Since $A$ is a domain, $f(Z, T) \neq 0$. We now prove a crucial step towards proving Theorem D.

**Proposition 3.5.1.** *Let $A$ be an affine domain as in* (3.5.1) *and $k_1 := k(\lambda_1, \dots, \lambda_m)$. Suppose that $A^{[l]} = k^{[l+m+2]}$ for some $l \geqslant 0$ and either* $\mathrm{ML}(A) = k$ *or* $\mathrm{DK}(A) = A$. *Then $k_1[Z, T] = k_1[f]^{[1]}$.*

*Proof.* Since $\alpha$ is $\mathbf{r}$-divisible with respect to $\{X_1 - \lambda_1, \dots, X_m - \lambda_m\}$ in $k_1^{[m]}$, for each $i \in \{1, \dots, m\}$ there exists $\alpha_i \in k_1^{[m-i+1]}$ such that

$$\alpha(X_1, \dots, X_m) = (X_1 - \lambda_1)^{r_1}\alpha_1(X_1 - \lambda_1, \dots, X_m - \lambda_m),$$

where $(X_1-\lambda_1)\nmid\alpha_1$ and for $2\leqslant i\leqslant m$,

$$\alpha_{i-1}(0,X_i-\lambda_i,\ldots,X_m-\lambda_m)=(X_i-\lambda_i)^{r_i}\alpha_i(X_i-\lambda_i,\ldots,X_m-\lambda_m)$$

with $(X_i-\lambda_i)\nmid\alpha_i$. Therefore, without loss of generality, we can assume that

$$A':=A\otimes_k k_1\cong\frac{k_1[X_1,\ldots,X_m,Y,Z,T]}{(X_1^{r_1}\alpha_1(X_1,\ldots,X_m)Y-f(Z,T)-h_1(X_1,\ldots,X_m,Z,T))},$$

where $h_1(X_1,\ldots,X_m,Z,T):=h(X_1+\lambda_1,\ldots,X_m+\lambda_m,Z,T)$. Note that, now $X_1^{r_1}\alpha_1$ is **r**-divisible in the system of coordinates $\{X_1,\ldots,X_m\}$ in $k_1^{[m]}$. Next $\mathrm{ML}(A')=k_1$ or $\mathrm{DK}(A')=A'$ according as $\mathrm{ML}(A)=k$ or $\mathrm{DK}(A)=A$.

Suppose $k$ is an infinite field. Then by Theorem 3.4.11(ii) or Theorem 3.4.12(ii) according as $\mathrm{ML}(A')=k_1$ or $\mathrm{DK}(A')=A'$, there exist a system of coordinates $\{Z_1,T_1\}$ of $k_1[Z,T]$ and $a_0,a_1\in k_1^{[1]}$ such that $f=a_0(Z_1)+a_1(Z_1)T_1$. Since $A^{[l]}=k^{[l+m+2]}$ note that $(A')^{[l]}=k_1^{[l+m+2]}$. Therefore, by Corollary 3.2.5 we have $k_1[Z,T]=k_1[f]^{[1]}$.

When $k$ is a finite field, then from the above paragraph we have $\overline{k}[Z,T]=\overline{k}[f]^{[1]}$. Therefore, by Lemma 2.4.10 $k[Z,T]=k[f]^{[1]}$ and hence $k_1[Z,T]=k_1[f]^{[1]}$. □

Next we prove an easy lemma.

**Lemma 3.5.2.** *Let $f=a_0(Z)+a_1(Z)T$ for some $a_0,a_1\in k^{[1]}$, be an irreducible polynomial of $k[Z,T]$ with $\left(\frac{k[Z,T]}{(f)}\right)^*=k^*$. Then $k[Z,T]=k[f]^{[1]}$. In particular, if $\frac{k[Z,T]}{(f)}=k^{[1]}$ then $k[Z,T]=k[f]^{[1]}$.*

*Proof.* Consider the following two cases.

**Case I :** If $a_1(Z)=0$, then $f=a_0(Z)$ is an irreducible polynomial. As $\left(\frac{k[Z,T]}{(f)}\right)^*=k^*$ it follows that $f$ is linear in $Z$, and hence $k[Z,T]=k[f]^{[1]}$.

**Case II :** If $a_1(Z)\neq 0$, then $\gcd(a_0,a_1)=1$ in $k[Z]$ as $f$ is irreducible in $k[Z,T]$. Hence $\frac{k[Z,T]}{(f)}=k\left[Z,\frac{1}{a_1(Z)}\right]$. Now since $\left(\frac{k[Z,T]}{(f)}\right)^*=k^*$ it follows that $a_1(Z)\in k^*$ and hence $f$ is linear in $T$. Therefore $k[Z,T]=k[f]^{[1]}$. □

We now prove an extended version of Theorem D.

**Theorem 3.5.3.** *Let $A$ and $H$ be as in* (3.5.1) *and for each $i$, $1\leqslant i\leqslant m$, $\lambda_i$ is separable over $k$, i.e., $k_1:=k(\lambda_1,\ldots,\lambda_m)$ is separable over $k$. Let $E:=k[x_1,\ldots,x_m]$ be a subring of $A$. Then the following statements are equivalent.*

(i) $k[X_1, \dots, X_m, Y, Z, T] = k[X_1, \dots, X_m, H]^{[2]}$.

(ii) $k[X_1, \dots, X_m, Y, Z, T] = k[H]^{[m+2]}$.

(iii) $A = k[x_1, \dots, x_m]^{[2]} = E^{[2]}$.

(iv) $A = k^{[m+2]}$.

(v) $k[Z, T] = k[f(Z, T)]^{[1]}$.

(vi) $A^{[l]} = k^{[l+m+2]}$ *for some* $l \geqslant 0$ *and* $\mathrm{ML}(A) = k$.

(vii) $f(Z, T)$ *is a line over* $k$ *in* $k[Z, T]$ *and* $\mathrm{ML}(A) = k$.

(viii) $A$ *is an* $\mathbb{A}^2$*-fibration over* $E$ *and* $\mathrm{ML}(A) = k$.

(ix) (a) *When* $m = 1$, $A$ *is a UFD,* $\mathrm{ML}(A) = k$ *and* $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = {k_1}^*$.

(b) *When* $m > 1$, $A \otimes_k k_1$ *is a UFD,* $\mathrm{ML}(A) = k$ *and* $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = {k_1}^*$.

(x) $A^{[l]} = k^{[l+m+2]}$ *for some* $l \geqslant 0$ *and* $\mathrm{DK}(A) = A$.

(xi) $f(Z, T)$ *is a line over* $k$ *in* $k[Z, T]$ *and* $\mathrm{DK}(A) = A$.

(xii) $A$ *is an* $\mathbb{A}^2$*-fibration over* $E$ *and* $\mathrm{DK}(A) = A$.

(xiii) (a) *When* $m = 1$, $A$ *is a UFD,* $\mathrm{DK}(A) = A$ *and* $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = {k_1}^*$.

(b) *When* $m > 1$, $A \otimes_k k_1$ *is a UFD,* $\mathrm{DK}(A) = A$ *and* $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = {k_1}^*$.

(xiv) $A \otimes_k \overline{k}$ *is a UFD,* $f(Z, T) \notin k$ *and there exists a non-trivial exponential map* $\psi$ *on* $A$ *such that* $x_1, \dots, x_m, a \in A^{\psi}$ *with* $w_1(a) > 0$ ($w_1$ is defined as in Theorem 3.4.12).

*Proof.* Since $\alpha(X_1, \dots, X_m)$ is **r**-divisible in the system of coordinates $\{X_1 - \lambda_1, \dots, X_m - \lambda_m\}$ for each $i \in \{1, \dots, m\}$, there exists $\alpha_i \in k_1^{[m-i+1]}$ such that

$$\alpha(X_1, \dots, X_m) = (X_1 - \lambda_1)^{r_1} \alpha_1(X_1 - \lambda_1, \dots, X_m - \lambda_m),$$

where $(X_1 - \lambda_1) \nmid \alpha_1$ and for $2 \leqslant i \leqslant m$,

$$\alpha_{i-1}(0, X_i - \lambda_i, \dots, X_m - \lambda_m) = (X_i - \lambda_i)^{r_i} \alpha_i(X_i - \lambda_i, \dots, X_m - \lambda_m)$$

with $(X_i - \lambda_i) \nmid \alpha_i$. Therefore, without loss of generality, we can assume that

$$A' := A \otimes_k k_1 \cong \frac{k_1[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1}\alpha_1(X_1, \dots, X_m)Y - f(Z,T) - h_1(X_1, \dots, X_m, Z, T))},$$

where $h_1(X_1, \dots, X_m, Z, T) := h(X_1 + \lambda_1, \dots, X_m + \lambda_m, Z, T)$. Note that $X_1^{r_1}\alpha_1$ is now $\mathbf{r}$-divisible in the system of coordinates $\{X_1, \dots, X_m\}$.

We are going to prove the above equivalence of statements in the following sequence:

$$\begin{array}{ccccccccccccccccc}
\text{(i)} & \Longrightarrow & \text{(ii)} & \Rightarrow & \text{(iv)} & \Rightarrow & \text{(vi)} & \Rightarrow & \text{(vii)} & \Leftrightarrow & \text{(viii)} & \Longrightarrow & \text{(ix)} & \Longrightarrow & \text{(v)} & \Longrightarrow & \text{(i)} \\
\Downarrow & \searrow & & & & & & & & & & & & \nearrow & \Uparrow & & \\
\text{(xiv)} & & \text{(iii)} & \Rightarrow & \text{(x)} & \Longrightarrow & \text{(xi)} & \Longleftrightarrow & \text{(xii)} & \Longrightarrow & \text{(xiii)} & & & & \text{(xiv)} & &
\end{array}$$

**(i) ⇒ (xiv)**: Since $k[X_1, \dots, X_m, Y, Z, T] = k[X_1, \dots, X_m, H]^{[2]}$ it follows that $A = k[x_1, \dots, x_m]^{[2]} = k[x_1, \dots, x_m, h_1, h_2]$ for some $h_1, h_2 \in A$. Therefore, $A \otimes_k \overline{k} = \overline{k}[x_1, \dots, x_m]^{[2]}$ is a UFD. Further since $A$ is a domain with $A^* = k^*$, it follows that $f(Z,T) \notin k$. Now $y \in A$ and $w_1(y) > 0$. Since $w_1(x_i) \leqslant 0$, for all $i \in \{1, \dots, m\}$, either $w_1(h_1) > 0$ or $w_1(h_2) > 0$. Without loss of generality, let $w_1(h_1) > 0$. Now we consider the exponential map $\phi : A \to A[U]$ defined as follows

$$\phi(h_2) = h_2 + U, \quad \phi(h_1) = h_1, \quad \phi(x_i) = x_i \quad \text{for all} \quad i \in \{1, \dots, m\}.$$

Then one can easily check that $A^{\phi} = k[x_1, \dots, x_m, h_1]$.

**(vi) ⇒ (vii)** and **(x) ⇒ (xi)**: By Proposition 3.5.1 we have $k_1[Z,T] = k_1[f]^{[1]}$. Since $k_1$ is a separable extension over $k$, by Lemma 2.4.10it follows that $k[Z,T] = k[f]^{[1]}$, i.e., $\frac{k[Z,T]}{(f(Z,T))} = k^{[1]}$.

**(vii) ⇔ (viii)** and **(xi) ⇔ (xii)**: Let $\mathfrak{p} := (x_1 - \lambda_1, \dots, x_m - \lambda_m)\overline{k}[x_1, \dots, x_m] \cap E \in \operatorname{Max}(E)$. Then $\alpha \in \mathfrak{p}$ and $\frac{E_{\mathfrak{p}}}{\mathfrak{p}E_{\mathfrak{p}}} \hookrightarrow k_1$ is a separable field extension of $k$. Hence from Remark 3.1.3 the assertions follow.

**(viii) ⇔ (vii) ⇒ (ix)** and **(xii) ⇔ (xi) ⇒ (xiii)** : Since $f(Z,T)$ is a line over $k$ in $k[Z,T]$ it follows that, $\frac{k_1[Z,T]}{(f(Z,T))} = k_1^{[1]}$ and $f(Z,T)$ is irreducible in $\overline{k}[Z,T]$. Hence $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = k_1^*$ and by Lemma 3.1.6, we have $A \otimes_k L$ is a UFD for every algebraic extension $L$ of $k$. Thus the assertions follow.

**(xiii) ⇒ (v)**:
**(a)**: Let $m = 1$. Since $A$ is a UFD and $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = {k_1}^*$, by Proposition 2.4.16 $f(Z,T)$ is irreducible in $k_1[Z,T]$ as $k(\lambda_1) = k_1$. Now as $\mathrm{DK}(A) = A$, we have $\mathrm{DK}(A') = A'$. Thus, by applying Theorem 3.4.12(ii) for $A'$, we have $f(Z,T) = a_0(Z) + a_1(Z)T$ for some $a_0, a_1 \in k_1^{[1]}$. Therefore by Lemma 3.5.2 $k_1[Z,T] = k_1[f(Z,T)]^{[1]}$ and hence by Lemma 2.4.10, $f$ is a coordinate in $k[Z,T]$ as $k_1$ is separable over $k$.
**(b)**: Henceforth $m > 1$. Now $A'$ is a UFD and $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = {k_1}^*$. Therefore, by Lemma 3.1.4 $f(Z,T)$ is irreducible in $k_1[Z,T]$. Now since $\mathrm{DK}(A) = A$ it follows that $\mathrm{DK}(A') = A'$. Therefore, by Theorem 3.4.12(i) there exists an integral domain $\tilde{A}$ such that

$$\tilde{A} = \frac{k_1[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} X_2^{r_2} \ldots X_m^{r_m} Y - f(Z,T))},$$

with $\mathrm{DK}(\tilde{A}) = \tilde{A}$. By Lemma 3.4.8, there exist $l \in \{1, \ldots, m\}$ and an integral domain $\tilde{A}_l$ such that

$$\tilde{A}_l \cong \frac{k_1(X_l)[X_1, \ldots, X_{l-1}, X_{l+1}, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} X_2^{s_2} \ldots X_m^{s_m} Y - f(Z,T))}$$

and $\mathrm{DK}(\tilde{A}_l) = \tilde{A}_l$. Therefore, by Proposition 3.4.7(i), there exist $Z_1, T_1 \in k_1(X_l)[Z,T]$ and $a_0, a_1 \in k_1(X_l)^{[1]}$ such that $k_1(X_l)[Z,T] = k_1(X_l)[Z_1,T_1]$ and

$$f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1.$$

Now $\left(\frac{k_1[Z,T]}{(f(Z,T))}\right)^* = k_1^*$, therefore $\left(\frac{k_1(X_l)[Z,T]}{(f(Z,T))}\right)^* = \left(\frac{k_1[Z,T]}{(f(Z,T))}[X_l] \otimes_{k_1[X_l]} k_1(X_l)\right)^* = k_1(X_l)^*$. Since $f$ is irreducible in $k_1[Z,T]$ it follows that $f$ is irreducible in $k_1(X_l)[Z,T]$. Thus, by Lemma 3.5.2, $k_1(X_l)[Z,T] = k_1(X_l)[f(Z,T)]^{[1]}$ and hence by Lemma 2.4.10, $f$ is a coordinate of $k[Z,T]$.
**(ix) ⇒ (v)**: When $m = 1$, since $A$ is a UFD and $\left(\frac{k_1[Z,T]}{f(Z,T)}\right)^* = {k_1}^*$, it follows by Proposition 2.4.16 that $f(Z,T)$ is irreducible in $k_1[Z,T]$. Now as $\mathrm{ML}(A) = k$ we have $\mathrm{ML}(A') = k_1$. Thus, by applying Theorem 3.4.11(ii) for $A'$, we have $f(Z,T) = a_0(Z) + a_1(Z)T$ for some $a_0, a_1 \in k_1^{[1]}$.

Similarly, when $m > 1$, $A'$ is a UFD and $\left(\frac{k_1[Z,T]}{f(Z,T)}\right)^* = {k_1}^*$. Therefore, by Lemma 3.1.4 $f(Z,T)$ is irreducible in $k_1[Z,T]$. Now since $\mathrm{ML}(A) = k$, it follows that $\mathrm{ML}(A') = k_1$. Therefore, by Theorem 3.4.11(i), there exists an integral domain $\tilde{A}$ such that

$$\tilde{A} := \frac{k_1[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} X_2^{r_2} \ldots X_m^{r_m} Y - f(Z,T))}$$

with $\mathrm{DK}(\tilde{A}) = \tilde{A}$. Next following the same arguments as in (xiii) $\Rightarrow$ (v), the implication follows.

**(xiv) $\Rightarrow$ (v)**: We can extend $\psi$ to a non-trivial exponential map $\phi$ on $A'$ (cf. Lemma 2.2.3(iv)) such that $x_1, \ldots, x_m, a \in (A')^{\phi}$. Thus $a \in \mathrm{DK}(A')$ and $w_1(a) > 0$. The proof of Theorem 3.4.12 shows that there exists an affine domain $\tilde{A}$ of the form

$$\tilde{A} = \frac{k_1[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} X_2^{r_2} \ldots X_m^{r_m} Y - f(Z,T))}$$

on which $\phi$ induces a non-trivial exponential map $\phi_1$. For each $g \in A'$, let $\tilde{g}$ denotes the image of $g$ in $\tilde{A}$. Again the proof of Theorem 3.4.12 shows that $\tilde{y} \in \tilde{A}^{\phi_1}$ and $\tilde{g} \in \tilde{A}^{\phi_1}$, for all $g \in (A')^{\phi}$. Therefore, $\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y} \in \tilde{A}^{\phi_1}$ and hence $k_1[\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{z}, \tilde{t}] \subsetneqq \mathrm{DK}(\tilde{A})$ (cf. Lemma 3.4.6). Now we consider two cases:

**Case I :** Let $k$ be an infinite field. Without loss of generality, by Proposition 3.4.7(i), we have $f(Z,T) = a_0(Z) + a_1(Z)T$ for some $a_0, a_1 \in k_1^{[1]}$. Now since $A \otimes_k \overline{k}$ is a UFD and $f \notin k$, by Lemma 3.1.4, we have $f(Z,T)$ is irreducible in $\overline{k}[Z,T]$.

If $a_1(Z) = 0$, then $f(Z,T) = a_0(Z)$ is irreducible in $\overline{k}[Z,T]$. Therefore, $f$ is linear in $Z$. Hence $k[Z,T] = k[f(Z,T)]^{[1]}$.

Next let us assume that $a_1(Z) \neq 0$. By Lemma 2.2.3(iii), $\phi_1$ induces a non-trivial exponential map on

$$\tilde{A}_1 := \tilde{A} \otimes_{k_1[\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y}]} k_1(\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y}) \cong \frac{k_1(\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y})[Z,T]}{(\beta - a_0(Z) - a_1(Z)T)},$$

for some $\beta \in k_1(\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y})$. Therefore, $\tilde{A}_1$ is non-rigid. Since $f$ is irreducible in $\overline{k}[Z,T]$, $\gcd(a_0(Z), a_1(Z)) = 1$ in $\overline{k}[Z,T]$. Then it follows that $\gcd(\beta - a_0(Z), a_1(Z)) = 1$ in $\overline{k}(\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y})[Z,T]$. Hence $\tilde{A}_1 \otimes_k \overline{k}$ is integral domain with $\mathrm{tr.deg}_k \tilde{A}_1 = 1$. Therefore, by Remark 2.2.5(ii), $\tilde{A}_1 = k_1(\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y})^{[1]}$ and hence it follows that $a_1(Z) \in k^*$. Thus $k[Z,T] = k[f(Z,T)]^{[1]}$.

**Case II :** Let $k$ be a finite field. Then similarly like in Case I, we can conclude that $\overline{k}[Z,T] = \overline{k}[f(Z,T)]^{[1]}$. Since $\overline{k}$ is a separable extension of $k$, by Lemma 2.4.10, we have $k[Z,T] = k[f(Z,T)]^{[1]}$.

**(v) $\Rightarrow$ (i)** : Follows from Lemma 3.2.1.

The rest of the equivalences follows trivially. □

Note that the family of hypersurfaces

$$a_1(X_1)\cdots a_m(X_m)Y - f(Z,T) - h(X_1,\ldots,X_m,Z,T),$$

where every prime divisor of $a_1(X_1)\cdots a_m(X_m)$ in $k[X_1,\ldots,X_m]$ divides $h$ and every $a_i(X_i)$ has a separable multiple root $\lambda_i$ over $k$, are included in the family of hypersurfaces mentioned in the above theorem.

Now we state a result connecting Theorem 3.5.3 and the Zariski Cancellation Problem.

**Corollary 3.5.4.** *Let $k$ be a field of positive characteristic. Suppose that $f(Z,T)$ is a non-trivial line over $k$ in $k[Z,T]$. Then an affine domain $A$ satisfying the hypotheses of Theorem 3.5.3 provides a counterexample to the ZCP and to the $\mathbb{A}^2$-fibration problem over $k^{[m]}$ in positive characteristic.*

*Proof.* Since $f$ is a line over $k$ in $k^{[2]}$, by [29, Theorem 3.7] we have $A^{[1]} = k^{[m+3]}$ and by Lemma 3.1.2, $A$ is an $\mathbb{A}^2$-fibration over $k[x_1,\ldots,x_m]$. Now, $k[Z,T] \neq k[f]^{[1]}$ therefore by Theorem 3.5.3, $A \neq k^{[m+2]}$. Hence $A$ is a counterexample to the ZCP. Again, by Theorem 3.5.3, $A \ncong_k k[x_1,\ldots,x_m]^{[2]}$. Thus $A$ is a non-trivial $\mathbb{A}^2$-fibration over $k[x_1,\ldots,x_m]$. □

**Remark 3.5.5.** Let $k$ be a field of positive characteristic and $\Omega_0$ be the set of all affine domains satisfying all the hypotheses of Theorem 3.5.3. Then the family of rings

$$\begin{aligned}\Omega := \{A(r_1,\ldots,r_m,f) \mid (r_1,\ldots,r_m) \in \mathbb{Z}^m_{\geqslant 2}\\ \text{and } f(Z,T) \text{ is a non-trivial line over } k \text{ in } k[Z,T]\}\end{aligned}$$

is contained in $\Omega_0$, where

$$A(r_1,\ldots,r_m,f) := \frac{k[X_1,\ldots,X_m,Y,Z,T]}{(X_1^{r_1}\cdots X_m^{r_m}Y - f(Z,T))}.$$

By [24, Corollary 4.4], $\Omega$ contains infinitely many rings which are pairwise non-isomorphic and are counterexamples to the ZCP in positive characteristic.

Thus $\Omega_0$ also contains infinitely many non-isomorphic counterexamples to ZCP in positive characteristic in higher dimension ($\geqslant 3$).

**Remark 3.5.6.** Let $k$ be a field of characteristic zero and let $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 1}$. Suppose $H := \alpha(X_1, \ldots, X_m)Y - f(Z,T) - h(X_1, \ldots, X_m, Z, T)$ be such that $\alpha$ is $\mathbf{r}$-divisible in the system of coordinates $\{X_1, \ldots, X_m\}$ in $k^{[m]}$ and every prime factor of $\alpha$ divides $h$ in $k[X_1, \ldots, X_m, Z, T]$. Let $k[X_1, \ldots, X_m, Y, Z, T]/(H) = k^{[m+2]}$. If $r_i = 1$ for some $i, 1 \leqslant i \leqslant m$, then $\alpha$ has a simple prime factor in $k[X_1, \ldots, X_m]$ and hence by Theorem 3.2.7(I), $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, H]^{[2]}$. So over a field of characteristic zero, the hypothesis $r \in \mathbb{Z}_{\geqslant 2}$ can be relaxed.

## 3.6 Generalisation of Theorem B and D

In this section we discuss generalisation of Theorems B and D over Noetherian domains. In this section, throughout $R$ will denote a Noetherian domain. Let $A_R$ be a domain defined as follows

$$A_R = \frac{R[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - f(Z,T) - h(X_1, \ldots, X_m, Z, T))} \tag{3.6.1}$$

such that $\alpha$ can be expressed as product of prime elements of $R[X_1, \ldots, X_m]$ and each of the prime factors of $\alpha$ divides $h$ in $R[X_1, \ldots, X_m, Y, Z, T]$ and let $H_R = \alpha(X_1, \ldots, X_m)Y - f(Z,T) - h(X_1, \ldots, X_m, Z, T)$. Let $x_1, \ldots, x_m, y, z, t$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in $A_R$.

We first discuss a partial generalisation of Theorem B (Theorem 3.2.7).

**Theorem 3.6.1.** *Let $R$ be a Noetherian integral domain such that $\mathbb{Q} \subseteq R$. Suppose that the image of $\alpha(0, 0, \ldots, 0)$ is not a unit in $k(p)$, for all $p \in$ Spec$(R)$ and one of the following condition is satisfied:*

(a) *$\alpha$ has a simple prime factor in $R[X_1, \ldots, X_m] \setminus R$.*

(b) *$\alpha$ has a multiple prime factor $q \in R[X_1, \ldots, X_m] \setminus R$ such that $q^2 \mid h$ in $R[X_1, \ldots, X_m, Z, T]$.*

*Then the following statements are equivalent:*

(i) $R[X_1, \ldots, X_m, Y, Z, T] = R[X_1, \ldots, X_m, H_R]^{[2]}$.

(ii) $R[X_1, \ldots, X_m, Y, Z, T] = R[H_R]^{[m+2]}$.

(iii) $A_R = R[x_1, \ldots, x_m]^{[2]}$.

(iv) $A_R = R^{[m+2]}$.

(v) $R[Z,T] = R[f(Z,T)]^{[1]}$.

*Proof.* Note that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow trivially and (v) $\Rightarrow$ (i) follows from Lemma 3.2.1. We now show the following equivalence which will complete our proof.

**(iv) $\Rightarrow$ (v)**: Let $p$ be an arbitrary prime ideal of $R$. Since $A_R = R^{[m+2]}$, we have $A_R \otimes_R k(p) = k(p)^{[m+2]}$. For any $a \in A_R$, let $\overline{a}$ denote the image of $a$ in $A_R \otimes_R k(p)$.

Since $\overline{\alpha(0,\ldots,0)} \notin k(p)^*$, there are only two possibilities, either $\overline{\alpha} = 0$ or $\overline{\alpha} \notin k(p)$. We now show that $k(p)[Z,T] = k(p)[\overline{f}]^{[1]}$ in both cases.

**Case I :** $\overline{\alpha} = 0$.

Then $\overline{h} = 0$ and hence

$$k(p)^{[m+2]} = A_R \otimes_R k(p) = \left(\frac{k(p)[Z,T]}{(\overline{f(Z,T)})}\right)^{[m+1]}.$$

Therefore, by Theorem 2.4.2 $k(p)[Z,T]/(\overline{f(Z,T)}) = k(p)^{[1]}$ and hence by Theorem 2.4.1, $k(p)[Z,T] = k(p)[\overline{f}]^{[1]}$.

**Case II :** $\overline{\alpha} \notin k(p)$.

If condition (a) is satisfied then either $\overline{\alpha}$ has a simple prime factor or $\overline{\alpha}$ and $\overline{h}$ has a common multiple prime factor. Next if condition (b) is satisfied then $\overline{\alpha}$ and $\overline{h}$ has a common multiple factor. Thus by Theorem 3.2.7, $k(p)[Z,T] = k(p)[\overline{f}]^{[1]}$.

Now we have $k(p)[Z,T] = k(p)[\overline{f}]^{[1]}$, for all $p \in \operatorname{Spec}(R)$ i.e., $f$ is a residual coordinate in $R[Z,T]$. Therefore by Theorem 2.4.8, $R[Z,T] = R[f]^{[1]}$. $\square$

Next we prove a partial generalisation of Theorem D (Theorem 3.5.3).

**Theorem 3.6.2.** *Let $R$ be a Noetherian domain containing a field $k$ and let $A_R$ be a domain as in* (3.6.1)*. Suppose the following conditions hold:*

(a) *There exists a separable field extension $L$ of $k$ such that either $ch.k = 0$ or $R \otimes_k L$ is a seminormal domain* (Definition 2.4.3)*.*

(b) *For $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 2}$ the polynomial $\alpha$ is $\mathbf{r}$-divisible in $(R \otimes_k L)^{[m]}$, i.e., there exist a system of coordinates $\{X_1, \ldots, X_m\}$ of $(R \otimes_k L)^{[m]}$ and*

$\eta_i \in (R \otimes_k L)^{[m+1-i]}$, $1 \leqslant i \leqslant m$ *such that*

$$\alpha = X_1^{r_1}(X_1\eta_1(X_1, \ldots, X_m) + \cdots + X_{m-1}^{r_{m-1}}(X_{m-1}\eta_{m-1}(X_{m-1}, X_m) + X_m^{r_m}\eta_m(X_m)) \ldots))$$

*in* $(R \otimes_k L)[X_1, \ldots, X_m]$.

(c) *The coefficients of* $\eta_m(X_m)$ *in* $R \otimes_k L$ *generate the unit ideal in* $R \otimes_k L$.

(d) *Every rank one projective module of* $R^{[1]}$ *is principal.*

*Then the following statements are equivalent.*

(i) $R[X_1, \ldots, X_m, Y, Z, T] = R[X_1, \ldots, X_m, H_R]^{[2]}$.

(ii) $R[X_1, \ldots, X_m, Y, Z, T] = R[H_R]^{[m+2]}$.

(iii) $A_R = R[x_1, \ldots, x_m]^{[2]}$.

(iv) $A_R = R^{[m+2]}$.

(v) $R[Z, T] = R[f(Z, T)]^{[1]}$.

*Proof.* Note that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow trivially and (v) $\Rightarrow$ (i) follows from Lemma 3.2.1. We now prove the following equivalence which will complete the proof.

**(iv) $\Rightarrow$ (v)** : Let $R_L = R \otimes_k L$ and $A_L = R_L \otimes_k A_R$. Let $p$ be an arbitrary prime ideal of $R_L$. Since $A_R = R^{[m+2]}$, we have $A_L \otimes_{R_L} k(p) = k(p)^{[m+2]}$. For any $a \in A_L$, let $\overline{a}$ denote the image of $a$ in $A_L \otimes_{R_L} k(p)$. Since the coefficients of $\eta_m(X_m)$ in $R_L$ generate the unit ideal in $R_L$, $\overline{\alpha}$ is a $\mathbf{r}$-divisible polynomial in $k(p)^{[m]}$, where $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 2}$. Hence by Theorem 3.5.3, we have $k(p)[Z, T] = k(p)[\overline{f}]^{[1]}$.

Since $p$ is an arbitrary prime ideal of $R_L$, it follows that $f$ is a residual coordinate in $R_L[Z, T]$ and hence by Theorem 2.4.8, $R_L[Z, T] = R_L[f]^{[1]}$. Therefore, by Theorem 2.4.9 $R[Z, T]$ is isomorphic to the symmetric algebra of a finitely generated rank one projective module over $R[f]$. Since rank one projective modules of $R[f]$ are principal it follows that $R[Z, T] = R[f]^{[1]}$. $\square$

Note that in the above proof, if for $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 2}$ the polynomial $\alpha$ is $\mathbf{r}$-divisible in $R^{[m]}$ then we don't need the condition on projective modules of $R^{[1]}$. Thus we have the following result.

**Corollary 3.6.3.** *Let $R$ be a Noetherian domain such that either $\mathbb{Q} \subseteq R$ or $R$ is a seminormal domain. For $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{Z}^m_{\geqslant 2}$, suppose $\alpha$ is a $\mathbf{r}$-divisible polynomial in $R^{[m]}$ i.e., there exist $\eta_i \in R[X_i, \dots, X_m]$, $1 \leqslant i \leqslant m$ such that*

$$\alpha = X_1^{r_1}(X_1\eta_1(X_1, \dots, X_m)+\cdots+X_{m-1}^{r_{m-1}}(X_{m-1}\eta_{m-1}(X_{m-1}, X_m)+X_m^{r_m}\eta_m(X_m))\dots))$$

*in $R[X_1, \dots, X_m]$. Moreover, if the coefficients of $\eta_m(X_m)$ in $R$ generate the unit ideal in $R$ then all the equivalence of Theorem 3.6.2 holds.*

When $R = k^{[n]}$ for some $n \in \mathbb{Z}_{\geqslant 0}$ then by Theorem 2.4.6, every finitely generated projective module over $k^{[n+1]}$ is free. Thus we have the following result.

**Corollary 3.6.4.** *Let $R = k[W_1, \dots, W_n](= k^{[n]})$ for some $n \in \mathbb{Z}_{\geqslant 0}$. Suppose there exist a separable field extension $L$ of $k$ and $\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{Z}^m_{\geqslant 2}$ such that $\alpha$ is a $\mathbf{r}$-divisible polynomial in $(R \otimes_k L)^{[m]}$, i.e., there exist $\eta_i \in (R \otimes_k L)[X_i, \dots, X_m]$, $1 \leqslant i \leqslant m$ such that*

$$\alpha = X_1^{r_1}(X_1\eta_1(X_1, \dots, X_m)+\cdots+X_{m-1}^{r_{m-1}}(X_{m-1}\eta_{m-1}(X_{m-1}, X_m)+X_m^{r_m}\eta_m(X_m))\dots))$$

*in $(R \otimes_k L)[X_1, \dots, X_m]$. Moreover, if the coefficients of $\eta_m(X_m)$ in $R \otimes_k L$ generate the unit ideal in $R \otimes_k L$, then all the equivalence of Theorem 3.6.2 holds.*

# Chapter 4

# On rigidity of Pham-Brieskorn surfaces

The main aim of this chapter is to prove Theorems E, F and G mentioned in the Introduction. Section 4.1 is dedicated to proving Theorem E (Theorem 4.1.4) and in Section 4.2 we prove Theorem F (Theorem 4.2.12). Next in Section 4.3 we discuss some auxillary results on rigidity including Theorem G and in Section 4.4 we discuss some applications of our results.

## 4.1 Theorem E: Stable Rigidity of the Pham-Brieskorn domain

We start this section by recalling a definition from [22, Section 2.8].

**Definition 4.1.1.** For an integral domain $A$, two non-zero elements $f, g$ of $A$ are said to be *relatively prime in* $A$ if $fA \cap gA = fgA$.

**Remark 4.1.2.** (cf. [23, Section 1]) Let $A$ be an integral domain. The following hold when $f, g \in A$ are relatively prime in $A$:

(i) If $fh_1 = gh_2$ for some $h_1, h_2 \in A$ then $g$ divides $h_1$ and $f$ divides $h_2$.

(ii) $f^m$ and $g^n$ are relatively prime in $A$ for all $m, n \in \mathbb{Z}_{\geqslant 1}$.

(iii) $f, g$ are relatively prime in $A^{[n]}$, for all $n \in \mathbb{Z}_{\geqslant 1}$.

(iv) $f, g$ are relatively prime in $A_1$, for any factorially closed subring $A_1$ of $A$ containing $f$ and $g$.

The following result on stable rigidity has been proved in [36], [23, Theorem 6.1(a)] and [22, Theorem 9.7] over a field $k$ of characteristic zero. We have modified the arguments in [22, Theorem 9.7] to give a characteristic free proof.

**Proposition 4.1.3.** *Let $A$ be a $k$-domain of characteristic $p \geqslant 0$. Suppose $x, y, z \in A \setminus \{0\}$ are pairwise relatively prime in $A$ satisfying $x^a + y^b + z^c = 0$, for some $a, b, c \in \mathbb{Z}_{\geqslant 1}$ with $p \nmid abc$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leqslant 1$. Then $k[x, y, z] \subseteq \mathrm{ML}(A)$. Moreover, $k[x, y, z] \subseteq \mathcal{R}(A)$.*

*Proof.* Let $\phi : A \to A[U]$ be a non-trivial exponential map on $A$ and for any $t \in A$, $\phi(t) := \sum_{i=0}^{\infty} \phi^{(i)}(t)U^i$ in $A[U]$. We want to show that $k[x, y, z] \subseteq A^{\phi}$. Suppose at least one of $x, y, z$ is in $A^{\phi}$, say $z \in A^{\phi}$. Then $x^a + y^b \in A^{\phi} \setminus \{0\}$. Since $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leqslant 1$, and hence $a, b, c \geqslant 2$, by Lemma 2.4.12, $x, y \in A^{\phi}$ i.e., $k[x, y, z] \in A^{\phi}$. Thus it suffices to show that at least one of $x, y, z$ is in $A^{\phi}$.

Suppose, if possible, $x, y, z \notin A^{\phi}$. Then there would exist $m \in \mathbb{Z}_{\geqslant 1}$ defined as follows:

$$m := \min\{i \in \mathbb{Z}_{\geqslant 1} \mid \phi^{(i)}(x) \neq 0 \text{ or } \phi^{(i)}(y) \neq 0 \text{ or } \phi^{(i)}(z) \neq 0\}.$$

Now $\phi^{(i)}(x) = \phi^{(i)}(y) = \phi^{(i)}(z) = 0$ for all $i$, $1 \leqslant i < m$ and $\phi^{(m)}(x^a + y^b + z^c) = 0$, so by Leibniz rule we would have,

$$ax^{a-1}\phi^{(m)}(x) + by^{b-1}\phi^{(m)}(y) + cz^{c-1}\phi^{(m)}(z) = 0. \tag{4.1.1}$$

Hence, by the definition of $m$, at least any two of $\phi^{(m)}(x), \phi^{(m)}(y), \phi^{(m)}(z)$ would be non-zero, say $\phi^{(m)}(x) \neq 0$ and $\phi^{(m)}(y) \neq 0$.

Let $M$ be the 3×3 matrix with elements from A defined by

$$M = \begin{bmatrix} x & y & z \\ a\phi^{(m)}(x) & b\phi^{(m)}(y) & c\phi^{(m)}(z) \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, by (4.1.1) and the given condition $x^a + y^b + z^c = 0$, we would have

$$M \begin{bmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{bmatrix} = z^{c-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since $x, y$ are relatively prime in $A$ and $x \nmid \phi^{(m)}(x)$ (by Lemma 2.2.3(iii)), it would follow that $bx\phi^{(m)}(y) - ay\phi^{(m)}(x) \neq 0$ and hence $\det(M) \neq 0$. Let

$\mathrm{Adj}(M) = (c_{ij})$. Then

$$\det(M)\begin{bmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{bmatrix} = z^{c-1}\mathrm{Adj}(M)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = z^{c-1}\begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} = z^{c-1}\begin{bmatrix} cy\phi^{(m)}(z) - bz\phi^{(m)}(y) \\ az\phi^{(m)}(x) - cx\phi^{(m)}(z) \\ bx\phi^{(m)}(y) - ay\phi^{(m)}(x) \end{bmatrix},$$

that is,

$$\begin{aligned} \det(M)x^{a-1} &= z^{c-1}(cy\phi^{(m)}(z) - bz\phi^{(m)}(y)) \text{ and} \\ \det(M)y^{b-1} &= z^{c-1}(az\phi^{(m)}(x) - cx\phi^{(m)}(z)). \end{aligned}$$

Next, since $x^{a-1}, y^{b-1}$ and $z^{c-1}$ are pairwise relatively prime (cf. Remark 4.1.2 (ii)), it follows that

$$\begin{array}{lll} x^{a-1} & \text{divides} & (cy\phi^{(m)}(z) - bz\phi^{(m)}(y)), \\ y^{b-1} & \text{divides} & (az\phi^{(m)}(x) - cx\phi^{(m)}(z)) \text{ and} \\ z^{c-1} & \text{divides} & (bx\phi^{(m)}(y) - ay\phi^{(m)}(x)) \ (= \det(M)). \end{array}$$

Therefore:

$$(a-1)\deg_\phi(x) \leqslant \deg_\phi(y) + \deg_\phi(z) - m, \tag{4.1.2}$$

$$(b-1)\deg_\phi(y) \leqslant \deg_\phi(z) + \deg_\phi(x) - m \text{ and} \tag{4.1.3}$$

$$(c-1)\deg_\phi(z) \leqslant \deg_\phi(x) + \deg_\phi(y) - m. \tag{4.1.4}$$

Let $j = \deg_\phi(x) + \deg_\phi(y) + \deg_\phi(z)$. Then from equations (4.1.2), (4.1.3) and (4.1.4) we get

$$j \leqslant (j-m)(\tfrac{1}{a} + \tfrac{1}{b} + \tfrac{1}{c}) \leqslant (j-1)(\tfrac{1}{a} + \tfrac{1}{b} + \tfrac{1}{c}) \leqslant (j-1),$$

a contradiction.

Therefore, at least one of $x, y, z$ and hence all of them are in $A^\phi$; i.e., $k[x, y, z] \subseteq A^\phi$.

Since $\phi$ is chosen arbitrarily, $k[x, y, z] \subseteq \mathrm{ML}(A)$.

As $\mathrm{ML}(A)$ is a factorially closed domain (cf. Lemma 2.2.3(i)), it follows that $x, y, z$ are also pairwise relatively prime in $\mathrm{ML}(A)$ (cf. Remark 4.1.2(iv)). Now a simple induction shows that $k[x, y, z] \subseteq \mathrm{ML}_n(A)$, for all $n \in \mathbb{Z}_{\geqslant 1}$. Therefore $k[x, y, z] \subseteq \mathcal{R}(A)$. □

Now Theorem E follows from the above proposition.

**Theorem 4.1.4.** *Let $k$ be a field of characteristic $p \geqslant 0$. For any $a, s_2, s_3 \in \mathbb{Z}_{\geqslant 1}$, $r, e \in \mathbb{Z}_{\geqslant 0}$ and $p \nmid as_2s_3$, the domain*

$$B_{(a,s_2p^r,s_3p^e)} := \frac{k[X,Y,Z]}{(X^a + Y^{s_2p^r} + Z^{s_3p^e})}$$

*is stably rigid when $\frac{1}{a} + \frac{1}{s_2} + \frac{1}{s_3} \leqslant 1$.*

*Proof.* Set $B := B_{(a,s_2p^r,s_3p^e)}$ and let $A = B^{[m]} = B[X_1, \ldots, X_m]$, for arbitrary $m \in \mathbb{Z}_{\geqslant 0}$. Let $x, y, z$ be the images of $X, Y$ and $Z$ in $B$ respectively. Now $x, y^{p^r}, z^{p^e}$ are all pairwise relatively prime in $A$ (cf. Remark 4.1.2 (ii) and (iii)). Further $x^a + (y^{p^r})^{s_2} + (z^{p^e})^{s_3} = 0$ in $A$ with $(\frac{1}{a} + \frac{1}{s_2} + \frac{1}{s_3}) \leqslant 1$ and $p \nmid as_2s_3$. Therefore, by Proposition 4.1.3, we have $x, y^{p^r}, z^{p^e} \in \mathrm{ML}(A)$. Hence $B = k[x,y,z] \subseteq \mathrm{ML}(A)$, as $\mathrm{ML}(A)$ is a factorially closed domain (cf. Lemma 2.2.3(i)). Therefore, $B \subseteq A^{\phi}$, for every non-trivial exponential map $\phi$ on $A$. Thus $B$ is stably rigid. □

## 4.2 Theorem F: Rigidity of the Pham-Brieskorn rings

Recall that for a field $k$ of characteristic $p \geqslant 0$ and for any integer $n \geqslant 3$, $\underline{a} := (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1}$ with $p \nmid gcd(a_1, \ldots, a_n)$,

$$B_{\underline{a}} = B_{(a_1,a_2,\ldots,a_n)} = \frac{k[X_1, X_2, \ldots, X_n]}{(X_1^{a_1} + X_2^{a_2} + \cdots + X_n^{a_n})}$$

denotes a Pham-Brieskorn domain. Note that $B_{(a_1,a_2,\ldots,a_n)} = B_{(a_{\sigma(1)},a_{\sigma(2)},\ldots,a_{\sigma(n)})}$ for every permutation $\sigma$ of the set of integer $\{1, 2, \ldots, n\}$. Let $x_i$ be the image of $X_i$ in $B_{(a_1,a_2,\ldots,a_n)}$, for $i \in \{1, 2, \ldots, n\}$. Thus $B_{\underline{a}} = k[x_1, x_2, \ldots, x_n]$. Let $(d_1, d_2, \ldots, d_n) := (L/a_1, L/a_2, \ldots, L/a_n)$, where $L = \mathrm{lcm}(a_1, a_2, \ldots, a_n)$. Then there exists a unique $\mathbb{Z}_{\geqslant 0}$-grading on $B_{\underline{a}}$ with the property that each $x_i$ is homogeneous of degree $d_i$, for all $i \in \{1, 2, \ldots, n\}$. We call it the *standard $\mathbb{Z}_{\geqslant 0}$-grading* on $B_{\underline{a}}$. The above mentioned grading is proper and admissible by Remark 2.3.6 and $\mathrm{gr}(B_{(a_1,a_2,\ldots,a_n)}) \cong B_{(a_1,a_2,\ldots,a_n)}$. For any $\phi \in \mathrm{EXP}(B_{\underline{a}})$, $\widehat{\phi}$ will denote its homogenization with respect to this standard $\mathbb{Z}_{\geqslant 0}$-grading (by Theorem 2.3.8 such a homogeneous exponential map $\widehat{\phi}$ exists) and for any $t \in B$, $\widehat{t}$ will denote its image in $\mathrm{gr}(B)(\cong B)$ with respect to the above mentioned grading. By Theorem 2.3.8, if $\phi$ is non-trivial then $\widehat{\phi}$ is also non-trivial.

### Proof of (i), (ii) and (iii) of Theorem F

Let $k$ be a field of characteristic $p \geqslant 0$ and $n \in \mathbb{Z}_{\geqslant 3}$, $\underline{a} := (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1}$. We recall the definition of $F_n$, $R_n$ and $T_n$.

$F_n := \{(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1} \mid p \nmid gcd(a_1, a_2, \ldots, a_n)\}$,
$T_n := \{(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1} \mid a_i = 1 \text{ for some } i, \text{ or } \exists\, i, j \in \{1, 2, \ldots, n\},\, i \neq j \text{ and } a_i = a_j = 2\}$
and
$R_n := \{(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1} \mid a_i = 1 \text{ for some } i, \text{ or } \exists\, i, j \in \{1, 2, \ldots, n\},\, i \neq j \text{ and } a_i = p^r, a_j = sp^e \text{ for some } r, s, e \in \mathbb{Z}_{\geqslant 1} \text{ with } r \leqslant e\}$.

(i) Suppose $(a_1, a_2, \ldots, a_n) \in R_n$.
Note that if $a_i = 1$, for some $i$, then $B_{\underline{a}}$ is a polynomial ring over $k$ and hence $B_{\underline{a}}$ is non-rigid.
Next without loss of generality, suppose $a_1 = p^r, a_2 = sp^e$ for some $r, s, e \in \mathbb{Z}_{\geqslant 1}$ with $r \leqslant e$. Then we have a non-trivial exponential map $\phi_T$ on $B_{\underline{a}}$ defined as follows:

$$\begin{aligned} \phi_T(x_1) &= x_1 - ((x_2 + T)^{sp^{e-r}} - x_2^{sp^{e-r}}) \\ \phi_T(x_2) &= x_2 + T \\ \phi_T(x_i) &= x_i, \text{ for all } i \geqslant 3. \end{aligned}$$

(ii) Suppose $(a_1, a_2, \ldots, a_n) \in T_n$ and $k$ is a field containing a square root of $-1$; say, $i \in k$ is such that $i^2 = -1$.
Without loss of generality, suppose $a_1 = a_2 = 2$. Then

$$\begin{aligned} B_{(2,2,a_3,\ldots,a_n)} &= k[X_1, X_2, X_3, \ldots, X_n]/(X_1^2 + X_2^2 + X_3^{a_3} + \cdots + X_n^{a_n}) \\ &\cong k[U, V, X_3 \ldots, X_n]/(UV + X_3^{a_3} + \cdots + X_n^{a_n}) \\ &= k[u, v, x_3, \ldots, x_n], \end{aligned}$$

where $U := (X_1 + iX_2)$, $V := (X_1 - iX_2)$ and $u, v$ denote the images of $U, V$ in $B_{\underline{a}}$ respectively. Then we have a non-trivial exponential map $\phi_T$ on $B_{\underline{a}}$ defined by,

$$\begin{aligned} \phi_T(u) &= u \\ \phi_T(v) &= -((x_3 + uT)^{a_3} + \cdots + (x_n + uT)^{a_n})/u \\ \phi_T(x_i) &= x_i + uT, \text{ for all } i \geqslant 3. \end{aligned}$$

(iii) Let $A := k[X_1, X_2, \ldots, X_n]/(X_1^{a_1} + X_2^{a_2} + \cdots + X_n^{a_n})^m$ for some $m \in \mathbb{Z}_{\geqslant 2}$. Note that all the non-domain Pham-Brieskorn rings i.e., $B_{(a_1,a_2,\ldots,a_n)}$ with $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1} \setminus F_n$, are of this form. Further, let $x_i$ denote the image of $X_i$ in $A$, for all $i \in \{1, 2, \ldots, n\}$. Then we have a non-trivial exponential map $\phi_T$ on $A$ defined as follows:

(a) When $p \nmid \prod_i a_i$, then

$$\phi_T(x_i) = x_i + (1/a_i)x_i(x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n})^{m-1}T,\ 1 \leqslant i \leqslant n.$$

(b) When $a_n = sp^e$ for some $s, e \in \mathbb{Z}_{\geqslant 1}$. Then there exists $r \in \mathbb{Z}_{\geqslant 0}$ such that $r$ is the largest integer strictly less than $m/p^e$. Now

$$\begin{aligned} \phi_T(x_i) &= x_i,\ 1 \leqslant i \leqslant n-1 \\ \phi_T(x_n) &= x_n + (x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n})^{r+1}T. \end{aligned}$$

## Proof of (iv) of Theorem F

For convenience we again here recall the definition of $F_3$, $R_3$, $T_3$ and $S_3$.

$F_3 = \{(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geqslant 1} \mid p \nmid gcd(a_1, a_2, a_3)\}$,
$T_3 = \{(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geqslant 1} \mid a_i = 1$ for some $i$, or
$\exists\, i, j \in \{1, 2, 3\}$, $i \neq j$ and $a_i = a_j = 2\}$,
$R_3 = \{(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geqslant 1} \mid a_i = 1$ for some $i$, or $\exists\, i, j \in \{1, 2, 3\}$,
$i \neq j$, and $a_i = p^r, a_j = sp^e$ for some $r, s, e \in \mathbb{Z}_{\geqslant 1}$ with $r \leqslant e\}$
and
$S_3 = \{(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geqslant 1} \mid \exists\, i, j, l$ all distinct with $a_i = 2, a_j = 2m, a_l = 2p^e$,
$m \in \mathbb{Z}_{\geqslant 2}$, $e \in \mathbb{Z}_{\geqslant 1}$ and $p \nmid 2m\}$.

Now from Section 4.2, it follows that for an algebraically closed field $k$, if $B_{(a,b,c)}$ is rigid then $(a, b, c) \in F_3 \setminus (R_3 \cup T_3)$. We will show step by step that the converse is true except when $(a, b, c) \in S_3$. Let $x, y, z$ respectively denote the images of $X, Y, Z$ in $k[X, Y, Z]/(X^a + Y^b + Z^c) = B_{(a,b,c)}$.

The first proposition of this section is a useful observation, where rigidity of a subring guarantees the rigidity of the Pham-Brieskorn domains under consideration.

**Proposition 4.2.1.** *Let $k$ be a field of characteristic $p > 0$. For any $n \in \mathbb{Z}_{\geqslant 3}$ let $\underline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 1}, s_2, \ldots, s_n, e_2, \ldots, e_n \in \mathbb{Z}_{\geqslant 1}$ be such that*

(i) $e_2 \leqslant e_3 \leqslant \cdots \leqslant e_n$

(ii) $a_i = s_i p^{e_i}$, $2 \leqslant i \leqslant n$

(iii) $p \nmid a_1 s_2 \ldots s_n$.

*Then every non-trivial exponential map* $\phi$ *on the domain* $B_{\underline{a}}$ $(= k[x_1, x_2, \ldots, x_n])$ *restricts to a non-trivial exponential map* $\varphi$ *on the subring* $A := k[x_1, x_2^{p^r}, \ldots, x_n^{p^r}]$, *where* $r := e_2$*. Moreover,* $A^{\varphi} = A \cap B_{\underline{a}}^{\phi}$*. Thus* $A$ *rigid implies* $B_{\underline{a}}$ *is also rigid.*

*Proof.* Let $y_i := x_i^{p^r}$ for $i \in \{2, \ldots, n\}$. Then $x_1^{a_1} + \sum_{i=2}^{n} y_i^{s_i p^{e_i - r}} = 0$ in $B_{\underline{a}}$. Hence

$$\begin{aligned} A = k[x_1, x_2^{p^r}, \ldots, x_n^{p^r}] &= k[x_1, y_2, \ldots, y_n] \\ &\cong \frac{k[X_1, Y_2, Y_3, \ldots, Y_n]}{(X_1^{a_1} + Y_2^{s_2} + Y_3^{s_3 p^{e_3 - r}} + \cdots + Y_n^{s_n p^{e_n - r}})} \\ &= B_{(a_1, s_2, s_3 p^{e_3 - r}, \ldots, s_n p^{e_n - r})}, \end{aligned}$$

where $r = e_2 \leqslant e_3 \leqslant \cdots \leqslant e_n$. It is well known that $A$ is normal (cf. [21, Proposition 11.2]). Suppose $\phi : B_{\underline{a}} \to B_{\underline{a}}[T]$ is an exponential map. Then it follows that $\phi(y_i) = \phi(x_i)^{p^r} \in A[T]$ for $i \in \{2, \ldots, n\}$. Let $K := \text{Frac}(A[T])$ and $L := \text{Frac}(B_{\underline{a}}[T])$. Note that

$$\begin{aligned} \phi(x_1)^{a_1} &= -(\phi(x_2)^{s_2} + \cdots + \phi(x_n)^{s_n p^{e_n - r}})^{p^r} \\ &= -(\phi(y_2)^{s_2} + \cdots + \phi(y_n)^{s_n p^{e_n - r}}) \in A[T] \subseteq K. \end{aligned} \tag{4.2.1}$$

Next we observe that $L/K$ is a finite, purely inseparable extension and $[L : K] = p^{(n-1)r}$. Now, as $\phi(x_1) \in L$, there exists $m \in \mathbb{Z}_{\geqslant 0}$ such that $\phi(x_1)^{p^m} \in K$. Since $gcd(a_1, p^m) = 1$ and $\phi(x_1)^{p^m}$, $\phi(x_1)^{a_1} \in K$ (from (4.2.1)), it follows that $\phi(x_1) \in K$. Therefore $\phi(x_1) \in K \cap B_{\underline{a}}[T] = A[T]$, since $A$ is a normal domain. Thus every exponential map $\phi$ on $B_{\underline{a}}$, restricts to an exponential map $\varphi$ on $A$ given by $\varphi(x_1) = \phi(x_1)$ and $\varphi(y_i) = \phi(x_i)^{p^r}$ for $2 \leqslant i \leqslant n$. Hence $A^{\varphi} = A \cap B_{\underline{a}}^{\phi}$. □

**Remark 4.2.2.** The converse of the above proposition is not true in general, i.e., $A$ non-rigid may not imply that $B_{\underline{a}}$ is non-rigid. For example, over an algebraically closed field $k$ of characteristic $p > 2$, $B_{(2,2p^r,2p^r)}$ is rigid for any $r \in \mathbb{Z}_{\geqslant 1}$ (cf. Lemma 4.2.10) but $A = B_{(2,2,2)}$ is non-rigid.

**Remark 4.2.3.** Suppose $(a, b, c) \in F_3 \setminus T_3$ and $p \nmid abc$. By Proposition 4.1.3, we know that $B_{(a,b,c)}$ is rigid if $(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \leqslant 1$. Without loss of generality if we assume $a \leqslant b \leqslant c$, then the remaining cases we need to consider are

$a = 2, b = 3$ and $c \in \{3, 4, 5\}$. Note that in every such case, we get two pairs among $a, b, c$ which are co-prime.

We shall now discuss the rigidity of Pham-Brieskorn domains $B_{(a,b,c)}$ where two pairs among $a, b, c$ are co-prime. We first prove two lemmas in this regard.

**Lemma 4.2.4.** *Let $k$ be a field of characteristic $p \geqslant 0$ and let $(a, b, c) \in F_3 \backslash R_3$ be such that $gcd(a, bc) = 1$. Then for any non-trivial exponential map $\phi$ on $B := B_{(a,b,c)} = k[x, y, z]$, $\{x, y, z\} \cap B^{\phi} = \varnothing$.*

*Proof.* Since $gcd(a, b) = 1$, $z$ is prime in $B$ and since $gcd(a, c) = 1$, $y$ is prime in $B$. Let $\phi$ be a non-trivial exponential map on $B$.

Suppose, if possible, that $x \in B^{\phi}$. Then $y^b + z^c \in B^{\phi} \setminus \{0\}$. We show that both $y$ and $z$ are in $B^{\phi}$, which will contradict the fact that $\phi$ is non-trivial.

**Case I :** Suppose $p$ does not divide one of $b$ and $c$, say $p \nmid c$. Then, by Lemma 2.4.12 or Lemma 2.4.13, according as $p \nmid b$ or $p \mid b$ we have $y, z \in B^{\phi}$.

**Case II :** Suppose $p$ divides both $b$ and $c$. Let $b = s_2 p^r$ and $c = s_3 p^e$ for some $s_2, s_3, e, r \in \mathbb{Z}_{\geqslant 1}$ with $e \geqslant r$ and $p \nmid s_2 s_3$. Now $y^b + z^c = (y^{s_2} + z^{s_3 p^{e-r}})^{p^r} \in B^{\phi} \setminus \{0\}$, hence by Lemma 2.2.3(i), $y^{s_2} + z^{s_3 p^{e-r}} \in B^{\phi} \setminus \{0\}$. If $e > r$ then $s_2 > 1$ since $(a, s_2 p^r, s_3 p^e) \notin R_3$, and hence by Lemma 2.4.13, $y, z \in B^{\phi}$. Next when $e = r$ then $s_2, s_3 \geqslant 2$, since $(a, s_2 p^r, s_3 p^e) \notin R_3$. Then by Lemma 2.4.12, $y, z \in B^{\phi}$. Hence $x \notin B^{\phi}$.

Now suppose, if possible, that $y \in B^{\phi}$. Then $x^a + z^c \in B^{\phi} \setminus \{0\}$. If $p \nmid c$ then, by Lemma 2.4.12 or Lemma 2.4.13, according as $p \nmid a$ or $p \mid a$ we have $x, z \in B^{\phi}$. But $\phi$ is non-trivial hence $p \mid c$. Then $p \nmid a$ since $gcd(a, c) = 1$. Let $c = s_3 p^e$ for some $s_3, e \in \mathbb{Z}_{\geqslant 1}$ with $p \nmid s_3$. By Lemma 2.2.3(vi) and Remark 2.2.5(i), $\phi$ induces a non-trivial exponential map on the integral domain $\widetilde{B} := B \otimes_{k[y]} \overline{k(y)} = \overline{k(y)}[X, Z]/(X^a + Z^c + y^b)$. Since $\operatorname{tr.deg}_{\overline{k(y)}} \widetilde{B} = 1$, by Lemma 2.2.3(v) we have $\widetilde{B} = \overline{k(y)}^{[1]}$. Now consider the irreducible polynomial,

$$\begin{aligned} F(X, Z) &:= X^a + Z^c + y^b \in \overline{k(y)}[X, Z] \\ &= X^a + (Z^{s_3} - \beta)^{p^e} \text{ for some } \beta \in \overline{k(y)}. \end{aligned}$$

Let $\gamma$ be a root of $Z^{s_3} - \beta$. Then $M := (X, Z - \gamma)$ is a maximal ideal of $\overline{k(y)}[X, Z]$ and $F \in M^2$. Thus $\widetilde{B}$ is not a normal domain, in particular $\widetilde{B} \neq \overline{k(y)}^{[1]}$. This leads to a contradiction. Hence $y \notin B^{\phi}$.

Similarly we can show that $z \notin B^{\phi}$. Thus our lemma follows. □

**Lemma 4.2.5.** *Let $k$ be an algebraically closed field of characteristic $p \geqslant 0$ and $(a,b,c) \in F_3 \setminus R_3$ with $gcd(a,bc) = 1$. Then for any non-trivial exponential map $\phi$ on $B := B_{(a,b,c)} = k[x,y,z]$, $y^{b/d} + \mu z^{c/d} \in B^{\widehat{\phi}}$ for some $\mu \in k^*$ and $d = gcd(b,c)$* (for the meaning of $\widehat{\phi}$ please refer to beginning of this section).

*Proof.* Let $\phi$ be a non-trivial exponential map on $B$ and let $f \in B^{\phi} \setminus k$. Using the relation $x^a = -(y^b + z^c)$, $f$ can be uniquely written as

$$f = \sum_{i=0}^{a-1} f_i(y,z)x^i \text{ for some polynomials } f_i(y,z) \in k[y,z] \text{ in } B.$$

By Lemma 4.2.4, $x \notin B^{\phi}$. Therefore $f_0(y,z) \notin k$. Now we consider the standard $\mathbb{Z}_{\geqslant 0}$-graded structure on $B$, given by

$$wt(x) = bc/d, \quad wt(y) = ac/d, \quad wt(z) = ab/d, \quad d = \gcd(b,c).$$

Let deg denote the degree function corresponding to the above grading.

If $\deg(x^{i_1}y^{j_1}z^{l_1}) = \deg(x^{i_2}y^{j_2}z^{l_2})$ for some $i_1, i_2, j_1, j_2, l_1, l_2 \in \mathbb{Z}_{\geqslant 0}$ with $0 \leqslant i_1 < i_2 < a$. Then $((j_1 - j_2)(c/d) + (l_1 - l_2)(b/d))a = (i_2 - i_1)(bc/d)$, which implies that $a|(i_2 - i_1)(bc/d)$. But $gcd(a,bc) = 1$ and $0 < i_2 - i_1 < a$. Therefore two different powers of $x$ cannot occur in $\widehat{f}$.

Thus $\widehat{f} = \widehat{f_i(y,z)}x^i$ for some $i$, $0 \leqslant i < a$. Now by Theorem 2.3.8, $\widehat{\phi}$ is non-trivial and $\widehat{f} \in B^{\widehat{\phi}}$. By Lemma 4.2.4 $x \notin B^{\widehat{\phi}}$, therefore $\widehat{f} = \widehat{f_0(y,z)}$. Next, $k$ is an algebraically closed field, so $\widehat{f} = \lambda y^r z^m \prod_{l \in \Lambda}(y^{b/d} + \mu_l z^{c/d})$ for some $\lambda, \mu_l \in k^*$, $r, m \in \mathbb{Z}_{\geqslant 0}$ and a finite set $\Lambda \subseteq \mathbb{Z}_{\geqslant 0}$, in $B$.

By Lemma 4.2.4 $y, z \notin B^{\widehat{\phi}}$ so $r = m = 0$. If $\widehat{f}$ has two distinct factors, $y^{b/d} + \mu_1 z^{c/d}$ and $y^{b/d} + \mu_2 z^{c/d}$, then $z \in B^{\widehat{\phi}}$, a contradiction. Thus $\widehat{f} = \lambda(y^{b/d} + \mu z^{c/d})^r$ for some $\lambda, \mu \in k^*$ and $r \in \mathbb{Z}_{\geqslant 1}$. Hence $y^{b/d} + \mu z^{c/d} \in B^{\widehat{\phi}}$ (cf. Lemma 2.2.3(i)). □

**Theorem 4.2.6.** *Let $k$ be a field of characteristic $p \geqslant 0$ and $(a,b,c) \in F_3 \setminus (R_3 \cup T_3)$ with $gcd(a,bc) = 1$. Then $B := B_{(a,b,c)} = k[x,y,z]$ is a rigid domain.*

*Proof.* In view of Remark 2.2.5(i), we assume $k$ to be an algebraically closed field. Suppose, if possible, that there exists a non-trivial exponential map $\varphi$ on $B$. By Lemma 4.2.5, $y^{b/d} + \mu z^{c/d} \in B^{\widehat{\varphi}} \setminus \{0\}$ for some $\mu \in k^*$ where $d = gcd(b,c)$. Note that $gcd(a,bc) = 1$ so $y, z$ are prime in $B$. For convenience let $b \leqslant c$. Next we consider different cases and arrive at contradiction in each of them and hence it will follow that $B$ is a rigid domain.

**Case I :** $d < b < c$.

Then $b/d, c/d \geqslant 2$ and since $y, z$ are primes in $B$, by Lemma 2.4.12 or

Lemma 2.4.13 it follows that $y, z \in B^{\widehat{\varphi}}$. Thus $\widehat{\varphi}$ is a trivial exponential map, a contradiction.

**Case II :** $d = b < c$.
Then $y + \mu z^{\frac{c}{b}} \in B^{\widehat{\varphi}}$ for some $\mu \in k^*$. Now for every $\lambda$ in $k^*$

$$\frac{B}{(y+\mu z^{c/b}-\lambda)B} = \frac{k[X,Y,Z]}{(X^a+Y^b+Z^c, Y+\mu Z^{c/b}-\lambda)} \cong \frac{k[X,Z]}{(X^a+P_\lambda(Z^{c/b}))},$$

where

$$P_\lambda(T) := (1+(-\mu)^b)T^b + \binom{b}{b-1}\lambda(-\mu T)^{b-1} + \cdots + \binom{b}{1}\lambda^{b-1}(-\mu T) + \lambda^b \in k[T]. \tag{4.2.2}$$

Then $P_\lambda(T) \in k^*$ if and only if $1 + (-\mu)^b = 0$ and $b = p^r$ for some $r \in \mathbb{Z}_{\geqslant 1}$. But $(a, b, c) \notin R_3$ and hence $y + \mu z^{c/b} - \lambda$ is a prime element of $B$ for all $\lambda \in k^*$. Therefore by Lemma 2.4.11, there exists a $\beta \in k^*$ such that $\widehat{\varphi}$ induces a non-trivial exponential map on the integral domain $\widetilde{B} := B/(y+\mu z^{c/b}-\beta)B$. Now

$$\widetilde{B} \cong \frac{k[X,Z]}{(X^a+P_\beta(Z^{c/b}))}.$$

Since $\operatorname{tr.deg}_k \widetilde{B} = 1$ hence by Remark 2.2.5(ii), $\widetilde{B} = k^{[1]}$. Thus $P_\beta(Z^{c/b})$ should have a linear term in $Z$. This is not possible as $c/b > 1$.

**Case III :** $d = b = c$ and $p|b$.
As in Case II, we get $\mu$, $\beta \in k^*$ such that $\widehat{\varphi}$ induces a non-trivial exponential map on the integral domain $\widetilde{B} := B/(y + \mu z - \beta)B$ and

$$\widetilde{B} \cong \frac{k[X,Z]}{(X^a+P_\beta(Z))} = k^{[1]}.$$

But $p|b$ hence $P_\beta(Z)$ does not have a linear term. Therefore, $\widetilde{B} \neq k^{[1]}$ a contradiction.

**Case IV :** $d = b = c$, $p \nmid b$ and $p|a$.
Following Case II, we get $\mu$, $\beta \in k^*$ such that $y + \mu z \in B^{\widehat{\varphi}}$ and

$$\widetilde{B} := B/(y+\mu z-\beta)B \cong \frac{k[X,Z]}{(X^a+P_\beta(Z))} = k^{[1]}.$$

Now $b = c \geqslant 3$ (as $(a, b, c) \notin T_3$) and $p \nmid b$ therefore $\deg_Z P_\beta(Z) \geqslant 2$. When $p \nmid (b-1)$ then the coefficient of $Z^{b-2}$ in $P'_\beta(Z)$ is non-zero. When $b = sp^r + 1$ for some $s \in \mathbb{Z}_{\geqslant 2}$, $r \in \mathbb{Z}_{\geqslant 1}$ with $p \nmid s$ then the coefficient of $Z^{(s-1)p^r}$ in $P'_\beta(Z)$ is non-zero. Thus $\deg_Z P'_\beta(Z) \geqslant 1$ except when $1+(-\mu)^b = 0$ and $b = c = p^r+1$ for some $r \in \mathbb{Z}_{\geqslant 1}$. Now when $\deg_Z P'_\beta(Z) \geqslant 1$ we can find $a_1, a_2 \in k$ such that $P'_\beta(a_2) = 0$ and $a_1^a + P_\beta(a_2) = 0$. Therefore by Jacobian criterion $\widetilde{B}$ is not normal hence cannot be $k^{[1]}$, a contradiction.

Let us now consider the case when $1 + (-\mu)^b = 0$ and $b = c = p^r + 1$ for some $r \in \mathbb{Z}_{\geqslant 1}$. Let $a = s_1 p^m$ for some $s_1, m \in \mathbb{Z}_{\geqslant 1}$ and $p \nmid s_1$. Now when $p > 2$ and $s_1 > 1$, then $1/s_1 + 1/(p^r+1) + 1/(p^r+1) \leqslant \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$ and when $p = 2$ with $s_1 \geqslant 3$ then $1/s_1 + 1/(p^r+1) + 1/(p^r+1) \leqslant \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, hence by Theorem 4.1.4, $B$ is rigid. Thus the only remaining case is when $s_1 = 1$.

Suppose $s_1 = 1$. Let $Z_1 = -\mu Z$ and $Y_1 = Y + \mu Z = Y - Z_1$, then

$$\begin{aligned} X^{p^m} + Y^{p^r+1} + Z^{p^r+1} &= X^{p^m} + (Y_1+Z_1)^{p^r+1} - (Z_1)^{p^r+1}, \text{ since } (-\mu)^{p^r+1} = -1 \\ &= X^{p^m} + Y_1^{p^r+1} + Y_1^{p^r} Z_1 + Y_1 Z_1^{p^r}. \end{aligned}$$

Therefore $B = B_{(p^m, p^r+1, p^r+1)} \cong k[X, Y_1, Z_1]/(X^{p^m} + Y_1^{p^r+1} + Y_1^{p^r} Z_1 + Y_1 Z_1^{p^r}) = k[x, y_1, z_1]$, where $x, y_1 (= y + \mu z), z_1 (= -\mu z)$ denote the images of $X, Y_1, Z_1$ in $B$ respectively. Now $y_1 \in B^{\widehat{\varphi}}$ hence by Lemma 2.2.3(vi), $\widehat{\varphi}$ induces a non-trivial exponential map on

$$B_1 := k(y_1)[X, Z_1]/(X^{p^m} + y_1 Z_1^{p^r} + y_1^{p^r} Z_1 + y_1^{p^r+1}).$$

Next $B_1 \otimes_{k(y_1)} \overline{k(y_1)} = \overline{k(y_1)}^{[1]}$, by [5, Proposition 4.2]. Since $\operatorname{tr.deg}_{k(y_1)} B_1 = 1$ hence by Remark 2.2.5(ii), $B_1 = k(y_1)^{[1]}$. But according to [5, Remark 4.5], $B_1$ is a non-trivial $\mathbb{A}^1$-form over $k(y_1)$ (i.e., $B_1 \neq k(y_1)^{[1]}$). This is a contradiction.

**Case V :** $d = b = c$, $p \nmid ab$.
By Theorem 4.1.4, if $(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \leqslant 1$ then $B$ is rigid. So the remaining case is when $a = 2$ and $b = c = 3$. As in Case II, we get $\mu$, $\beta \in k^*$ such that

$$\widetilde{B} := B/(y + \mu z - \beta) \cong \frac{k[X, Z]}{(X^2 + P_\beta(Z))} = k^{[1]},$$

with $P_\beta(Z) = (1-\mu^3)Z^3 + 3\beta\mu^2 Z^2 - 3\beta^2 \mu Z + \beta^3 \in k[Z]$.

If $1 - \mu^3 \neq 0$, then $\deg_Z(P_\beta(Z)) = 3$. Now since $\widetilde{B} = k^{[1]}$, there exists a surjective $k$-algebra homomorphism, $\Psi : k[X, Z] \to k[T] (= k^{[1]})$ such that $\deg_T(\Psi(X)) = \deg_Z(P_\beta(Z)) = 3$ and $\deg_T(\Psi(Z)) = 2$. So all the conditions of Epimorphism theorem (cf. Theorem 2.4.1) are satisfied. But $gcd(2, 3) = 1$, which leads to a contradiction. So $\mu^3 = 1$. Therefore $X^2 + P_\beta(Z) = X^2 + 3\beta\mu^2 Z^2 - 3\beta^2 \mu Z + \beta^3 = X^2 + 3\beta(\mu Z - \beta/2)^2 + \beta^3/4$. Then $X^2 + P_\beta(Z)$ has no linear term in the new coordinates $X$ and $Z_1 := (\mu Z - \beta/2)$ and hence $\widetilde{B} \neq k^{[1]}$. This leads to a contradiction. $\square$

Now we consider all those Pham-Brieskorn domains $B_{(a,b,c)}$ over a field $k$ of characteristic $p \geqslant 0$ such that $p \nmid abc$.

**Theorem 4.2.7.** *Let $k$ be a field of characteristic $p \geqslant 0$. Then $B_{(a,b,c)}$ is rigid when $(a,b,c) \in F_3 \setminus T_3$ with $p \nmid abc$.*

*Proof.* By Remark 4.2.3 and Theorem 4.2.6 the theorem follows. $\square$

Next we turn our attention to all those Pham-Brieskorn domains $B_{(a,b,c)}$ over a field $k$ of characteristic $p > 0$ where only one among $a, b, c$ has $p$ as a factor.

**Theorem 4.2.8.** *Let $k$ be a field of characteristic $p > 0$ and $B := B_{(a,b,c)}$ with $c = sp^e$ for some $a, b, s, e \in \mathbb{Z}_{\geqslant 1}$ with $p \nmid abs$. Then $B$ is rigid when $(a,b,c) \notin (T_3 \cup S_3)$.*

*Proof.* If $(\frac{1}{a} + \frac{1}{b} + \frac{1}{s}) \leqslant 1$ then by Theorem 4.1.4, we know that $B$ is rigid.

Next if $s = 1$, then $gcd(ab, c) = 1$ and hence by Theorem 4.2.6, $B$ is rigid.

So henceforth we assume that $(\frac{1}{a} + \frac{1}{b} + \frac{1}{s}) > 1$ and $s > 1$.

For convenience let $a \leqslant b$. Then the remaining cases are $\{a,b,s\} = \{2,3,5\}$ or $\{2,3,4\}$ or $\{2,3,3\}$ or $a = 2, s = 2, b > 2$. Now we consider two cases and show that in each case $B$ is rigid.

**Case I :** $\{a,b,s\} = \{2,3,5\}$ or $\{2,3,4\}$ or $\{2,3,3\}$.
Then as $p \nmid abs$, in every case we get two pairs among $a, b, c$, which are coprime. Therefore by Theorem 4.2.6, $B$ is rigid.

**Case II :** $a = 2 = s$ and $b$ is an odd integer greater than 1 since $(a,b,c) \notin S_3$ i.e., $(a,b,c) = (2, 2m+1, 2p^e)$ for some $m \in \mathbb{Z}_{\geqslant 1}$.
Then $gcd(ac, b) = 1$ since $2 \nmid b$ and $p \nmid b$. Thus the result again follows from Theorem 4.2.6. $\square$

Next we consider all those Pham-Brieskorn domains $B := B_{(a,b,c)}$ over a field $k$ of characteristic $p > 0$ such that $p$ occurs as factors in both $b$ and $c$. Let $b = s_2p^r$ and $c = s_3p^e$ for some $s_2, s_3, e, r \in \mathbb{Z}_{\geqslant 1}$ such that $p \nmid as_2s_3$ and $e \geqslant r$.

**Lemma 4.2.9.** *Let $B = B_{(a,s_2p^r,s_3p^e)}$ for some $s_2, s_3, e, r \in \mathbb{Z}_{\geqslant 1}$, $a \in \mathbb{Z}_{\geqslant 2}$ such that $p \nmid as_2s_3$ and $e \geqslant r$. Suppose $\phi$ is an exponential map on $B = k[x,y,z]$. Then $x, y^{s_2} + z^{s_3p^{e-r}} \in B^{\phi}$. Hence it implies that $x, y^{s_2} + z^{s_3p^{e-r}} \in \mathrm{ML}(B)$.*

*Proof.* Note that if $\phi_1 := \phi \otimes id$ is the extension of $\phi$ on $B \otimes_k \overline{k}$ and $x, y^{s_2} + z^{s_3 p^{e-r}} \in (B \otimes_k \overline{k})^{\phi_1}$, then $x, y^{s_2} + z^{s_3 p^{e-r}} \in (B \otimes_k \overline{k})^{\phi_1} \cap B = B^{\phi}$. Therefore we assume henceforth $k$ is an algebraically closed field.

Let $\phi$ be a non-trivial exponential map on $B$ and $f \in B^{\phi} \setminus k$. Let $p = (x, h(y,z))B$, where $h(y,z)$ is an irreducible factor of $y^{s_2} + z^{s_3 p^{e-r}}$. Then $p$ is a prime ideal of $B$ of height one and $B_p$ is not a normal domain. Now by Lemma 2.2.3(vi), $\phi$ induces a non-trivial exponential map on $B_1 := B \otimes_{k[f]} k(f)$. Since $\operatorname{tr.deg}_{k(f)} B_1 = 1$, by Lemma 2.2.3(v), $B_1 = L^{[1]}$, where $L$ is the algebraic closure of $k(f)$ in $B_1$. Hence $B_1$ is a normal domain. Therefore $pB_1 = B_1$, i.e., $p \cap k[f] \neq (0)$. Hence $p \cap k[f] = (f - \lambda_p)k[f]$ for some $\lambda_p \in k$. Let $f = \sum_{i=0}^{a-1} f_i(y,z)x^i$ for some polynomials $f_i(y,z) \in k[y,z]$. Then as $f - \lambda_p \in p$, it follows that $\theta_p := f_0 - \lambda_p \in (h)B$. Thus if $h_1, h_2, \ldots, h_n$ are all the distinct irreducible factors of $y^{s_2} + z^{s_3 p^{e-r}}$ for some $n \in \mathbb{Z}_{\geqslant 1}$ then we obtain $\theta_{p_1}, \theta_{p_2}, \ldots, \theta_{p_n} \in k[y,z]$ such that $\prod_{i=1}^{n} \theta_{p_i} \in (h_1 h_2 \ldots h_n)B$. Therefore there exists $m \in \mathbb{Z}_{\geqslant 1}$ such that

$$(\prod_{i=1}^{n} f_0 - \lambda_{p_i})^m = (\prod_{i=1}^{n} \theta_{p_i})^m \in ((y^{s_2} + z^{s_3 p^{e-r}})^{p^r})B \subseteq (x)B. \tag{4.2.3}$$

Hence $(\prod_{i=1}^{n} f - \lambda_{p_i})^m \in (x)B$, by (4.2.3). Since $(\prod_{i=1}^{n}(f - \lambda_{p_i}))^m \in B^{\phi}$ so $x \in B^{\phi}$ (cf. Lemma 2.2.3(i)) and hence $y^{s_2} + z^{s_3 p^{e-r}} \in B^{\phi}$. □

**Lemma 4.2.10.** *Let $k$ be a field of characteristic $p > 0$. Let $B_1 = B_{(2, 2mp^r, s_3 p^e)}$ for some $m, s_3, r, e \in \mathbb{Z}_{\geqslant 1}$ such that $e \geqslant r$, $p \nmid 2ms_3$ and $(2, 2mp^r, s_3 p^e) \in F_3 \setminus R_3$. Then $B_1$ is rigid.*

*Proof.* Without loss of generality, we assume, $k$ to be an algebraically closed field (cf. Remark 2.2.5(i)). Let $\phi$ be an exponential map on $B_1$. Then by Lemma 4.2.9, $x, y^{2m} + z^{s_3 p^{e-r}} \in B_1^{\phi}$. We will show that $y, z \in B_1^{\phi}$. Then $\phi$ is a trivial exponential map. So $B_1$ is rigid.

**Case I :** $s_3$ is odd.
Note that if $s_3 = 1$ then $e > r$ as $(2, 2mp^r, s_3 p^e) \in F_3 \setminus R_3$. Now $gcd(2, s_3 p^e) = 1$ hence $y$ is prime in $B_1$. Since $y^{2m} + z^{s_3 p^{e-r}} \in B_1^{\phi}$, it follows that $y, z \in B_1^{\phi}$, by Lemma 2.4.12 or Lemma 2.4.13, according as $e = r$ or $e > r$.

**Case II :** $s_3$ is even, say $s_3 = 2s_4$ for some $s_4 \in \mathbb{Z}_{\geqslant 1}$ and $i \in k$ be such that $i^2 = -1$.

Now $y^{2m} + z^{2s_4p^{e-r}} \in B_1^{\phi}$. Hence $y^m + iz^{s_4p^{e-r}}$, $y^m - iz^{s_4p^{e-r}} \in B_1^{\phi}$ (cf. Lemma 2.2.3(i)). Therefore $y, z \in B_1^{\phi}$. □

**Theorem 4.2.11.** *Let $k$ be a field of characteristic $p > 0$. Suppose $B = B_{(a,s_2p^r,s_3p^e)}$ for some $s_2, s_3, e, r \in \mathbb{Z}_{\geqslant 1}$ such that $p \nmid as_2s_3$ and $e \geqslant r$. Then $B$ is a rigid domain whenever $(a, s_2p^r, s_3p^e) \in F_3 \setminus R_3$.*

*Proof.* Without loss of generality, we assume, $k$ to be an algebraically closed field (cf. Remark 2.2.5(i)). We consider the subring $A := k[x, y^{p^r}, z^{p^r}] = B_{(a,s_2,s_3p^{e-r})}$ of $B$. Based on $e, r$ we consider two cases:

**Case I :** Let $e = r$. Since $(a, s_2p^r, s_3p^r) \in F_3 \setminus R_3$ we have $s_2, s_3 \geqslant 2$.
By Lemma 4.2.9, $x, y^{s_2} + z^{s_3} \in \mathrm{ML}(B)$. Since $s_2, s_3 \geqslant 2$, therefore by Lemma 2.4.12, $x, y, z \in \mathrm{ML}(B)$ i.e., $\mathrm{ML}(B) = B$. So $B$ is rigid.

**Case II :** Let $e > r$ then since $(a, s_2p^r, s_3p^e) \in F_3 \setminus R_3$ we have $s_2 \geqslant 2$.
When $(a, s_2, s_3p^{e-r}) \in F_3 \setminus (T_3 \cup S_3)$, then $A$ is rigid by Theorem 4.2.8 and hence by Proposition 4.2.1, $B$ is also rigid. In the remaining cases, one among $a$ and $s_2$ is 2 and the other is even. Hence by Lemma 4.2.10, $B$ is rigid. □

Now Theorem B(iv) follows.

**Theorem 4.2.12.** *$B_{(a,b,c)}$ is rigid when $(a, b, c) \in F_3 \setminus (R_3 \cup T_3 \cup S_3)$.*

*Proof.* By combining Theorems 4.2.7, 4.2.8 and 4.2.11 the theorem follows. □

## 4.3 Theorem G: Some Auxiliary Results on Rigidity

In this section we prove Theorem G. We first prove a technical lemma.

**Lemma 4.3.1.** *Let $R$ be a $\mathbb{Z}_{\geqslant 0}$-graded, finitely generated $k$-domain. For arbitrary $r \in \mathbb{Z}_{\geqslant 1}$ and $h, \widehat{h} \in R$ let*

$$C := \frac{R[X]}{(X^r + h)} \quad \text{and} \quad \widehat{C} := \frac{R[X]}{(X^r + \widehat{h})}$$

*be two integral domains, where $\widehat{h}$ is the highest degree homogeneous summand of $h$. Then we can define an admissible $\mathbb{Z}$-filtration on $C$ such that $\mathrm{gr}(C) \cong \widehat{C}$.*

*Proof.* Let $d$ denote the degree function induced on $R$ by the given $\mathbb{Z}_{\geqslant 0}$-grading (cf. Remark 2.3.3). We can assume that $r$ divides $d(h)$, if necessary we can change the grading by multiplying the degree function suitably and set $a := d(h)/r$. We regard $R$ as a subring of $C$. Let $x$ denote the image of $X$ in $C$. By using the relation $x^r = -h$, every $g$ in $C$ has a unique representation as

$$g = \sum_{j=0}^{r-1} g_j x^j \quad \text{for some} \quad g_j \text{ in } R. \tag{4.3.1}$$

Let $\deg : C \setminus \{0\} \to \mathbb{Z}_{\geqslant 0}$ be a function defined by,

$$\deg(g) = \max_{0 \leqslant j < r} \{d(g_j) + ja\}.$$

It is clear that deg is a semi-degree function. We now show that for non-zero $f, g \in C$ with $\deg(f) = l$ and $\deg(g) = m$ we have $\deg(fg) = l + m$. This would imply that the above defined deg is a degree function on $C$.

Let $f, g \in C \setminus \{0\}$ with $\deg(f) = l$ and $\deg(g) = m$. Let $f_0$ and $g_0$ respectively denote the highest degree homogeneous summands of $f$ and $g$ in $C$. Let

$$f_0 = \sum_{j=0}^{r-1} f_{0j} x^j \quad \text{and} \quad g_0 = \sum_{i=0}^{r-1} g_{0i} x^i$$

be the unique representation of $f_0$ and $g_0$ respectively, where $f_{0j}$, $0 \leqslant j < r$, are either zero or homogeneous elements of $R$ with $\deg(f_{0j}) = l - ja$ and $g_{0i}$, $0 \leqslant i < r$, are either zero or homogeneous elements of $R$ with $\deg(g_{0i}) = m - ia$. Suppose, if possible, that $\deg(fg) < l + m$. Then we would have $\deg(f_0 g_0) < l + m$, i.e.,

$$\sum_{i+j=0}^{r-1} f_{0j} g_{0i} x^{i+j} + \sum_{i+j=r}^{2r-2} f_{0j} g_{0i} (-\widehat{h}) x^{i+j-r} = 0,$$

i.e., $(\sum_{j=0}^{r-1} f_{0j} x^j)(\sum_{i=0}^{r-1} g_{0i} x^i) = 0$ in $\widehat{C}$ (we use the same symbol $x$ for the image of $X$ in $\widehat{C}$). Now as $\widehat{C}$ is an integral domain, it would follow that either $\sum_{j=0}^{r-1} f_{0j} x^j = 0$ or $\sum_{i=0}^{r-1} g_{0i} x^i = 0$, say $\sum_{i=0}^{r-1} g_{0i} x^i = 0$ in $\widehat{C}$. Since $\{1, x, \ldots, x^{r-1}\}$ is a basis of $\widehat{C}$ over $R$, it would follow that $g_0 = 0$ in $C$, a contradiction. Thus $\deg(fg) = l + m$.

Since $R$ is a finitely generated $k$-domain and since $R$ is a graded ring with respect to the degree function deg, by (4.3.1), it follows that this filtration on $C$ is admissible with respect to a homogeneous generating set of $R$ and $x$.

Let $\overline{C}$ denote the associated $\mathbb{Z}$-graded domain determined by the above filtration. For any $g \in C$, let $\overline{g}$ denote its image in $\overline{C}$. Note that $R$ can be regarded as a graded subring of $\overline{C}$.

Let $\psi : R[X] \to \overline{C}$ be the surjective $R$-algebra homomorphism defined by $\psi(X) = \overline{x}$. Since $x^r + \widehat{h} = -(h - \widehat{h})$ and since $\deg(h - \widehat{h}) < \deg(x^r + \widehat{h})$, we have $\overline{x}^r + \widehat{h} = 0$ in $\overline{C}$. Since $\overline{C}$ is an integral domain with $\operatorname{tr.deg}_R \overline{C} = 0$ and $(X^r + \widehat{h})$ is a prime ideal of $R[X]$, it follows that $\psi$ induces an isomorphism $\overline{\psi} : R[X]/(X^r + \widehat{h}) \to \overline{C}$. □

In particular, when $R = k^{[2]}$, we have the following result which will be needed subsequently.

**Corollary 4.3.2.** *Let $r \in \mathbb{Z}_{\geqslant 1}$ and $h(Y,Z) \in k[Y,Z]$ be such that $X^r + h(Y,Z)$ is a prime element of $k[X,Y,Z]$. Suppose $k[Y,Z]$ has a $\mathbb{Z}_{\geqslant 0}$-graded structure such that $X^r + \widehat{h}(Y,Z)$ is also a prime element of $k[X,Y,Z]$, where $\widehat{h}(Y,Z)$ is the highest degree homogeneous summand of $h$. Then we can define an admissible $\mathbb{Z}$-filtration on*

$$C := \frac{k[X,Y,Z]}{(X^r + h(Y,Z))} \quad \textit{such that} \quad \operatorname{gr}(C) \cong \frac{k[X,Y,Z]}{(X^r + \widehat{h}(Y,Z))}.$$

The next lemma has been proved by Crachiola and Maubach over a field of characteristic zero ([12, Lemma 5.1]). We prove it for arbitrary characteristic.

**Lemma 4.3.3.** *Let $a,b,c \in \mathbb{Z}_{\geqslant 2}$ with $gcd(a,b,c) = 1$. Then $D := k[X,Y,Z]/(X^a + Y^b Z^c)$ is a rigid domain.*

*Proof.* Let $x, y, z$ denote the images of $X, Y$ and $Z$ in $D$ respectively. Suppose, if possible, that there exists a non-trivial exponential map $\phi$ on $D$ and let $f \in D^{\phi} \setminus k$.

We first show that $\{x,y,z\} \cap D^{\phi} = \varnothing$. If $x \in D^{\phi}$ then $y, z \in D^{\phi}$, since $D^{\phi}$ is factorially closed (cf. Lemma 2.2.3(i)), contradicting the fact that $\phi$ is non-trivial. So $x \notin D^{\phi}$. Next suppose, if possible, that $y \in D^{\phi}$. Then by Lemma 2.2.3(vi), $\phi$ would induce a non-trivial exponential map on $\widetilde{D} := k(y)[X,Z]/(X^a + y^b Z^c)$. Then, denoting $L$ to be the algebraic closure of $k(y)$ in $\widetilde{D}$, by Lemma 2.2.3(v), we would have $\widetilde{D} = L^{[1]}$. However, when $c \geqslant a$ then $x/z$ is in the integral closure of $\widetilde{D}$ in Frac$\widetilde{D}$ but not in $\widetilde{D}$ and when $c < a$ then $z/x$ plays the same role, showing that $\widetilde{D}$ is not a normal domain In particular, $\widetilde{D} \neq L^{[1]}$, a contradiction. Therefore $y \notin D^{\phi}$. Similarly we can conclude that $z \notin D^{\phi}$.

We consider two different $\mathbb{Z}$-gradings on the $k$-algebra $D$, given by

$$wt_1(x) = c, \quad wt_1(y) = 0, \quad wt_1(z) = a$$
$$wt_2(x) = b, \quad wt_2(y) = a, \quad wt_2(z) = 0.$$

By Remark 2.3.6, both the gradings define admissible $\mathbb{Z}$-filtrations on $D$. Now we are going to apply the gradings one by one. For each $g \in D$, let $\widehat{g}$ denote the image of $g$ under the composite map $D \to gr_1(D) \to gr_2(gr_1(D))$. For $l = 1, 2$, let $\deg_l$ denote the degree function corresponding to the weight $wt_l$. By Theorem 2.3.8, $\phi$ induces a non-trivial exponential map $\overline{\phi}$ on $D(\cong gr_1(D))$ with respect to $\deg_1$, and then $\overline{\phi}$ induces another non-trivial exponential map $\widehat{\phi}$ on $D(\cong gr_2(gr_1(D)))$ with respect to $\deg_2$, such that $gr_2(gr_1(f)) = \widehat{f} \in D^{\widehat{\phi}}$. Then $\widehat{f} \notin k$ and with the help of the relation $x^a = -y^b z^c$,

$$\widehat{f} = \sum_{i \in \Gamma} h_i(y,z)x^i \quad \text{for some non-zero polynomials } h_i(y,z) \in k[y,z]$$

with $\Gamma \subseteq \{0, 1, \ldots, a-1\}$.

We now show that $\Gamma$ is a singleton set. Suppose, if possible, there exist distinct $i, j \in \Gamma$. Then $\deg_l(h_i(y,z)x^i) = \deg_l(h_j(y,z)x^j)$ for $l = 1, 2$. Let $\alpha_l = \deg_z(h_l)$ and $\beta_l = \deg_y(h_l)$ for $l = i, j$. Since $\deg_1(h_i(y,z)x^i) = \deg_1(h_j(y,z)x^j)$, we would have

$$\alpha_i a + ic = \alpha_j a + jc \quad \text{and hence} \quad (\alpha_i - \alpha_j)a = (j-i)c, \tag{4.3.2}$$

and $\deg_2(h_i(y,z)x^i) = \deg_2(h_j(y,z)x^j)$, would imply

$$\beta_i a + ib = \beta_j a + jb \quad \text{and hence} \quad (\beta_i - \beta_j)a = (j-i)b. \tag{4.3.3}$$

From (4.3.2) and (4.3.3) it would follow that there exists a prime factor $p_1$ of $a$ such that $p_1|c$ and $p_1|b$, and hence $p_1|gcd(a,b,c)$. But $gcd(a,b,c) = 1$, a contradiction. Thus $|\Gamma| = 1$.

Again since $x \notin D^{\widehat{\phi}}$ by the earlier argument, it would follow that $\widehat{f} = h_0(y,z) = \lambda y^n z^m$, for some $\lambda \in k^*$, $m, n \in \mathbb{Z}_{\geqslant 0}$ with at least one of them strictly greater than 0. Therefore either $y \in D^{\widehat{\phi}}$ or $z \in D^{\widehat{\phi}}$ which is not possible. Thus there does not exist any non-trivial exponential map on $D$, hence $D$ is rigid. $\square$

We now prove Theorem G, with the help of Corollary 4.3.2 and Lemma 4.3.3.

**Theorem 4.3.4.** *Let $k$ be a field and $a, b, c \in \mathbb{Z}_{\geqslant 2}$ be such that $gcd(a, b, c) = 1$. Then for any $F(Y) \in k[Y]$, $D := k[X, Y, Z]/(X^a + Y^b Z^c + F(Y))$ is a rigid domain.*

*Proof.* Let $x, y, z$ be the images of $X, Y$ and $Z$ in $D$ respectively. We define a $\mathbb{Z}_{\geqslant 0}$-graded structure on $k[Y, Z]$, given by $wt(Y) = 0$ and $wt(Z) = a$. Then the highest degree homogeneous summand of $Y^b Z^c + F(Y)$ is $Y^b Z^c$ and $X^a + Y^b Z^c$ is irreducible (as $gcd(a, b, c) = 1$) and hence prime in $k[X, Y, Z]$. Therefore, by Corollary 4.3.2, the degree function $\deg : D \setminus \{0\} \to \mathbb{Z}_{\geqslant 0}$ defined by

$$\deg(x) = c, \quad \deg(y) = 0, \quad \deg(z) = a$$

admits an admissible $\mathbb{Z}$-filtration on $D$ such that

$$\mathrm{gr}(D) \cong k[X, Y, Z]/(X^a + Y^b Z^c).$$

By Lemma 4.3.3, $\mathrm{gr}(D)$ is rigid and hence, by Theorem 2.3.8, $D$ is also rigid. □

## 4.4 Applications

As an application of Theorem F and results of Section 4.3, we now prove rigidity of certain surfaces and threefolds. We first consider Asanuma threefold, which gave the first counterexample to the Zariski Cancellation Problem in positive characteristic for $n = 3$ ([27]).

**Proposition 4.4.1.** *Let $k$ be a field of characteristic $p > 0$. Let*

$$B = \frac{k[X, Y, Z, T]}{(X^m Y^n + T^{qp^r} + Z^{p^e})},$$

*for some $m, n, q, r, e \in \mathbb{Z}_{\geqslant 1}$ with $p \nmid q$ and $e > r \geqslant 1$. Let $x, y, z, t$ respectively denote the images of $X, Y, Z, T$ in $B$. Then the following statements are true:*

(i) *If $p \nmid m$ and $m, q \geqslant 2$, then there does not exist any non-trivial exponential map $\phi$ on $B$ such that $y \in B^{\phi}$.*

(ii) *If $m, n \geqslant 2$ and $gcd(m, n, p) = 1$, then there does not exist any non-trivial exponential map $\phi$ on $B$ such that $t \in B^{\phi}$.*

(iii) *If* $m, n \geqslant 2$ *and* $gcd(m, n, pq) = 1$, *then there does not exist any non-trivial exponential map* $\phi$ *on* $B$ *such that* $z \in B^{\phi}$.

*Proof.* Let $\phi$ be a non-trivial exponential map on $B$.

***(i)*** Suppose, if possible, that $y \in B^{\phi}$. Then, by Lemma 2.2.3(vi) and Remark 2.2.5(i), $\phi$ induces a non-trivial exponential map on $\widetilde{B} := \overline{k(y)}[X, Z, T]/(y^n X^m + T^{qp^r} + Z^{p^e})$. Since $e > r$, $q > 1$ and $m > 1$, we have $(m, qp^r, p^e) \in F_3 \setminus R_3$. Hence, by Theorem 4.2.11, $\widetilde{B}$ is rigid, a contradiction.

***(ii)*** Suppose, if possible, that $t \in B^{\phi}$. Then again by Lemma 2.2.3(vi), $\phi$ induces a non-trivial exponential map on $\widetilde{B} := k(t)[X, Y, Z]/(X^m Y^n + Z^{p^e} + t^{qp^r})$. Since $m, n, p^e \geqslant 2$ and $gcd(m, n, p^e) = 1$, by Theorem 4.3.4, $\widetilde{B}$ is rigid, a contradiction.

***(iii)*** Suppose, if possible, that $z \in B^{\phi}$. Then, as before, we get a non-trivial exponential map on $\widetilde{B} := k(z)[X, Y, T]/(X^m Y^n + T^{qp^r} + z^{p^e})$. Now $m, n, qp^r \geqslant 2$ and $gcd(m, n, qp^r) = 1$, hence by Theorem 4.3.4, $\widetilde{B}$ is rigid, a contradiction. □

**Remark 4.4.2.** Proposition 4.4.1(i) provides an alternative proof of [27, Lemma 3.5] in some cases.

The next result relates the rigidity of Pham-Brieskorn surfaces with its translates.

**Proposition 4.4.3.** *Let* $A = k[X, Y, Z]/(X^a + Y^b + Z^c + \lambda)$ *for some* $(a, b, c) \in F_3$ *with* $\lambda \in k$. *Then* $A$ *is a rigid domain whenever* $B_{(a,b,c)}$ *is rigid.*

*Proof.* Let $x, y, z$ be the images of $X, Y$ and $Z$ in $A$ respectively. We define a $\mathbb{Z}_{\geqslant 0}$-graded structure on $k[Y, Z]$, given by $wt(Y) = ac$ and $wt(Z) = ab$. Then the highest degree homogeneous summand of $Y^b + Z^c + \lambda$ is $Y^b + Z^c$. Now $X^a + Y^b + Z^c$ is a prime element of $k[X, Y, Z]$ because $(a, b, c) \in F_3$. Therefore by Corollary 4.3.2, the function $\deg : A \setminus \{0\} \to \mathbb{Z}_{\geqslant 0}$ defined by

$$\deg(x) = bc, \quad \deg(y) = ac, \quad \deg(z) = ab$$

admits an admissible $\mathbb{Z}$-filtration on $A$ such that

$$\operatorname{gr}(A) \cong \frac{k[X, Y, Z]}{(X^a + Y^b + Z^c)} = B_{(a,b,c)}.$$

By Theorem 2.3.8, every non-trivial exponential map on $A$ induces a non-trivial exponential map on $gr(A) = B_{(a,b,c)}$. Therefore $A$ is rigid whenever $B_{(a,b,c)}$ is rigid. $\square$

As we have seen in [27], the non existence of non-trivial exponential maps turns out to be crucial in settling the triviality or otherwise of important affine domains. Below we give a few sufficient conditions for non existence of certain non-trivial exponential maps on certain Pham-Brieskorn threefolds.

**Proposition 4.4.4.** *Let $k$ be a field of characteristic $p \geqslant 0$. For $(a,b,c,d) \in F_4 \setminus (R_4 \cup T_4)$ let $B = k[X,Y,Z,T]/(X^a+Y^b+Z^c+T^d) = B_{(a,b,c,d)} = k[x,y,z,t]$ (where $x,y,z,t$ respectively denote the images of $X,Y,Z$ and $T$ in $B$). Then for the following conditions on $b,c,d$, there does not exist any non-trivial exponential map on $B$ fixing $x$.*

(i) $(b,c,d) \in F_3 \setminus S_3$.

(ii) *$b = s_2p^m$, $c = s_3p^r$ and $d = s_4p^e$ for some $s_2,s_3,s_4,m,r,e \in \mathbb{Z}_{\geqslant 1}$ with $p \nmid s_2s_3s_4$, $m \leqslant r \leqslant e$ and $(s_2, s_3p^{r-m}, s_4p^{e-m}) \in F_3 \setminus (T_3 \cup S_3)$.*

*Proof.* Suppose, if possible, $\phi$ is a non-trivial exponential map on $B$ with $x \in B^{\phi}$. We arrive at a contraction in each case.

***(i)*** By Lemma 2.2.3(vi) and Remark 2.2.5(i), $\phi$ induces a non-trivial exponential map on

$$\widetilde{B} := \frac{\overline{k(x)}[Y,Z,T]}{(Y^b+Z^c+T^d+x^a)}.$$

By the given condition $(b,c,d) \in F_3 \setminus (R_3 \cup T_3 \cup S_3)$, hence by Theorem 4.2.12, $B_{(b,c,d)}$ is rigid. Therefore by Proposition 4.4.3, $\widetilde{B}$ is rigid. This leads to a contradiction.

***(ii)*** Since $(a,b,c,d) \in F_4$, $p \nmid a$. Let $A := k[x, y^{p^m}, z^{p^m}, t^{p^m}]$ be a subring of $B$. Then

$$A \cong \frac{k[X,Y,Z,T]}{(X^a+Y^{s_2}+Z^{s_3p^{r-m}}+T^{s_4p^{e-m}})} = B_{(a,s_2,s_3p^{r-m},s_4p^{e-m})}.$$

By Proposition 4.2.1, $\phi$ induces a non-trivial exponential map $\phi_1$ on $A$ such that $x \in A \cap B^{\phi} = A^{\phi_1}$. Since $(a,b,c,d) \in F_4 \setminus (R_4 \cup T_4)$ and $(s_2, s_3p^{r-m}, s_4p^{e-m}) \in F_3 \setminus (T_3 \cup S_3)$ we have $(s_2, s_3p^{r-m}, s_4p^{e-m}) \in F_3 \setminus (R_3 \cup T_3 \cup S_3)$. By (i) such a $\phi_1$ cannot exist on $A$, a contradiction. $\square$

Finally, we deduce a partial result on the rigidity of a certain Pham-Brieskorn surface when $k$ is not algebraically closed.

**Proposition 4.4.5.** *Let $k$ be a field of characteristic $p \neq 2$ and also not containing a square root of $-1$. We consider $B := B_{(2,2,c)} = k[x,y,z]$ for an odd integer $c > 1$, then $B$ is rigid.*

*Proof.* Suppose, if possible, that there exists a non-trivial exponential map $\phi$ on $B$ and let $f \in B^{\phi} \setminus k$. Let $\phi_1 = \phi \otimes_k id$ be the extension of $\phi$ on $B_1 := k[x,y,z] \otimes_k \overline{k} \cong \overline{k}[x,y,z]$. Then $f \in B_1^{\phi_1} \setminus \overline{k}$. Since $c$ is odd, as in the the proof of Lemma 4.2.5, we get $\widehat{f} = \lambda_1(x+\mu y)^r \in B_1^{\widehat{\phi_1}} \subseteq B_1$ for some $\lambda_1 \in k^*$, $\mu \in \overline{k}^*$ and $r \in \mathbb{Z}_{\geqslant 1}$. Therefore $x + \mu y \in B_1^{\widehat{\phi_1}}$ (cf. Lemma 2.2.3(i)). Now for every $\lambda$ in $\overline{k}^*$

$$\frac{B_1}{(x+\mu y-\lambda)B_1} = \frac{\overline{k}[X,Y,Z]}{(X^2+Y^2+Z^c, X+\mu Y-\lambda)} \cong \frac{\overline{k}[Y,Z]}{(Z^c+P_\lambda(Y))},$$

where $P_\lambda(Y) := (1+\mu^2)Y^2 - 2\lambda\mu Y + \lambda^2 \in \overline{k}[Y]$. Thus $P_\lambda(Y) \notin \overline{k}$. Hence $x+\mu y-\lambda$ is a prime element of $B_1$ for all $\lambda \in \overline{k}^*$. Therefore, by Lemma 2.4.11, there exists $\beta \in \overline{k}^*$ such that $\widehat{\phi_1}$ induces a non-trivial exponential map on the integral domain $\widetilde{B}_1 := B_1/(x+\mu y-\beta)B_1 \cong \overline{k}[Y,Z]/(Z^c+P_\beta(Y))$. Since $\operatorname{tr.deg}_k \widetilde{B}_1 = 1$, so by Lemma 2.2.3(v), $\widetilde{B}_1 = \overline{k}^{[1]}$. Suppose, if possible, $\mu^2+1 \neq 0$. Then we would have

$$\begin{aligned} Z^c + P_\beta(Y) &= Z^c + (1+\mu^2)Y^2 - 2\beta\mu Y + \beta^2 \\ &= Z^c + (\mu^2+1)(Y-\lambda_1)^2 + \lambda_2, \end{aligned}$$

where $\lambda_1 := \mu\beta/(\mu^2+1)$ and $\lambda_2 := \beta^2(2\mu+1)/(\mu^2+1)^2$. So $Z^c + P_\beta(Y)$ would have no linear term in the new coordinates $Z$ and $Y_1 := Y - \lambda_1$. Hence $\widetilde{B}_1 \neq \overline{k}^{[1]}$, a contradiction. Therefore $\mu^2+1 = 0$, say $\mu = i$, a square root of $-1$ in $\overline{k}$. Note that $\widehat{\phi_1}$ is the extension of $\widehat{\phi}$ on $B_1$. Therefore $\widehat{f} = \lambda_1(x+iy)^r \in B^{\widehat{\phi}} \subseteq B$ for $\lambda_1 \in k^*$ and $r > 0$. So $i \in k$, a contradiction. Therefore $B_{(2,2,c)}$ is rigid for every odd integer $c > 1$. □

Note that if $k$ is algebraically closed then by Theorem F(ii), the above surfaces are non-rigid.

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## List of Publications

- Parnashree Ghosh, Neena Gupta and Ananya Pal, *On embedding of linear hypersurfaces*, https://arxiv.org/abs/2405.07205, submitted

- Neena Gupta and Ananya Pal, *On rigidity of Pham-Brieskorn surfaces*, J. Algebra **657** (2024), 748–772.