Second Moment of Degree Three L-Functions

Sampurna Pal

Indian Statistical Institute

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Author:
Sampurna Pal

Supervisor:
Prof. Ritabrata Munshi

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Contents

Chapter 1

Introduction

1.1 Integral Moments of L-functions in t-aspect

In analytic number theory, understanding the integral moments of L-functions is of fundamental interest. Any L-function $L(f, s)$ associated to an automorphic form f, has an analytic continuation over the whole complex plane (with a possible exception of a pole at $s = 1$). Thus, we may define the $2k$th moment of $L(f, s)$ at the critical line in t-aspect as

$$M_f^k(T) := \int_0^T \left| L\left(f, \frac{1}{2} + it\right) \right|^{2k} dt \quad \text{for } k \in \mathbb{N}. \tag{1.1}$$

Except for a few families of L-functions, we still lack an understanding of these integral moments of L-functions in terms of asymptotic estimation or even in terms of lower and upper bounds.

One of the very first results regarding these moments is due to Hardy and Littlewood who in [HL16] derived an asymptotic expression of the second moment ($k = 1$) of the Riemann zeta function:

$$M_\zeta^1(T) = \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim \ T \log T.$$

This asymptotic expression was further refined by Atkinson [At49].

Then, Ingham [In27] obtained an asymptotic formula for the fourth moment ($k = 2$) of the Riemann zeta function:

$$M_\zeta^2(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \sim \frac{1}{2\pi^2}\, T\log^4 T.$$

The asymptotic expansion of the fourth moment was subsequently improved by Heath-Brown [Hb79] and generalized to weighted fourth moments by Motohashi (see [Mo]).

Let

$$E(z,w) = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}} \frac{y^w}{|cz+d|^{2w}} \qquad \text{for } \Re(w) > 1$$

be the Eisenstein series and let $L(E(\cdot,w),s)$ be the L-function associated to $E(z,w)$. For $\Re(s) > 1$, $L(E(\cdot,1/2),s)$, the L-function associated to $E(z,1/2)$, can be represented by the Dirichlet series

$$L(E(\cdot,1/2),s) = \sum_{n=1}^{\infty} d(n)n^{-s}$$

where $d(n)$ is the number of divisors of n. Thus, $L(E(\cdot,1/2),s)$ is the square of the Riemann zeta function, i.e. $L(E(\cdot,1/2),s) = \zeta(s)^2$. Hence, the fourth moment of the Riemann zeta function is actually the second moment of the L-function associated to the Eisenstein series $E(z,1/2)$:

$$M_\zeta^2(T) = M^1_{E(\cdot,1/2)} = \int_0^T \left|L\left(E(\cdot,1/2),\frac{1}{2}+it\right)\right|^2 dt.$$

The Eisenstein series $E(z,w)$s are non-cuspidal automorphic forms for $SL(2,\mathbb{Z})$. Thus, a natural generalization of the fourth moment of the Riemann zeta function would be the second moment of the L-functions associated to cuspidal automorphic forms for $SL(2,\mathbb{Z})$: a holomorphic cusp form or a Hecke-Maaß cusp form for $SL(2,\mathbb{Z})$. Good [Go82] derived an asymptotic expansion for the second moment of the L-functions associated to holomorphic cusp forms. Let f be a holomorphic cusp form for $SL(2,\mathbb{Z})$ of weight k with the Fourier expansion $f(z) = \sum_{n=1}^{\infty} a_n e(nz)$. Here $e(z)$ is the standard notation for $e^{2\pi iz}$. The L-function associated to f is represented by the Dirichlet series

$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}$ for $\Re(s) > \frac{k+1}{2}$. He proved

$$\int_0^T \left| L\left(f, \frac{k}{2} + it\right)\right|^2 dt = 2c_- T\left(\log\left(\frac{T}{2\pi e}\right) + c_0\right) + O((T\log T)^{\frac{2}{3}}), \tag{1.2}$$

for some constants c_-, c_0 computed explicitly. Here, the power saving error term was important as it resulted in a subconvex bound (actually a bound of the Weyl strength) of $L(f,s)$. But any asymptotic estimates of higher moments of degree two L-functions are still unknown.

Beyond the fourth moment, no asymptotic expression is known for the higher moments ($k \geq 3$) of the Riemann zeta function. But it is conjectured that there exists some constant C_k such that

$$M_\zeta^k(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \sim C_k\, T \log^{k^2} T.$$

These conjectured C_ks have been explicitly computed by Conrey and Ghosh [CGh84], Conrey and Gonek [CGo01], and through random matrix theory by Keating and Snaith [KS00]. Diaconu, Goldfeld and Hoffstein also computed these constants with a different approach in [DGH03]. Recently, Conrey et al. computed the lower order terms of these asymptotic expressions [CFKRS05].

Though we do not have an asymptotic expression of $M_\zeta^k(T)$ for $k \geq 3$, we do know of an upper bound for the twelfth moment of the Riemann zeta function due to Heath-Brown [Hb78]

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{12} dt \ll_\varepsilon T^{2+\varepsilon}.$$

From this result, with a simple application of Cauchy's inequality, one can get non-trivial upper bounds of $M_\zeta^k(T)$ for $k = 3, 4, 5$. In particular, this implies a non-trivial upper bound of the sixth moment of the Riemann zeta function

$$M_\zeta^3(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{6} dt \ll_\varepsilon T^{5/4+\varepsilon}. \tag{1.3}$$

Let $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ and $E_{P_{\min}}(z, \nu)$ be the minimal parabolic Eisenstein series for $SL(3, \mathbb{Z})$ (see (10.4.1), [Gf]). Then, the L-function associated to $E_{P_{\min}}(z, \nu)$ is a product of three copies of shifted Riemann zeta function:

$$L(E_{P_{\min}}(\cdot, \nu), s) = \zeta(s + \nu_1 + 2\nu_2 - 1)\zeta(s - 2\nu_1 - \nu_2 + 1)\zeta(s + \nu_1 - \nu_2).$$

Thus, the L-function associated to $E_{P_{\min}}(z, \nu)$ for $\nu = (1/3, 1/3)$ is $\zeta(s)^3$. Hence, the sixth moment of the Riemann zeta function is the second moment of the L-function associated to $E_{P_{\min}}(z, (1/3, 1/3))$, a non-cuspidal automorphic form for $SL(3, \mathbb{Z})$. So, the cuspidal analogue of the sixth moment of the Riemann zeta function would be the second moment of the L-function associated to a Hecke-Maaß cusp form for $SL(3, \mathbb{Z})$.

Let F be a Hecke-Maass cusp form for $SL(3, \mathbb{Z})$ of type $\nu = (\nu_1, \nu_2) \in \mathbb{C}$ with normalized $(A(1, 1) = 1)$ Fourier coefficients $A(m, n)$ and let $L(F, s)$ be the Godement-Jacquet L-function associated to F and defined by the Dirichlet series

$$L(F, s) = \sum_{n=1}^{\infty} A(1, n)n^{-s} \quad \text{for} \quad \Re(s) > 1.$$

Then we denote its t-aspect second moment at the critical line as

$$M_F(T) = \int_T^{2T} \left| L\left(F, \frac{1}{2} + it\right) \right|^2 dt. \tag{1.4}$$

This is essentially the same as $M_F^1(T)$ but with a dyadic range.

We can trivially bound the second moment $M_F(T)$ by $M_F(T) \ll_\varepsilon T^{3/2+\varepsilon}$ for any $\varepsilon > 0$. With the application of the approximate functional equation (Lemma 2.1), we can truncate the sum over n of the Dirichlet series in the integrand of (1.4) up to $N = T^{3/2+\varepsilon}$ with a negligible error term. After opening up the absolute square and evaluating the t integral, we get

$$M_F(T) \ll T^{-1/2} \left| \sum_{h \ll H} \sum_{n \sim N} A(1, n)A(1, n + h) \right|, \tag{1.5}$$

for $H \ll T^{1/2+\varepsilon}$. The double sum can be bounded above by $(NH)^{1+\varepsilon}$ by Cauchy's inequality and the Ramanujan bound on average (2.6). Thus, we get the upper bound $M_F(T) \ll_\varepsilon T^{3/2+\varepsilon}$. On the other hand, the generalized Lindelöf hypothesis suggests that $M_F(T) \ll T^{1+\varepsilon}$.

As the same arguments hold for $E_{P_{\min}}(z, (1/3, 1/3))$ too, the sixth moment of the Riemann zeta function can thusly be trivially bounded by $M_\zeta^3(T) \ll T^{3/2+\varepsilon}$. So the sixth moment bound $M_\zeta^3(T) \ll T^{5/4+\varepsilon}$ (1.3) obtained by Heath-Brown is a "non-trivial" upper bound. But the arithmetic structure of the Riemann zeta function used in ([Hb78]) is not inherited by the Hecke-Maaß cusp forms for $SL(3, \mathbb{Z})$. Thus, we cannot naturally extend the method of ([Hb78]) to obtain a non-trivial upper bound for $M_F(T)$, the second moment of $L(F, s)$. So, it was an important open problem to improve upon the trivial upper bound of $M_F(T) \ll_\varepsilon T^{3/2+\varepsilon}$ for the second moment of general degree three L-functions.

The main result of this dissertation establishes the first non-trivial upper bound for the second moment of degree three automorphic L-function:

Theorem 1. *Let F be a Hecke-Maaß cusp form for $SL(3, \mathbb{Z})$ and let $L(F, s)$ be the the Godement-Jacquet L-function associated to F. Then,*

$$\int_T^{2T} \left| L\left(F, \frac{1}{2} + it\right) \right|^2 dt \ll_{F,\varepsilon} T^{\frac{3}{2} - \frac{3}{32} + \varepsilon}. \tag{1.6}$$

Remark. After the appearance of the first version of this work [Pal23], Dasgupta, Leung and Young [DLY24] have further improved the exponent of Theorem 1. Consequently, they also improved the exponents in the following corollaries.

Though the problem of estimating the full second moment $M_F(T)$ for a Hecke-Maaß cusp form F for $SL(3, \mathbb{Z})$ was open until this work, different variations of this expression have been extensively studied. To achieve a subconvex bound for a self-dual form F, Li [Li11] proved

$$\int_{-\infty}^{\infty} e^{-\frac{(t-T)^2}{T^{3/4}}} \left| L\left(F, \frac{1}{2} - it\right) \right|^2 dt \ll_{F,\varepsilon} T^{\frac{11}{8} + \varepsilon}.$$

Let u_j be an orthonormal basis of Hecke-Maaß cusp form for $SL(2,\mathbb{Z})$ corresponding to Laplace eigenvalue $\frac{1}{4}+t_j^2$. Then Young [Yo11] proved

$$\int_{-T^{1-\varepsilon}}^{T^{1-\varepsilon}} \left|L\left(F\otimes u_j, \frac{1}{2}+it\right)\right|^2 dt \ll T^{1+\varepsilon},$$

for all but $o(T^2)$ many t_j in $T \leq t_j \leq 2T$. In a recent pre-print, Aggarwal, Leung and Munshi [ALM22] derived an upper bound for the second moment in a short interval. For $T^{1/2} < M < T^{1-\varepsilon}$ they proved

$$\int_{T-M}^{T+M} \left|L\left(F, \frac{1}{2}+it\right)\right|^2 dt \ll T^{\varepsilon}\left(\frac{T^{\frac{9}{4}}}{M^{\frac{3}{2}}} + \frac{M^3}{T^{\frac{21}{20}}} + M^{\frac{7}{4}}T^{\frac{3}{40}} + M^{\frac{15}{14}}T^{\frac{15}{28}}\right). \tag{1.7}$$

As the result is intended for the short moment, we note that it fails to give any non-trivial upper bound for the full second moment ($M = T$).

One way to obtain a non-trivial estimate of (1.4) would have been obtaining a non-trivial bound of the shifted convolution sum (1.5), specifically for $H \ll T^{1/2}$, which itself is a hard problem. We evade the issue by splitting the range of the t-integral $[T, 2T]$ into shorter ranges of length ξ and then evaluating the t-integral (see 3.6). This increases the size of H in (1.5) and introduces an extra oscillatory term. At this stage, we also introduce an extra averaging of the length of the new integrals ξ in the range $[X, 2X]$ (to be determined optimally), which is also essential for obtaining the non-trivial bound as it will be apparent at the end. This is the crucial input of our work.

Now, we use the delta method of Duke, Friedlander and Iwaniec to separate the oscillations and we treat these sums with the Voronoi summation formula or the Poisson summation formula. We then apply the duality principle of the large sieve like [ALM22] to interchange the order of the summations. At this point, we apply the Cauchy's inequality and Poisson summation formula twice, and each time, we get a character sum and an oscillatory integral. We then carefully evaluate these character sums and oscillatory integrals to recover the trivial bound of $M_F(T)$. But we do have an oscillatory term left, reminiscent of the extra oscillatory term introduced at the beginning. We also have the extra integral we introduced at the beginning, precisely the averaging over the length of the integral, ξ. Thus, we extract an extra saving out of that oscillatory integral to get the non-trivial bound in Theorem 1.

1.2 Applications

1.2.1 Subconvex bound for $GL(3)$ L-functions:

Let $L(f,s)$ be a degree d L-function attached to an automorphic form f with analytic conductor $\mathfrak{q}(f,s)$. The analytic conductor $\mathfrak{q}(f,s)$ can be bounded by $\mathfrak{q}(f,s) \leq \mathfrak{q}(f)(|s|+3)^d$. As we are only interested in the t-aspect, we may ignore the part $\mathfrak{q}(f)$, which only depends on the automorphic form f but not on s. Phragmen-Lindelof convexity principle implies that at the critical line ($s = \frac{1}{2} + it$), the L-function can be bounded by

$$L(f,s) \ll_\varepsilon \mathfrak{q}(f,s)^{1+\varepsilon} \ll_{f,\varepsilon} (1+|t|)^{d/4+\varepsilon}.$$

This bound is called the convexity bound of $L(f,s)$ in t-aspect. The upper bound with any improvement over the exponent of the form $\frac{d}{4} - \eta$ for some $\eta > 0$, is called a subconvex bound of $L(f,s)$. On the other extreme, Lindelof hypothesis conjectures that $L(f,s) \ll_{f,\varepsilon} (1+|t|)^\varepsilon$.

For the Riemann zeta function, the first subconvex bound was obtained by Hardy and Littlewood by a method of Weyl, though it was first published by Landau [La24]:

$$\zeta(1/2+it) \ll_\varepsilon (1+|t|)^{1/6+\varepsilon}.$$

This bound (even in the context of other L-functions) is called the Weyl bound. There have been periodical improvements of this Weyl bound (called sub-Weyl bound), the latest being $1/6 - 1/84$ by Bourgain [Bo17].

For degree two L-functions, precisely for holomorphic cusp forms of full level, Good [Go82] established the first subconvexity bound in the t-aspect by obtaining an asymptotic expression for the second moment (see 1.2). Let f be a normalized holomorphic cusp form for $SL(2,\mathbb{Z})$ and let $L(f,s)$ be the L-function associated to f. He obtained the Weyl bound:

$$|L(f,1/2+it)| \ll_{\varepsilon,f} (|t|+1)^{1/3+\varepsilon}.$$

In a recent pre-print, Holowinsky, Munshi, Sharma and Streipel [HMSS25] obtained a sub-Weyl bound for $L(f,s)$, an L-function associated to a holomorphic or Hecke-Maaß cusp form for $SL(2,\mathbb{Z})$:

$$|L(f,1/2+it)| \ll_{\varepsilon,f} (|t|+1)^{1/3-1/174+\varepsilon}.$$

Let F be a Hecke-Maaß cusp form for $SL(3,\mathbb{Z})$. For degree three L-functions, the first subconvex bound in t-aspect was established by Li [Li11] for self-dual forms for $SL(3,\mathbb{Z})$. She obtained

$$L(F,1,2+it) \ll_{F,\varepsilon} (1+|t|)^{\frac{11}{16}+\varepsilon},$$

where F is self-dual. It was extended to any general automorphic form for $SL(3,\mathbb{Z})$ by Munshi [Mu15] with the same exponent. In a recent pre-print Aggarwal, Leung and Munshi [ALM22] improved the subconvex bound of [Mu15] by obtaining

$$L(F,1,2+it) \ll_{F,\varepsilon} (1+|t|)^{\frac{5}{8}+\varepsilon},$$

by bounding a short second moment (1.7). We also note that Nelson [Ne21] has obtained the t-aspect subconvexity bound for standard L-function of any degree d.

Theorem 1 also implies a t-aspect subconvex bound for degree three L-functions:

Corollary 1.1. *Let F be a normalized Hecke-Maaßcusp form for $SL(3,\mathbb{Z})$ and $L(F,s)$ be the L-function associated to F. Then,*

$$L(F,1/2+it) \ll_{F,\varepsilon} (1+|t|)^{3/4-3/64+\varepsilon}. \tag{1.8}$$

Remark. The bound we obtain is weaker than the first bound obtained in [Mu15].

1.2.2 Subconvex bound for self-dual $GL(3)$ L-functions:

Let F be a self-dual Hecke-Maaß cusp form for $SL(3,\mathbb{Z})$ and let f_j be an orthonormal basis of even Hecke-Maaß cusp form for $SL(2,\mathbb{Z})$ of Laplacian eigenvalue $\frac{1}{4}+t_j^2$. With these assumptions Li, in her pioneering work [Li11], proved the subconvex bounds

$$L(F,1/2+it) \ll_{F,\varepsilon} (1+|t|)^{11/16+\varepsilon}, \quad L(F\otimes f_j,1/2) \ll_{F,\varepsilon} (1+|t_j|)^{11/8+\varepsilon}.$$

These bounds were later improved by McKee, Sun and Ye [MSY18] and by Nunes [Nu17]. Recently, Lin, Nunes and Qi [LNQ22] reached the limit of the moment method of Li. Let f_j be a Hecke-Maaß cusp form (not necessarily even) for $SL(2,\mathbb{Z})$ of Laplacian eigenvalue $\frac{1}{4}+t_j^2$. Then they proved

$$L(F,1/2+it) \ll_{F,\varepsilon} (1+|t|)^{3/5+\varepsilon}, \quad L(F\otimes f_j,1/2) \ll_{F,\varepsilon} (1+|t_j|)^{6/5+\varepsilon}.$$

Their bounds rely on an upper bound of the second moment of $GL(3)$ L-functions. Thus, our non-trivial upper bound of the second moment of $GL(3)$ L-functions (Theorem 1) leads to the following improvements of the above bounds :

Corollary 1.2. *Let F be a self-dual Hecke-Maaß cusp form for $SL(3,\mathbb{Z})$ and let f_j be a Hecke-Maaß cusp form for $SL(2,\mathbb{Z})$ of Laplacian eigenvalue $\frac{1}{4}+t_j^2$. Then,*

$$L(F,1/2+it) \ll_{F,\varepsilon} (1+|t|)^{45/77+\varepsilon} \quad \text{and} \quad L(F\otimes f_j,1/2) \ll_{F,\varepsilon} (1+|t_j|)^{90/77+\varepsilon}.$$

1.2.3 Rankin-Selberg Problem

Let f be a normalized holomorphic Hecke cusp form or a Hecke-Maass cusp form for $SL(2,\mathbb{Z})$. Let $\lambda_f(n)$ be its n-th Hecke eigenvalue. The goal of the Rankin-Selberg problem is to bound the error term for the second moment of $\lambda_f(n)$, i.e.

$$\Delta(X) := \sum_{n\leq X} \lambda_f(n)^2 - cX.$$

Let $L(f,s)$ be the L-function associated with f and let $L(f\otimes f,s)$ be the Rankin-Selberg convolution defined by $L(f\otimes f,s)=\zeta(2s)\sum_{n\geq 1}\lambda_f(n)^2 n^{-s}$. $L(f\otimes f,s)$ is also related to the symmetric square lift of f by $L(f\otimes f,s)=\zeta(s)L(\mathrm{Sym}^2 f,s)$. Then, we can explicitly write c as $c=L(\mathrm{Sym}^2 f,1)/\zeta(2)$.

Rankin [Ra39] and Selberg [Se40] established the longstanding upper bound of

$$\Delta(X)\ll X^{3/5}.$$

Huang [Hu21] lowered the upper bound of Rankin and Selberg establishing

$$\Delta(X)\ll_\varepsilon X^{3/5-1/560+\varepsilon}.$$

Theorem 1 implies a further improvement of this bound:

Corollary 1.3. *Let $\Delta(X)$ be as defined above. Then,*

$$\Delta(X)\ll_{f,\varepsilon} X^{45/77+\varepsilon}\sim X^{3/5-6/385+\varepsilon}.$$

1.2.4 Zero density estimate for $GL(3)$ L-functions

Let $L(f,s)$ be any L-function associated to an automorphic form f. Now by $N(\sigma,T)$ we denote the number of zeros $\rho=\beta+i\gamma$ of $L(f,s)$ in the region $\beta\geq\sigma$ and $|\gamma|\leq T$ where $\frac{1}{2}\leq\sigma\leq 1$. In the absence of any asymptotic expression, upper bounds of $N(\sigma,T)$ of the form

$$N(\sigma,T)\ll T^{A(\sigma)(1-\sigma)+\varepsilon}$$

are usually referred to as zero-density estimates in the literature. Zero density estimates for the Riemann zeta function have been studied extensively, a detailed account of which can be found in Chapter 11 of [Iv]. For higher degree L-functions Kaczorowski and Perelli [KP03] obtained zero density estimates for L-functions of arbitrary degree:

$$N(\sigma,T)\ll T^{C(1-\sigma)+\varepsilon}.$$

Though their result is very general, the exponent C is not optimal for the degree three L-functions. For K, an algebraic number field of degree n, Heath-Brown [Hb77] obtained zero-density estimate for a general $\zeta_K(s)$ function. This was further improved by Paul and Sankaranarayanan [PS20].

Later, Mukhopadhyay and Srinivas [MS07] obtained zero density estimates for L-functions of arbitrary degree depending on the upper bound of the second moment for that L-function:

$$\int_0^T |L(f,1/2+it)|^2 dt \ll T^{\alpha+\varepsilon} \implies N(\sigma,T) \ll \begin{cases} T^{2\alpha(1-\sigma)+\varepsilon} & \text{for } \frac{1}{\alpha} \le \sigma \le 1 \\ T^{\frac{2}{\sigma}(1-\sigma)+\varepsilon} & \text{for } \frac{2}{3} \le \sigma \le \frac{1}{\alpha} \end{cases}. \tag{1.9}$$

In [YZ13], Ye and Zhang obtained zero density estimates for L-functions of arbitrary degree depending on the upper bound of any $2k$th moment. Considering only the second moment bound, we may restate their result as

$$\int_0^T |L(f,1/2+it)|^2 dt \ll T^{\alpha+\varepsilon} \implies N(\sigma,T) \ll T^{\frac{2(1+\alpha)(1-\sigma)}{3-2\sigma}+\varepsilon} \text{ for } \frac{1}{2} \le \sigma \le 1. \tag{1.10}$$

Then if $L(F,s)$ be a $GL(3)$ L-function and as both of these results depends on the upper bound of t-aspect moments of $L(F,s)$, our non-trivial bound of the second moment of $GL(3)$ L-functions (1.6) implies the following improvement of the zero density estimates for $GL(3)$ L-functions.

Corollary 1.4. *Let $L(F,s)$ be an L-function associated to an Hecke-Maaß cusp for for $SL(3,\mathbb{Z})$ and let $N(\sigma,T)$ denote the number of zeros $\rho=\beta+i\gamma$ of $L(f,s)$ in the region $\beta \ge \sigma$ and $|\gamma| \le T$ where $\frac{1}{2} \le \sigma \le 1$. Then,*

$$N(\sigma,T) \ll \begin{cases} T^{(3-3/16)(1-\sigma)+\varepsilon}, & \text{for } \frac{32}{45} \le \sigma \le 1 \\ T^{\frac{(77/16)(1-\sigma)}{(3-2\sigma)}+\varepsilon}, & \text{for } \frac{1}{2} \le \sigma \le 1 \end{cases}.$$

1.3 Proof of the corollaries

1.3.1 Proof of Corollary 1.1

Proof. This corollary is an easy consequence of Theorem 1 if we follow the proof of [Go82]. By residue theorem we can express $L(F, 1/2 + iT_0)$ as

$$L\left(F, \frac{1}{2} + iT_0\right)^2 = \frac{1}{2\pi i} \int_{(1/\log T_0)} L\left(F, \frac{1}{2} + iT_0 + s\right)^2 \frac{e^{s^2}}{s} ds$$
$$- \frac{1}{2\pi i} \int_{(-1/\log T_0)} L\left(F, \frac{1}{2} + iT_0 + s\right)^2 \frac{e^{s^2}}{s} ds.$$

Then, using the functional equation and Stirling's approximation in the second integral, we get

$$\left|L\left(F, \frac{1}{2} + iT_0\right)\right|^2 \ll \int_{-\infty}^{\infty} \left|L\left(F, \frac{1}{2} + \frac{1}{\log T_0} + i(T_0 + t)\right)\right|^2 \times \frac{(1 + |t + T_0|^{\frac{3}{\log T_0}})e^{-t^2}}{(t^2 + 1/\log^2 T_0)^{1/2}} dt. \tag{1.11}$$

In a similar fashion for $1/2 < \sigma_1 < 1$ we obtain

$$|L(F, \sigma_1 + iT_1)|^2 \ll 1 + \int_{-\infty}^{\infty} \left|L\left(F, \frac{1}{2} + i(T_1 + \tau)\right)\right|^2 \frac{e^{-\tau^2}}{(\tau^2 + (\frac{1}{2} - \sigma_1)^2)^{\frac{1}{2}}} d\tau. \tag{1.12}$$

In (1.12), we replace σ_1 by $1/2 + 1/\log T_0$ and T_1 by $T_0 + t$ and apply this bound in (1.11) and estimate the t integral to get

$$\left|L\left(F, \frac{1}{2} + iT_0\right)\right|^2 \ll \log T_0 + \log T_0 \int_{-\infty}^{\infty} \left|L\left(F, \frac{1}{2} + i(T_0 + \tau)\right)\right|^2 e^{-\frac{\tau^2}{2}} d\tau.$$

Finally, truncating the integral at $\tau \in [-\log T_0, \log T_0]$ and estimating the rest of the range trivially, we prove the corollary. □

1.3.2 Proof of Corollary 1.2

Proof. This corollary follows directly from the proof of [LNQ22] with minor changes. In the proof of [LNQ22], they have assumed the trivial bound of the second moment:

$$\int_{-U}^{U} \left|L\Big(F, \frac{1}{2}+it\Big)\right|^2 dt \ll_{F,\varepsilon} U^{\frac{3}{2}+\varepsilon}$$

Instead, we proceed with the assumption

$$\int_{-U}^{U} \left|L\Big(F, \frac{1}{2}+it\Big)\right|^2 dt \ll_{F,\varepsilon} U^{\frac{3}{2}-\delta+\varepsilon}, \quad \delta > 0.$$

Thus, we can replace the bound of (52) of [LNQ22] by

$$MT^{\varepsilon}\sqrt{R^{+}+U}\cdot U^{3/4-\delta/2} \ll \frac{T^{5/4-\delta/2+\varepsilon}}{M^{1/4-\delta/2}},$$

for the + case. Upper bounds for the diagonal case ($MT^{1+\varepsilon}$) and the - case ($\frac{T^{1+\varepsilon}}{M^{1/2}} + \frac{T^{5/4+\varepsilon}}{M}$) remain unchanged. Hence, we can replace the upper bound of Theorem 1.1 of [LNQ22] by

$$\sum_{|t_j-T|\le M} L(F\otimes f_j, 1/2) + \int_{T-M}^{T+M} |L(F, 1/2+it)|^2 \ll MT^{1+\varepsilon} + \frac{T^{5/4-\delta/2+\varepsilon}}{M^{1/4-\delta/2}}.$$

Then, we choose $M = T^{\frac{1-2\delta}{5-2\delta}}$. Thus, we derive the upper bounds

$$L(F, 1/2+it) \ll_{F,\varepsilon} (1+|t|)^{\frac{3-2\delta}{5-2\delta}+\varepsilon} \quad \text{and} \quad L(F\otimes f_j, 1/2) \ll_{F,\varepsilon} (1+|t_j|)^{\frac{6-4\delta}{5-2\delta}+\varepsilon}.$$

Finally, we apply Theorem 1 and put $\delta = \frac{3}{32}$ to conclude the proof. □

1.3.3 Proof of Corollary 1.3

Proof. We bound $\sum_{n\le 2X} \lambda_f(n)^2$ from above by taking a smooth dyadic partition with the help of a smooth function $V(x)$ which is supported in $[1-X^{-A}, 2+X^{-A}]$ and satisfies the decay condition $V^{(j)} \ll X^{jA}$ for $j \geq 1$, $0 \leq V(x) \leq 1$ and $V(x) = 1$ for $x \in [1,2]$:

$$\sum_{n \leq 2X} \lambda_f(n)^2 \leq \sum_{j=0}^{\infty} \sum_n \lambda_f(n)^2 V\left(\frac{n}{2^{-j}X}\right).$$

We similarly bound $\sum_{n \leq 2X} \lambda_f(n)^2$ from below by taking a smooth dyadic partition with the help of a smooth function $W(x)$ which is supported in $[1,2]$ and satisfies the decay condition $W^{(j)} \ll X^{jA}$ for $j \geq 1$, $0 \leq W(x) \leq 1$ and $W(x) = 1$ for $x \in [1 + X^{-A}, 2 - X^{-A}]$:

$$\sum_{n \leq 2X} \lambda_f(n)^2 \geq \sum_{j=0}^{\infty} \sum_n \lambda_f(n)^2 W\left(\frac{n}{2^{-j}X}\right).$$

By the inverse Mellin transform, we may express the sum $\sum_n \lambda_f(n)^2 V(n/X)$ (for some $c > 2$) as

$$\zeta(2) \sum_n \lambda_f(n)^2 V(n/X) = \frac{1}{2\pi i} \int\limits_{\Re(s)=c} L(f \otimes f, s) \tilde{V}(s) X^s ds,$$

where $\tilde{V}(s) = \int\limits_0^{\infty} V(x) x^{s-1} dx$. Now, we shift the line of integral to $\Re(s) = 1/2$ and encounter a pole at $s = 1$ we get the residue $L(\mathrm{Sym}^2 f, 1)\tilde{V}(1)X$. By the residue theorem, we get

$$\begin{aligned}\sum_n \lambda_f(n)^2 V(n/X) =& \frac{L(\mathrm{Sym}^2 f, 1)\tilde{V}(1)X}{\zeta(2)} \\ &+ \frac{1}{\zeta(2)2\pi} \int\limits_{-\infty}^{\infty} L\left(f \otimes f, \frac{1}{2} + it\right) \tilde{V}\left(\frac{1}{2} + it\right) X^{1/2+it} dt.\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{j=0}^{\infty} \sum_n \lambda_f(n)^2 V\left(\frac{n}{2^{-j}X}\right) \\ =& \frac{L(\mathrm{Sym}^2 f, 1)\tilde{V}(1)X}{\zeta(2)} \sum_{j=0}^{\infty} 2^{-j} \\ &+ \sum_{j=0}^{\infty} \frac{1}{2\pi\zeta(2)} \int\limits_{-\infty}^{\infty} L\left(f \otimes f, \frac{1}{2} + it\right) \tilde{V}\left(\frac{1}{2} + it\right) (2^{-j}X)^{1/2+it} dt \\ =& \frac{L(\mathrm{Sym}^2 f, 1)\tilde{V}(1)(2X)}{\zeta(2)} \\ &+ O\left(X^{\varepsilon} \sup_{X_1 \ll X} \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} L\left(f \otimes f, \frac{1}{2} + it\right) \tilde{V}\left(\frac{1}{2} + it\right) (X_1)^{1/2+it} dt\right).\end{aligned}$$

The same expression holds if V is replaced by W. As $\tilde{V}(1) = 1 + O(X^{-A})$ and $\tilde{W}(1) = 1 + O(X^{-A})$, we obtain the same main term $\frac{2L(\mathrm{Sym}^2 f,1)\cdot X}{\zeta(2)}$ and we derive

$$\Delta(2X) \ll X^{\varepsilon} \sup_{X_1 \ll X} X_1^{1/2} \int_{-\infty}^{\infty} L\left(f \otimes f, \frac{1}{2} + it\right) \tilde{V}\left(\frac{1}{2} + it\right) X_1^{it} dt + O(X^{1-A}). \tag{1.13}$$

We may truncate the range of t. If we integrate $\tilde{V}(1/2+it)$ by parts repeatedly and use the fact $V^j \ll X^{jA}$, then we observe $\tilde{V}(1/2+it)$ is arbitrarily small unless $t \ll X^{A+\varepsilon}$. In that range, we take a smooth dyadic partition of the t-integral. So we consider $t \sim T$ for $T \ll X^{A+\varepsilon}$. For $t \sim T$, by integrating $\int_0^\infty V(x) x^{-1/2+it} dx$ by parts we note that

$$\tilde{V}(1/2+it) = -\int_0^\infty V'(x) \frac{x^{1/2+it}}{1/2+it} \ll T^{-1}.$$

Putting it in (1.13) and by Cauchy's inequality we derive

$$\begin{aligned}\Delta(2X) &= O\left(X^{\varepsilon} \sup_{X_1 \ll X} X_1^{1/2} \sup_{T \ll X^{A+\varepsilon}} \int_{t \sim T} L\left(f \otimes f, \frac{1}{2} + it\right) \tilde{V}\left(\frac{1}{2} + it\right) X_1^{it} dt\right) + O(X^{1-A}) \\ &= O\Bigg[X^{\varepsilon} \sup_{X_1 \ll X} X_1^{1/2} \sup_{T \ll X^{A+\varepsilon}} T^{-1} \left(\int_{t\sim T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt\right)^{1/2} \\ &\qquad \times \left(\int_{t \sim T} \left|L\left(\mathrm{Sym}^2 f, \frac{1}{2} + it\right)\right|^2 dt\right)^{1/2}\Bigg] + O(X^{1-A}).\end{aligned}$$

Assuming $\int_T^{2T} \left|L\left(F, \frac{1}{2} + it\right)\right|^2 dt \ll_{F,\varepsilon} T^{\frac{3}{2}-\delta+\varepsilon}$ and using $\int_{t\sim T} |\zeta(1/2+it)|^2 dt \ll T^{1+\varepsilon}$, we derive

$$\begin{aligned}\Delta(2X) =& O\left(X^{\varepsilon} \sup_{X_1 \ll X} X_1^{1/2} \sup_{T \ll X^{A+\varepsilon}} T^{-1+3/4-\delta/2+1/2}\right) + O(X^{1-A}) \\ =& O(X^{1/2+A(1/4-\delta/2)+\varepsilon}) + O(X^{1-A}).\end{aligned}$$

We equate these two upper bounds to choose $A = 1/(5/2-\delta)$. Thus we derive

$$\Delta(2X) \ll X^{1-\frac{1}{5/2-\delta}+\varepsilon} \sim X^{\frac{3-2\delta}{5-2\delta}}.$$

Then by Theorem 1 we replace $\delta = \frac{3}{32}$ to conclude the proof. □

1.3.4 Proof of Corollary 1.4

Proof. Once we replace $\alpha = 3/2$ by $\alpha = \frac{3}{2} - \frac{3}{32}$ (1) in (1.9) and (1.10), the proof of the corollary is immediate. □

1.4 Notations

We follow the popular notation of $e(x) := e^{2\pi i x}$ and ε would denote an arbitrarily small positive real number. $f = O(A)$ and $f \ll A$ would denote $|f| \leq CAT^{\varepsilon}$ for some constant $C > 0$ depending only on the form F and ε. For some $n > 0$, $x \sim n$ would imply $c_1 n < x < c_2 n$ where $0 < c_1 < c_2$ and $x \in \mathbb{R}$. In general, we will reserve this notation for the range of sum and integrals. $f \asymp g$ would imply $T^{-\varepsilon} < |f/g| < T^{\varepsilon}$ when g is not zero. We will reserve the letter V, W for compactly supported smooth function. They will additionally satisfy $x^j V^{(j)}(x) \ll 1$ for $j \in \mathbb{N}$ unless specified otherwise. The definition of these smooth functions V, W will vary from place to place, like when any non-oscillatory smooth function satisfying the above derivative condition would be absorbed by these smooth functions. For brevity, often we will omit the underlying smooth function of the form $V\left(\frac{x}{n}\right)$ in a sum or integral by denoting $\sum_{x\sim n}$ or $\int\limits_{x\sim n}$. Any contribution is called negligible if the term is $O_A(T^{-A})$ for any $A > 0$. We will also use the following standard notations:

- $e(z) = e^{2\pi i z}$,
- $e_q(n) = e(n/q) = e^{2\pi i n/q}$,
- $S(m, n; c) = \sum^{*}_{a \bmod c} e_c(\bar{a}m + an)$ is the Kloosterman sum,
- $\mathfrak{c}_q(n) = \sum^{*}_{a \bmod q} e_q(an)$ is the Ramanujan sum.

Chapter 2

Preliminaries

In this chapter, we will briefly revisit the theory of the Hecke-Maass cusp forms for $SL(3,\mathbb{Z})$ from [Gf]. For a detailed introduction of the theory and proof of the results stated below, one may refer to [Gf].

2.1 Hecke-Maass cusp forms for $SL(3,\mathbb{Z})$

2.1.1 Definition

Before going into the theory of Maaß cusp forms, we will recall a few important definitions of the matrix group.

- $GL(3,\mathbb{R})$ is the general linear group of degree 3, i.e. the multiplicative group of all invertible 3×3 matrices with coefficients in $\mathbb{R}$
- $SL(3,\mathbb{Z})$ is the discrete subgroup of $GL(3,\mathbb{R})$ consisting of all 3×3 matrices with coefficients in $\mathbb{Z}$ and determinant 1
- $O(3,\mathbb{R})$ is the orthogonal group of degree 3, i.e.

$$O(3,\mathbb{R}) = \{g \in GL(3,\mathbb{R}) : g.g^t = I\}$$

where g^t is the transpose of g and I is the identity matrix. Here $\cdot$ denotes matrix multiplication.

- Z_3: is the center of $(GL(3,\mathbb{R}))$.

The *generalized upper half plane* $\mathfrak{h}^3$ is defined to be the set of all $z \in GL(3,\mathbb{R})$ such that z can be written as $z = x \cdot y$ where

$$x = \begin{pmatrix} 1 & x_2 & x_{1,3} \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

and $x_1, x_2, x_{1,3} \in \mathbb{R}$ and $y_1, y_2 > 0$. The Iwasawa decomposition states that

$$GL(3,\mathbb{R}) = \mathfrak{h}^3 \cdot O(3,\mathbb{R}) \cdot Z_3.$$

As Z_3 is isomorphic to $\mathbb{R}^\times$, we can express $\mathfrak{h}^3$ as

$$\mathfrak{h}^3 \cong GL(3,\mathbb{R})/(O(3,\mathbb{R}) \cdot \mathbb{R}^\times).$$

Let $\mathfrak{gl}(3,\mathbb{R})$ be the Lie algebra of $GL(3,\mathbb{R})$ and $\mathfrak{D}^3$ be the center of the universal enveloping algebra of $\mathfrak{gl}(3,\mathbb{R})$. For $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ and $z = x \cdot y \in \mathfrak{h}^3$, we define

$$I_\nu : \mathfrak{h}^3 \to \mathbb{C} : \quad I_\nu(z) = y_1^{\nu_1+2\nu_2} y_2^{2\nu_1+\nu_2}. \tag{2.2}$$

Then, $I_\nu(z)$ is an eigenfunction of every $D \in \mathfrak{D}^3$. If we denote the eigenvalues by λ_D, we have

$$DI_\nu(z) = \lambda_D \cdot I_\nu(z). \tag{2.3}$$

Now, we can define the Maass cusp form for $SL(3,\mathbb{Z})$.

Definition 2.1. A smooth function $f \in L^2(SL(3,\mathbb{Z}) \setminus \mathfrak{h}^3)$ is called a Maass cusp form for $SL(3,\mathbb{Z})$ of type $(\nu_1, \nu_2) \in \mathbb{C}^2$ if it satisfies

1. $f(\gamma z) = f(z)$, for all $\gamma \in SL(3,\mathbb{Z})$ and $z \in \mathfrak{h}^3$.

2. $Df(z) = \lambda_D f(z)$, for all $D \in \mathfrak{D}^3$ and λ_D is the eigenvalue corresponding to $I_\nu(z)$.

3. $\int\limits_{(SL(3,\mathbb{Z})\cap U)\backslash U} f(uz)du = 0$, for all subgroups U of upper triangular real valued 3×3 matrices of the form:
$$U = \left\{ \begin{pmatrix} I_{m_1} & * & * \\ & \ddots & * \\ & & I_{m_k} \end{pmatrix} \right\}$$

where $(m_1, \ldots, m_k)$ is a partition of 3, i.e. $m_1 + \cdots m_k = 3$, where $0 \le m_j \le 3$, $j = 1, 2, \ldots, k$.

2.1.2 Fourier expansion

Let $m = (m_1, m_2) \in \mathbb{Z}^2$ and let $u \in U_3(\mathbb{R})$, i.e.

$$u = \begin{pmatrix} 1 & u_2 & u_{1,3} \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad u_1, u_2, u_{1,3} \in \mathbb{R}.$$

Then, the character $\psi_m : U_3(\mathbb{R}) \to \mathbb{C}^\times$ is defined by

$$\psi_m(u) = e(m_1 u_1 + m_2 u_2).$$

Let $w_3 = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$ and $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$. Then,

$$W(z, \nu; \psi_m) = \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} I_\nu(w_3 \cdot u \cdot z)\overline{\psi_m(u)} du_1 du_2 du_{1,3} \tag{2.4}$$

is called the Jacquet Whittaker function for $SL(3,\mathbb{Z})$. Here, $z \in \mathfrak{h}^3$ and $I_\nu(z)$ is as defined in (2.2).

Let F be a Maaß cusp form of type $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ for $SL(3, \mathbb{Z})$ as defined in (2.1). Then $F(z)$ has an Fourier expansion

$$F(z) = \sum_{\gamma \in U_2(\mathbb{Z}) \backslash SL(2,\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1, m_2)}{|m_1 m_2|} \times W\left(\begin{pmatrix} |m_1 m_2| & & \\ & m_1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \psi_{(1, \frac{m_2}{|m_2|})} \right). \tag{2.5}$$

where $W(z, \nu, \psi)$ are the Jacquet Whittaker function defined above. $A(m_1, m_2)$ is called the (m_1, m_2)th Fourier coefficient of F. For $m_1 \geq 1$ and $m_2 \neq 0$,

$$A(m_1, m_2) = A(m_1, -m_2) \quad \text{and} \quad \frac{A(m_1, m_2)}{|m_1 m_2|} = O(1).$$

These Fourier coefficients also satisfy Ramanujan type bound on average

$$\sum_{m_1^2 m_2 \leq N} |A(m_1, m_2)|^2 \ll N^{1+\varepsilon}. \tag{2.6}$$

Let $\tilde{F}(z) := F(w_3 \cdot (z^{-1})^t \cdot w_3)$. Then, $\tilde{F}(z)$ is also a Maaß cusp form of type (ν_2, ν_1) for $SL(3, \mathbb{Z})$. $\tilde{F}(z)$ is called the dual form of F. The Fourier coefficients of $\tilde{F}$ are related to F in the following sense

$$A_{\tilde{F}}(m_1, m_2) = A_F(m_2, m_1) = \overline{A_F(m_1, m_2)}.$$

2.1.3 Hecke relation

Let $n \geq 1$ be an integer. Then T_n, defined by

$$T_n f(z) = \frac{1}{n} \sum_{\substack{abc=n \\ 0 \leq c_1, c_2 \leq c \\ 0 \leq b_1 < b}} f\left(\begin{pmatrix} a & b_1 & c_1 \\ 0 & b & c_2 \\ 0 & 0 & c \end{pmatrix} \cdot z \right)$$

is an operator on the space of Maaß cusp forms for $SL(3,\mathbb{Z})$. Then T_n for $n \geq 1$ are called the Hecke operators. Let F be a Maaß cusp form for $SL(3,\mathbb{Z})$, which is also a simultaneous eigenfunction of all the Hecke operators. Then, F is called a Hecke-Maaß cusp form for $SL(3,\mathbb{Z})$.

Let F be a Hecke-Maaß cusp form for $SL(3,\mathbb{Z})$ and $A(m_1,m_2)$ be its (m_1,m_2)th Fourier coefficients. If $A(1,1)=0$, it would force F to vanish identically (see Theorem 6.4.11, [Gf]). If F is not identically zero, we may divide it by $A(1,1)$ and assume that $A(1,1)=1$. If $A_F(1,1)=1$, F is called a normalized Hecke-Maaß cusp form.

If F is a normalized Hecke-Maaß cusp form for $SL(3,\mathbb{Z})$, $A(n,1)$ is actually the eigenvalue of T_n, i.e.,

$$T_n f = A(n,1)\cdot f, \qquad \text{for } n \geq 1.$$

Then, the Fourier coefficients $A(m_1,m_2)$s also satisfy the following multiplicative relations (see Theorem 6.4.11)

$$A(1,n)A(m_1,m_2) = \sum_{\substack{d_0 d_1 d_2 = n \\ d_1 | m_1 \\ d_2 m_2}} A\left(\frac{m_1 d_2}{d_1}, \frac{m_2 d_0}{d_2}\right), \tag{2.7}$$

$$A(m_1,1)A(1,m_2) = \sum_{d|(m_1,m_2)} A\left(\frac{m_1}{d}, \frac{m_2}{d}\right). \tag{2.8}$$

2.1.4 L-function associated to F

Let F be a normalized Hecke-Maaß cusp form for $SL(3,\mathbb{Z})$ with Fourier coefficient $A(m_1,m_2)$. Then, we can associate an L-function to F:

Definition 2.2. Let F be a normalized Hecke-Maaß cusp form for $SL(3,\mathbb{Z})$ with Fourier coefficient $A(m_1,m_2)$. Then, the Godement-Jacquet L-function $L(F,s)$ is defined by the Dirichlet series

$$L(F,s) = \sum_{n=1}^{\infty} A(1,n)n^{-s}, \quad \text{for } \Re(s) > 1. \tag{2.9}$$

For $\Re(s) > 1$, $L(F,s)$ has an Euler product representation

$$L(F,s) = \prod_p (1 - A(p,1)p^{-s} + A(1,p)p^{-2s} - p^{-3s})^{-1}.$$

Remark. The trivial bound of the Fourier coefficients ($A(1,n) \ll |n|$) implies that the abscissa of convergence of the Dirichlet series (2.9) is actually $\Re(s) = 2$. But, the Ramanujan-type bound on average : $\sum_{m_1^2 m_2 \leq N} |A(m_1,m_2)|^2 \ll N^{1+\varepsilon}$, implies that the Dirichlet series (2.9) is absolutely convergent for $\Re(s) > 1$.

The Langlands parameters of F, denoted by $(\alpha_1, \alpha_2, \alpha_3)$, are defined as

$$\alpha_1 = -\nu_1 - 2\nu_2 + 1, \ \alpha_2 = -\nu_1 + \nu_2, \ \alpha_3 = 2\nu_1 + \nu_2 - 1. \tag{2.10}$$

and the dual form $\tilde{F}$ has Langlands parameters $(-\alpha_3, -\alpha_2, -\alpha_1)$. Then, the L-function $L(F,s)$ satisfies the functional equation

$$G(F,s)L(F,s) = G(\tilde{F}, 1-s)L(\tilde{F}, 1-s),$$

where

$$G(F,s) = \pi^{-3s/2}\Gamma\left(\frac{s-\alpha_1}{2}\right)\Gamma\left(\frac{s-\alpha_2}{2}\right)\Gamma\left(\frac{s-\alpha_3}{2}\right).$$

Hence, $L(F,s)$ can be holomorphically continued over the whole complex plane.

2.1.5 Approximate Functional Equation

$L(F,s)$, the L-function associated to a normalized Hecke-Maaß cusp form F for $SL(3,\mathbb{Z})$, also satisfy an approximate functional equation in the critical strip. One may refer to Theorem 5.3 and Proposition 5.4 of [IK] for the proofs of the following lemmas.

Lemma 2.1 (Approximate Functional Equation)**.** *For $0 < \Re(s) < 1$ and any bounded even function $g(u)$, holomorphic in the strip $-4 < \Re(u) < 4$ and normalized as $g(0) = 1$ and $X > 0$,*

$$L(F,s) = \sum_{n=1}^{\infty} \frac{A(1,n)}{n^s} V_s\left(\frac{n}{X}\right) + \frac{G(\tilde{F},s)}{G(F,s)} \sum_{n=1}^{\infty} \frac{A(n,1)}{n^{1-s}} \tilde{V}_{1-s}(nX),$$

where

$$V_s(y) = \frac{1}{2\pi i}\int_{(3)} y^{-u}\frac{G(F, s+u)}{G(F+s)}g(u)\frac{du}{u}$$

and

$$\tilde{V}_s(y) = \frac{1}{2\pi i}\int_{(3)} y^{-u}\frac{G(\tilde{F}, s+u)}{G(\tilde{F}, s)}g(u)\frac{du}{u}.$$

Remark. We will choose $X = 1$.

Lemma 2.2 (Upper Bound of $V_s(y)$)**.** *For α satisfying $\Re(s+\alpha_i) > \alpha > 0$ for $i = 1, 2, 3$ and for any $A > 0$, we have the following bounds on $V_s(y)$ (exact bounds hold for $\tilde{V}_s(y)$ too) and it's derivatives w.r.t y:*

$$y^j V_s^{(j)} = \delta_j + O\left(\frac{y}{\sqrt{\mathfrak{q}_\infty}}\right)^{\alpha},$$

$$y^j V_s^{(j)} \ll \left(1 + \frac{y}{\sqrt{\mathfrak{q}_\infty}}\right)^{-A}$$

and if $\Re(s) = \frac{1}{2}$, we also have

$$\frac{G(\tilde{F}, s)}{G(F, s)} \ll 1.$$

Here, $\delta_j = 1$ if $j = 0$ and 0 otherwise.

2.1.6 Voronoi summation formula for $SL(3,\mathbb{Z})$:

Following the notations set at the beginning, let F be an $SL(3,\mathbb{Z})$ Maass form with $(m,n)^{th}$ Fourier coefficient $A(m,n)$ and let $\tilde{F}$ be its dual form with Fourier coefficients $A(n,m)$. Then, we have a summation formula for $A(1,m)$ twisted by additive characters. We precisely follow the expression of Corollary 3.7 of [GL06], which we summarize in the lemma below.

Lemma 2.3 (Voronoi type summation formula). *Let $\psi(x) \in C_c^\infty(0,\infty)$ and let $a, \bar{a}, q \in \mathbb{Z}$ with $(a,q) = 1$. If we denote*

- $\tilde{\psi}(s) = \int_0^\infty \psi(x) x^s \frac{dx}{x}$
- $$\Psi_k(x) = \int_{\Re(s)=\sigma} (\pi^3 x)^{-s} \frac{\Gamma\left(\frac{1+s+2k+\alpha_1}{2}\right)\Gamma\left(\frac{1+s+2k+\alpha_2}{2}\right)\Gamma\left(\frac{1+s+2k+\alpha_3}{2}\right)}{\Gamma\left(\frac{-s-\alpha_1}{2}\right)\Gamma\left(\frac{-s-\alpha_2}{2}\right)\Gamma\left(\frac{-s-\alpha_3}{2}\right)} \times \tilde{\psi}(-s-k) ds, \qquad \text{for } k = 0,1$$
- $\Psi_{0,1}^{\pm}(x) = \Psi_0(x) \pm \frac{\pi^{-3} q^3}{m_1^2 m_2 i} \Psi_1(x),$

then we have

$$\sum_{m=1}^\infty A(1,m) e\left(\frac{m\bar{a}}{q}\right)\psi(m) = \frac{q\pi^{-5/2}}{4i} \sum_{\pm} \sum_{m_1|q} \sum_{m_2>0} \frac{A(m_2,m_1)}{m_1 m_2} \times S(a, \pm m_2; q m_1^{-1}) \Psi_{0,1}^{\pm}\left(\frac{m_2 m_1^2}{q^3}\right).$$

Here $S(a,b,q)$ denotes the Kloosterman sum

$$S(a,b,q) = \sum_{x \bmod q}^{*} e\left(\frac{ax + b\bar{x}}{q}\right).$$

In the above lemma $\Psi_{0,1}^{\pm}\left(\frac{m_2 m_1^2}{q^3}\right) = \Psi_0\left(\frac{m_2 m_1^2}{q^3}\right) \pm \frac{\pi^{-3}q^3}{m_1^2 m_2 i} \Psi_1\left(\frac{m_2 m_1^2}{q^3}\right)$ consists of four terms. But we will only estimate $\Psi_0(x)$ with the help of the following lemma by [Li09] (Lemma 6.1) and consider only the term $\Psi_0(x)$. The estimate of $\Psi_1(x)$ is quite similar; thus are the remaining three terms.

Lemma 2.4. *Suppose $\psi(x)$ is a smooth function compactly supported on $[X, 2X]$ and $\Psi_0(x)$ is defined as above, then for any fixed integer $K \geq 1$ and $xX \gg 1$, we have*

$$\Psi_0(x) = 2\pi^4 xi \int_0^\infty \psi(y) \sum_{j=1}^K \frac{c_j \cos(6\pi x^{1/3} y^{1/3}) + d_j \sin(6\pi x^{1/3} y^{1/3})}{(\pi^3 xy)^{j/3}} dy + O\left((xX)^{\frac{-K+2}{3}}\right),$$

where c_j and d_j are constants depending on α_i, in particular, $c_1 = 0$, $d_1 = \frac{-2}{\sqrt{3\pi}}$.

In the above integral, we may write the integrand as

$$\psi(y)\sum_{j=1}^{K}\left(\frac{c_j - id_j}{2}\right)\frac{e(3x^{1/3}y^{1/3})}{(\pi^3 xy)^{j/3}} + \psi(y)\sum_{j=1}^{K}\left(\frac{c_j + id_j}{2}\right)\frac{e(-3x^{1/3}y^{1/3})}{(\pi^3 xy)^{j/3}}.$$

We will only consider the first sum as the analysis of the second sum is similar. So we will treat

$$2\pi^4 xi\int_0^\infty \psi(y)\sum_{j=1}^{K}\frac{e(3x^{1/3}y^{1/3})}{(\pi^3 xy)^{j/3}}\ dy. \tag{2.11}$$

We also note that the oscillatory part $e(3x^{1/3}y^{1/3})$ is independent of the sum over j and the non-oscillatory terms $(\pi^3xy)^{-j/3}$ decrease with increasing j provided $xy \gg 1$. So $\sum_{j=1}^{K}(\pi^3xy)^{-j/3}$ is asymptotic to the term $(\pi^3xy)^{-1/3}$, i.e. $j = 1$. Hence we take K sufficiently large so that the error term $O\Big((xX)^{\frac{-K+2}{3}}\Big)$ can be dropped from further consideration. For such K, we will consider only the term $j = 1$.

For $xy \ll 1$, as the term $e(3x^{1/3}y^{1/3})$ is non-oscillatory, it can be absorbed into the smooth function of $\psi(y)$. Thus, we will only consider

$$\Psi_0(x) \asymp \sum_{\pm} 2\pi^4 c_{\pm} xi\int_0^\infty \psi(y)\frac{e(\pm 3x^{1/3}y^{1/3})}{(\pi^3 xy)^{1/3}}\ dy + O(T^{-A}). \tag{2.12}$$

2.2 Poisson summation formula

Let $f(x)$ be a compactly supported smooth function and $C(n)$ be a periodic function modulo q. Then, by the Poisson summation formula, we get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} C(n)f(n) &= \sum_{b \bmod q} C(b)\sum_{n_1\in\mathbb{Z}} f(b+n_1q)\\ &= \sum_{b \bmod q} C(b)\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} f(b+yq)e(-ny)dy\\ &=\frac{1}{q}\sum_{n\in\mathbb{Z}}\sum_{b \bmod q} C(b)e_q(nb)\int_{\mathbb{R}} f(y)e\left(-\frac{ny}{q}\right)dy.\end{aligned} \tag{2.13}$$

In particular, if $C(n) = e_q(an)$, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} e_q(an)f(n) =& \frac{1}{q}\sum_{n\in\mathbb{Z}}\sum_{b \bmod q} e_q((a+n)b)\int_{\mathbb{R}} f(y)e\left(-\frac{ny}{q}\right)dy \\ =& \sum_{\substack{n\in\mathbb{Z}\\ n\equiv -a \bmod q}} \int_{\mathbb{R}} f(y)e\left(-\frac{ny}{q}\right)dy. \end{aligned} \tag{2.14}$$

2.3 Stationary Phase Analysis

To treat the oscillatory integrals of one variable, we will use the following lemmas.

When the phase function does not have a stationary point, we will use the following lemma from [Mu15].

Lemma 2.5. *Let $g(x)$ be a compactly supported smooth function supported in $[a,b]$ satisfying $g^j(x) \ll_{a,b,j} 1$. Let $f(x)$ be a real valued smooth function satisfying $|f'(x)| \geq \Theta_f$ and $|f^{(j)}(x)| \ll \Theta_f$ for $j \geq 2$. Then, for any $j \in \mathbb{N}$, we have*

$$\int_a^b e(f(x))g(x)dx \ll_{a,b,j,\varepsilon} \Theta_f^{-j+\varepsilon}. \tag{2.15}$$

Remark. Frequently in this paper, we will mention that an oscillatory integral is negligibly small by repeated integration by parts if the first derivative of its phase function is greater than N^ε for any ε throughout the support of the smooth function $u(x)$. That is actually a direct consequence of this lemma.

When the phase function has a unique stationary point, we will use the following lemma from [BKY13] by Blomer, Khan, and Young. So, we restate Proposition 8.2 of [BKY13] below.

Lemma 2.6. *Let $0 < \delta < 1/10, \Theta_g, \Theta_f, \Omega_g, L, \Omega_f > 0$ and let $Z := \Omega_f + \Theta_f + \Theta_g + L + 1$ and we also assume that*

$$\Theta_f \geq Z^{3\delta}, \quad L \geq \Omega_g \geq \frac{\Omega_f Z^{\delta/2}}{\Theta_f^{1/2}}. \tag{2.16}$$

Let $g(x)$ be a compactly supported smooth function with support in a length L and satisfying the derivative $g^{(j)}(x) \ll \Theta_g \Omega_g^{-j}$ and let x_0 be the unique point such that $f'(x_0) = 0$, where $f(x)$ is a smooth function satisfying

$$f''(x) \gg \Theta_f \Omega_f^{-2}, \quad f^{(j)}(x) \ll \Theta_f \Omega_f^{-j}, \quad \forall j \in \mathbb{N}. \tag{2.17}$$

Then the oscillatory integral $I = \int_{-\infty}^{\infty} g(x)e(f(x))dx$ would have the asymptotic expression (for arbitrary $A > 0$)

$$I = \frac{e(f(x_0))}{\sqrt{g''(x_0)}} \sum_{n \leq 3\delta^{-1}A} p_n(x_0) + O_{A,\delta}(Z^{-A}), \tag{2.18}$$

where

$$\begin{gathered} p_n(x_0) = \frac{e^{\pi i/4}}{n!} \left(\frac{i}{2f''(x_0)} \right)^n G^{(2n)}(x_0) \\ \text{where } G(x) = g(x)e(f(x) - f(x_0) - f''(x_0)(x - x_0)^2/2). \end{gathered} \tag{2.19}$$

Each p_n is a rational function in derivatives of f satisfying

$$\frac{d^j}{dx_0^j} p_n(x_0) \ll \Theta_g (\Omega_g^{-j} + \Omega_f^{-j})((\Omega_g^2 \Theta_f / \Omega_f^2)^{-n} + \Theta_f^{-n/3}). \tag{2.20}$$

Remark. As observed in [BKY13], from (2.16) and (2.20), in the asymptotic expression (2.18), every term is smaller than the preceding term. So it is enough to consider the leading term in the asymptotic provided we verify 2.16

But to deal with the multi-variable oscillatory integrals, we will use Theorem 7.7.1 and Lemma 7.7.5 of [Ho]. Here, we will only mention the $\mathbb{R}^2$ version of the lemma stated in [HMQ23], which suffices our requirement for a two-variable oscillatory integral.

Lemma 2.7 (Theorem 7.7.1)**.** *Let $K \subset \mathbb{R}^2$ be a compact set and let X be an open subset of $\mathbb{R}^2$ containing K and let k be a non-negative integer. Let $u \in C_c^k(K)$ and $f \in C^{k+1}(X)$ where f is a real valued bounded function. Then, for $\lambda > 0$,*

$$\left| \int_K e(\lambda f(x)) u(x) dx \right| \leq C\lambda^{-k} \sum_{j_1 + j_2 \leq k} \sup |\partial_1^{j_1} \partial_2^{j_2} u| |f'|^{j_1 + j_2 - 2k}.$$

where C is bounded as long as f stays bounded in $C^{k+1}(X)$.

Lemma 2.8 (Lemma 7.7.5)**.** *Let $K \subset \mathbb{R}^2$ be a compact set and let X be an open subset of $\mathbb{R}^2$ containing K. Let $u \in C_c^2(K)$ and $f \in C^4(X)$ where f is a real valued function. Then if there is a point $x_0 \in K$ such that $f'(x_0) = 0$ and $\det(H_f(x_0)) \neq 0$ and $f'(x) \neq 0 \quad ; \forall x \in K \setminus x_0$ then for $\lambda > 0$ we have*

$$\left| \int_K e(\lambda f(x)) u(x) dx - \frac{u(x_0) e(\lambda f(x_0))}{\lambda \sqrt{-\det(H_f(x_0))}} \right| \leq \frac{C}{\lambda^2} \left(1 + |\det f''(x_0)|^{-3}\right) \sum_{j_1 + j_2 < 4} \sup |\partial_1^{j_1} \partial_2^{j_2} u|. \tag{2.21}$$

Chapter 3

Proof of Theorem 1: Delta method

3.1 Sketch of the proof

To prove Theorem 1, we start with the integral of $M_F(T)$ (1.4) though we take the range to be $[2T, 3T]$. By approximate functional equation, we can truncate the length of the Dirichlet series of $L(F, 1/2+it)$ to $n \sim N$ for $N \ll T^{3/2+\varepsilon}$. For the sketch, we will only consider $N = T^{3/2}$. So, we start with (see 3.3)

$$M(T) = \int_{2T}^{3T} \left| \sum_{n \sim N} A(n) n^{-it} \right|^2 dt.$$

In Section 3.2, we split the t integral into smaller integrals of length ξ and then take an average over ξ (see 3.4), in the form of an integral on the range $[X, 2X]$, where X would be optimally chosen at the end.

$$\int_{2T}^{3T} |\cdots|^2 dt \rightsquigarrow \sum_{\frac{T}{\xi} \leq r < \frac{2T}{\xi}} \int_{T+r\xi}^{T+(r+1)\xi} |\cdots|^2 dt \ll \sum_{\frac{T}{2X} \leq r < \frac{2T}{X}} \frac{1}{X} \int_{\xi \sim X} \int_{T+r\xi}^{T+(r+1)\xi} |\cdots|^2 dt$$

We now open up the absolute square to get

$$\sum_h \sum_n A(1,n)\overline{A(1,n+h)} \int_t \cdots .$$

Then we evaluate the t-integral (see Lemma 3.2), which restricts the range of h to be $h \ll H = NT^{\varepsilon}/X$, as the t-integral would be negligible otherwise. We essentially have

$$M_F(T) \ll N^{-1} \sum_{r \sim T/X} \mathfrak{M},$$

where

$$\mathfrak{M} = \int_{\xi \sim X} \sum_{h \ll H} \sum_{n \sim N} A(1,n)\overline{A(1,n+h)} e\left(\frac{(T+r\xi)h}{2\pi n}\right).$$

Comparing it with (1.5), we note that the fragmentation of the integral has increased the size of h, introduced an extra oscillatory factor, and along with it, there is an extra integral over ξ. Ultimately, these two elements would give us the extra saving required for the non-trivial bound of Theorem 1.

Now, we will separate the oscillations of $\mathfrak{M}$ with the circle method. In Section 3.3, we would use the delta method of Duke, Friedlander, and Iwaniec (Lemma 3.4) with modulus $Q = \sqrt{N/(T/X)} = \sqrt{X}T^{1/4}$.

$$\begin{aligned}\mathfrak{M} = \int_{x \sim 1} \frac{1}{Q} \sum_{q \sim Q} \frac{1}{q} \sideset{}{^*}\sum_{a \bmod q} \int_{\xi \sim X} \sum_{h \ll H} e_q(-ah) \\ \times \left[\sum_{n \sim N} A(1,n) e_q(-an) e\left(\frac{(T+r\xi)}{2\pi} \times \frac{h}{n}\right)\right] \\ \times \left[\sum_m A(1,m) e_q(am) e\left(\frac{(m-n-h)x}{qQ}\right) W\left(\frac{m-n-h}{Q^2}\right)\right].\end{aligned} \tag{3.1}$$

Here, $W(x)$ is an even smooth function supported in $[-1,1]$ and $W(0)=1$. If we trivially evaluate $\mathfrak{M}$, we get

$$\mathfrak{M} \ll X \cdot H \cdot N \cdot Q^2 = Q^2N^2 \implies M_F(T) \ll T^3.$$

So in $\mathfrak{M}$, we need to save $T^{3/2}$ and a little more to get a non-trivial bound; that is we need to prove $\mathfrak{M} \ll \frac{Q^2N^2}{T^{3/2+\delta}}$ for some $\delta > 0$.

To treat these m and n-sums above, we employ the Voronoi type summation formula for $SL(3,\mathbb{Z})$ (Lemma 2.3). In Section 3.4.1, we apply the Voronoi summation formula first on the m-sum and then on the n-sum. Though we have $A(m_2, m_1)$ in the RHS of the Voronoi summation formula along with the condition $m_1|q$, for brevity, we would only consider the "generic case" $m_1 = 1$ in this sketch and write m_2 as m. In the m-sum, we encounter an oscillatory integral, and by repeated integration by parts, it restricts the range of $m \ll (T/Q)^{3+\varepsilon}$ (see Lemma 3.5). Then, by Taylor series approximation of the phase function, the m sum transforms into

$$\frac{Q^2}{N} \sum_{m \ll (T/Q)^3} A(m) S(\bar{a}, m; q) e\left(\frac{3m^{1/3}n^{1/3}}{q}\right).$$

Thus in the m-sum we save

$$\frac{\text{initial bound}}{\text{final bound}} = \frac{Q^2}{\frac{Q^3}{qN} \cdot \frac{T^3}{Q^3} \cdot q^{1/2}} \sim \frac{X^{5/4}}{T^{7/8}}.$$

We take the above oscillatory term into the n-sum and apply the Voronoi summation formula. There, we also face an oscillatory integral I_2 (see 3.20) of trivial size T. So, the n sum transforms to

$$\frac{1}{T} \sum_{n \ll (T/Q)^3} A(n) S(-\bar{a}, n; q) I_2.$$

Thus in the n-sum we save

$$\frac{\text{initial bound}}{\text{final bound}} = \frac{N}{\frac{1}{T} \cdot \frac{T^3}{Q^3} \cdot q^{1/2} \cdot T} \sim \frac{X^{5/4}}{T^{7/8}}.$$

Finally, we put the oscillatory integral I_2 in the h-sum and apply the Poisson summation formula on

$$\sum_{h \ll H} e_q(ah) I_2(h).$$

Thus, we get a double integral I_3 (see 3.23). This restrict the dual variable h in the range $h \sim \frac{qT}{N} \sim \frac{\sqrt{X}}{T^{1/4}}$. After careful evaluation of the double integral, the h sum transforms into (see 3.28)

$$T^{3/2}\sum_{\substack{h\sim\frac{\sqrt{X}}{T^{1/4}}\\ h\equiv a \bmod q}} e\left(\frac{3(T+r\xi)^{1/3}(m^{1/3}-n^{1/3})}{(2\pi)^{1/3}q^{2/3}h^{1/3}}\right).$$

In the h-sum we save

$$\frac{\text{initial bound}}{\text{final bound}} = \frac{HT}{T^{3/2}\cdot\frac{1}{q}\cdot\frac{\sqrt{X}}{T^{1/4}}} \sim \frac{T^{3/2}}{X}.$$

Hence we evaluate the total saving on $\mathfrak{M}$ to be $X^{3/2}/T^{1/4}$ and by Cauchy's inequality and the symmetry of the n and the m sum, $\mathfrak{M}$ can be bounded by

$$\mathfrak{M} \ll \frac{T^{\varepsilon}}{T}\int_{\xi\sim X}\sum_{q\sim Q}\sum_{h\sim\sqrt{X}/T^{1/4}} \times \left|\sum_{m\sim(T/Q)^3} A(m)S(\bar{h},m;q)e\left(\frac{3(T+r\xi)^{1/3}m^{1/3}}{(2\pi)^{1/3}q^{2/3}h^{1/3}}\right)\right|^2.$$

In Section 4.1, similar to [ALM22], we use the duality principle of large sieve to interchange the order of sums in the expression of $\mathfrak{M}$. The Ramanujan bound on average (2.6):

$$\sum_{m_1^2 m_2\le M}|A(m_1,m_2)|^2 \ll M^{1+\varepsilon},$$

would imply that $\mathfrak{M}$ is bounded above by

$$\mathfrak{M} \ll \frac{T^{+\varepsilon}}{T}\sum_{m\sim(T/Q)^3}|A(m)|^2\Delta \ll \frac{T^{2+\varepsilon}}{Q^3}\Delta,$$

where

$$\Delta = \sup_{\substack{\alpha\\ ||\alpha||_2=1}}\sum_{m\sim(T/Q)^3}\left|\int_{\xi\sim X}\sum_{q\sim Q}\sum_{h}\alpha(\xi,q,h)S(\bar{h},m;q)\times e\left(\frac{3(T+r\xi)^{1/3}m^{1/3}}{(2\pi)^{1/3}q^{2/3}h^{1/3}}\right)d\xi\right|^2.$$

Trivially we have $\Delta \ll T^{5/2}X$. To bound Δ, we first open up the absolute square and apply the Poisson summation formula to the m-sum. We note that we denote the two copies of the variables (ξ,q,h) as (ξ_1,q_1,h_1) and (ξ_2,q_2,h_2). This produces a character sum $\mathfrak{C}$ and an oscillatory integral $\mathfrak{J}$ (see 4.4).

$$\sum_{m\sim(T/Q)^3} S(\bar{h}_1,m;q_1)S(\bar{h}_2,m;q_2)e(\cdots) = \frac{T^3}{Q^3 q_1 q_2}\sum_{m\in\mathbb{Z}}\mathfrak{C}\,\mathfrak{J},$$

where

$$\mathfrak{C} = \sum_{\beta \bmod q_1 q_2} S(\bar{h}_1, \beta; q_1) S(\bar{h}_2, \beta; q_2) e_{q_1 q_2}(m\beta),$$

and $$\mathfrak{J} = \int_{y \sim 1} e(\cdots\ y^{1/3}) e\left(- \frac{mT^3 y}{Q^3 q_1 q_2} \right) dy.$$

At this point, we will consider two cases of this expression: "Diagonal" ($m = 0$) and "Off-diagonal" ($m \neq 0$). We will analyze them separately.

In Section 4.2, we consider the diagonal case ($m = 0$). In this case, the character sum turns out to be a Ramanujan sum along with the condition $q_1 = q_2$:

$$\mathfrak{C} = q_1^2 \mathfrak{c}_{q_1}(\bar{h}_1 - \bar{h}_2) \delta_{q_1 = q_2}.$$

We analyze the oscillatory integral with repeated integration by parts (Lemma 2.5) and observe that it is arbitrarily small unless

$$\xi_1 - \xi_2 \ll \frac{X^2}{T}.$$

By careful analysis of these two conditions, we see that we save

$$Q \frac{\sqrt{X}}{T^{1/4}} \cdot \frac{T}{X} \sim T$$

in the diagonal case (see 4.7). Before the application of the duality principle, we have saved $\frac{X^{3/2}}{T^{1/4}}$. So, this saving is sufficient as long as $\frac{X^{3/2}}{T^{1/4}} T \gg T^{3/2} \iff X \gg T^{1/2}$.

In the off-diagonal case, both the character sum and the oscillatory integrals are more complicated. In Section 4.3, we first analyze the oscillatory integral $\mathfrak{J}$ with stationary phase analysis (see Lemma 4.3). For $m \ll \frac{X^{3/2}}{T^{3/4}}$ we have

$$\mathfrak{J} \asymp \frac{1}{\sqrt{T/X}} e\left(2\sqrt{\frac{q_1 q_2}{m}} (\Xi_2 - \Xi_1)^{3/2} \right), \text{ where }, \quad \Xi_k = \left(\frac{(T + r\xi_k)}{2\pi q_k^2 h_k} \right)^{1/3} \sim \frac{T^{1/4}}{\sqrt{X}},$$

for $k = 1, 2$. We also note that the oscillation is of the size T/X.

Next, we simplify the character sum $\mathfrak{C}$. For simplicity if we assume that $(q_1, q_2) = 1$, by Chinese remainder theorem and reciprocity we can transform $\mathfrak{C}$ into

$$\mathfrak{C} \asymp q_1 q_2 \, e\left(\frac{q_2 \bar{q}_1}{m h_1} + \frac{q_1 \bar{q}_2}{m h_2}\right).$$

In reality there are additional terms depending on the g.c.d. of q_1 and q_2 and it requires careful evaluation. By the Poisson summation formula on the m-sum (in off-diagonal case) we save

$$\frac{\text{initial bound}}{\text{final bound}} = \frac{\frac{T^3}{Q^3} \cdot Q^{1/2} Q^{1/2}}{\frac{T^3}{Q^3} \cdot \frac{X^{3/2}}{T^{3/4}} \cdot \frac{\sqrt{X}}{\sqrt{T}}} \sim \frac{T^{3/2}}{X^{3/2}}.$$

So, if we look at the expression, we essentially have

$$\sum_m \int_{\xi_1} \sum_{h_1} \sum_{q_1} \alpha(\xi_1, q_1, h_1) \int_{\xi_2} \sum_{h_2} \sum_{q_2} \bar{\alpha}(\xi_2, q_2, h_2) e\left(\frac{q_2 \bar{q}_1}{m h_1} + \frac{q_1 \bar{q}_2}{m h_2}\right) e(\cdots).$$

We then apply the Cauchy's inequality on the (m, ξ_1, q_1, h_1)-sum to get rid of $\alpha(\xi_1, q_1, h_1)$ and open up the resulting absolute square and denote the resulting two copies of (ξ_2, q_2, h_2) variables as (ξ_2, q_2, h_2) and (ξ_2', q_2', h_2'). We now apply the Poisson summation formula on the q_1 sum. This produces another character sum $\mathcal{C}$ and oscillatory integral (see 5.4). We take the ξ_1 integral along with the oscillatory integral and denote the two variable oscillatory integral as $\mathcal{I}$.

$$\int_{\xi_1} \sum_{q_1 \sim Q} e\left(\frac{q_2 \bar{q}_1}{m h_1} + \frac{q_1 \bar{q}_2}{m h_2} - \frac{q_2' \bar{q}_1}{m h_1} - \frac{q_1 \bar{q}_2'}{m h_2'}\right) e(\cdots) = \frac{Q}{m h_1 h_2 h_2'} \sum_{q_1 \in \mathbb{Z}} \mathcal{C} \, \mathcal{I}.$$

If we continue with our assumption $(q_1, q_2 q_2') = 1$, then by the Chinese remainder theorem character sum $\mathcal{C}$ appears to be a product of a Kloosterman sum and a congruence relation:

$$\mathcal{C} \asymp S(\cdots, \cdots; m h_1) \cdot h_2 h_2' \cdot \delta(\cdots \bmod h_2 h_2').$$

So we may bound $\mathcal{C}$ by $\sqrt{m h_1}$ on average. But in reality, our analysis of $\mathcal{C}$ depends on the common divisors of q_1, q_2, q_2'. On the other hand, we analyze the double integral $\mathcal{I}$ by multi-variable stationary phase analysis (see Lemma 5.2) and save $\frac{T}{X}$.

So, due to the Poisson summation formula on q_1 we save

$$\frac{\text{initial bound}}{\text{final bound}} = \frac{QX}{\frac{Q}{\frac{X^3}{T^{3/2}}}\frac{X^{3/2}}{T^{3/4}} \cdot \frac{X}{\sqrt{T}}\frac{X^2}{T}} \sim \frac{T^{3/4}}{X^{1/2}}.$$

After the duality principle, the total saving in the off-diagonal case is $\frac{T^{3/2}}{X^{3/2}} \cdot \frac{T^{3/8}}{X^{1/4}} \sim \frac{T^{15/8}}{X^{7/4}}$. With the optimal choice of $X = T^{1/2}$, this saving equals the diagonal saving. Hence, we are at the threshold, i.e., we recover the trivial bound of $M_F(T)$. So, any additional saving in the off-diagonal will lead to a non-trivial bound of $M_F(T)$.

We now have an oscillatory term, which is the exponential term arising from the stationary phase analysis of $\mathcal{I}$ and its oscillation is of the size T/X. We apply Cauchy's inequality, keeping everything but the $\int\limits_{\xi_2'} \alpha_2' e(\cdots)$ inside. So, we get

$$\int\limits_{\xi_2'} \alpha_2' \int\limits_{\xi_2''} \alpha_2'' \int_{\xi_2} e(\cdots).$$

Then, by repeated integration by parts (Lemma 2.5) on the ξ_2 integral, we restrict the range of

$$|\xi_2' - \xi_2''| \ll \frac{X}{T/X}$$

(see 5.39). Thus, we save T/X. As we have applied Cauchy's inequality twice, the effective saving in $M_F(T)$ is $\frac{T^{1/4}}{X^{1/4}}$.

Hence, the total saving in the off-diagonal (after duality) is $\frac{T^{15/8}}{X^{7/4}} \cdot \frac{T^{1/4}}{X^{1/4}} = \frac{T^{17/8}}{X^2}$. We equate this saving with the diagonal saving of T to optimally choose $X = T^{9/16}$. Hence, in $M_F(T)$, we save $X^{3/2}T^{3/4} = T^{3/2+3/32}$ which gives us

$$M_F(T) \ll T^{3/2-3/32+\varepsilon}.$$

3.2 Setting up for delta method

3.2.1 Adjusting the range of n:

We will consider the limit of integration for the second moment to be $[2T, 3T]$ instead of $[T, 2T]$, which would not affect the end result. We start with the application of the approximate functional equation on $M_F(T)$

$$M_F(T) = \int_{2T}^{3T} \left| \sum_{n=1}^{\infty} \frac{A(1,n)}{n^{1/2+it}} V_{1/2+it}(n) + \frac{G(\tilde{F}, s)}{G(F, s)} \sum_{n=1}^{\infty} \frac{A(n,1)}{n^{1/2-it}} \tilde{V}_{1/2-it}(n) \right|^2 dt.$$

As $\frac{G(\tilde{F},s)}{G(F,s)} \ll 1$ for $\Re(s) = 1/2$ and both the sums are essentially the same for $\Re(s) = 1/2$, we can consider the first sum and get

$$M_F(T) \ll \int_{2T}^{3T} \left| \sum_{n=1}^{\infty} \frac{A(1,n)}{n^{1/2+it}} V_{1/2+it}(n) \right|^2 dt.$$

As we are only considering t-aspect here, by Lemma 2.2 we may truncate the n sum to $n \ll t^{3/2+\varepsilon}$ with a negligible error term, getting

$$M_F(T) \ll \int_{2T}^{3T} \left| \sum_{n \ll t^{3/2+\varepsilon}} \frac{A(1,n)}{n^{1/2+it}} \right|^2 dt.$$

Finally, we take a smooth dyadic partition for the sum over n with a smooth function $V_1(x)$ supported in [1,2]. As t is in the range $[2T, 3T]$, we can just evaluate the sum over a range $[N, 2N]$ where $N \ll T^{3/2+\varepsilon}$. Thus, we arrive at the expression

$$M(T) = \int_{2T}^{3T} \left| \sum_{n \in \mathbb{N}} A(1,n) n^{-it} V_1\left(\frac{n}{N}\right) \right|^2 dt, \text{ and } M_F(T) \ll \sup_{N \ll T^{3/2+\varepsilon}} N^{-1+\varepsilon} M(T). \quad (3.2)$$

For small values of N ($N \ll T^{15/11+\varepsilon}$), we will use the trivial bound of $M(T) \ll N^{2+\varepsilon}$. Hence,

$$M_F(T) \ll \sup_{T^{15/11+\varepsilon} \ll N \ll T^{3/2+\varepsilon}} N^{-1+\varepsilon} M(T) + T^{15/11+\varepsilon}. \quad (3.3)$$

3.2.2 Breaking down the integral

We fix a variable $X > 0$ here, whose optimal value will be chosen later. We also choose a non-negative smooth function $V(x)$ supported on $[-1/2, 3/2]$ and 1 in the range $[0, 1]$ to split the integral of (3.3) into small integrals of the size ξ for $X \le \xi \le 2X$. We further take a smooth averaging of ξ in a dyadic range $[X, 2X]$. We will often omit the underlying dyadic smooth functions by the notation $n \sim N$ for brevity. Then, we take an average over ξ, where we vary ξ continuously in the range $[X, 2X]$. We summarize the splitting of the integral in the following lemma.

Lemma 3.1. *Let $V(x)$ be a non-negative smooth function supported on $[-1/2, 3/2]$ and 1 in the range $[0, 1]$, satisfying $V^j(x) \ll 1$ and let $V_2(x)$ be a non-negative smooth function supported in $[1, 2]$ satisfying the normalization condition $\int\limits_{\mathbb{R}} V_2(x) = 1$ and satisfying $V_2^j(x) \ll 1$. Then,*

$$M(T) \ll X^{-1} \sum_{T/X \le r \le 2T/X} \mathcal{M}(r, T),$$

where

$$\mathcal{M}(r, T) := \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \left| \sum_{n} A(1, n) n^{-it} V_1\left(\frac{n}{N}\right) \right|^2 V\left(\frac{t - T - r\xi}{\xi}\right) V_2\left(\frac{\xi}{X}\right) dt d\xi. \tag{3.4}$$

Proof. Our first step would be breaking down the limit of the integration $[2T, 3T]$ of

$$M(T) = \int\limits_{2T}^{3T} \left| \sum_{n\in\mathbb{N}} A(1, n) n^{-it} V_1\left(\frac{n}{N}\right) \right|^2 dt$$

into smaller fragments of the form $[T + r\xi, T + (r + 1)\xi]$, for $\lfloor \frac{T}{\xi} \rfloor \le r \le \lfloor \frac{2T}{\xi} \rfloor$, with the help of the smooth functions $V\left(\frac{t-T-r\xi}{\xi}\right)$ as defined above. We use

$$\mathbb{1}_{2T\le t\le 3T} \le \sum_{\lfloor \frac{T}{\xi} \rfloor \le r \le \lfloor \frac{2T}{\xi} \rfloor} V\left(\frac{t - T - r\xi}{\xi}\right), \quad t \in \mathbb{R}$$

as $V(x)$ is non-negative in it's support and 1 in $[0,1]$. Here, the inequality is derived from the properties of the smooth function and positivity of the integrand. So, we have

$$\mathbb{1}_{2T\leq t\leq 3T} \leq \frac{1}{X} \sum_{\lfloor \frac{T}{2X}\rfloor \leq r\leq \lfloor \frac{2T}{X}\rfloor} \int_{\mathbb{R}} V\left(\frac{t-T-r\xi}{\xi}\right) V_2\left(\frac{\xi}{X}\right) d\xi, \qquad t\in\mathbb{R}. \tag{3.5}$$

Since $V(x)\geq 0$ and 1 in $[0,1]$, for a fixed $t\in[2T,3T]$, we have

$$\sum_{\lfloor \frac{T}{2X}\rfloor \leq r\leq \lfloor \frac{2T}{X}\rfloor} V\left(\frac{t-T-r\xi}{\xi}\right) \geq \sum_{\lfloor \frac{T}{2X}\rfloor \leq r\leq \lfloor \frac{2T}{X}\rfloor} \mathbb{1}_{\frac{t-T}{r+1}\leq \xi\leq \frac{t-T}{r}} \geq \mathbb{1}_{X\leq\xi\leq 2X}.$$

As $V_2(x)$ is supported in $[1,2]$, for a fixed $t\in[2T,3T]$,

$$\sum_{\lfloor \frac{T}{2X}\rfloor \leq r\leq \lfloor \frac{2T}{X}\rfloor} \int_{\mathbb{R}} V\left(\frac{t-T-r\xi}{\xi}\right) V_2\left(\frac{\xi}{X}\right) \geq \int_{\mathbb{R}} \mathbb{1}_{X\leq\xi\leq 2X} V_2\left(\frac{\xi}{X}\right) d\xi = X.$$

For $t\in\mathbb{R}\setminus[2T,3T]$, this expression is always non-negative. Thus, we arrive at the inequality (3.5). Applying this argument in $M(T)$ (3.2), we derive the statement of the lemma. □

Remark. $M_F(T) \ll N^{-1}M(T) \ll (NX)^{-1}\sum_{r\sim T/X}\mathcal{M}(r,T)$.

3.2.3 t- integral

Lemma 3.2. *Let $H := \frac{N}{X}$. For some smooth function $U(x,y)$ of two variables supported in $[-HT^{\varepsilon}, HT^{\varepsilon}]\times[N,2N]$ and satisfying*

$$x^i y^j \frac{\partial^i\partial^j}{\partial x^i \partial y^j} U(x,y) \ll_{i,j,\varepsilon} \left(\frac{x}{H}\right)^i T^{(i+j)\varepsilon}$$

we can modify $\mathcal{M}(r,T)$ as

$$\begin{aligned}\mathcal{M}(r,T) = \int_{\mathbb{R}} \xi \sum_h \sum_n A(1,n)\overline{A(1,n+h)} e\left(\frac{(T+r\xi)}{2\pi}\log\left(1+\frac{h}{n}\right)\right) \\ \times U(h,n) V_1\left(\frac{n}{N}\right) V_2\left(\frac{\xi}{X}\right) d\xi + O(T^{-A}).\end{aligned} \tag{3.6}$$

Proof. We open the absolute square in (3.4)

$$\mathcal{M}(r,T) = \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\sum_n A(1,n)n^{-it}V_1\left(\frac{n}{N}\right)\right|^2 V\left(\frac{t-T-r\xi}{\xi}\right)V_2\left(\frac{\xi}{X}\right)dtd\xi,$$

and then perform a change of variable $u := (t-T-r\xi)/\xi$ on the t-integral to get

$$\begin{aligned}\mathcal{M}(r,T) = \int_{\mathbb{R}} \sum_m\sum_n A(1,n)\overline{A(1,m)}\xi\left(\frac{n}{m}\right)^{-i(T+r\xi)} V_1\left(\frac{n}{N}\right)V_1\left(\frac{m}{N}\right)\\ \times\left[\int_{\mathbb{R}}\left(\frac{n}{m}\right)^{-iu\xi}V(u)du\right]V_2\left(\frac{\xi}{X}\right)d\xi.\end{aligned}$$

Now, we put $m = n+h$, and we can write

$$\begin{aligned}\mathcal{M}(r,T) = \int_{\mathbb{R}} \xi\sum_h\sum_n A(1,n)\overline{A(1,n+h)}\left(1+\frac{h}{n}\right)^{i(T+r\xi)} V_1\left(\frac{n}{N}\right)V_1\left(\frac{n+h}{N}\right)\\ \times\left[\int_{\mathbb{R}}\left(1+\frac{h}{n}\right)^{iu\xi}V(u)du\right]V_2\left(\frac{\xi}{X}\right)d\xi.\end{aligned}$$

If we apply repeated integration by parts (Lemma 2.5) on the u-integral inside the third bracket, as $\xi \sim X$, it would be negligibly small unless

$$|X\log(1+h/n)| \ll T^{\varepsilon} \implies |h| \ll \frac{NT^{\varepsilon}}{X}. \tag{3.7}$$

We denote $H := \frac{N}{X}$. Let $V_3(u)$ be an even smooth function supported in $[-3/2, 3/2]$ with value 1 in $-[1,1]$ satisfying $V_3^j(y) \ll 1$. Then, let

$$U(h,n) := \left[\int_{\mathbb{R}}\left(1+\frac{h}{n}\right)^{iu\xi}V(u)du\right]V_3\left(\frac{h}{HT^{\varepsilon}}\right)V_1\left(\frac{n+h}{N}\right)$$

Then,

$$x^iy^j\frac{\partial^i\partial^j}{\partial x^i\partial y^j}U(x,y) \ll_{i,j,\varepsilon} \left(\frac{x}{H}\right)^i T^{(i+j)\varepsilon}.$$

Additionally, the function $V_3\left(\frac{h}{HT^{\varepsilon}}\right)$ forces the support of x of $U(x,y)$ in the range $[-HT^{\varepsilon}, HT^{\varepsilon}]$. □

Remark. Here $\mathcal{M}(r,T) \ll N^2X$. Consequently $M_F(T) \ll \frac{NT}{X}$. So we need to save T/X in $\mathcal{M}(r,T)$ and a little more.

3.2.4 Further simplification

Lemma 3.3. *If $X \gg T^{1/2+\varepsilon}$, then for some (different) smooth function $U(x,y)$ satisfying the same condition as previous $U(x,y)$, we have*

$$\mathcal{M}(r,T) = \int_{\mathbb{R}} \xi \sum_h \sum_n A(1,n)\overline{A(1,n+h)} \times e\left(\frac{(T+r\xi)}{2\pi} \times \frac{h}{n}\right) U(h,n) V_1\left(\frac{n}{N}\right) V_2\left(\frac{\xi}{X}\right) d\xi. \tag{3.8}$$

Proof. In (3.6)

$$\mathcal{M}(r,T) = \int_{\mathbb{R}} \xi \sum_h \sum_n A(1,n)\overline{A(1,n+h)} e\left(\frac{(T+r\xi)}{2\pi} \log\left(1+\frac{h}{n}\right)\right) \times U(h,n) V_1\left(\frac{n}{N}\right) V_2\left(\frac{\xi}{X}\right) d\xi,$$

the exponential term has argument of the form $\frac{(T+r\xi)}{2\pi}\log(1+h/n)$. Upon expressing $\log(1+h/n)$ in terms of the Taylor series, we note that the error term

$$\frac{(T+r\xi)}{2\pi}[\log(1+h/n) - h/n] \ll \frac{h}{H}$$

provided

$$T\frac{h^2}{n^2} \ll \frac{h}{H} \impliedby T \cdot \frac{NT^\varepsilon}{NX} \ll X \iff X \gg T^{1/2+\varepsilon}.$$

Thus, assuming $X \gg T^{1/2+\varepsilon}$ (though at a later stage, this condition would be proven to be a necessary one), we can drop the higher order terms of the Taylor series by incorporating them into the smooth function $U(h,n)$ and only use the term

$$e\left(\frac{(T+r\xi)}{2\pi} \times \frac{h}{n}\right).$$

□

From (3.8), we just extract out X corresponding to the size of ξ and we focus on the new expression

$$\mathfrak{M} := \int_{\mathbb{R}} \sum_h \sum_n A(1,n)\overline{A(1,n+h)}e\left(\frac{(T+r\xi)}{2\pi} \times \frac{h}{n}\right)U(h,n)V_1\left(\frac{n}{N}\right)V_2\left(\frac{\xi}{X}\right)d\xi. \quad (3.9)$$

Here, we will apply the δ-method of Duke, Friedlander, and Iwaniec in the next section.

Remark. We note that $\mathcal{M}(r,T) \ll X \cdot \sup_{H \ll \frac{NT^{\varepsilon}}{X}} \mathfrak{M}$. And

$$M_F(T) \ll N^{-1}M(T) \ll (NX)^{-1} \sum_{r \sim T/X} \mathcal{M}(r,T) \ll N^{-1} \sum_{r \sim T/X} \sup_{H \ll \frac{NT^{\varepsilon}}{X}} \mathfrak{M}.$$

3.3 Application of the delta method

To separate the oscillations in the expression of $\mathfrak{M}$ in (3.9), we will employ the circle method, specifically the δ-method of Duke, Friedlander, and Iwaniec (Chapter 20 of [IK]) with the conductor lowering trick of Munshi. Here, the δ symbol represents the function

$$\delta(n) : \mathbb{Z} \to \{0,1\}, \quad \text{such that} \quad \delta(n) = \begin{cases} 0 \text{ if } n \neq 0 \\ 1 \text{ if } n = 0 \end{cases}.$$

We will use the expression of $\delta(n)$ mentioned in [Mu22]. We summarize the expression and a few of its properties, as stated in [Mu22], in the following lemma.

Lemma 3.4.

$$\delta(n) = \frac{1}{Q} \sum_{1 \le q \le Q} \frac{1}{q} \sideset{}{^*}\sum_{a \bmod q} e\left(\frac{an}{q}\right) \int_{-\infty}^{\infty} g(q,x)e\left(\frac{nx}{qQ}\right)dx,$$

where the sum over a is over the reduced residue class of q (signified by $*$) and for any $\alpha > 1$, $g(q,x)$ satisfies

$$g(q,x) = 1 + O\left(\frac{Q}{q}\left(\frac{q}{Q} + |x|\right)^{\alpha}\right), \quad g(q,x) \ll |x|^{-\alpha}, \tag{3.10}$$

$$x^j \frac{\partial^j}{\partial x^j} g(q,x) \ll \log Q \min\left\{\frac{Q}{q}, \frac{1}{|x|}\right\}, \tag{3.11}$$

$$\int_{\mathbb{R}} (|g(q,x)| + |g(q,x)|^2) dx \ll Q^{\varepsilon} \tag{3.12}$$

In particular, $g(q,x)$ is supported in $[-T^{\varepsilon}, T^{\varepsilon}]$ with negligible error term. When $q \ll Q^{1-\varepsilon}$ and $|x| \ll Q^{-\varepsilon}$, by (3.10), $g(q,x)$ can be taken to be 1 with a negligible error term. When $q \ll Q^{1-\varepsilon}$ and $|x| \gg Q^{-\varepsilon}$, by (3.11), we have

$$\frac{\partial^j}{\partial x^j} g(q,x) \ll \log Q \frac{1}{|x|^{j+1}} \ll Q^{j\varepsilon}.$$

In $\mathfrak{M}$ (3.9), we introduce an additional sum over m accompanied with

$$\delta(m-n-h) W\left(\frac{m-n-h}{Q^2}\right).$$

Here $\delta(m-n-h)$ represents the condition $m = n+h$ and W is an even smooth function supported in $[-1,1]$ with $W(0) = 1$. Then, we apply Lemma 3.4 with modulus $Q_0 = \sqrt{N/(T/X)}$. We introduce a smooth dyadic partition of unity in the q sum and separate the n sum and m sum to get

$$\begin{aligned} \mathfrak{M} \ll & \frac{T^{\varepsilon}}{Q_0} \sup_{Q \ll Q_0} \sum_{q \sim Q} \frac{1}{q} \sideset{}{^*}\sum_{a \bmod q} \int_{\mathbb{R}} V_2\left(\frac{\xi}{X}\right) \sum_h e_q(-ah) \\ & \times \left[\sum_n \bar{A}(1,n) e_q(-an) e\left(\frac{(T + r\xi)}{2\pi} \times \frac{h}{n}\right) U(h,n) V\left(\frac{n}{N}\right)\right] \\ & \times \left[\sum_m A(1,m) e_q(am) \int_{\mathbb{R}} g(q,x) e\left(\frac{(m-n-h)x}{qQ_0}\right) W\left(\frac{m-n-h}{Q_0^2}\right) dx\right] d\xi. \end{aligned} \tag{3.13}$$

Remark. At this point, we note that trivially,

$$\mathfrak{M} \ll XHNQ^2 \times \frac{1}{Q} \times Q \times \frac{1}{Q} \times Q \asymp Q^2NHX \asymp Q^2N^2.$$

Hence,

$$M_F(T) \ll N^{-1} \sum_{r\sim T/X} \sup_{H \ll \frac{NT^\varepsilon}{X}} \mathfrak{M} \ll \frac{T}{NX} \cdot Q^2N^2 \sim N^2$$

So, in $\mathfrak{M}$, now we need to save $N^{1+\delta}$ for some $\delta > 0$.

3.4 Dualization

3.4.1 Voronoi summation in the m-sum

We will apply the Voronoi summation formula (Lemma 2.3) to the m-sum

$$\begin{aligned} S_{\mathcal{M}} &= \sum_m A(1,m) e_q(am)\psi(m), \\ \text{where} \quad \psi(y) &= \int_{\mathbb{R}} g(q,x) e\left(\frac{(y-n-h)x}{qQ_0}\right) W\left(\frac{y-n-h}{Q_0^2}\right) dx. \end{aligned} \tag{3.14}$$

As stated in Lemma 2.3, we will only consider the term corresponding to Ψ_0. Following (2.12), we get

$$\begin{aligned} \Psi_0\left(\frac{m_1^2 m_2}{q^3}\right) &\asymp \sum_{\pm} c_{\pm} \left(\frac{m_1^2 m_2}{q^3}\right)^{2/3} I_{1,\pm}, \\ \text{where} \quad I_{1,\pm} :&= \int_{-\infty}^{\infty} g(q,x) \int_0^{\infty} y_1^{-1/3} \\ &\times e\left(\frac{x(y_1-n-h)}{qQ_0} \pm \frac{3(m_2 m_1^2 y_1)^{1/3}}{q}\right) W\left(\frac{y_1-n-h}{Q_0^2}\right) dy_1 dx. \end{aligned} \tag{3.15}$$

With a change of variable $y = \frac{y_1-n-h}{qQ_0}$, we get

$$I_{1,\pm} = qQ_0 \int_{-\infty}^{\infty} (n+h+qQ_0y)^{-1/3} \times e\left(\pm\frac{3(m_2m_1^2)^{1/3}(n+h+qQ_0y)^{1/3}}{q}\right) W\left(\frac{yq}{Q_0}\right) \int_{-\infty}^{\infty} e(xy)g(q,x)dxdy.$$

When $q \gg Q_0^{1-\varepsilon}$, $W(\frac{y}{Q_0/q})$ implies $y \ll T^{\varepsilon}$. Thus, $e\left(xy\right)$ can be considered as a smooth function in x, y_1 and q and can be dropped from further consideration. When $q \ll Q_0^{1-\varepsilon}$, we evaluate the x integral

$$\int_{-\infty}^{\infty} g(q,x)e\left(xy\right)dx.$$

When $q \ll Q_0^{1-\varepsilon}$, by repeated integration by parts (Lemma 2.5), this integral is negligibly small unless $y_1 \ll T^{\varepsilon}$. In that range, $e\left(xy\right)$ can be considered to be a smooth functions in x and y and can be dropped from further consideration. Thus, for any q, we get

$$I_{1,\pm} = qQ_0 \int_{|y|\ll T^{\varepsilon}} (n+h+qQ_0y)^{-1/3} \times e\left(\pm\frac{3(m_2m_1^2)^{1/3}(n+h+qQ_0y)^{1/3}}{q}\right) W\left(\frac{yq}{Q_0}\right) \int_{-\infty}^{\infty} g(q,x)dxdy + O(T^{-A}).$$

At this point, by repeated integration by parts (Lemma 2.7), we observe that the y integral is negligibly small unless $m_1^2m_2 \ll \frac{N^{2+\varepsilon}}{Q_0^3}$. Then, by the Taylor series expansion, we observe that

$$\begin{aligned}&\frac{3(m_2m_1^2)^{1/3}(n+h+Q_0qy)^{1/3}}{q}\\ =&\frac{3(m_2m_1^2)^{1/3}(n+h)^{1/3}}{q}\cdot\left(1+\frac{Q_0qy}{n+h}\right)^{1/3}\\ =&\frac{3(m_2m_1^2)^{1/3}(n+h)^{1/3}}{q}+\frac{(m_2m_1^2)^{1/3}(n+h)^{1/3}}{q}\cdot\frac{Q_0qy}{n+h}+\cdots.\end{aligned}$$

The exponential terms corresponding to all but the first term of this Taylor series expansion can be absorbed into the smooth function $U(h,n)$ as

$$\frac{(m_2m_1^2)^{1/3}(n+h)^{1/3}}{q}\cdot\left(\frac{Q_0qy}{n+h}\right) \ll N^{\varepsilon}\frac{N}{qQ_0}\cdot\frac{qQ_0}{N} \ll N^{\varepsilon}.$$

Hence,

$$I_{1,\pm} \asymp \frac{qQ_0}{N^{1/3}} e\left(\pm\frac{3(m_2m_1^2)^{1/3}(n+h)^{1/3}}{q}\right)\int g(q,x)V(\cdots)dx + O(T^{-A}).$$

Now, we also expand $(1+h/n)^{1/3}$ by it's Taylor series expansion

$$\left(1+\frac{h}{n}\right)^{1/3} = 1+\frac{h}{3n}-\frac{h^2}{9n^2}+\cdots .$$

We note that if

$$\frac{(m_1^2m_2)^{1/3}n^{1/3}h^2}{n^2q} \ll \frac{hT^{\varepsilon}}{H},$$

the exponential terms corresponding to all but the first two terms of the Taylor series expansion of

$$\frac{3(m_2m_1^2)^{1/3}(n+h)^{1/3}}{q} = \frac{3(m_2m_1^2)^{1/3}n^{1/3}}{q}\cdot(1+h/n)^{1/3}$$

can be absorbed into the smooth function $U(h,n)$. This is true as

$$q\geq 1 \text{ and } X \gg \sqrt{T} \implies \frac{(m_1^2m_2)^{1/3}n^{1/3}h^2}{n^2q} \ll \frac{h}{H}\cdot\frac{N^{2/3}}{Q_0}\cdot\frac{N^{1/3}}{X^2q} \ll \frac{T^{5/4}}{X^{5/2}}\frac{hT^{\varepsilon}}{H}.$$

Finally, we get

$$I_{1,\pm} \asymp \frac{qQ_0}{N^{1/3}} e\left(\pm\frac{3(m_2m_1^2)^{1/3}(n^{1/3}+h/3n^{2/3})}{q}\right)\int g(q,x)V(\cdots)dx + O(T^{-A}). \quad (3.16)$$

We summarize the result in the lemma below.

Lemma 3.5. *$S_{\mathcal{M}}$ (3.14) is negligibly small unless $m_1^2m_2 \ll \frac{N^{2+\varepsilon}}{Q^3}$ and we have*

$$S_{\mathcal{M}} \asymp \frac{Q_0}{N^{1/3}}\sum_{\pm}\sum_{\substack{m_1|q \\ m_1^2m_2\ll\frac{N^{2+\varepsilon}}{Q_0^3}}}\sum_{m_2>0}\frac{A(m_2,m_1)\cdot m_1^{1/3}}{m_2^{1/3}}S(\bar{a},\pm m_2, qm_1^{-1})$$
$$\times\, e\left(\pm\frac{3(m_2m_1^2)^{1/3}(n^{1/3}+\frac{h}{3n^{2/3}})}{q}\right)\int g(q,x)dx + O(T^{-A}). \quad (3.17)$$

Remark. Initial size of $S_{\mathcal{M}}$ is $S_{\mathcal{M}} \ll Q^2 \sim X\sqrt{T}$. Final size of $S_{\mathcal{M}}$ is

$$S_{\mathcal{M}} \ll \frac{Q}{N^{1/3}} \cdot \frac{N^{4/3}}{Q^2} \cdot Q^{1/2} \sim \frac{T^{11/8}}{X^{1/4}}.$$

Hence, in the m sum, we save

$$\frac{\text{Initial size}}{\text{Final size}} = \frac{X\sqrt{T}}{\frac{T^{11/8}}{X^{1/4}}} \sim \frac{X^{5/4}}{T^{7/8}}.$$

3.4.2 Voronoi summation formula on the n-sum

After computing the m sum, the n sum, for a fixed $m = m_1^2 m_2$, becomes

$$\begin{aligned} S_{\mathcal{N}} &:= \sum_n \bar{A}(1,n) e_q(-an)\psi(n) \\ \text{where} \quad \psi(u) &:= e\left(\frac{(T+r\xi)}{2\pi} \times \frac{h}{u} \pm \frac{3(m_1^2 m_2)^{1/3}(u^{1/3} + h/3u^{2/3})}{q}\right) U(h,u)V(u/N). \end{aligned} \tag{3.18}$$

Similar to the m sum, we will apply the Voronoi summation formula to the n-sum and only consider the part corresponding to Ψ_0. For that, we use (2.12) with $x = n_2 n_1^2/q^3$. So,

$$\begin{aligned} \Psi_0\left(\frac{n_1^2 n_2}{q^3}\right) \asymp \left(\frac{n_1^2 n_2}{q^3}\right)^{2/3} \sum_{\pm} c_{\pm} \int_0^{\infty} e\Bigg(\frac{(T+r\xi)}{2\pi} \times \frac{h}{u} \pm \frac{(m_1^2 m_2)^{1/3} h}{u^{2/3} q} \pm \frac{3(n_1^2 n_2)^{1/3} u^{1/3}}{q} \\ \pm \frac{3(m_1^2 m_2)^{1/3} u^{1/3}}{q}\Bigg) \times \frac{1}{u^{1/3}} U(h,u)V(u/N)\,du + O(T^{-A}). \end{aligned} \tag{3.19}$$

Let us denote the oscillatory integral by

$$\begin{aligned} I_2 := \int_0^{\infty} e\Bigg(\frac{(T+r\xi)}{2\pi} \times \frac{h}{u} \pm \frac{(m_1^2 m_2)^{1/3} h}{q u^{2/3}} \\ + \frac{3(\pm(m_1^2 m_2)^{1/3} \pm (n_1^2 n_2)^{1/3}) u^{1/3}}{q}\Bigg) \frac{1}{u^{1/3}} U(h,u)V(u/N)\,du. \end{aligned} \tag{3.20}$$

Note that all the choices of $\pm$s are allowed, but we fix one to begin with.

As $(m_1^2 m_2) \ll \frac{N^{2+\varepsilon}}{Q_0^3}$, by repeated integration by parts (Lemma 2.5), this integral is negligibly small unless $(n_1^2 n_2) \ll \frac{N^{2+\varepsilon}}{Q_0^3}$. So, (3.13) transforms into

$$S_{\mathcal{N}} \asymp \frac{1}{q} \sum_{\pm} \sum_{n_1 | q} n_1^{1/3} \sum_{\substack{n_2 > 0 \\ n_1^2 n_2 \ll \frac{N^{2+\varepsilon}}{Q_0^3}}} \frac{A(n_1, n_2)}{n_2^{1/3}} S(\bar{a}, \pm n_2; q n_1^{-1}) \cdot I_2 + O(T^{-A}) \tag{3.21}$$

where I_2 is as defined in (3.20).

Remark. Initial size of $S_{\mathcal{N}} \ll N$. Final size of $S_{\mathcal{N}}$:

$$S_{\mathcal{N}} \ll \frac{1}{Q} \cdot \frac{N^{4/3}}{Q^2} \cdot Q^{1/2} \cdot N^{2/3} \sim \frac{N^2}{Q^{5/2}}.$$

Hence, in the n sum, we save

$$\frac{\text{Initial size}}{\text{Final size}} = \frac{N}{\frac{N^2}{Q^{5/2}}} \sim \frac{Q^{5/2}}{N} \sim \frac{X^{5/4} T^{5/8}}{T^{3/2}} \sim \frac{X^{5/4}}{T^{7/8}}.$$

3.4.3 Poisson summation formula on the h sum

Now, we evaluate the following h sum:

$$S_{\mathcal{H}} = \sum_{h} e_q(-ah) I_2(h) \tag{3.22}$$

where, I_2 is the integral (3.20) and we have written $I_2(h)$ to show the dependency on h. Here, we will use the Poisson Summation formula (2.14) and get (here we have denoted the dual variable by h too)

$$S_{\mathcal{H}} = \sum_{h \equiv a \bmod q} \int_{\mathbb{R}} I_2(y) e(-hy/q) dy = \sum_{h \equiv a \bmod q} I_3,$$

where the double integral I_3 is given by

$$I_3 := \int_0^\infty e\left(\frac{3(\pm (m_1^2 m_2)^{1/3} \pm (n_1^2 n_2)^{1/3}) u^{1/3}}{q} \right) \frac{1}{u^{1/3}} V(u/N) \cdot I_{h,2} \, du, \tag{3.23}$$

where

$$I_{h,2} = \int_{\mathbb{R}} e\left(\frac{(T+r\xi)}{2\pi} \times \frac{y}{u} \pm \frac{(m_1^2 m_2)^{1/3} y}{q u^{2/3}}\right) e\left(\frac{-hy}{q}\right) U(y,u) dy.$$

We recall that

$$\frac{\partial^j}{\partial x^j} U(x,u) \ll \left(\frac{T^\varepsilon}{H}\right)^j.$$

So, we make a change of variable: $y \mapsto \frac{yu}{X}$ and get

$$I_{h,2} = \frac{u}{X} \int_{\mathbb{R}} e\left(\frac{(T+r\xi)}{2\pi} \times \frac{y}{X} \pm \frac{(m_1^2 m_2)^{1/3} y u^{1/3}}{qX}\right) e\left(-\frac{hyu}{qX}\right) U\left(\frac{yu}{X}, u\right) dy. \quad (3.24)$$

Now,

$$\frac{\partial^j}{\partial y^j} U\left(\frac{yu}{X}, u\right) \ll \left(\frac{uT^\varepsilon}{XH}\right)^j \ll T^{j\varepsilon}.$$

Hence, by repeated integration by parts (Lemma 2.5), we must have

$$\left|\frac{(T+r\xi)}{2\pi X} \pm \frac{(m_1^2 m_2)^{1/3} u^{1/3}}{qX} - \frac{hu}{qX}\right| \ll T^{2\varepsilon}$$
$$\iff \left|\frac{q(T+r\xi)}{2\pi u} \pm \frac{(m_1^2 m_2)^{1/3}}{u^{2/3}} - h\right| \ll \frac{T^{2\varepsilon} qX}{N}.$$

Otherwise, the integral would be negligibly small. We also have $(a,q) = 1$ and $h \equiv a \bmod q$. So, we cannot have $h = 0$. Hence, the contribution of S_h would be negligibly small unless $|h| \geq 1$. As $u \sim N$, $\frac{(m_1^2 m_2)^{1/3}}{u^{2/3}} \ll \frac{T^\varepsilon}{Q_0}$ and $X < T$, this forces $h \sim \frac{qT}{N}$ and consequently $q \gg \frac{N^{1-\varepsilon}}{T}$. With h free in this range, we get that the integral is negligibly small unless

$$\left|\frac{(T+r\xi)}{2\pi X} \pm \frac{(m_1^2 m_2)^{1/3} u^{1/3}}{qX} - \frac{hu}{qX}\right| \ll T^{2\varepsilon}$$
$$\iff \left|\frac{(T+r\xi)q}{2\pi h} \pm \frac{(m_1^2 m_2)^{1/3} u^{1/3}}{h} - u\right| \ll T^{2\varepsilon}\frac{qX}{h} \ll \frac{NT^{2\varepsilon}}{T/X}.$$

We have already observed that $|h| \geq 1$. Hence, $\frac{(m_1^2 m_2)^{1/3} u^{1/3}}{h} \ll \frac{N}{Q_0}$. If we assume $Q_0 > T/X$ (which we will verify with our final choice of Q_0 in Section 5.7), the integral is negligibly small unless

$$\left|\frac{(T+r\xi)q}{2\pi h}-u\right| \ll T^{2\varepsilon}\frac{qX}{h} \ll \frac{NT^{2\varepsilon}}{T/X}. \tag{3.25}$$

In this range, $\frac{X}{u}\cdot I_{h,2}$ is a smooth function on h, q and ξ satisfying $y^j\frac{\partial^j}{\partial y^j}(\frac{X}{u}I_{h,2}) \ll T^{j\varepsilon}$ in each variable (but not on u) with absolute value bounded by 1. Then, we go back to the I_3 integral

$$I_3 = \frac{1}{X}\int_0^\infty e\left(\frac{3(\pm(m_1^2m_2)^{1/3}\pm(n_1^2n_2)^{1/3})u^{1/3}}{q}\right)u^{2/3}V(u/N)\cdot\left(\frac{X}{u}\cdot I_{h,2}\right)\, du, \tag{3.26}$$

and make a change of variable $u_1 = u - \frac{(T+r\xi)q}{2\pi h}$. Thus, $|u_1| \ll \frac{NT^\varepsilon}{T/X}$. Now, we have

$$I_3 \asymp \frac{N^{2/3}}{X}\int e\left(\frac{3(\pm(m_1^2m_2)^{1/3}\pm(n_1^2n_2)^{1/3})(u_1+\frac{(T+r\xi)q}{2\pi h})^{1/3}}{q}\right)U(u_1)du + O(T^{-A})$$

for some $U(x)$ supported in $[-\frac{NT^\varepsilon X}{T}, \frac{NT^\varepsilon X}{T}]$ satisfying $U^j(x) \ll (NX/T)^{-j}T^{j\varepsilon}$. Then, by repeated integration by parts (Lemma 2.5), the integral I_3 is negligibly small unless

$$|\pm(m_1^2m_2)^{1/3}\pm(n_1^2n_2)^{1/3}| \ll \frac{qT^\varepsilon T}{N^{1/3}X}.$$

Now, we expand $\left(1+\frac{u_1}{\frac{(T+r\xi)q}{2\pi h}}\right)^{1/3}$ by Taylor series expansion

$$\left(1+\frac{u_1}{\frac{(T+r\xi)q}{2\pi h}}\right)^{1/3} = 1+\frac{u_1}{\frac{3(T+r\xi)q}{2\pi h}}+\cdots$$

The first term leads to the following phase function:

$$\frac{3(T+r\xi)^{1/3}(\pm(m_1^2m_2)^{1/3}\pm(n_1^2n_2)^{1/3})}{(2\pi)^{1/3}q^{2/3}h^{1/3}}.$$

The second and higher-order terms can be incorporated into the smooth functions as

$$\frac{3(T+r\xi)^{1/3}(\pm(m_1^2m_2)^{1/3}\pm(n_1^2n_2)^{1/3})}{(2\pi)^{1/3}q^{2/3}h^{1/3}}\cdot\frac{u_1}{\frac{3(T+r\xi)q}{2\pi h}} \ll \frac{N^{1/3}}{q}\cdot\frac{qT^\varepsilon T}{N^{1/3}X}\cdot\frac{X}{T} \ll T^\varepsilon.$$

Here, we have used the fact $|\pm(m_1^2m_2)^{1/3}\pm(n_1^2n_2)^{1/3}| \ll \frac{qT^\varepsilon T}{N^{1/3}X}$. So, I_3 can be written as

$$I_3 \asymp \frac{N^{2/3}T^\varepsilon}{X}\cdot\frac{NX}{T}\cdot e\left(\frac{3(T+r\xi)^{1/3}(\pm(m_1^2m_2)^{1/3}\pm(n_1^2n_2)^{1/3})}{(2\pi)^{1/3}q^{2/3}h^{1/3}}\right)V(\cdots)+O(T^{-A}). \tag{3.27}$$

Putting it back into $S_{\mathcal{H}}$, we evaluate

$$\begin{aligned} S_{\mathcal{H}} &= \sum_{\substack{h\sim\frac{qT}{N}\\ h\equiv a \bmod q}} I_3 \\ &\asymp \frac{N^{5/3}}{T}\sum_{\substack{h\sim\frac{qT}{N}\\ h\equiv a \bmod q}} e\left(\frac{3(T+r\xi)^{1/3}(\pm(n_1^2n_2)^{1/3}\pm(m_1^2m_2)^{1/3})}{(2\pi)^{1/3}q^{2/3}h^{1/3}}\right)V(\cdots)+O(T^{-A}). \end{aligned} \tag{3.28}$$

Remark. Initial size of $s_{\mathcal{H}}$ is $S_{\mathcal{H}} \ll HN^{2/3} \sim \frac{N^{5/3}}{X}$. Final size of $S_{\mathcal{H}}$ is

$$S_{\mathcal{H}} \ll \frac{N^{5/3}}{T}\cdot\frac{QT}{N}\cdot\frac{1}{Q} \sim N^{2/3}.$$

So, in the h sum, we save

$$\frac{\text{Initial size}}{\text{Final size}} = \frac{N^{5/3}/X}{N^{2/3}} \sim \frac{N}{X} \sim \frac{T^{3/2}}{X}.$$

Hence, in total, after the Voronoi summation formula on the n and m sum and the Poisson summation formula on the h sum, we save

$$\frac{X^{5/4}}{T^{7/8}}\cdot\frac{X^{5/4}}{T^{7/8}}\cdot\frac{T^{3/2}}{X} \sim \frac{X^{3/2}}{T^{1/4}}.$$

3.4.4 Final Equation After Voronoi

We put (3.17), (3.21), (3.28) into (3.13) and arrive at the expression

$$\mathfrak{M} \ll \frac{T^{\varepsilon}}{Q_0} \cdot \frac{Q_0}{N^{1/3}} \cdot \frac{N^{5/3}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sum_{q \sim Q} \int_{\xi \sim X} \sum_{h \sim \frac{qT}{N}} \int g(q,x)dx$$
$$\sum_{\substack{m_1 | q \\ m_1^2 m_2 \ll \frac{N^{2+\varepsilon}}{Q_0^3}}} \sum_{m_2 > 0} \frac{A(m_2,m_1) \cdot m_1^{1/3}}{m_2^{1/3}} S(\bar{a}, \pm m_2, q m_1^{-1}) e\left(\frac{3(T+r\xi)^{1/3}(\pm(m_1^2 m_2)^{1/3})}{(2\pi)^{1/3} q^{2/3} h^{1/3}} \right) \cdot$$
$$\sum_{\substack{n_1 | q \\ n_1^2 n_2 \ll \frac{N^{2+\varepsilon}}{Q_0^3}}} \sum_{n_2 > 0} \frac{A(n_1,n_2) n_1^{1/3}}{n_2^{1/3}} S(\bar{a}, \pm n_2; q n_1^{-1}) e\left(\frac{3(T+r\xi)^{1/3}(\pm(n_1^2 n_2)^{1/3})}{(2\pi)^{1/3} q^{2/3} h^{1/3}} \right) \tag{3.29}$$

Now, we evaluate the x integral by (3.12), we take a dyadic partition of both the $m_1^2 m_2$ sum and the $n_1^2 n_2$ sum and then by Cauchy's inequality and the symmetry of the n_1, n_2 and m_1, m_2 sum, we get

$$\mathfrak{M} \ll \frac{N^{4/3} T^{\varepsilon}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{-2/3} \sum_{q \sim Q} \int_{\xi \sim X} \sum_{h \sim \frac{qT}{N}}$$
$$\times \left| \sum_{\substack{m_1 | q \\ m_1^2 m_2 \sim M_0}} \sum_{m_2 > 0} A(m_2,m_1) \cdot m_1 \cdot S(\bar{h}, m_2, q m_1^{-1}) e\left(\frac{3(T+r\xi)^{1/3}(m_1^2 m_2)^{1/3}}{(2\pi)^{1/3} q^{2/3} h^{1/3}} \right) \right|^2 \tag{3.30}$$

Here, we have only considered the + sign case as the analysis of the other cases is exactly similar.

Remark. Initial size of $\mathfrak{M}$ is $\mathfrak{M} \ll Q^2 N^2$. Final size of $\mathfrak{M}$ is

$$\mathfrak{M} \ll \frac{N^{4/3}}{T} \cdot \frac{1}{Q^2} \cdot \frac{N^{8/3}}{Q^4} \cdot Q \cdot X \cdot \frac{QT}{N} \cdot Q \sim \frac{N^3 X}{Q^3}.$$

Hence, in total, we have saved

$$\frac{Q^2 N^2}{N^3 X / Q^3} \sim \frac{Q^5}{NX} \sim \frac{X^{5/2} T^{5/4}}{T^{3/2} X} \sim \frac{X^{3/2}}{T^{1/4}},$$

which reconfirms the saving calculated at the end of the previous section.

Chapter 4

Proof of Theorem 1: Duality

4.1 Duality principle of the large sieve

To interchange the order of summations in the above expression, we will use the duality principle of the large sieve. We restate the duality principle stated in Chapter 7.1 of [IK].

Lemma 4.1. *For $1 \leq m \leq M$ and $1 \leq n \leq N$ let $\alpha_m, \beta_n \in \mathbb{C}$ and $(\phi(m,n))_{\substack{1\leq m\leq M\\ 1\leq n\leq N}}$ be a complex matrix . Then, for some $\Delta > 0$, if we have*

$$\sum_{n} \Big| \sum_{m} \alpha_m \phi(m,n) \Big|^2 \ll \Delta ||\alpha||^2,$$

for the same Δ we would have

$$\sum_{m} \Big| \sum_{n} \beta_n \phi(m,n) \Big|^2 \ll \Delta ||\beta||^2.$$

Using the above lemma, we can prove that

Lemma 4.2.

$$\mathfrak{M} \ll \frac{N^{4/3} T^{\varepsilon}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \times \Delta,$$

where

$$\Delta = \sup_{\substack{\alpha \\ ||\alpha||^2=1}} \sum_{\substack{m_1>0 \\ m_1^2 m_2 \sim M_0}} \sum_{m_2>0} \left| \int_{\xi \sim X} \sum_{q\sim Q} \sum_h \alpha(\xi,q,h) S(\bar{h}, m_2; qm_1^{-1}) \right. \\ \left. \times\, m_1 \mathbb{1}_{\{m_1|q\}} e\left(\frac{3(T+r\xi)^{1/3} m_1^{2/3} m_2^{1/3}}{(2\pi)^{1/3} q^{2/3} h^{1/3}} \right) d\xi \right|^2. \tag{4.1}$$

where $\alpha(\xi,q,h)$ *varies over all the complex vectors such that* $\int_{\xi\sim X} \sum_{q\sim Q} \sum_{h\sim \frac{qT}{N}} |\alpha|^2 = 1$.

Proof. In the expression of $\mathfrak{M}$ in (3.30) if we think of

$$\int_{\xi\sim X} \sum_{q\sim Q} \sum_h$$

as $\sum_m$ of Lemma 4.1 ,

$$\sum_{\substack{m_1>0 \\ m_1^2 m_2 \sim (T/Q)^3}} \sum_{m_2>0}$$

as $\sum_n$ of Lemma 4.1, $\beta := A(m_2, m_1)$ and

$$\phi := m_1 \mathbb{1}_{\{m_1|q\}} S(\bar{h}, m_2; qm_1^{-1}) e\left(\frac{3(T+r\xi)^{1/3} (m_1^2 m_2)^{1/3}}{(2\pi)^{1/3} q^{2/3} h^{1/3}} \right),$$

then from lemma 4.1 we can derive that

$$\mathfrak{M} \ll \frac{N^{4/3} T^{\varepsilon}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{-2/3} \sum_{\substack{m_1>0 \\ m_1^2 m_2 \sim M_0}} \sum_{m_2>0} |A(m_2,m_1)|^2 \times \Delta.$$

Finally, we use the Ramanujan type bound on average (2.6) to conclude the lemma. □

Remark. Trivial size of Δ is

$$\Delta \ll \frac{N^2}{Q^3} \cdot X \cdot Q \cdot \frac{QT}{N} \cdot Q \sim NXT.$$

4.1.1 After large sieve

At this point, we aim to achieve a non-trivial upper bound for Δ. First of all, as $m_1|q$, everywhere in the expression, we will replace q by $m_1 q$ where the new q is in the range

$q \sim Q/m_1$. Now, we open up the absolute square in (4.1) and denote the m_2 sum as $S_{\mathcal{M}_2}$. Thus, we get

$$\Delta \ll T^{\varepsilon} \sup_{||\alpha||_2=1} \sum_{m_1 \ll Q} m_1^2 \int_{\xi_1 \sim X} \sum_{q_1 \sim Q/m_1} \sum_{h_1 \sim \frac{QT}{N}} \int_{\xi_2 \sim X} \sum_{q_2 \sim Q/m_1} \sum_{h_2 \sim \frac{QT}{N}} \alpha_1 \bar{\alpha}_2 \times S_{\mathcal{M}_2},$$

$$\text{where } S_{\mathcal{M}_2} = \sum_{m_2 \sim \frac{M_0}{m_1^2}} S(\bar{h}_1, m_2; q_1) S(\bar{h}_2, m_2; q_2) \times e\left(\frac{3(T + r\xi_2)^{1/3} (m_1^2 m_2)^{1/3}}{(2\pi)^{1/3} (q_2 m_1)^{2/3} h_2^{1/3}} - \frac{3(T + r\xi_1)^{1/3} (m_1^2 m_2)^{1/3}}{(2\pi)^{1/3} (q_1 m_1)^{2/3} h_1^{1/3}} \right). \tag{4.2}$$

We note that, for $i = 1, 2$, $\alpha(\xi_i, q_i m_1, h_i)$ is written as α_i in short. Let us define

$$\Xi(\xi, q, h) = \frac{(T + r\xi)^{1/3}}{(2\pi)^{1/3} q^{2/3} h^{1/3}}$$

and for brevity, we will often write $\Xi_1 = \Xi(\xi_1, q_1 m_1, h_1)$ or $\Xi_2 = \Xi(\xi_2, q_2 m_2, h_2)$. We note that

$$|\Xi_2 - \Xi_1| \ll \frac{T^{1/3}}{Q^{2/3} \cdot (QT/N)^{1/3}} \sim \frac{N^{1/3}}{Q}.$$

Then, we take a smooth dyadic partition of $\Xi_2 - \Xi_1$ of the form $V\left(\frac{\Xi_2 - \Xi_1}{X_1}\right)$ where $T^{-A} \ll |X_1| \ll \frac{N^{1/3}}{Q}$. The contribution from $|\Xi_2 - \Xi_1| \ll T^{-A}$ would be negligibly small. So,

$$\Delta \ll T^{\varepsilon} \sup_{||\alpha||_2=1} \sup_{T^{-A} \ll X_1 \ll \frac{N^{1/3}}{Q}} \left| \sum_{m_1 \ll Q} m_1^2 \int_{\xi_1 \sim X} \sum_{q_1 \sim \frac{Q}{m_1}} \sum_{h_1 \sim \frac{QT}{N}} \int_{\xi_2 \sim X} \sum_{q_2 \sim \frac{Q}{m_1}} \sum_{h_2 \sim \frac{QT}{N}} \alpha_1 \bar{\alpha}_2 \times V\left(\frac{\Xi_2 - \Xi_1}{X_1} \right) \cdot S_{\mathcal{M}_2} \right| + O(T^{-A}). \tag{4.3}$$

Here, we only deal with $X_1 > 0$ as the case of the negative X_1 is symmetrical.

Now, we intend to apply the Poisson Summation formula on $S_{\mathcal{M}_2}$. We note that

$$S(\bar{h}_1, m_2; q_1) S(\bar{h}_2, m_2; q_2)$$

is of modulus q_1q_2. Applying Poisson summation formula (2.13) on the $S_{\mathcal{M}_2}$ sum, we get

$$S_{\mathcal{M}_2} = \frac{1}{q_1q_2}\sum_{m\in\mathbb{Z}}\left(\sum_{\beta \bmod q_1q_2} S(\bar{h}_1,\beta;q_1)S(\bar{h}_2,\beta;q_2)e_{q_1q_2}(m\beta)\right) \times \int_{y\sim\frac{M_0}{m_1^2}} e\left(3(\Xi_2-\Xi_1)m_1^{2/3}y^{1/3}\right)e(-my/q_1q_2)dy.$$

We make a change of variable $y \mapsto \frac{M_0y}{m_1^2}$ and arrive at

$$S_{\mathcal{M}_2} = \frac{M_0}{m_1^2q_1q_2}\sum_{m\in\mathbb{Z}}\mathfrak{C}\,\mathfrak{J},$$

where

$$\mathfrak{C} = \sum_{\beta \bmod q_1q_2} S(\bar{h}_1,\beta;q_1)S(\bar{h}_2,\beta;q_2)e_{q_1q_2}(m\beta),$$
$$\mathfrak{J} = \int_{y\sim 1} e\left(3(\Xi_2-\Xi_1)M_0^{1/3}y^{1/3}\right)\times e\left(-\frac{mM_0y}{m_1^2q_1q_2}\right)dy. \tag{4.4}$$

Depending on m, we will divide the analysis of the above expression into two separate cases: "Diagonal" when $m=0$ and "Off-Diagonal" when $m\neq 0$.

Remark. Before Poisson, the trivial size of $S_{\mathcal{M}_2}$ was $\frac{N^2Q}{Q^3}\sim\frac{NT}{X}$.

4.2 Diagonal

In this case, we have $m=0$. At first, we will calculate the contribution of the character sum, which is

$$\mathfrak{C} = \sum_{\beta \bmod q_1q_2} S(\bar{h}_1,\beta;q_1)S(\bar{h}_2,\beta;q_2) = q_1q_2 \sideset{}{^*}\sum_{\substack{x \bmod q_1\\ xq_2+yq_1\equiv 0 \bmod q_1q_2}} \sideset{}{^*}\sum_{y \bmod q_2} e\left(\frac{\bar{h}_1\bar{x}}{q_1}\right)e\left(\frac{\bar{h}_2\bar{y}}{q_2}\right). \tag{4.5}$$

To have a solution of the congruence relation, we must have $q_2|yq_1$, which would imply $q_2|q_1$. Similarly, we get that $q_1|q_2$. So we would have $q_1=q_2$. In that case, the character sum becomes

$$\mathfrak{C} = q_1^2 \sum_{x \bmod q_1}^{*} e\left(\frac{(h_1 - h_2)x}{q_1}\right) = q_1^2 \mathfrak{c}_{q_1}(h_1 - h_2)\delta_{q_1 = q_2},$$

where $\mathfrak{c}_{q_1}$ is the Ramanujan sum modulo q_1. Only when $h_1 \equiv h_2 \bmod q_1$, we have $\mathfrak{c}_{q_1}(h_1 - h_2) \ll q_1$. We will use the bound when $m_1 \ll \frac{N}{T}$. In any other case, we may write $\mathfrak{c}_{q_1}(h_1 - h_2) \ll (h_1 - h_2, q_1)$. We will use this bound for $\frac{N}{T} \ll m_1 \ll Q$. From the above arguments on the character sum, we can deduce that

$$\sum_{h_1} c_{q_1}(h_1 - h_2) \ll \begin{cases} \ll \frac{Q^{1+\varepsilon}}{m_1} \text{ when } m_1 \ll \frac{N}{T}, \\ \ll \frac{Q^{1+\varepsilon}T}{N} \text{ when } \frac{N}{T} \ll m_1 \ll Q. \end{cases}$$

We also note that

$$\mathfrak{J} = \int_{y \sim 1} e\left(\frac{3(T + r\xi_2)^{1/3} M_0^{1/3} y^{1/3}}{(2\pi)^{1/3} m_1^{2/3} q_2^{2/3} h_2^{1/3}} - \frac{3(T + r\xi_1)^{1/3} M_0^{1/3} y^{1/3}}{(2\pi)^{1/3} m_1^{2/3} q_1^{2/3} h_1^{1/3}}\right) dy$$

is negligible (by repeated integration by parts (Lemma 2.5)) unless

$$\begin{aligned} &\left| \frac{3(T + r\xi_2)^{1/3} M_0^{1/3} y^{1/3}}{(2\pi)^{1/3} m_1^{2/3} q_2^{2/3} h_2^{1/3}} - \frac{3(T + r\xi_1)^{1/3} M_0^{1/3} y^{1/3}}{(2\pi)^{1/3} m_1^{2/3} q_1^{2/3} h_1^{1/3}} \right| \ll N^{\varepsilon}. \\ \iff &\left| \frac{(T + r\xi_2)^{1/3}}{h_2^{1/3}} - \frac{(T + r\xi_1)^{1/3}}{h_1^{1/3}} \right| \ll \frac{Q^{2/3} N^{\varepsilon}}{M_0^{1/3}} \\ \iff &\left| \frac{T + r\xi_2}{h_2} - \frac{T + r\xi_1}{h_1} \right| \ll \frac{Q^{2/3} N^{\varepsilon}}{M_0^{1/3}} \cdot \frac{T^{2/3}}{(QT/N)^{2/3}} \asymp \frac{N^{2/3} N^{\varepsilon}}{M_0^{1/3}} \end{aligned}$$

For the remaining range, we take $\mathfrak{J} \ll 1$. From the above condition we can derive that there exists a $\tilde{\xi_1}$ depending on ξ_1, h_1, h_2 such that

$$|\xi_2 - \tilde{\xi_1}| \ll \frac{N^{2/3+\varepsilon}}{M_0^{1/3}} \cdot \frac{QT}{N} \cdot \frac{X}{T} \asymp \frac{QX}{M_0^{1/3} N^{1/3}}.$$

Symmetrically, if we fix ξ_2, h_1, h_2 then for a fixed $\tilde{\xi_2}$ we would have $|\xi_1 - \tilde{\xi_2}| \ll \frac{QX}{M_0^{1/3}N^{1/3}}$. By Cauchy's inequality for $m = 0$, Δ is bounded by

$$\begin{aligned}\Delta_{m=0} \ll & \sum_{m_1 \ll Q} \int_{\xi_1} \sum_{q_1 \sim Q/m_1} \sum_{h_1 \sim \frac{QT}{N}} \int_{\xi_2} \sum_{q_2 \sim Q/m_1} \sum_{h_2 \sim \frac{QT}{N}} \alpha_1 \bar{\alpha}_2 \times \frac{M_0}{q_1 q_2} \mathfrak{C}\, \mathfrak{J} \\ \ll & M_0 \Bigg(\int_{\xi_1} \sum_{m_1 \ll Q} \sum_{q_1 \sim Q/m_1} \sum_{h_1 \sim \frac{QT}{N}} |\alpha_1|^2 \sum_{h_2 \sim \frac{QT}{N}} \int_{|\xi_2 - \tilde{\xi}_1| \ll \frac{QX}{M_0^{1/3}N^{1/3}}} |c_{q_1}(h_1 - h_2)| \Bigg)^{1/2} \\ & \times \Bigg(\int_{\xi_2} \sum_{m_1 \ll Q} \sum_{q_2 \sim Q/m_1} \sum_{h_2 \sim \frac{QT}{N}} |\alpha_2|^2 \sum_{h_1 \sim \frac{QT}{N}} \int_{|\xi_1 - \tilde{\xi}_2| \ll \frac{QX}{M_0^{1/3}N^{1/3}}} |c_{q_2}(h_1 - h_2)| \Bigg)^{1/2}.\end{aligned}$$

We also recall that $||\alpha_1||^2 = 1$. Hence,

$$\int_{\xi_1} \sum_{m_1 \ll Q} \sum_{q_1 \sim Q/m_1} \sum_{h_1} |\alpha(q_1 m_1, h_1, \xi_1)|^2 \ll Q^{\varepsilon}. \tag{4.6}$$

Thus we get

$$\Delta_{m=0} \ll M_0 N^{\varepsilon} \cdot \frac{QX}{M_0^{1/3} N^{1/3}} \cdot \left(Q + \frac{QT}{N} \right) \ll \frac{Q^2 X M_0^{2/3} N^{\varepsilon}}{N^{1/3}}. \tag{4.7}$$

Hence, the contribution of the diagonal part in $\mathfrak{M}$ is

$$\begin{aligned}\mathfrak{M} \ll & \frac{N^{4/3}T^{\varepsilon}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \times \frac{Q^2 X M_0^{2/3} N^{\varepsilon}}{N^{1/3}} \\ \ll & \frac{N^3 X T^{\varepsilon}}{T Q_0^3} \sim \frac{N^{3/2}\sqrt{T} T^{\varepsilon}}{\sqrt{X}}. \end{aligned} \tag{4.8}$$

Remark. We require, $\mathfrak{M} \ll NX\sqrt{T}$. Hence, the bound in the diagonal part is smaller than the expected bound as long as $X \gg T^{1/2+\varepsilon}$.

4.3 Off-Diagonal

For the $m \neq 0$ case, we will start with the analysis of oscillatory integral $\mathfrak{J}$ in (4.4). We recall

$$\mathfrak{J} = \int_{y\sim 1} e\left(3(\Xi_2 - \Xi_1)M_0^{1/3}y^{1/3}\right) \times e\left(-\frac{mM_0 y}{m_1^2 q_1 q_2}\right)dy \tag{4.9}$$

and $\Xi_2 - \Xi_1 \sim X_1$. We will now assume that $X_1 \gg \frac{N^\varepsilon}{M_0^{1/3}}$.

When $X_1 \ll \frac{N^\varepsilon}{M_0^{1/3}}$, we can absorb the integral $\mathfrak{J}$ into the smooth functions if $|m| \ll \frac{Q^2 N^\varepsilon}{M_0}$ and $\mathfrak{J}$ would be negligibly small (by repeated integration by parts (Lemma 2.5)) for the other range of m. This case would be considered separately in Section 4.23.

4.3.1 Integral $\mathfrak{J}$

Lemma 4.3. *Let $X_1 \gg \frac{N^\varepsilon}{M_0^{1/3}}$. Then, if $m \sim \frac{Q^2 X_1}{M_0^{2/3}}$, we have*

$$\mathfrak{J} \asymp \frac{1}{\sqrt{X_1 M_0^{1/3}}} \times e\left(2\sqrt{\frac{q_1 q_2 m_1^2}{m}}(\Xi_2 - \Xi_1)^{3/2}\right) + O(N^{-A}).$$

In the rest of the range of m, $\mathfrak{J}$ is negligibly small.

Proof. The exponential integral $\mathfrak{J}$ is of the form $\mathfrak{J} = \int_{y\sim 1} e(f(y))$ and $f(y) = 3Ay^{1/3} - By$, where

$$A = (\Xi_2 - \Xi_1)M_0^{1/3} \text{ and } B = \frac{mM_0}{m_1^2 q_1 q_2}.$$

We get

$$\begin{aligned} f'(y) &= \frac{A}{y^{2/3}} - B \\ f''(y) &= \frac{2A}{3y^{5/3}} \\ \text{if} \quad f'(y_0) = 0, \quad y_0 &= \left(\frac{A}{B}\right)^{3/2} \\ f(y_0) &= \frac{2A^{3/2}}{B^{1/2}} = 2\sqrt{\frac{q_1 q_2 m_1^2}{m}}(\Xi_2 - \Xi_1)^{3/2}. \end{aligned}$$

As $X_1 \gg \frac{N^\varepsilon}{M_0^{1/3}}$, there would be a stationary point only if $m \sim \frac{Q^2 X_1}{M_0^{2/3}}$. If there is no stationary point, by repeated integration by parts (Lemma 2.5), we note that the integral is negligibly small. Thus, by Lemma 2.6, we conclude the proof. □

4.3.2 Preliminary Analysis of Character Sum

We will simplify the Character sum $\mathfrak{C}$ to make it conducive for the application of the Poisson Summation formula. For that, we will introduce a new set of notations that will be used in the rest of this paper.

Notation: Let $d = (q_1, q_2)$. Then let $q_1 = u_1 v_1$ and $q_2 = u_2 v_2$ such that $v_1, v_2 | d^\infty$ and u_1, u_2, d are mutually coprime, i.e. v_1 is the part of q_1 corresponding to the prime divisors of d and u_1 is the rest of q_1 which is consequently coprime to d . Hence, u_1, u_2 and $v_1 v_2$ are mutually coprime. We further denote $\tilde{v}_1 = v_1/d$ and $\tilde{v}_2 = v_2/d$. Later in this section, we will observe that $d|m$. Thus we denote $\tilde{m} = m/d$.

Then, we may further split d into $d = d_0 \cdot d_1 \cdot d_2$ in the following manner

$$d_1 = (d, \tilde{v}_1^\infty), \; d_2 = (d, \tilde{v}_2^\infty) \text{ and } d_0 = \frac{d}{d_1 d_2}.$$

We note that $(d_0, \tilde{v}_1 \tilde{v}_2) = 1$. We will observe that $(\tilde{m}, \tilde{v}_1 \tilde{v}_2) = 1$. Then, $(\tilde{m}, d) | d_0$. We recall that $\tilde{m} = m/d$. Now, we denote $d_m = (\tilde{m}, d_0^\infty)$ and $m^* = \frac{m}{d d_m}$.

We also note that for a fixed d, the numbers of $\tilde{v}_1$, $\tilde{v}_2$ and d_m is bounded by d^ε for any $\varepsilon > 0$.

We break up the Kloosterman sums in $\mathfrak{C}$

$$\mathfrak{C} = \sum_{\beta \bmod q_1 q_2} S(\bar{h}_1, \beta; q_1) S(\bar{h}_2, \beta; q_2) e_{q_1 q_2}(m\beta),$$

by the following lemma. We omit the proof of this lemma as it is an easy consequence of the Chinese Remainder Theorem.

Lemma 4.4. *If* $(m, n) = 1$ *then*

$$S(a, b, mn) = S(a\bar{m}, b\bar{m}; n) S(a\bar{n}, b\bar{n}; m),$$

where $m\bar{m} \equiv 1 \bmod n$ *and* $n\bar{n} \equiv 1 \bmod m$*.*

As u_1, u_2 and $v_1 v_2$ are mutually coprime, we split the character sum $\mathfrak{C}$ as:

$$\mathfrak{C} = q_1 q_2 \mathfrak{C}_1 \mathfrak{C}_2 \mathfrak{C}_3$$

where

$$\begin{aligned}
\mathfrak{C}_1 &= \frac{1}{u_1} \sum_{\beta_1 \bmod u_1} S(\bar{h}_1 \bar{v}_1, \beta_1 \bar{v}_1; u_1) e_{u_1}(m\bar{u}_2 \overline{v_1 v_2} \beta_1), \\
\mathfrak{C}_2 &= \frac{1}{u_2} \sum_{\beta_2 \bmod u_2} S(\bar{h}_2 \bar{v}_2, \beta_2 \bar{v}_2; u_2) e_{u_2}(m\bar{u}_1 \overline{v_1 v_2} \beta_2), \\
\mathfrak{C}_3 &= \frac{1}{v_1 v_2} \sum_{\beta_3 \bmod v_1 v_2} S(\bar{h}_1 \bar{u}_1, \beta_3 \bar{u}_1; v_1) S(\bar{h}_2 \bar{u}_2, \beta_3 \bar{u}_2; v_2) e_{v_1 v_2}(m\bar{u}_1 \bar{u}_2 \beta_3).
\end{aligned}$$

Here, $(v_1 v_2) \cdot \overline{v_1 v_2} \equiv 1 \bmod u_1 u_2$. We will now simplify each $\mathfrak{C}_{\mathfrak{i}}$.

- $\mathfrak{C}_1$ and $\mathfrak{C}_2$: We evaluate

$$\begin{aligned}
\mathfrak{C}_1 &= \frac{1}{u_1} \sum_{\beta_1 \bmod u_1} S(\bar{h}_1 \bar{v}_1, \beta_1 \bar{v}_1; u_1) e_{u_1}(m\bar{u}_2 \overline{v_1 v_2} \beta_1), \\
&= \sum_{x \bmod u_1}^{*} e\left(\frac{\bar{x}\overline{h_1 v_1}}{u_1}\right) \cdot \frac{1}{u_1} \sum_{\beta_1 \bmod u_1} e\left(\frac{\beta_1 (x\overline{v_1} + m\overline{u_2 v_2 v_1})}{u_1}\right) \\
&= e\left(-\frac{\bar{m}\bar{h}_1 \bar{v}_1 u_2 v_2}{u_1}\right).
\end{aligned}$$

By reciprocity, we can write $\mathfrak{C}_1$ as

$$\mathfrak{C}_1 = e\left(\frac{\bar{u_1}u_2\tilde{v}_2}{mh_1\tilde{v}_1}\right) \times e\left(-\frac{u_2\tilde{v}_2}{u_1mh_1\tilde{v}_1}\right).$$

As

$$\frac{u_2\tilde{v}_2}{u_1mh_1\tilde{v}_1} \ll T^{-\varepsilon},$$

the second term can be absorbed into the smooth functions, and we can drop it from further consideration. Similarly, we get

$$\mathfrak{C}_2 = e\left(\frac{\bar{u_2}u_1\tilde{v}_1}{mh_2\tilde{v}_2}\right) \times e\left(-\frac{u_1\tilde{v}_1}{u_2mh_2\tilde{v}_2}\right) = e\left(\frac{\bar{u_2}u_1\tilde{v}_1}{mh_2\tilde{v}_2}\right)V(\cdots).$$

We also get that $(m, u_1u_2) = 1$. • $\mathfrak{C}_3$:

$$\begin{aligned}\mathfrak{C}_3 &= \frac{1}{v_1v_2}\sum_{\beta_3 \bmod v_1v_2} S(\bar{h_1}\bar{u_1}, \beta_3\bar{u_1}; v_1)S(\bar{h_2}\bar{u_2}, \beta_3\bar{u_2}; v_2)e_{v_1v_2}(m\bar{u_1}\bar{u_2}\beta_3)\\ &= \frac{1}{v_1v_2}\sideset{}{^*}\sum_{x \bmod v_1} e\left(\frac{\bar{h_1}\bar{u_1}\bar{x}}{v_1}\right)\sideset{}{^*}\sum_{y \bmod v_2} e\left(\frac{\bar{h_2}\bar{u_2}\bar{y}}{v_2}\right)\sum_{\beta_3 \bmod v_1v_2} e\left(\frac{\beta_3(x\bar{u_1}v_2 + y\bar{u_2}v_1 + m\bar{u_1}\bar{u_2})}{v_1v_2}\right)\\ &= \sideset{}{^*}\sum_{\substack{x \bmod v_1\\ u_2v_2x+u_1v_1y\equiv -m \bmod v_1v_2}}\sideset{}{^*}\sum_{y \bmod v_2} e\left(\frac{\bar{h_1}\bar{u_1}\bar{x}}{v_1}\right)e\left(\frac{\bar{h_2}\bar{u_2}\bar{y}}{v_2}\right).\end{aligned} \tag{4.10}$$

We note that $u_2v_2x + u_1v_1y \equiv -m \bmod v_1v_2$ implies $d = (v_1, v_2)|m$ and $(\tilde{m}, \tilde{v}_1\tilde{v}_2) = 1$. Once we change the x and y variables to x_1 and x_2, $x_1 \equiv u_2x \bmod v_1$ and $x_2 \equiv u_1y \bmod v_2$, from the simplifications above, we can express the character sum $\mathfrak{C}$ as

$$\mathfrak{C} = q_1q_2 \cdot \mathfrak{C}_0 \cdot V(\cdots)$$

where

$$\mathfrak{C}_0 = e\left(\frac{\tilde{v}_2 u_2 \overline{u_1}}{m h_1 \tilde{v}_1}\right) e\left(\frac{\tilde{v}_1 u_1 \overline{u_2}}{m h_2 \tilde{v}_2}\right) \sideset{}{^*}\sum_{\substack{x_1 \bmod d\tilde{v}_1 x_2 \bmod d\tilde{v}_2 \\ x_1\tilde{v}_2 + x_2\tilde{v}_1 + \tilde{m} \equiv 0 \bmod d\tilde{v}_1\tilde{v}_2}} \sideset{}{^*}\sum e\left(\frac{\overline{x_1}\overline{h}_1 u_2 \overline{u_1}}{d\tilde{v}_1}\right) e\left(\frac{\overline{x_2}\overline{h}_2 u_1 \overline{u_2}}{d\tilde{v}_2}\right). \tag{4.11}$$

We recall that $d = (q_1, q_2)$ and we have split d into $d = d_0 \cdot d_1 \cdot d_2$ in the following manner

$$d_1 = (d, \tilde{v}_1^\infty),\ d_2 = (d, \tilde{v}_2^\infty) \text{ and } d_0 = \frac{d}{d_1 d_2}.$$

We note that $(d_0, \tilde{v}_1\tilde{v}_2) = 1$. Then, we may split the sum over x_1 and x_2 by CRT in the following fashion:

$$\sideset{}{^*}\sum_{x_1 \bmod d\tilde{v}_1} \mapsto \sideset{}{^*}\sum_{x_{1,0} \bmod d_0} \sideset{}{^*}\sum_{x_{1,1} \bmod d_1\tilde{v}_1} \sideset{}{^*}\sum_{x_{1,2} \bmod d_2},$$

where $$x_1 = x_{1,0} \cdot \frac{d\tilde{v}_1}{d_0}\overline{\frac{d\tilde{v}_1}{d_0}} + x_{1,1} \cdot \frac{d\tilde{v}_1}{d_1\tilde{v}_1}\overline{\frac{d\tilde{v}_1}{d_1\tilde{v}_1}} + x_{1,2} \cdot \frac{d\tilde{v}_1}{d_2}\overline{\frac{d\tilde{v}_1}{d_2}},$$

and

$$\sideset{}{^*}\sum_{x_2 \bmod d\tilde{v}_2} \mapsto \sideset{}{^*}\sum_{x_{2,0} \bmod d_0} \sideset{}{^*}\sum_{x_{2,1} \bmod d_1} \sideset{}{^*}\sum_{x_{2,2} \bmod d_2\tilde{v}_2},$$

where $$x_2 = x_{2,0} \cdot \frac{d\tilde{v}_2}{d_0}\overline{\frac{d\tilde{v}_2}{d_0}} + x_{2,1} \cdot \frac{d\tilde{v}_2}{d_1}\overline{\frac{d\tilde{v}_2}{d_1}} + x_{2,2} \cdot \frac{d\tilde{v}_2}{d_2\tilde{v}_2}\overline{\frac{d\tilde{v}_2}{d_2\tilde{v}_2}}. \tag{4.12}$$

With these notations, we can also split the congruence relation

$$x_1\tilde{v}_2 + x_2\tilde{v}_1 + \tilde{m} \equiv 0 \bmod d\tilde{v}_1\tilde{v}_2$$

into

$$x_{1,0}\tilde{v}_2 + x_{2,0}\tilde{v}_1 + \tilde{m} \equiv 0 \bmod d_0 \implies x_{1,0} \equiv -\overline{\tilde{v}_2}(x_{2,0}\tilde{v}_1 + \tilde{m}) \bmod d_0 \tag{4.13}$$

$$x_{1,1}\tilde{v}_2 + x_{2,1}\tilde{v}_1 + \tilde{m} \equiv 0 \bmod d_1\tilde{v}_1 \implies \overline{x_{1,1}} \equiv -\tilde{v}_2\overline{(x_{2,1}\tilde{v}_1 + \tilde{m})} \bmod d_1\tilde{v}_1 \tag{4.14}$$

$$x_{1,2}\tilde{v}_2 + x_{2,2}\tilde{v}_1 + \tilde{m} \equiv 0 \bmod d_2\tilde{v}_2 \implies \overline{x_{2,2}} \equiv -\tilde{v}_1\overline{(x_{1,2}\tilde{v}_2 + \tilde{m})} \bmod d_2\tilde{v}_2 \tag{4.15}$$

We recall that $(\tilde{m}, \tilde{v}_1\tilde{v}_2) = 1$, $(\tilde{m}, d)|d_0$ and $\tilde{m} = m/d$. We also recall the notations

$d_m = (\tilde{m}, d_0^\infty)$ and $m^* = \frac{m}{dd_m}$. Now, we may split the character sum into the following four clusters of coprime moduli ($m^*h_1h_2,\ d_0d_m,\ d_1\tilde{v}_1$ and $d_2\tilde{v}_2$):

$$\mathfrak{C}_0 = \mathfrak{C}_h \cdot \mathfrak{C}_m \cdot \mathfrak{C}_1 \cdot \mathfrak{C}_2 \tag{4.16}$$

where $\mathfrak{C}_h,\ \mathfrak{C}_m,\ \mathfrak{C}_1$ and $\mathfrak{C}_2$ are defined below.

$$\begin{aligned}\mathfrak{C}_h =& e\left(\frac{u_2\overline{u_1}\cdot\tilde{v}_2\overline{d_m d\tilde{v}_1}}{m^*h_1}\right) e\left(\frac{u_1\overline{u_2}\cdot\tilde{v}_1\overline{d_m d\tilde{v}_2}}{m^*h_2}\right)\\ =& e\left(\frac{u_2\overline{u_1}\cdot\tilde{v}_2\overline{\tilde{v}_1}\cdot\overline{\frac{mh_1}{m^*h_1}}}{m^*h_1}\right) e\left(\frac{u_1\overline{u_2}\cdot\tilde{v}_1\overline{\tilde{v}_2}\cdot\overline{\frac{mh_2}{m^*h_2}}}{m^*h_2}\right).\end{aligned} \tag{4.17}$$

$$\begin{aligned}\mathfrak{C}_m =& e\left(\frac{u_2\overline{u_1}\tilde{v}_2\overline{\frac{mh_1\tilde{v}_1}{d_0d_m}}}{d_0d_m}\right) e\left(\frac{u_1\overline{u_2}\tilde{v}_1\overline{\frac{mh_2\tilde{v}_2}{d_0d_m}}}{d_0d_m}\right)\\ & \sideset{}{^*}\sum_{\substack{x_{1,0} \bmod d_0 \\ x_{1,0}\equiv-\overline{\tilde{v}_2}(x_{2,0}\tilde{v}_1+\tilde{m}) \bmod d_0}} \ \sideset{}{^*}\sum_{x_{2,0} \bmod d_0} e\left(\frac{\overline{x}_{1,0}\overline{h}_1u_2\overline{u_1}\overline{\frac{d\tilde{v}_1}{d_0}}}{d_0}\right) e\left(\frac{\overline{x}_{2,0}\overline{h}_2u_1\overline{u_2}\overline{\frac{d\tilde{v}_2}{d_0}}}{d_0}\right)\\ =& \sideset{}{^*}\sum_{x_{2,0} \bmod d_0} e\left(\frac{u_2\overline{u_1}\tilde{v}_2\overline{\frac{mh_1\tilde{v}_1}{d_0d_m}}}{d_0d_m}\right) e\left(\frac{u_1\overline{u_2}\tilde{v}_1\overline{\frac{mh_2\tilde{v}_2}{d_0d_m}}}{d_0d_m}\right)\\ & \times e\left(-\frac{\overline{(x_{2,0}\tilde{v}_1+\tilde{m})}\tilde{v}_2\overline{h}_1u_2\overline{u_1}\overline{\frac{d\tilde{v}_1}{d_0}}}{d_0}\right) e\left(\frac{\overline{x}_{2,0}\overline{h}_2u_1\overline{u_2}\overline{\frac{d\tilde{v}_2}{d_0}}}{d_0}\right).\end{aligned}$$

Now, we make a change of variable $x_0 m^* \equiv x_{2,0}\tilde{v}_1 \bmod d_0$. We recall the notation $m = d\tilde{m} = dd_m\tilde{m}^*$. Then,

$$\begin{aligned}
\mathfrak{C}_m &= \sum_{x_0 \bmod d_0}^{*} e\left(\frac{u_2\overline{u_1}\tilde{v}_2\overline{\tilde{v}_1}\cdot\overline{\frac{mh_1}{d_0d_m}}}{d_0d_m}\right) e\left(\frac{u_1\overline{u_2}\tilde{v}_1\overline{\tilde{v}_2}\cdot\overline{\frac{mh_2}{d_0d_m}}}{d_0d_m}\right)\\
&\quad\times e\left(-\frac{u_2\overline{u_1}\tilde{v}_2\overline{\tilde{v}_1}\overline{(x_0m^*+d_mm^*)}\,\overline{\frac{dh_1}{d_0}}}{d_0}\right) e\left(\frac{u_1\overline{u_2}\tilde{v}_1\overline{\tilde{v}_2}\bar{x}_0\overline{m^*}\,\overline{\frac{dh_2}{d_0}}}{d_0}\right)\\
&= \sum_{x_0 \bmod d_0}^{*} e\left(\frac{u_2\overline{u_1}\tilde{v}_2\overline{\tilde{v}_1}[1-d_m\cdot\overline{(x_0+d_m)}]\cdot\overline{\frac{mh_1}{d_0d_m}}}{d_0d_m}\right) e\left(\frac{u_1\overline{u_2}\tilde{v}_1\overline{\tilde{v}_2}\cdot(1+\bar{x}_0d_m)\overline{\frac{mh_2}{d_0d_m}}}{d_0d_m}\right)\\
&= \sum_{x_0 \bmod d_0}^{*} e\left(\frac{u_2\overline{u_1}\tilde{v}_2\overline{\tilde{v}_1}\cdot\overline{(1+\bar{x}_0d_m)}\cdot\overline{\frac{mh_1}{d_0d_m}}}{d_0d_m}\right) e\left(\frac{u_1\overline{u_2}\tilde{v}_1\overline{\tilde{v}_2}\cdot(1+\bar{x}_0d_m)\overline{\frac{mh_2}{d_0d_m}}}{d_0d_m}\right)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{C}_1 =& e\left(\frac{u_2\overline{u_1}\tilde{v}_2\overline{\frac{mh_1}{d_1}}}{d_1\tilde{v}_1}\right) \sum_{\substack{x_{1,1} \bmod d_1\tilde{v}_1 \\ \overline{x_{1,1}}\equiv-\tilde{v}_2\overline{(x_{2,1}\tilde{v}_1+\tilde{m})} \bmod d_1\tilde{v}_1}}^{*} \sum_{x_{2,1} \bmod d_1}^{*} e\left(\frac{u_2\overline{u_1}\overline{x_{1,1}}\,\overline{h}_1\cdot\overline{\frac{d}{d_1}}}{d_1\tilde{v}_1}\right) e\left(\frac{u_1\overline{u_2}\overline{x}_{2,1}\overline{h}_2\overline{\frac{\tilde{v}_2d}{d_1}}}{d_1}\right)\\
=& \sum_{x_{2,1} \bmod d_1}^{*} e\left(\frac{u_2\overline{u_1}\tilde{v}_2\overline{\frac{mh_1}{d_1}}}{d_1\tilde{v}_1}\right) e\left(\frac{-u_2\overline{u_1}\tilde{v}_2\overline{(x_{2,1}\tilde{v}_1+\tilde{m})}\cdot\overline{\frac{dh_1}{d_1}}}{d_1\tilde{v}_1}\right) e\left(\frac{u_1\overline{u_2}\overline{\tilde{v}_2}\cdot\bar{x}_{2,1}\overline{\frac{dh_2}{d_1}}}{d_1}\right).
\end{aligned}$$

Now, we change the variable $x_{2,1}$ to x_1, where $x_{2,1} \equiv x_1\tilde{m} \bmod d_1$. Then,

$$\begin{aligned}
\mathfrak{C}_1 &= \sum_{x_1 \bmod d_1}^{*} e\left(\frac{u_2\overline{u_1}\tilde{v}_2\overline{\frac{mh_1}{d_1}}}{d_1\tilde{v}_1}\right) e\left(\frac{-u_2\overline{u_1}\tilde{v}_2\overline{(x_1\tilde{v}_1+1)}\cdot\overline{\frac{mh_1}{d_1}}}{d_1\tilde{v}_1}\right) e\left(\frac{u_1\overline{u_2}\overline{\tilde{v}_2}\cdot\overline{x}_1\overline{\frac{mh_2}{d_1}}}{d_1}\right)\\
&= \sum_{x_1 \bmod d_1}^{*} e\left(\frac{u_2\overline{u_1}\tilde{v}_2[1-\overline{(x_1\tilde{v}_1+1)}]\cdot\overline{\frac{mh_1}{d_1}}}{d_1\tilde{v}_1}\right) e\left(\frac{u_1\overline{u_2}\overline{\tilde{v}_2}\cdot\overline{x}_1\overline{\frac{mh_2}{d_1}}}{d_1}\right)\\
&= \sum_{x_1 \bmod d_1}^{*} e\left(\frac{u_2\overline{u_1}\tilde{v}_2\tilde{v}_1\overline{\overline{x}_1+\tilde{v}_1}\cdot\overline{\frac{mh_1}{d_1}}}{d_1\tilde{v}_1}\right) e\left(\frac{u_1\overline{u_2}\overline{\tilde{v}_2}\cdot\overline{x}_1\overline{\frac{mh_2}{d_1}}}{d_1}\right)\\
&= \sum_{x_1 \bmod d_1}^{*} e\left(\frac{u_2\overline{u_1}\tilde{v}_2\overline{(\overline{x}_1+\tilde{v}_1)}\cdot\overline{\frac{mh_1}{d_1}}}{d_1}\right) e\left(\frac{u_1\overline{u_2}\overline{\tilde{v}_2}\cdot\overline{x}_1\overline{\frac{mh_2}{d_1}}}{d_1}\right)
\end{aligned}$$

Similarly, we derive

$$\mathfrak{C}_2 = e\left(\frac{u_2\overline{u_1}\overline{\tilde{v}_1}\cdot\overline{x}_2\cdot\overline{\frac{mh_1}{d_2}}}{d_2}\right)\sideset{}{^*}\sum_{x_2 \bmod d_2} e\left(\frac{u_1\overline{u_2}\tilde{v}_1(\overline{\bar{x}_2+\tilde{v}_2})\cdot\overline{\frac{mh_2}{d_2}}}{d_2}\right)$$

Hence, we can write

$$\mathfrak{C}_0 = \sideset{}{^*}\sum_{x_0 \bmod d_0}\sideset{}{^*}\sum_{x_1 \bmod d_1}\sideset{}{^*}\sum_{x_2 \bmod d_2} e\left(\frac{\overline{u_1}u_2A_1}{mh_1}\right)e\left(\frac{u_1\overline{u_2}A_2}{mh_2}\right) \tag{4.18}$$

where,

$$A_1 \equiv \begin{cases} \tilde{v}_2\overline{\tilde{v}_1} & (\bmod\ m^*h_1), \\ \tilde{v}_2\overline{\tilde{v}_1}[\overline{1+\bar{x}_0d_m}] & (\bmod\ d_0d_m), \\ \tilde{v}_2(\overline{\bar{x}_1+\tilde{v}_1}) & (\bmod\ d_1), \\ \overline{\tilde{v}_1x_2} & (\bmod\ d_2), \end{cases} \tag{4.19}$$

and

$$A_2 \equiv \begin{cases} \tilde{v}_1\overline{\tilde{v}_2} & (\bmod\ m^*h_2), \\ \tilde{v}_1\overline{\tilde{v}_2}[1+\bar{x}_0d_m] & (\bmod\ d_0d_m), \\ \overline{\tilde{v}_2x_1} & (\bmod\ d_1), \\ \tilde{v}_1(\overline{\bar{x}_2+\tilde{v}_2}) & (\bmod\ d_2). \end{cases} \tag{4.20}$$

4.3.3 Conclusion of the m sum derivation

So, Δ from (4.3) can be written as

$$\begin{aligned}\Delta \ll T^{\varepsilon}\sup_{||\alpha||_2=1}\sup_{X_1\ll\frac{N^{1/3}}{Q}}\sum_{m_1\ll Q} m_1^2\int_{\xi_1\sim X}\sum_{q_1\sim\frac{Q}{m_1}}\sum_{h_1\sim\frac{QT}{N}}\int_{\xi_2\sim X}\sum_{q_2\sim\frac{Q}{m_1}}\sum_{h_2\sim\frac{QT}{N}}\alpha_1\bar{\alpha}_2 \\ \times V\left(\frac{\Xi_2-\Xi_1}{X_1}\right)S_{\mathcal{M}_2},\end{aligned} \tag{4.21}$$

where, for $m \neq 0$ and $X_1 \gg \frac{N^{\varepsilon}}{M_0^{1/3}}$, we have

$$\begin{aligned} S_{\mathcal{M}_2} =& \frac{M_0}{m_1^2 q_1 q_2} \sum_{m \sim \frac{Q^2 X_1}{M_0^{2/3}}} \mathfrak{C} \cdot \mathfrak{J} + O(T^{-A}) \\ =& \frac{M_0}{m_1^2 q_1 q_2} \cdot q_1 q_2 \cdot \sum_{m \sim \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{\sqrt{X_1 M_0^{1/3}}} \cdot \mathfrak{C}_0 \cdot e\left(2\sqrt{\frac{q_1 q_2 m_1^2}{m}} (\Xi_2 - \Xi_1)^{3/2} \right) + O(T^{-A}), \end{aligned} \tag{4.22}$$

and for $m \neq 0$ and $X_1 \ll \frac{N^{\varepsilon}}{M_0^{1/3}}$, we have

$$S_{\mathcal{M}_2} = \frac{M_0}{m_1^2} \sum_{m \ll \frac{Q^2 N^{\varepsilon}}{M_0}} \mathfrak{C}_0. \tag{4.23}$$

We recall,

$$\mathfrak{C}_0 = \sideset{}{^*}\sum_{x_0 \bmod d_0} \sideset{}{^*}\sum_{x_1 \bmod d_1} \sideset{}{^*}\sum_{x_2 \bmod d_2} e\left(\frac{\bar{u}_1 u_2 A_1}{m h_1} + \frac{\bar{u}_2 u_1 A_2}{m h_2} \right), \tag{4.24}$$

where A_1 and A_2 are as defined in (4.19) and (4.20).

Remark. Initial size of $S_{\mathcal{M}_2}$ was $S_{\mathcal{M}_2} \ll \frac{NT}{X} \sim \frac{T^{5/2}}{X}$. We also note that in the generic case, $m \sim \frac{X^{3/2}}{T^{3/4}}$. Final size of $S_{\mathcal{M}_2}$ is

$$S_{\mathcal{M}_2} \ll \frac{N^2}{Q^3} \cdot \frac{X^{3/2}}{T^{3/4}} \cdot \frac{\sqrt{X}}{\sqrt{T}} \sim T\sqrt{X}.$$

In the off-diagonal, after the Poisson summation formula, we save

$$\frac{T^{5/2}/X}{T\sqrt{X}} \sim \frac{T^{3/2}}{X^{3/2}}.$$

Chapter 5

Proof of Theorem 1: Off-diagonal

5.1 Cauchy's inequality and Poisson Summation Formula on q_1

We now come back to (4.3) and modify the sum over m, q_1 and q_2 into

$$\sum_{m\sim M_1}\sum_{q_1\sim Q/m_1}\sum_{q_2\sim Q/m_1} \rightsquigarrow \sum_{d\ll M_1}\sum_{\tilde{m}\sim\frac{Q^2X_1}{M_0^{2/3}d}}\sum_{\substack{\tilde{v}_1,\tilde{v}_2|d^\infty\\(v_1,v_2)=d\\(\tilde{m},u_1\tilde{v}_1u_2\tilde{v}_2)=1}}\sum_{u_1\sim\frac{Q}{d\tilde{v}_1m_1}}\sum_{u_2\sim\frac{Q}{d\tilde{v}_2m_1}},$$

which is just a simple interpretation of the notations defined in Section 4.3.2. We have denoted the range of m as M_1, i.e. $M_1 := \frac{Q^2X_1}{M_0^{2/3}}$. We then apply Cauchy's inequality on all the sums except ξ_2, h_2 and u_2 in (4.3). We will be mindful of the dyadic partition $V\left(\frac{\Xi_2-\Xi_1}{X_1}\right)$ but, for brevity, would often omit it from writing.

We recall that $m = d\cdot d_m\cdot m^*$ where $d_m = (m/d, d^\infty)$ and $(d_m, \tilde{v}_1\tilde{v}_2) = 1$. We recall that, $(dd_m, m^*h_1h_2) = 1$ because $(q_1, h_1) = 1$, $(q_2, h_2) = 1$. Now, we define a set of notations similar to the section 4.3.2 to group the prime factors of $m^*h_1h_2$ to maintain coprimality.

Let $\hat{h}_2 := (h_2, (m^*h_1)^\infty)$ and $\tilde{h}_2 := \frac{h_2}{\hat{h}_2}$. Now, for a fixed $\hat{h}_2$, we denote $\hat{h}_1 := (h_1, (m^*\hat{h}_2)^\infty)$ and $\tilde{h}_1 = \frac{h_1}{\hat{h}_1}$. As a result dd_m, $m^*\hat{h}_1\hat{h}_2$, $\tilde{h}_1$, $\tilde{h}_2$ are mutually coprime.

When $X_1 \gg \frac{N^\varepsilon}{M_0^{1/3}}$, we get

$$\Delta \ll \sup_{||\alpha||_2=1} \sup_{X_1 \ll \frac{N^{1/3}}{Q}} \frac{M_0^{5/6}}{\sqrt{X_1}} \cdot S_0^{1/2} \Bigg(\sum_{m_1 \ll Q} \sum_{d \ll M_1} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \int_{\xi_1 \sim X} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \ll \frac{M_1}{d}} \sum_{\substack{\hat{h}_2 \ll \frac{QT}{N} \\ \hat{h}_2 | (m^* h_1)^\infty}} \sum_{u_1 \sim \frac{Q}{d\tilde{v}_1 m_1}} \left| \int_{\xi_2 \sim X} \sum_{\tilde{h}_2 \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} \alpha_2 \cdot \mathfrak{C}_0 \cdot e\left(2\sqrt{\frac{q_1 q_2 m_1^2}{m}} (\Xi_2 - \Xi_1)^{3/2} \right) \right|^2 \Bigg)^{1/2} \tag{5.1}$$

where

$$S_0 = \sum_{m_1 \ll Q} \sum_{d \ll M_1} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (v_1, v_2)=d}} \sum_{u_1 \sim \frac{Q}{d\tilde{v}_1 m_1}} \int_{\xi_1 \sim X} \sum_{h_1 \sim \frac{QT}{N}} |\alpha(m_1 d\tilde{v}_1 u_1, \xi_1, h_1)|^2 \sum_{\substack{\tilde{m} \ll \frac{M_1}{d} \\ (\tilde{m}, \tilde{v}_1 \tilde{v}_2)=1}} d \sum_{\substack{\hat{h}_2 \ll \frac{QT}{N} \\ \hat{h}_2 | (m^* h_1)^\infty}} 1$$
$$\ll M_1 N^\varepsilon \asymp \frac{Q^2 X_1 N^\varepsilon}{M_0^{2/3}}.$$

Here, we have used the fact that

$$\sum_{m_1} \sum_{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \sum_{u_i \sim \frac{Q}{d\tilde{v}_i m_1}} \int_{\xi_i \sim X} \sum_{h_i \sim \frac{QT}{N}} |\alpha(m_1 d\tilde{v}_i u_i, \xi_i, h_i)|^2 \ll N^\varepsilon. \tag{5.2}$$

Hence,

$$\Delta \ll \sup_{||\alpha||_2=1} \sup_{\frac{N^\varepsilon}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} N^\varepsilon Q \sqrt{M_0} \Bigg(\sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \int_{\xi_1 \sim X} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\substack{\hat{h}_2 \ll \frac{QT}{N} \\ \hat{h}_2 | (m^* h_1)^\infty}} \sum_{u_1 \sim \frac{Q}{d\tilde{v}_1 m_1}} \left| \int_{\xi_2 \sim X} \sum_{\tilde{h}_2 \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} \alpha_2 \cdot \mathfrak{C}_0 \cdot e\left(2\sqrt{\frac{q_1 q_2 m_1^2}{m}} (\Xi_2 - \Xi_1)^{3/2} \right) \right|^2 \Bigg)^{1/2}. \tag{5.3}$$

We open up the absolute square in (5.3), denote all the copies of variables by $'$ in superscript (after opening the absolute square, variable u_2 has two copies. We denote one by u_2 and another one by u_2') and the u_1 sum is of the form

$$\mathcal{E} := \sum_{u_1 \sim \frac{Q}{d\tilde{v}_1 m_1}} \mathfrak{C}_0 \cdot \overline{\mathfrak{C}_0'} \cdot e\left(2\sqrt{\frac{q_1 q_2 m_1^2}{m}} (\Xi_2 - \Xi_1)^{3/2} - 2\sqrt{\frac{q_1 q_2' m_1^2}{m}} (\Xi_2' - \Xi_1)^{3/2} \right).$$

We note that, $q_1 = d \cdot \tilde{v}_1 \cdot u_1$, $q_2 = d \cdot \tilde{v}_2 \cdot u_2$, $q_2' = d \cdot \tilde{v}_2 \cdot u_2'$, and

$$\Xi(\xi, q, h) = \frac{(T + r\xi)^{1/3}}{(2\pi)^{1/3} q^{2/3} h^{1/3}}, \ \Xi_2' = \Xi(\xi_2', q_2' m_1, h_2'), \ \Xi_2 = \Xi(\xi_2, q_2 m_1, h_2).$$

$\mathfrak{C}_0'$ is a copy of $\mathfrak{C}_0$, i.e.

$$\mathfrak{C}_0' = \sideset{}{^*}\sum_{x_0' \bmod d_0} \sideset{}{^*}\sum_{x_1' \bmod d_1} \sideset{}{^*}\sum_{x_2' \bmod d_2} e\left(\frac{\bar{u}_1 u_2' A_1'}{m h_1} + \frac{\bar{u}_2' u_1 A_2'}{m h_2'} \right),$$

where

$$A_1' \equiv \begin{cases} \tilde{v}_2 \overline{\tilde{v}_1} & (\bmod\ m^* h_1), \\ \tilde{v}_2 \overline{\tilde{v}_1} \overline{[1 + \bar{x}_0' d_m]} & (\bmod\ d_0 d_m), \\ \tilde{v}_2 (\overline{\bar{x}_1' + \tilde{v}_1}) & (\bmod\ d_1), \\ \overline{\tilde{v}_1 x_2'} & (\bmod\ d_2), \end{cases}$$

and

$$A_2' \equiv \begin{cases} \tilde{v}_1 \overline{\tilde{v}_2} & (\bmod\ m^* h_2'), \\ \tilde{v}_1 \overline{\tilde{v}_2} [1 + \bar{x}_0' d_m] & (\bmod\ d_0 d_m), \\ \overline{\tilde{v}_2 x_1'} & (\bmod\ d_1), \\ \tilde{v}_1 (\overline{\bar{x}_2' + \tilde{v}_2}) & (\bmod\ d_2). \end{cases}$$

Now, we will apply the Poisson summation formula to this u_1 sum. We note that $\mathfrak{C}_0 \bar{\mathfrak{C}}_0'$ is of modulus $m h_1 h_2 h_2'$. We note that $h_2' = \hat{h}_2 \tilde{h}_2'$. Then, by the Poisson summation formula (2.13), we can write

$$\mathcal{E} = \frac{1}{mh_1h_2h_2'} \cdot \sum_{\tilde{u}_1 \in \mathbb{Z}} \sum_{\substack{\alpha \bmod mh_1h_2h_2' \\ (\alpha, mh_1)=1}} \mathfrak{C}_0(\alpha)\bar{\mathfrak{C}}_0'(\alpha) e_{mh_1h_2h_2'}(\tilde{u}_1\alpha) \times \int_{y_1 \sim \frac{Q}{m_1v_1}} e\left(\frac{-\tilde{u}_1 y_1}{mh_1h_2h_2'}\right)$$
$$\times e\left(2\sqrt{\frac{(v_1y_1)q_2m_1^2}{m}}(\Xi_2 - \Xi_1(y_1))^{3/2} - 2\sqrt{\frac{(v_1y_1)q_2'm_1^2}{m}}(\Xi_2' - \Xi_1(y_1))^{3/2}\right) dy_1.$$

Then, the sum over α gives a character sum $\mathcal{C}$, and the rest of $\mathcal{E}$ gives an oscillatory integral. We also bring the ξ_1 integral inside to get a two-variable oscillatory integral $\mathcal{I}$. Then, we make a change of variable $y = \frac{y_1 Q}{m_1 v_1}$. Once we write out all the notations in their original form, we get

Lemma 5.1.

$$\int_{\xi_1 \sim X} \mathcal{E} \ d\xi_1 = \frac{Q}{m_1v_1mh_1h_2h_2'} \sum_{\tilde{u}_1 \in \mathbb{Z}} \mathcal{C}\, \mathcal{I}, \tag{5.4}$$

where

$$\mathcal{C} = \sideset{}{^*}\sum_{x_0 \bmod d_0} \sideset{}{^*}\sum_{x_1 \bmod d_1} \sideset{}{^*}\sum_{x_2 \bmod d_2} \sideset{}{^*}\sum_{x_0' \bmod d_0} \sideset{}{^*}\sum_{x_1' \bmod d_1} \sideset{}{^*}\sum_{x_2' \bmod d_2}$$
$$\times \sum_{\substack{\alpha \bmod mh_1h_2h_2' \\ (\alpha, mh_1)=1}} e\left(\frac{\bar{\alpha}u_2A_1}{mh_1} + \frac{\bar{u}_2\alpha A_2}{mh_2}\right) e\left(-\frac{\bar{\alpha}u_2'A_1'}{mh_1} - \frac{\bar{u}_2'\alpha A_2'}{mh_2'}\right) e\left(\frac{\tilde{u}_1\alpha}{mh_1h_2h_2'}\right) \tag{5.5}$$

and

$$\mathcal{I} = \int_{\xi_1 \sim X} \int_{y \sim 1} e\left(2\sqrt{\frac{Qyq_2m_1}{m}}\left(\frac{(T+r\xi_2)^{1/3}}{(2\pi q_2^2 m_1^2 h_2)^{1/3}} - \frac{(T+r\xi_1)^{1/3}}{(2\pi (Qy)^2 h_1)^{1/3}}\right)^{3/2}\right.$$
$$\left. - 2\sqrt{\frac{Qyq_2'm_1}{m}}\left(\frac{(T+r\xi_2')^{1/3}}{(2\pi q_2'^2 m_1^2 h_2')^{1/3}} - \frac{(T+r\xi_1)^{1/3}}{(2\pi (Qy)^2 h_1)^{1/3}}\right)^{3/2}\right) \tag{5.6}$$
$$\times e\left(-\frac{\tilde{u}_1 yQ}{dm_1\tilde{v}_1mh_1h_2h_2'}\right) V\left(\frac{\Xi_2 - \Xi_1}{X_1}\right) V\left(\frac{\Xi_2' - \Xi_1}{X_1}\right) dy d\xi_1.$$

Remark. Initial size of $\int_{\xi \sim X} \mathcal{E}$ is XQ.

5.2 Analysis of $\mathcal{C}$:

We recall that $m = d \cdot d_m \cdot m^*$ where $d_m = (m/d, d^\infty)$ and $(d_m, \tilde{v}_1\tilde{v}_2) = 1$. We recall that, $(dd_m, m^*h_1h_2) = 1$ because $(q_1, h_1) = 1$, $(q_2, h_2) = 1$.

We also note that $h_2 = \hat{h}_2\tilde{h}_2$ and $h_2' = \hat{h}_2\tilde{h}_2'$. We also recall that $(\tilde{h}_2\tilde{h}_2', (mh_1)^\infty) = 1$. Now, for a fixed $\hat{h}_2$, we recall the notation $\hat{h}_1 := (h_1, (m^*\hat{h}_2)^\infty)$ and $\tilde{h}_1 = \frac{h_1}{\hat{h}_1}$. As a result $dd_m,\ m^*\hat{h}_1\hat{h}_2^2,\ \tilde{h}_1,\ \tilde{h}_2$ are mutually coprime.

Then, we may write

$$mh_1h_2h_2' = (dd_m) \cdot (m^*\hat{h}_1\hat{h}_2^2) \cdot (\tilde{h}_1) \cdot (\tilde{h}_2\tilde{h}_2')$$

where every group inside the parentheses is mutually coprime. We further denote $\mathcal{D} = dd_m$ and $\mathcal{M} = m^*\hat{h}_1\hat{h}_2^2$.

We recall

$$\mathcal{C} = \sideset{}{^*}\sum_{x_0 \bmod d_0} \sideset{}{^*}\sum_{x_1 \bmod d_1} \sideset{}{^*}\sum_{x_2 \bmod d_2} \sideset{}{^*}\sum_{x_0' \bmod d_0} \sideset{}{^*}\sum_{x_1' \bmod d_1} \sideset{}{^*}\sum_{x_2' \bmod d_2}$$
$$\times \sum_{\substack{\alpha \bmod mh_1h_2h_2' \\ (\alpha, mh_1)=1}} e\left(\frac{\bar{\alpha}u_2A_1}{mh_1} + \frac{\bar{u}_2\alpha A_2}{mh_2}\right) e\left(-\frac{\bar{\alpha}u_2'A_1'}{mh_1} - \frac{\bar{u}_2'\alpha A_2'}{mh_2'}\right) e\left(\frac{\tilde{u_1}\alpha}{mh_1h_2h_2'}\right),$$

where

$$A_1 \equiv \begin{cases} \tilde{v}_2\overline{\tilde{v}_1} & (\mathrm{mod}\ m^*h_1), \\ \tilde{v}_2\overline{\tilde{v}_1}[\overline{1+\bar{x}_0 d_m}] & (\mathrm{mod}\ d_0 d_m), \\ \tilde{v}_2(\overline{\bar{x}_1+\tilde{v}_1}) & (\mathrm{mod}\ d_1), \\ \overline{\tilde{v}_1 x_2} & (\mathrm{mod}\ d_2), \end{cases}$$

and

$$A_2 \equiv \begin{cases} \tilde{v}_1\overline{\tilde{v}_2} & (\mathrm{mod}\ m^*h_2), \\ \tilde{v}_1\overline{\tilde{v}_2}[1+\bar{x}_0 d_m] & (\mathrm{mod}\ d_0 d_m), \\ \overline{\tilde{v}_2 x_1} & (\mathrm{mod}\ d_1), \\ \tilde{v}_1(\overline{\bar{x}_2+\tilde{v}_2}) & (\mathrm{mod}\ d_2). \end{cases}$$

And now we have

$$A_1' \equiv \begin{cases} \tilde{v}_2\overline{\tilde{v}_1} & (\mathrm{mod}\ m^*h_1), \\ \tilde{v}_2\overline{\tilde{v}_1}[\overline{1+\bar{x}_0' d_m}] & (\mathrm{mod}\ d_0 d_m), \\ \tilde{v}_2(\overline{\bar{x}_1'+\tilde{v}_1}) & (\mathrm{mod}\ d_1), \\ \overline{\tilde{v}_1 x_2'} & (\mathrm{mod}\ d_2), \end{cases}$$

and

$$A_2' \equiv \begin{cases} \tilde{v}_1\overline{\tilde{v}_2} & (\mathrm{mod}\ m^*h_2'), \\ \tilde{v}_1\overline{\tilde{v}_2}[1+\bar{x}_0' d_m] & (\mathrm{mod}\ d_0 d_m), \\ \overline{\tilde{v}_2 x_1'} & (\mathrm{mod}\ d_1), \\ \tilde{v}_1(\overline{\bar{x}_2'+\tilde{v}_2}) & (\mathrm{mod}\ d_2). \end{cases}$$

In particular, A_1, A_1', A_2, A_2' modulo $\mathcal{M}, \tilde{h}_1, \tilde{h}_2, \tilde{h}_2'$ does not depend on x_i and x_i' and $A_1 \equiv A_1' \bmod m^*h_1$. Then, by Chinese remainder theorem, we can write

$$\mathcal{C} = \mathcal{C}_d \cdot \mathcal{C}_m \cdot \mathcal{C}_{h1} \cdot \mathcal{C}_{h2}$$

where

$$\mathcal{C}_d = \sideset{}{^*}\sum_{x_0 \bmod d_0} \sideset{}{^*}\sum_{x_1 \bmod d_1} \sideset{}{^*}\sum_{x_2 \bmod d_2} \sideset{}{^*}\sum_{x_0' \bmod d_0} \sideset{}{^*}\sum_{x_1' \bmod d_1} \sideset{}{^*}\sum_{x_2' \bmod d_2} \cdot$$
$$\sideset{}{^*}\sum_{\beta \bmod dd_m} e_{dd_m}(\bar{\beta} \cdot \overline{m^* h_1}(A_1 u_2 - A_1' u_2') + \beta(\bar{u}_2 A_2 \overline{m^* h_2} - \bar{u}_2' A_2' \overline{m^* h_2'} + \tilde{u}_1 \overline{m^* h_1 h_2 h_2'}))$$
$$= \sideset{}{^*}\sum_{x_0 \bmod d_0} \sideset{}{^*}\sum_{x_1 \bmod d_1} \sideset{}{^*}\sum_{x_2 \bmod d_2} \sideset{}{^*}\sum_{x_0' \bmod d_0} \sideset{}{^*}\sum_{x_1' \bmod d_1} \sideset{}{^*}\sum_{x_2' \bmod d_2}$$
$$\times S(\overline{m^* h_1}(A_1 u_2 - A_1' u_2'), \bar{u}_2 A_2 \overline{m^* h_2} - \bar{u}_2' A_2' \overline{m^* h_2'} + \tilde{u}_1 \overline{m^* h_1 h_2 h_2'}; dd_m),$$

$$\mathcal{C}_m = \sideset{}{^*}\sum_{\beta \bmod \mathcal{M}} e_{\mathcal{M}}(\bar{\beta} A_1 \overline{\mathcal{D}\tilde{h}_1} \hat{h}_2^2 (u_2 - u_2') + \beta \overline{\mathcal{D}}(\bar{u}_2 A_2 \hat{h}_1 \hat{h}_2 \overline{\tilde{h}_2} - \bar{u}_2' A_2' \hat{h}_1 \hat{h}_2 \overline{\tilde{h}_2'} + \tilde{u}_1 \overline{\tilde{h}_1 \tilde{h}_2 \tilde{h}_2'}))$$
$$= S(A_1 \overline{\mathcal{D}\tilde{h}_1} \hat{h}_2^2 (u_2 - u_2'), \overline{\mathcal{D}} \hat{h}_1 (\bar{u}_2 A_2 \hat{h}_2 \overline{\tilde{h}_2} - \bar{u}_2' A_2' \hat{h}_2 \overline{\tilde{h}_2'} + \tilde{u}_1 \overline{h_1 \tilde{h}_2 \tilde{h}_2'}); m^* \hat{h}_1 \hat{h}_2^2)$$
$$= S(A_1 \overline{\mathcal{D}\tilde{h}_1} \hat{h}_2^2 (u_2 - u_2'), \overline{\mathcal{D}\tilde{h}_2 \tilde{h}_2'} \hat{h}_1 (\bar{u}_2 A_2 h_2' - \bar{u}_2' A_2' h_2 + \tilde{u}_1 \overline{h_1}); m^* \hat{h}_1 \hat{h}_2^2),$$

$$\mathcal{C}_{h1} = \sideset{}{^*}\sum_{\beta \bmod \tilde{h}_1} e_{\tilde{h}_1}(\bar{\beta} A_1 \overline{\mathcal{D} m^* \hat{h}_1}(u_2 - u_2') + \beta \tilde{u}_1 \overline{m^* \hat{h}_1 h_2 h_2'})$$
$$= S(A_1 \overline{\mathcal{D} m^* \hat{h}_1}(u_2 - u_2'), \tilde{u}_1 \overline{m^* \hat{h}_1 h_2 h_2'}; \tilde{h}_1) \qquad \text{when } \tilde{u}_1 \neq 0$$
$$= \mathfrak{c}_{\tilde{h}_1}(u_2 - u_2') \qquad \text{when } \tilde{u}_1 = 0,$$

where $\mathfrak{c}_q(a) = \sum^*_{n \bmod q} e_q(an)$ is the Ramanujan sum and

$$\mathcal{C}_{h2} = \sum_{\beta \bmod \tilde{h}_2 \tilde{h}_2'} e_{\tilde{h}_2 \tilde{h}_2'}(\beta(A_2 \bar{u}_2 \overline{m \hat{h}_2} \tilde{h}_2' - A_2' \bar{u}_2' \overline{m \hat{h}_2} \tilde{h}_2 + \tilde{u}_1 \overline{m h_1 \hat{h}_2^2}))$$
$$= \tilde{h}_2 \tilde{h}_2' \ \ \delta(A_2 \bar{u}_2 \overline{\hat{h}_2} \tilde{h}_2' - A_2' \bar{u}_2' \overline{\hat{h}_2} \tilde{h}_2 + \tilde{u}_1 \overline{h_1 \hat{h}_2^2} \equiv 0 \bmod \tilde{h}_2 \tilde{h}_2') \tag{5.7}$$
$$= \tilde{h}_2 \tilde{h}_2' \ \ \delta(\bar{u}_2 A_2 h_2' - \bar{u}_2' A_2' h_2 + \tilde{u}_1 \overline{h_1} \equiv 0 \bmod \tilde{h}_2 \tilde{h}_2'). \tag{5.8}$$

To estimate the contribution of the Kloosterman sums $\mathcal{C}_m$ and $\mathcal{C}_{h1}$, we will use Weil's bound for Kloosterman sum (see [IK, Corollary 11.12]):

$$S(a, b; c) \ll_\varepsilon c^\varepsilon (a, b, c)^{1/2} c^{1/2}.$$

To bound the Ramanujan sums, we will use the bound (see [IK, (3.5)])

$$\mathfrak{c}_q(a) \leq (a, q).$$

And we will always bound the character sum $\mathcal{C}_d$ by it's trivial bound

$$\mathcal{C}_d \ll N^{\varepsilon} d^3 d_m.$$

When $\tilde{u}_1 = 0$, the last congruence relation (5.8) forces

$$\tilde{h}_2 | \tilde{h}_2' \text{ and } \tilde{h}_2' | \tilde{h}_2 \implies \tilde{h}_2 = \tilde{h}_2'$$

and

$$u_2 \hat{h}_2 \equiv u_2' \hat{h}_2 \bmod \tilde{h}_2.$$

Now, let us denote the range of $u_3 = u_2 - u_2'$ by $|u_3| \ll \frac{Q_2}{m_1 d \tilde{v}_2}$, where Q_2 might be different from Q. We also note that $dd_m \cdot m^* \hat{h}_1 \hat{h}_2^2 \tilde{h}_1 = m h_1 \hat{h}_2^2$.

1. When $u_3 = 0$ and $\tilde{u}_1 = 0$, we get $\tilde{h}_2 = \tilde{h}_2'$ and we use the trivial bound for all the character sums $\mathcal{C}_d$, $\mathcal{C}_m$, $\mathcal{C}_{h1}$ and $\mathcal{C}_{h2}$ and get

$$|\mathcal{C}| \ll N^{\varepsilon} d^3 d_m \cdot m^* \hat{h}_1 \hat{h}_2^2 \cdot \tilde{h}_1 \cdot \tilde{h}_2 \tilde{h}_2' = N^{\varepsilon} d^2 m h_1 h_2^2. \tag{5.9}$$

2. When $0 \neq |u_3| \ll \frac{Q_2}{m_1 d \tilde{v}_2}$ and $\tilde{u}_1 = 0$, we have $\tilde{h}_2 = \tilde{h}_2'$. We use the trivial bound in $\mathcal{C}_d$ and $\mathcal{C}_{h2}$. We use Weil's bound in $\mathcal{C}_m$

$$|\mathcal{C}_m| \ll N^{\varepsilon} \sqrt{m^* \hat{h}_1 \hat{h}_2^2} \cdot (\hat{h}_2^2 u_3, m^* \hat{h}_1 \hat{h}_2^2)^{1/2}$$

and note that $\mathcal{C}_{h1}$ is a Ramanujan sum $\mathfrak{c}_{\tilde{h}_1}(u_3)$. Thus, we get

$$\sum_{0 \neq |u_3| \ll \frac{Q_2}{m_1 d \tilde{v}_2}} |\mathcal{C}| \ll N^{\varepsilon} d^3 d_m \sqrt{m^* \hat{h}_1} h_2^2 \cdot \frac{Q_2}{m_1 d \tilde{v}_2}. \tag{5.10}$$

3. Combining (5.9) and (5.10), for $\tilde{u}_1 = 0$, we get $\tilde{h}_2 = \tilde{h}_2'$ and

$$\sum_{|u_3| \ll \frac{Q_2}{m_1 d\tilde{v}_2}} |\mathcal{C}| \ll N^{\varepsilon} d^3 d_m \sqrt{m^* \hat{h}_1} h_2 h_2' \left(\frac{Q_2}{m_1 d\tilde{v}_2} + \sqrt{m^* \hat{h}_1} \tilde{h}_1 \right). \tag{5.11}$$

4. When $u_3 = 0$ but $0 \neq |\tilde{u}_1| \ll Q_3$ (but small), we choose to use a weaker bound of the form

$$\sum_{0 \neq |\tilde{u}_1| \ll Q_3} |\mathcal{C}| \ll N^{\varepsilon} d^3 d_m m^* \hat{h}_1 h_2 h_2' \cdot Q_3, \tag{5.12}$$

where we have only used the Ramanujan sum structure of $\mathcal{C}_{h1}$.

5. When u_3 and $\tilde{u}_1$ both are small but both are non-zero, we use

$$\sum_{0 \neq |u_3| \ll \frac{Q_2}{m_1 d\tilde{v}_2}} \sum_{0 \neq |\tilde{u}_1| \ll Q_3} |\mathcal{C}| \ll N^{\varepsilon} d^3 d_m \sqrt{m^* h_1} h_2 h_2' \cdot \frac{Q_2}{m_1 d\tilde{v}_2} \cdot Q_3. \tag{5.13}$$

6. For u_3 small and $\tilde{u}_1 \neq 0$, combining (5.12) and (5.13), we will use

$$\sum_{|u_3| \ll \frac{Q_2}{m_1 d\tilde{v}_2}} \sum_{0 \neq |\tilde{u}_1| \ll Q_3} |\mathcal{C}| \ll N^{\varepsilon} d^3 d_m \sqrt{m^* \hat{h}_1} h_2 h_2' \cdot Q_3 \cdot \left(\frac{Q_2 \sqrt{\tilde{h}_1}}{m_1 d\tilde{v}_2} + \sqrt{m^* \hat{h}_1} \right). \tag{5.14}$$

7. Now, we will consider the case of non-zero frequency ($\tilde{u}_1 \neq 0$). Suppose, $u_3 = u_2 - u_2' \neq 0$ and let $|u_3| \ll \frac{Q_2}{m_1 \tilde{v}_2 d}$ where Q_2 might be different from Q. Then, we

will use the following bounds for the character sums:

$$\begin{aligned}
|\mathcal{C}_d| \ll & d^3 d_m, \\
|\mathcal{C}_m| \ll & N^\varepsilon \sqrt{m^* \hat{h}_1 \hat{h}_2^2} (\hat{h}_2^2 u_3, \hat{h}_1(\bar{u}_2 A_2 h_2' - \bar{u}_2' A_2' h_2 + \tilde{u}_1 \overline{h_1}); m^* \hat{h}_1 \hat{h}_2^2)^{1/2} \\
\ll & N^\varepsilon \sqrt{m^* \hat{h}_1 \hat{h}_2^2} \sum_{\substack{\ell_1 | m^* \hat{h}_1 \hat{h}_2^2 \\ \ell_1 | \hat{h}_2^2 u_3 \\ \hat{h}_1(\bar{u}_2 A_2 h_2' - \bar{u}_2' A_2' h_2 + \tilde{u}_1 \overline{h_1}) \equiv 0 \bmod \ell_1}} \ell_1^{1/2}, \\
|\mathcal{C}_{h1}| \ll & N^\varepsilon \sqrt{\tilde{h}_1} (u_3, \tilde{h}_1)^{1/2} \ll N^\varepsilon \sqrt{\tilde{h}_1} \sum_{\substack{\ell_2 | \tilde{h}_1 \\ \ell_2 | u_3}} \ell_2^{1/2}, \\
|\mathcal{C}_{h2}| \ll & \tilde{h}_2 \tilde{h}_2' \delta(\bar{u}_2 A_2 h_2' - \bar{u}_2' A_2' h_2 + \tilde{u}_1 \overline{h_1} \equiv 0 \bmod \tilde{h}_2 \tilde{h}_2').
\end{aligned}$$

Let Q_1 be a positive parameter which may depend on Q, X and T. Then,

$$\begin{aligned}
& \sum_{0 < |u_3| \ll \frac{Q_2}{m_1 \bar{v}_2 d}} u_3^\beta \sum_{u_1 \sim Q_1} |\mathcal{C}_d \mathcal{C}_m \mathcal{C}_{h1} \mathcal{C}_{h2}| \\
\ll & N^\varepsilon d^3 d_m \sqrt{m^* h_1 \hat{h}_2^2} \tilde{h}_2 \tilde{h}_2' \sum_{\ell_1 | m^* h_1 \hat{h}_2^2} \ell_1^{1/2} \sum_{\ell_2 | \tilde{h}_1} \ell_2^{1/2} \\
& \times \sum_{\substack{0 < |u_3| \ll \frac{Q_2}{m_1 \bar{v}_2 d} \\ \ell_2 \cdot \frac{\ell_1}{(\ell_1, \hat{h}_2^2)} | u_3}} u_3^\beta \sum_{\substack{u_1 \sim Q_1 \\ \bar{u}_2 A_2 h_2' - \bar{u}_2' A_2' h_2 + \tilde{u}_1 \overline{h_1} \equiv 0 \bmod \tilde{h}_2 \tilde{h}_2' \cdot \frac{\ell_1}{(\ell_1, \hat{h}_1)}}} 1 \\
\ll & N^\varepsilon d^3 d_m \sqrt{m^* h_1 \hat{h}_2^2} \tilde{h}_2 \tilde{h}_2' \sum_{\ell_1 | m^* h_1 \hat{h}_2^2} \ell_1^{1/2} \sum_{\ell_2 | \tilde{h}_1} \ell_2^{1/2} \\
& \times \sum_{\substack{0 < |u_3| \ll \frac{Q_2}{m_1 \bar{v}_2 d} \\ \ell_2 \cdot \frac{\ell_1}{(\ell_1, \hat{h}_2^2)} | u_3}} u_3^\beta \left(\frac{Q_1 (\ell_1, \hat{h}_1)}{\ell_1 \tilde{h}_2 \tilde{h}_2'} + 1 \right).
\end{aligned}$$

We note that, for $\beta \geq -1$,

$$\sum_{\substack{0 < n < N \\ d | n}} n^\beta = d^\beta \sum_{0 < m < N/d} m^\beta \ll N^\varepsilon d^\beta \cdot (N/d)^{\beta+1} \ll_\varepsilon N^\varepsilon N^{\beta+1} d^{-1}.$$

Hence,

$$\begin{aligned}
&\sum_{0<|u_3|\ll\frac{Q_2}{m_1\tilde{v}_2 d}} u_3^\beta \sum_{u_1\sim Q_1} |\mathcal{C}_d\mathcal{C}_m\mathcal{C}_{h1}\mathcal{C}_{h2}| \\
\ll& N^\varepsilon d^3 d_m \sqrt{m^* h_1 \hat{h}_2^2}\tilde{h}_2\tilde{h}_2' \sum_{\ell_1|m^*\hat{h}_1\hat{h}_2^2} \ell_1^{1/2} \sum_{\ell_2|\tilde{h}_1} \ell_2^{1/2} \\
&\times \left(\ell_2\cdot\frac{\ell_1}{(\ell_1,\hat{h}_2^2)}\right)^{-1}\left(\frac{Q_2}{m_1\tilde{v}_2 d}\right)^{\beta+1}\left(\frac{Q_1(\ell_1,\hat{h}_1)}{\ell_1\tilde{h}_2\tilde{h}_2'}+1\right). \\
\ll& N^\varepsilon d^3 d_m \sqrt{m^* h_1 \hat{h}_2^2}\tilde{h}_2\tilde{h}_2' \cdot \left(\frac{Q_2}{m_1\tilde{v}_2 d}\right)^{\beta+1} \sum_{\ell_1|m^*\hat{h}_1\hat{h}_2^2}\left(\frac{Q_1(\ell_1,\hat{h}_1)(\ell_1,\hat{h}_2^2)}{\ell_1^{3/2}\tilde{h}_2\tilde{h}_2'}+\frac{(\ell_1,\hat{h}_2^2)}{\ell_1^{1/2}}\right) \\
\ll& N^\varepsilon d^3 d_m \sqrt{m^* h_1 \hat{h}_2^2}\tilde{h}_2\tilde{h}_2' \cdot \left(\frac{Q_2}{m_1\tilde{v}_2 d}\right)^{\beta+1}\left(\frac{Q_1\sqrt{\hat{h}_1}}{\tilde{h}_2\tilde{h}_2'}+\sqrt{\hat{h}_2^2}\right) \\
\ll& N^\varepsilon d^3 d_m \sqrt{m^* h_1}\cdot\left(\frac{Q_2}{m_1\tilde{v}_2 d}\right)^{\beta+1}\left(Q_1\sqrt{\hat{h}_1\hat{h}_2^2}+h_2h_2'\right)
\end{aligned}$$

So, when $\tilde{u}_1 \neq 0$, $u_3 = u_2 - u_2' \neq 0$ and $\beta \geq -1$, we have

$$\sum_{0<|u_3|\ll\frac{Q_2}{m_1\tilde{v}_2 d}} u_3^\beta \sum_{u_1\sim Q_1} |\mathcal{C}| \ll N^\varepsilon d^3 d_m \sqrt{m^* h_1} h_2 h_2' \left(\frac{Q_2}{m_1\tilde{v}_2 d}\right)^{\beta+1}\left(\frac{Q_1\sqrt{\hat{h}_1\hat{h}_2^2}}{h_2h_2'}+1\right). \tag{5.15}$$

Here, we have assumed that Q_1 does not depend on u_3. But if it appears with a non-negative exponent, we may replace u_3 by its upper bound $\frac{Q_2}{m_1\tilde{v}_1 d}$ in Q_1. The bound stays the same if the role of u_2 and u_2' is reversed.

Remark. In the generic case, the effective size of $\mathcal{C}$ on average is $\mathcal{C} \ll \sqrt{mh_1} \ll \frac{X}{\sqrt{T}}$.

5.3 Analysis of $\mathcal{I}$:

We will now analyze the two variables oscillatory integral $\mathcal{I}$ of (5.6).

$$\begin{aligned}\mathcal{I} = \int_{\xi_1\sim X}\int_{y\sim 1} e\Bigg(2\sqrt{\frac{Qyq_2m_1}{m}}\left(\frac{(T+r\xi_2)^{1/3}}{(2\pi q_2^2m_1^2h_2)^{1/3}}-\frac{(T+r\xi_1)^{1/3}}{(2\pi(Qy)^2h_1)^{1/3}}\right)^{3/2}\\ -2\sqrt{\frac{Qyq_2'm_1}{m}}\left(\frac{(T+r\xi_2')^{1/3}}{(2\pi q_2'^2m_1^2h_2')^{1/3}}-\frac{(T+r\xi_1)^{1/3}}{(2\pi(Qy)^2h_1)^{1/3}}\right)^{3/2}\Bigg)\\ \times e\left(-\frac{\tilde{u_1}yQ}{dm_1\tilde{v}_1mh_1h_2h_2'}\right)V\left(\frac{\Xi_2-\Xi_1}{X_1}\right)V\left(\frac{\Xi_2'-\Xi_1}{X_1}\right)dyd\xi_1. \qquad (5.16)\end{aligned}$$

We recall that $\Xi_i \sim \frac{T^{1/3}}{(Q^2\cdot\frac{QT}{N})^{1/3}} \sim \frac{N^{1/3}}{Q}$ and the notation $\int_{y\sim 1}$ represents the integral $\int V(y)dy$, where $V(y)$ is a dyadic smooth function supported in $[1,2]$. We also derive the partial derivatives of Ξ_1 with respect to ξ_1 and y

$$\begin{aligned}\frac{\partial\Xi_1}{\partial y} &= \frac{-2}{3}\cdot\frac{(T+r\xi_1)^{1/3}}{(2\pi Q^2h_1)^{1/3}y^{5/3}} = \frac{-2\Xi_1}{3y},\\ \frac{\partial\Xi_1}{\partial\xi_1} &= \frac{1}{3}\cdot\frac{r\cdot(T+r\xi_1)^{-2/3}}{(2\pi(Qy)^2h_1)^{1/3}} = \frac{r\Xi_1}{3(T+r\xi_1)}.\end{aligned}$$

Then, we note that

$$\frac{\partial^i\partial^j}{\partial\xi_1^i\partial y^j}V\left(\frac{\Xi_2-\Xi_1(\xi_1,y)}{X_1}\right) \ll \frac{1}{X_1^{i+j}}\frac{\partial^i\partial y^j\Xi_1}{\partial\xi_1^i\partial y^j} \sim \left(\frac{r\Xi_1}{X_1T}\right)^i\left(\frac{\Xi_1}{X_1y}\right)^j \sim \left(\frac{N^{1/3}}{X_1XQ}\right)^i\left(\frac{N^{1/3}}{X_1Q}\right)^j.$$

Let us denote $Y_1 = \frac{X_1}{N^{1/3}/Q}$. Then, $Y_1 \le 1$. Now, let us make the following changes of variables

$$\xi_1 \mapsto XY_1z_1 \quad\text{and}\quad y\mapsto y_1Y_1, \qquad d\xi_1dy = XY_1^2dz_1dy_1.$$

Before applying the lemma, we will recall and define a few notations to de-clutter the upcoming calculations.

$$\begin{aligned}
\Xi_1 &= \frac{(T+rXY_1z_1)^{1/3}}{(2\pi(QY_1y_1)^2h_1)^{1/3}},\\
\Xi_2 &= \frac{(T+r\xi_2)^{1/3}}{(2\pi q_2^2m_1^2h_2)^{1/3}},\\
\Xi_2' &= \frac{(T+r\xi_2')^{1/3}}{(2\pi q_2'^2m_1^2h_2')^{1/3}},\\
M(z_1,y_1) &:= \frac{(T+r\xi_2)^{1/3}}{(2\pi q_2^2m_1^2h_2)^{1/3}} - \frac{(T+rXY_1z_1)^{1/3}}{(2\pi(QY_1y_1)^2h_1)^{1/3}} = \Xi_2 - \Xi_1,\\
M'(z_1,y_1) &:= \frac{(T+r\xi_2')^{1/3}}{(2\pi q_2'^2m_1^2h_2')^{1/3}} - \frac{(T+rXY_1z_1)^{1/3}}{(2\pi(QY_1y_1)^2h_1)^{1/3}} = \Xi_2' - \Xi_1,\\
L_k &:= \sqrt{q_2m_1}M^k - \sqrt{q_2'm_1}M'^k,\\
q_1 &= \frac{\tilde{u_1}}{m_1\tilde{v}_1d},\\
F &= \frac{q_1Q}{mh_1h_2h_2'} = \tilde{u}_1(m_1\tilde{v}_1dmh_1h_2h_2')^{-1}.
\end{aligned} \tag{5.17}$$

With these notations, let us also define the smooth function

$$U(z_1,y_1) := V(Y_1z_1)V(Y_1y_1)V\left(\frac{\Xi_2-\Xi_1}{X_1}\right)V\left(\frac{\Xi_2'-\Xi_1}{X_1}\right).$$

$U(z_1,y_1)$ is supported in a compact set $K \subset [Y_1^{-1}, 2Y_1^{-1}] \times [Y_1^{-1}, 2Y_1^{-1}]$. We also note that

$$\begin{aligned}
\frac{\partial \Xi_1}{\partial y_1} &= \frac{-2}{3}\cdot\frac{(T+rXY_1z_1)^{1/3}}{(2\pi Q^2Y_1^2h_1)^{1/3}y_1^{5/3}} = \frac{-2\Xi_1}{3y_1},\\
\frac{\partial \Xi_1}{\partial z_1} &= \frac{1}{3}\cdot\frac{rXY_1\cdot(T+rXY_1z_1)^{-2/3}}{(2\pi(QY_1y)^2h_1)^{1/3}} = \frac{rXY_1\Xi_1}{3(T+rXY_1z_1)}.
\end{aligned}$$

Hence,

$$\frac{\partial^i\partial^j}{\partial z_1^i\partial z_1^j}V\left(\frac{\Xi_2-\Xi_1}{X_1}\right) \ll \frac{1}{X_1^{i+j}}\frac{\partial^i\partial^j\Xi_1}{\partial z_1^i\partial z_1^j} \ll \left(\frac{Y_1\Xi_1}{X_1}\right)^i\left(\frac{\Xi_1}{X_1y_1}\right)^j \ll 1 \tag{5.18}$$

by the observation $Y_1 \sim \frac{X_1}{\Xi_1}$ and $y_1 \sim Y_1^{-1}$. This implies

$$\frac{\partial^i\partial^j}{\partial z_1^i\partial z_1^j}U(z_1,y_1) \ll T^{(i+j)\varepsilon} \tag{5.19}$$

With the above set of notations we can write

$$\mathcal{I} = \frac{XX_1^2Q^2}{N^{2/3}} \int_{z_1}\int_{y_1} e(f(z_1,y_1))U(z_1,y_1)dz_1dy_1 \tag{5.20}$$

where the phase function is

$$f(z_1,y_1) = 2\sqrt{QY_1y_1/m}L_{3/2} - FY_1y_1.$$

Then, we deduce that

Lemma 5.2. *Let us denote*

$$Q_1 = m_1\tilde{v}_1 d \cdot \frac{mh_1h_2h_2'}{Q} \cdot \frac{m_1(q_2'-q_2)X_1^{3/2}}{\sqrt{m}}, \qquad Y_1 = \frac{X_1Q}{N^{1/3}}. \tag{5.21}$$

When $|q_2 - q_2'| \gg \frac{QN^\varepsilon}{m_1M_0^{1/3}Y_1X_1}$, $\mathcal{I}$ *(5.20) is negligibly small unless*

$$\tilde{u}_1 \sim Q_1$$

In that range, we have $\Xi_2' - \Xi_2 \sim \frac{(q_2-q_2')X_1}{Q/m_1}$ *and when* $\tilde{u}_1 \neq 0$,

$$\mathcal{I} \asymp X \cdot \frac{Q^2(q_2-q_2')^{-1}}{m_1M_0^{1/3}N^{1/3}} \cdot e\left(\frac{h_1h_2h_2'q_2q_2'm_1(\Xi_2'-\Xi_2)^3}{\tilde{u}_1(m_1\tilde{v}_1d)^{-1}(q_2-q_2')}\right) V\left(\frac{\tilde{u}_1}{Q_1}\right) V\left(\frac{\Xi_2'-\Xi_2}{\frac{(q_2-q_2')X_1}{Q/m_1}}\right) + O\left(\frac{XQ^2|q_2-q_2'|^{-2}}{M_0^{2/3}X_1^2m_1^2}\right). \tag{5.22}$$

Proof. We will use Lemma 2.7 and Lemma 2.8 to prove this lemma. We note that the main term in (2.21) is independent of the choice of λ. So we will choose it at the end. We also note that in the notation of Lemma 2.8, $U(z_1,y_1)$ is supported in some compact subset $K \subset [Y_1^{-1}, 2Y_1^{-1}]^2$ and satisfies $\partial_1^{j_1}\partial_2^{j_2}U \ll T^{(j_1+j_2)\varepsilon}$ for any $j_1, j_2 \geq 0$.

Then, we will calculate the first-order partial derivatives of f. For convenience, we evaluate the first derivatives of the functions above with respect to z_1 and y_1:

$$\begin{aligned}
\frac{\partial \Xi_1}{\partial y_1} =& \frac{-2}{3} \cdot \frac{(T + rXY_1z_1)^{1/3}}{(2\pi Q^2 Y_1^2 h_1)^{1/3} y_1^{5/3}} = \frac{-2\Xi_1}{3y_1}, \\
\frac{\partial \Xi_1}{\partial z_1} =& \frac{1}{3} \cdot \frac{rXY_1 \cdot (T + rXY_1z_1)^{-2/3}}{(2\pi(QY_1y)^2 h_1)^{1/3}} = \frac{rXY_1\Xi_1}{3(T + rXY_1z_1)}, \\
\frac{\partial L_k}{\partial y_1} =& kL_{k-1} \cdot \frac{-\partial \Xi_1}{\partial y_1} = kL_{k-1} \cdot \frac{2\Xi_1}{3y_1}, \\
\frac{\partial L_k}{\partial z_1} =& kL_{k-1} \cdot \frac{-\partial \Xi_1}{\partial z_1} = -kL_{k-1} \cdot \frac{rXY_1\Xi_1}{3(T + rXY_1z_1)}
\end{aligned}$$

We recall,

$$f(z_1, y_1) = 2\sqrt{QY_1y_1/m} L_{3/2} - FY_1y_1.$$

Then,

$$\begin{aligned}
\frac{\partial f}{\partial y_1} &= \sqrt{\frac{QY_1}{my_1}} L_{3/2} + 2\sqrt{\frac{QY_1}{my_1}} L_{1/2} \times \Xi_1 - FY_1, \\
\frac{\partial f}{\partial z_1} &= -\sqrt{\frac{QY_1y_1}{m}} L_{1/2} \times \frac{rXY_1\Xi_1}{(T + rXY_1z_1)}.
\end{aligned} \tag{5.23}$$

Let $S \subset K$, where $\frac{\partial f}{\partial z_1} \gg N^{\varepsilon}$. In any subset of S, the integral is negligibly small by 2.7. Now, in S^c,

$$\frac{\partial f}{\partial z_1} \ll N^{\varepsilon} \implies |L_{1/2}| \ll N^{\varepsilon} \frac{T\sqrt{m}}{rXY_1\Xi_1\sqrt{Q}} \sim N^{\varepsilon} \frac{\sqrt{m}}{X_1\sqrt{Q}}.$$

As $M, M' \sim X_1$, we note that

$$\sqrt{q_2m_1} M^{1/2} + \sqrt{q_2'm_1} M'^{1/2} \sim \sqrt{QX_1}.$$

Then, multiplying $L_{1/2}$ (5.17) by $\frac{1}{m_1}(\sqrt{q_2m_1}M^{1/2}+\sqrt{q_2'm_1}M'^{1/2})$, we have

$$\begin{aligned}\frac{1}{m_1}|L_{1/2}(\sqrt{q_2m_1}M^{1/2}+\sqrt{q_2'm_1}M'^{1/2})| &= |q_2M-q_2'M'|\\ &= |q_2(\Xi_2-\Xi_1)-q_2'(\Xi_2'-\Xi_1)|\\ &= |(q_2\Xi_2-q_2'\Xi_2')-(q_2-q_2')\Xi_1|\\ &= |(q_2-q_2')(\Xi_2-\Xi_1)-q_2'(\Xi_2'-\Xi_2)|\\ &\ll N^{\varepsilon}\frac{\sqrt{m}}{X_1\sqrt{Q}m_1}\cdot\sqrt{QX_1}\sim\frac{QN^{\varepsilon}}{m_1M_0^{1/3}}.,\end{aligned}$$

and we already had $X_1\le\Xi_2-\Xi_1\le 2X_1$. Thus, we have

$$\frac{(q_2-q_2')X_1}{q_2'}-\frac{N^{\varepsilon}}{M_0^{1/3}}\ll\Xi_2'-\Xi_2\ll\frac{2(q_2-q_2')X_1}{q_2'}+\frac{N^{\varepsilon}}{M_0^{1/3}}. \tag{5.24}$$

We emphasize on the fact that when $|q_2-q_2'|\gg\frac{QN^{2\varepsilon}}{m_1M_0^{1/3}Y_1X_1}$, $\Xi_2'-\Xi_2\sim\frac{(q_2-q_2')X_1}{Q/m_1}$. In particular, $\Xi_2'-\Xi_2$ and q_2-q_2' always have the same sign. With this condition, we will analyze $\frac{\partial f}{\partial y}$ (5.23). We note that

$$\begin{aligned}L_{3/2}+2L_{1/2}\Xi_1 =&\sqrt{q_2m_1}\cdot\Xi_2M^{1/2}-\sqrt{q_2'm_1}\cdot\Xi_2'M'^{1/2}+L_{1/2}\Xi_1\\ =&(\Xi_2+\Xi_1)(\sqrt{q_2m_1}M^{1/2}-\sqrt{q_2'm_1}M'^{1/2})-(\Xi_2'-\Xi_2)\sqrt{q_2'm_1}M'^{1/2}\\ =&(\Xi_2+\Xi_1)L_{1/2}-(\Xi_2'-\Xi_2)\sqrt{q_2'm_1}M'^{1/2}\end{aligned} \tag{5.25}$$

Hence, in S^c, the integral is negligibly small unless $\frac{\partial f}{\partial y}\ll N^{\varepsilon}$, i.e.

$$\left|\frac{Y_1\sqrt{Q}}{\sqrt{m}}\cdot(\Xi_2+\Xi_1)L_{1/2}-\frac{Y_1\sqrt{Q}}{\sqrt{m}}\cdot(\Xi_2'-\Xi_2)\sqrt{q_2'm_1}M'^{1/2}-FY_1\right|\ll N^{\varepsilon}. \tag{5.26}$$

As

$$|L_{1/2}|\ll N^{\varepsilon}\frac{\sqrt{m}}{X_1\sqrt{Q}}\implies\frac{Y_1\sqrt{Q}}{\sqrt{m}}\cdot(\Xi_2+\Xi_1)L^{1/2}\ll N^{\varepsilon},$$

in S^c, the integral is negligibly small unless

$$\left|\frac{Y_1\sqrt{Q}}{\sqrt{m}}\cdot(\Xi_2'-\Xi_2)\sqrt{q_2'm_1}M'^{1/2}+FY_1\right|\ll N^{\varepsilon}$$

If $|q_2 - q_2'| \gg \frac{QN^{2\varepsilon}}{m_1 M_0^{1/3} Y_1 X_1}$, we have

$$\frac{Y_1\sqrt{Q}}{\sqrt{m}} \cdot (\Xi_2' - \Xi_2)\sqrt{q_2' m_1} M'^{1/2} \sim \frac{Y_1 Q\sqrt{X_1} M_0^{1/3}}{Q\sqrt{X_1}} \frac{m_1(q_2 - q_2')X_1}{Q} \gg N^{2\varepsilon}.$$

Thus, the integral is negligibly small unless

$$F \sim \frac{m_1(q_2' - q_2)X_1^{3/2}}{\sqrt{m}}. \tag{5.27}$$

Once we recall $F = \tilde{u}_1 Q(m_1\tilde{v}_1 dmh_1h_2h_2')^{-1}$, we observe that the integral is negligibly small unless $\tilde{u}_1 \sim Q_1$ (5.21) provided $|q_2 - q_2'| \gg \frac{QN^{2\varepsilon}}{m_1 M_0^{1/3} Y_1 X_1}$.

Now, let us assume $\tilde{u}_1 \neq 0$. Then, in this range, we have a unique stationary point. Let us denote the zero of f' by (z_0, y_0). Then,

$$\begin{aligned} &\frac{\partial f}{\partial z_1} = 0 \iff L_{1/2} = 0 \iff q_2 M = q_2' M' \iff \Xi_1(z_0) = \frac{q_2\Xi_2 - q_2'\Xi_2'}{q_2 - q_2'}, \\ \text{and} \quad &\frac{\partial f}{\partial y_1} = 0 \iff \sqrt{\frac{QY_1}{my_0}} L_{3/2} = FY_1 \iff \sqrt{y_0} = \sqrt{\frac{Q}{mY_1}} F^{-1} L_{3/2}. \end{aligned} \tag{5.28}$$

From (5.28), we explicitly calculate y_0:

$$\begin{aligned} M^{3/2}(z_0, y_0) &= \frac{q_2'^{3/2}(\Xi_2' - \Xi_2)^{3/2}}{(q_2 - q_2')^{3/2}}, \\ L_{3/2}(z_0, y_0) &= -\frac{\sqrt{q_2 q_2' m_1}(\Xi_2' - \Xi_2)^{3/2}}{(q_2 - q_2')^{1/2}}, \\ y_0^{1/2} &= -\frac{\sqrt{Qm}h_1h_2h_2'}{q_1 Q\sqrt{Y_1}} \times \frac{\sqrt{q_2 q_2' m_1}(\Xi_2' - \Xi_2)^{3/2}}{(q_2 - q_2')^{1/2}}. \end{aligned} \tag{5.29}$$

Then, we calculate $f(z_0, y_0)$:

$$\begin{aligned} f(z_0, y_0) &= 2\sqrt{QY_1 y_0/m} L_{3/2} - FY_1 y_0 = FY_1 y_0, \\ &= \frac{h_1h_2h_2' q_2 q_2' m_1(\Xi_2' - \Xi_2)^3}{\tilde{u}_1(m_1\tilde{v}_1 d)^{-1}(q_2 - q_2')}. \end{aligned}$$

As $y_0^{1/2} \sim Y_1^{-1/2}$, we must have

$$\tilde{u}_1 \sim m_1\tilde{v}_1 d \cdot \frac{\sqrt{m}h_1h_2h_2' X_1^{3/2} m_1(q_2' - q_2)}{Q}.$$

Now, we will calculate the Hessian matrix

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial z_1^2} & \frac{\partial^2 f}{\partial z_1 \partial y} \\ \frac{\partial^2 f}{\partial y \partial z_1} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}.$$

We derive

$$\begin{aligned}
\frac{\partial^2 f}{\partial z_1 \partial y_1} &= \sqrt{\frac{QY_1}{my_1}} \times \frac{rXY_1\Xi_1}{6(T + rXY_1z_1)} \times (L_{1/2} - 2\Xi_1 L_{-1/2}), \\
\frac{\partial^2 f}{\partial y_1^2} &= -\sqrt{\frac{QY_1}{my_1}} \times \frac{1}{y_1}\left(\frac{L_{3/2}}{2} + \frac{4L_{1/2}}{3}\Xi_1 - \frac{2L_{-1/2}\Xi_1^2}{3}\right), \\
\frac{\partial^2 f}{\partial z_1^2} &= \sqrt{\frac{QY_1y_1}{m}} \times \frac{r^2X^2Y_1^2}{6(T + rXY_1z_1)^2}(4\Xi_1 L_{1/2} + \Xi_1^2 L_{-1/2}), \\
\det(H_f(z_1, y_1)) &= \frac{-QY_1}{12my_1} \times \frac{r^2X^2Y_1^2}{(T + rXY_1z_1)^2} \times [\Xi_1^2 L_{3/2} L_{-1/2} \\
&\quad + L_{1/2}\Xi_1(L_{3/2} + 33L_{1/2}\Xi_1 - 12L_{-1/2}\Xi_1^2)].
\end{aligned}$$

At the stationary point (z_0, y_0), we have $L_{1/2} = 0$. Hence,

$$\det(H_f(z_0, y_0)) = \frac{-QY_1}{12my_0} \times \frac{r^2X^2Y_1^2}{(T + rXY_1z_0)^2} \times (\Xi_1^2 L_{3/2} L_{-1/2}).$$

We will try to simplify the Hessian further using (5.29). We note that

$$\begin{aligned}
M(z_0, y_0) &= \frac{q_2'(\Xi_2' - \Xi_2)}{(q_2 - q_2')}, \\
M'(z_0, y_0) &= \frac{q_2(\Xi_2' - \Xi_2)}{(q_2 - q_2')}, \\
L_{-1/2}(z_0, y_0) &= \frac{\sqrt{m_1}(q_2 - q_2')^{3/2}}{\sqrt{q_2 q_2'}(\Xi_2' - \Xi_2)^{1/2}} \\
L_{3/2}(z_0, y_0) &= -\frac{\sqrt{q_2 q_2' m_1}(\Xi_2' - \Xi_2)^{3/2}}{(q_2 - q_2')^{1/2}},
\end{aligned}$$

Hence,

$$\begin{aligned}\det(H_f(z_0,y_0)) =& \frac{QY_1}{12my_0}\times\frac{r^2X^2\Xi_1^2}{(T+rXY_1z_0)^2}\times(m_1(q_2-q_2')(\Xi_2'-\Xi_2))\\ \sim& -\frac{1}{12m}\times\frac{r^2X^2Y_1^4\Xi_1^2}{(T+rXY_1z_0)^2}\times(X_1m_1^2(q_2-q_2')^2)\\ \sim& \frac{M_0^{2/3}N^{2/3}Y_1^4}{Q^4}m_1^2(q_2-q_2')^2.\end{aligned}$$

which concludes the calculation of the main term of Lemma 2.8.

Now, we choose our λ to be

$$\lambda = \frac{M_0^{1/3}N^{1/3}Y_1^2}{Q^2}m_1|q_2-q_2'| \sim \frac{M_0^{1/3}Y_1m_1X_1|q_2-q_2'|}{Q}$$

and we have $|\partial_1^i\partial_2^jU| \ll 1$. Thus, we get the error term, which completes the proof of our lemma.

□

Remark. The case of $|q_2-q_2'| \ll \frac{QN^\varepsilon}{m_1M_0^{1/3}Y_1X_1}$ is considered in Section 5.6.3 and the contribution of the error term in (5.22) is evaluated in Section 5.6.4. The case of the zero frequency ($\tilde{u}_1=0$) is evaluated in Section 5.6.2. Thus, in the rest of the work we will consider the main term of (5.22) and only in the case of $\tilde{u}_1\neq 0$ and $|q_2-q_2'| \gg \frac{QN^\varepsilon}{m_1M_0^{1/3}Y_1X_1}$ and $X_1 \gg \frac{N^\varepsilon}{M_0^{1/3}}$.

Remark. In the generic case, dual length q_1 is of size

$$Q_1 \ll \frac{mh_1h_2h_2'}{Q}\cdot\frac{QN^{1/2}}{Q^{3/2}\cdot\frac{X^{3/4}}{T^{3/8}}} \sim \frac{mh_1h_2h_2'}{Q}\cdot\frac{T}{X} \sim \frac{X^{3/2}}{T^{3/4}}.$$

The main term of the $\mathcal{I}$ integral is of the size $\frac{X\cdot Q}{N/Q} \sim \frac{X}{T/X}$, and the error term is of the size $\frac{X}{(T/X)^2}$. Now, we calculate the saving from the Poisson on $q_1(\tilde{u}_1)$. Initial size of $\int\limits_{\xi_1\sim X}\mathcal{E}$ was XQ. Final size is

$$\frac{Q}{mh_1h_2h_2'}\cdot Q_1\cdot\sqrt{mh_1}\cdot\frac{X}{T/X} \sim \frac{T}{X}\cdot\frac{X}{\sqrt{T}}\cdot\frac{X}{T/X} \sim \frac{X^2}{\sqrt{T}}.$$

So, in this step, we save

$$\frac{\text{Initial length}}{\text{Final length}} = \frac{X^{3/2}T^{1/4}}{X^2/\sqrt{T}} \sim \frac{T^{3/4}}{\sqrt{X}}.$$

We calculate the saving after the duality:

$$\frac{T^{3/2}}{X^{3/2}} \cdot \frac{T^{3/8}}{X^{1/4}} = \frac{T^{15/8}}{X^{7/4}}.$$

This saving is equal to the diagonal saving if $\frac{T^{15/8}}{X^{7/4}} = T \iff X = \sqrt{T}$. So, we are at the boundary. To save more, we exploit the oscillatory term coming from the double integral $\mathcal{I}$ (Lemma 5.2).

5.4 Preparation for ξ_2 integral

Form (5.3) we recall that

$$\Delta \ll T^{\varepsilon} \sup_{||\alpha||_2=1} \sup_{\frac{N^{\varepsilon}}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{M_0} \Bigg(\sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^{\infty} \\ (\tilde{v}_1, \tilde{v}_2)=1}} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\substack{\hat{h}_2 \ll \frac{QT}{N} \\ \hat{h}_2 | (m^* h_1)^{\infty}}}$$
$$\int_{\xi_2 \sim X} \sum_{\tilde{h}_2 \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} \alpha_2 \int_{\xi_2' \sim X} \sum_{\tilde{h}_2' \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} \alpha_2' \frac{Q}{m_1 v_1 m h_1 h_2 h_2'} \sum_{\tilde{u}_1 \in \mathbb{Z}} \mathcal{C}\mathcal{I} \Bigg)^{1/2}. \tag{5.30}$$

When $X_1 \gg \frac{N^{\varepsilon}}{M_0^{1/3}}$ and $|q_2 - q_2'| \gg \frac{QN^{\varepsilon}}{m_1 M_0^{1/3} Y_1 X_1}$, if we only consider the main term of (5.22), we get

$$\Delta \ll T^{\varepsilon} \sup_{||\alpha||_2=1} \sup_{\frac{N^{\varepsilon}}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{M_0} \Bigg(\sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^{\infty} \\ (\tilde{v}_1, \tilde{v}_2)=1}} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\substack{\hat{h}_2 \ll \frac{QT}{N} \\ \hat{h}_2 | (m^* h_1)^{\infty}}}$$
$$\sum_{\tilde{h}_2 \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} \sum_{\tilde{h}_2' \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} \frac{Q}{m_1 v_1 m h_1 h_2 h_2'} \sum_{\tilde{u}_1 \sim Q_1} \mathcal{C} \cdot \frac{XQ^2 (q_2 - q_2')^{-1}}{m_1 M_0^{1/3} N^{1/3}}$$
$$\int_{\xi_2 \sim X} \alpha_2 \int_{\xi_2' \sim X} \alpha_2' e\left(\frac{h_1 h_2 h_2' q_2 q_2' m_1 (\Xi_2' - \Xi_2)^3}{\tilde{u}_1 (m_1 \tilde{v}_1 d)^{-1} (q_2 - q_2')} \right) V\left(\frac{\Xi_2' - \Xi_2}{\frac{(q_2 - q_2') X_1}{Q/m_1}} \right) \Bigg)^{1/2}.$$

If we denote $u_3 := u_2 - u_2'$, and recall that $q_2 = m_1 d\tilde{v}_2 u_2$ and $q_2' = m_1 d\tilde{v}_2 u_2'$, Now, we will apply Cauchy's inequality, keeping all but the ξ_2' integral inside. Hence, we get

$$\Delta \ll N^{\varepsilon} \sup_{||\alpha||_2=1} \sup_{\frac{N^{\varepsilon}}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{XM_0} \cdot |S_3|^{1/4}|S_4|^{1/4}, \tag{5.31}$$

where

$$S_3 = \sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^{\infty} \\ (\tilde{v}_1, \tilde{v}_2)=1}} \int_{\xi_2 \sim X} \sum_{h_2 \sim \frac{QT}{N}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} |\alpha_2|^2 \sum_{\hat{h}_2 | h_2}$$
$$\sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\tilde{h}_2' \sim \frac{QT}{N\hat{h}_2}} \frac{Q}{m_1 \tilde{v}_1 d m h_1 h_2 h_2'} \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} \sum_{\tilde{u}_1 \sim Q_1} |\mathcal{C}| \cdot \frac{Q^2 |q_2 - q_2'|^{-1}}{m_1 M_0^{1/3} N^{1/3}} \tag{5.32}$$

and

$$S_4 = \sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^{\infty} \\ (\tilde{v}_1, \tilde{v}_2)=1}} \sum_{h_2' \sim \frac{QT}{N}} \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} \sum_{\hat{h}_2 | h_2'} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}}$$
$$\times \sum_{\tilde{h}_2 \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} \times \frac{Q}{m_1 \tilde{v}_1 d m h_1 h_2 h_2'} \sum_{\tilde{u}_1 \sim Q_1} |\mathcal{C}| \cdot \frac{Q^2 |q_2 - q_2'|^{-1}}{m_1 M_0^{1/3} N^{1/3}}$$
$$\times \int_{\xi_2 \sim X} \left| \int_{\xi_2' \sim X} \alpha_2' e\left(\frac{h_1 h_2 h_2' q_2 q_2' m_1 (\Xi_2' - \Xi_2)^3}{\tilde{u}_1 (m_1 \tilde{v}_1 d)^{-1} (q_2 - q_2')} \right) V\left(\frac{\Xi_2' - \Xi_2}{\frac{(q_2 - q_2')X_1}{Q/m_1}} \right) \right|^2. \tag{5.33}$$

In both cases, we have changed the order of the $\hat{h}_2, \tilde{h}_2(\tilde{h}_2')$ and $h_1, \tilde{m}$ sum and used the notation $h_2 = \hat{h}_2 \tilde{h}_2$ and $h_2' = \hat{h}_2 \tilde{h}_2'$. Though not stated, we recall the condition $\hat{h}_1 = (h_1, (m^* \hat{h}_2)^{\infty})$.

Now, we will evaluate S_3. For simplicity, let us denote $H_1 = \frac{QT}{N}$ and $M_1 = \frac{Q^2 X_1}{M_0^{2/3}}$. So $h_1, h_2, h_2' \sim H_1$ and $m = \tilde{m} d = m^* d d_m \sim M_1$. These notations are temporary and introduced to keep the calculation decluttered. If we recall $q_2 = m_1 \tilde{v}_2 d u_2$, $q_2' = m_1 \tilde{v}_2 d u_2'$

and write $u_3 = u_2 - u_2'$, we can rewrite S_3 as

$$S_3 \ll \sum_{m_1 \ll Q} \sum_{d \ll M_1} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \int_{\xi_2 \sim X} \sum_{h_2 \sim H_1} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} |\alpha_2|^2 \sum_{\hat{h}_2 | h_2'} \sum_{h_1 \sim H_1} \sum_{\tilde{m} \sim \frac{M_1}{d}} \sum_{\tilde{h}_2' \sim \frac{H_1}{\hat{h}_2}} \times \frac{Q}{m_1 \tilde{v}_1 d^2 m h_1 h_2 h_2'} \times \frac{Q^2}{m_1 \tilde{v}_2 d M_0^{1/3} N^{1/3}} \sum_{0 \neq u_3 \ll \frac{Q}{d\tilde{v}_2 m_1}} |u_3|^{-1} \sum_{\tilde{u}_1 \sim Q_1} |\mathcal{C}|. \tag{5.34}$$

We recall

$$|Q_1| = m_1 \tilde{v}_1 d \cdot \frac{m h_1 h_2 h_2'}{Q} \cdot \frac{m_1 |q_2' - q_2| X_1^{3/2}}{\sqrt{m}} \ll m_1 \tilde{v}_1 d X_1^{3/2} \sqrt{M_1} H_1^3.$$

Now, if we evaluate the character sum in S_3 by (5.15), we get

$$\begin{aligned} &\frac{Q}{m_1 \tilde{v}_1 d^2 m h_1 h_2 h_2'} \times \frac{Q^2}{m_1 \tilde{v}_2 d M_0^{1/3} N^{1/3}} \sum_{0 \neq u_3 \ll \frac{Q}{d\tilde{v}_2 m_1}} |u_3|^{-1} \sum_{\tilde{u}_1 \sim Q_1} |\mathcal{C}| \\ \ll & N^\varepsilon \frac{Q}{m_1 \tilde{v}_1 d^2 m h_1 h_2 h_2'} \times \frac{Q^2}{m_1 \tilde{v}_2 d M_0^{1/3} N^{1/3}} \cdot d^3 d_m \sqrt{m^* h_1} h_2 h_2' \left(\frac{Q_1 \sqrt{\hat{h}_1 \hat{h}_2^2}}{h_2 h_2'} + 1 \right) \\ \ll & N^\varepsilon \frac{Q^3 \sqrt{d_m}}{m_1^2 \tilde{v}_1 \tilde{v}_2 \sqrt{m h_1} M_0^{1/3} N^{1/3} \sqrt{d}} \left(m_1 \tilde{v}_1 d X_1^{3/2} \sqrt{M_1} H_1 \sqrt{\hat{h}_1} \hat{h}_2 + 1 \right) \\ \ll & N^\varepsilon \frac{\sqrt{d_m} \cdot Q^3}{M_0^{1/3} N^{1/3}} \left(\frac{\sqrt{d} X_1^{3/2} \sqrt{H_1} \sqrt{\hat{h}_1} \hat{h}_2}{m_1 \tilde{v}_2} + \frac{1}{m_1^2 \tilde{v}_1 \tilde{v}_2 \sqrt{M_1 H_1 d}} \right) \end{aligned}$$

We observe that for a fixed $\hat{h}_2$ and m^*,

$$\sum_{h_1 \sim H_1} \hat{h}_1 = \sum_{h_1 \sim H_1} (h_1, (m^* \hat{h}_2)^\infty) \ll H_1^{1+\varepsilon} \tag{5.35}$$

and we also have

$$\sum_{\tilde{m} \sim \frac{M_1}{d}} d_m = \sum_{\tilde{m} \sim \frac{M_1}{d}} (\tilde{m}, d^\infty) \ll \frac{M_1^{1+\varepsilon}}{d}. \tag{5.36}$$

Thus, we get

$$S_3 \ll \sum_{m_1 \ll Q} \sum_{d \ll M_1} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \int_{\xi_2 \sim X} \sum_{h_2 \sim H_1} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} |\alpha_2|^2 \sum_{\hat{h}_2 | h_2}$$
$$\times \frac{N^\varepsilon \cdot Q^3}{M_0^{1/3} N^{1/3}} \sum_{\tilde{h}_2' \sim \frac{H_1}{\hat{h}_2}} \sum_{\tilde{m} \sim \frac{M_1}{d}} \sqrt{d_m} \sum_{h_1 \sim H_1} \left(\frac{\sqrt{d} X_1^{3/2} \sqrt{H_1} \sqrt{\hat{h}_1} \hat{h}_2}{m_1 \tilde{v}_2} + \frac{1}{m_1^2 \tilde{v}_1 \tilde{v}_2 \sqrt{M_1 H_1 d}} \right)$$
$$\ll \sum_{m_1 \ll Q} \sum_{d \ll M_1} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \int_{\xi_2 \sim X} \sum_{h_2 \sim H_1} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} |\alpha_2|^2$$
$$\times \frac{N^\varepsilon \cdot Q^3}{M_0^{1/3} N^{1/3}} \frac{M_1}{d} H_1^2 \left(\frac{\sqrt{d} X_1^{3/2} \sqrt{H_1}}{m_1 \tilde{v}_2} + \frac{1}{m_1^2 \tilde{v}_1 \tilde{v}_2 \sqrt{M_1 H_1 d}} \right)$$
$$\ll \frac{N^\varepsilon \cdot Q^3}{M_0^{1/3} N^{1/3}} \left(X_1^{3/2} H_1^{5/2} M_1 + H_1^{3/2} M_1^{1/2} \right)$$
$$\ll N^\varepsilon \left(\frac{Q^{15/2} X_1^{5/2} T^{5/2}}{M_0 N^{17/6}} + \frac{Q^{11/2} T^{3/2} X_1^{1/2}}{M_0^{2/3} N^{11/6}} \right). \tag{5.37}$$

We have used the fact that

$$\sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \int_{\xi_2 \sim X} \sum_{h_2 \sim \frac{QT}{N}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} |\alpha(m_1 d \tilde{v}_2 u_2, \xi_2, h_2)|^2 \ll N^\varepsilon.$$

5.5 ξ_2 integral

In S_4 (5.33), we open up the absolute square to get

$$S_4 = \sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \sum_{h_2' \sim \frac{QT}{N}} \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} \sum_{\hat{h}_2 | h_2'} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\tilde{h}_2 \sim \frac{QT}{N \hat{h}_2}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}}$$
$$\times \frac{Q}{m_1 v_1 m h_1 h_2 h_2'} \sum_{\tilde{u}_1 \sim Q_1} |\mathcal{C}| \cdot \frac{Q^2 |q_2 - q_2'|^{-1}}{m_1 M_0^{1/3} N^{1/3}} \int_{\xi_2' \sim X} \alpha_2' \int_{\xi_2'' \sim X} \bar{\alpha}_2'' \, \mathcal{I}_2,$$

where $\mathcal{I}_2$ is the ξ_2 integral:

$$\mathcal{I}_2 := \int\limits_{\xi_2 \sim X} e\left(\frac{[(\Xi_2' - \Xi_2)^3 - (\Xi_2'' - \Xi_2)^3]}{\tilde{u}_1 (m_1 \tilde{v}_1 d h_1 h_2 h_2' q_2 q_2' m_1)^{-1} (q_2 - q_2')} \right) V\left(\frac{\Xi_2' - \Xi_2}{\frac{(q_2 - q_2')X_1}{Q/m_1}} \right) V\left(\frac{\Xi_2'' - \Xi_2}{\frac{(q_2 - q_2')X_1}{Q/m_1}} \right) d\xi_2. \tag{5.38}$$

We note that $\Xi_2' = \frac{(T + r\xi_2')^{1/3}}{(2\pi q_2'^2 m_1^2 h_2')^{1/3}}$ and $\Xi_2'' = \frac{(T + r\xi_2'')^{1/3}}{(2\pi q_2'^2 m_1^2 h_2')^{1/3}}$. Let us make a change of variable

$$\xi_2 \mapsto X \cdot Y_2 z_2, \quad \text{where} \quad Y_2 = \frac{m_1 (q_2 - q_2') X_1}{N^{1/3}} \quad \text{and} \quad \Xi_2 = \frac{(T + rXY_2 z_2)^{1/3}}{(2\pi q_2^2 m_1^2 h_2)^{1/3}}$$

With this change, we note that the smooth function

$$U(z_2) = V(Y_2 z_2) V\left(\frac{\Xi_2'' - \Xi_2}{\frac{(q_2 - q_2')X_1}{Q/m_1}} \right) V\left(\frac{\Xi_2'' - \Xi_2}{\frac{(q_2 - q_2')X_1}{Q/m_1}} \right)$$

satisfies the decay relation $\partial^j U(z_2) \ll 1$. We simplify the phase function and get

$$\mathcal{I}_2 = XY_2 \int\limits_{z_2} e\left(\frac{(\Xi_2' - \Xi_2'')[3\Xi_2^2 - 3\Xi_2(\Xi_2' + \Xi_2'') + (\Xi_2'^2 + \Xi_2'\Xi_2'' + \Xi_2''^2)]}{\tilde{u}_1 (m_1 \tilde{v}_1 d h_1 h_2 h_2' q_2 q_2' m_1)^{-1} (q_2 - q_2')} \right) U(z_2) dz_2$$

where $\Xi_2 = \frac{(T + rXY_2 z_2)^{1/3}}{(2\pi q_2^2 m_1^2 h_2)^{1/3}}$. Let us denote the phase function by $f(z_2)$:

$$f(z_2) := \frac{(\Xi_2' - \Xi_2'')[3\Xi_2(z_2)^2 - 3\Xi_2(z_2)(\Xi_2' + \Xi_2'') + (\Xi_2'^2 + \Xi_2'\Xi_2'' + \Xi_2''^2)]}{\tilde{u}_1 (m_1 \tilde{v}_1 d h_1 h_2 h_2' q_2 q_2' m_1)^{-1} (q_2 - q_2')}.$$

Then we get

$$\begin{aligned} \frac{d\Xi_2}{dz_2} &= \frac{1}{3} \cdot \frac{rXY_2}{(T + rXY_2 z_2)^{2/3}} \cdot \frac{1}{(2\pi q_2^2 m_1^2 h_2)^{1/3}} = \frac{1}{3} \cdot \frac{rXY_2}{(T + rXY_2 z_2)} \cdot \Xi_2 \\ f'(z_2) &= \frac{h_1 h_2 h_2' q_2 q_2' m_1 (\Xi_2' - \Xi_2'')}{q_1 (q_2 - q_2')} \cdot \frac{rXY_2 \Xi_2}{(T + rXY_2 z_2)} \cdot [2\Xi_2 - (\Xi_2' + \Xi_2'')] \end{aligned}$$

As $\Xi_2' - \Xi_2 \sim \frac{X_1(q_2 - q_2')}{Q/m_1}$ and $\Xi_2'' - \Xi_2 \sim \frac{X_1(q_2 - q_2')}{Q/m_1}$, we get

$$2\Xi_2 - (\Xi_2' + \Xi_2'') \sim -\frac{X_1(q_2 - q_2')}{Q/m_1}.$$

We also recall $\Xi_i \sim \frac{N^{1/3}}{Q}$. Hence,

$$\begin{aligned} |f'(z_2)| \sim & \frac{h_1 h_2 h_2' q_2 q_2' m_1 (\Xi_2' - \Xi_2'')}{\tilde{u}_1 (m_1 \tilde{v}_1 d)^{-1} (q_2 - q_2')} \cdot \frac{r X Y_2 \Xi_2}{(T + r X Y_2 z_2)} \cdot \frac{X_1 |q_2 - q_2'|}{Q/m_1} \\ \sim & \frac{h_1 h_2 h_2' (\Xi_2' - \Xi_2'')}{\tilde{u}_1 (m_1 \tilde{v}_1 d)^{-1}} \cdot X_1^2 m_1 |q_2 - q_2'|. \end{aligned}$$

As $\tilde{u}_1 \sim Q_1$ where

$$Q_1 = m_1 \tilde{v}_1 d \cdot \frac{m h_1 h_2 h_2'}{Q} \cdot \frac{m_1 (q_2' - q_2) X_1^{3/2}}{\sqrt{m}} \quad \text{and} \quad m \sim \frac{Q^2 X_1}{M_0^{2/3}},$$

the integral $\mathcal{I}_2$ is negligibly small unless

$$\begin{aligned} |\Xi_2' - \Xi_2''| \ll & \frac{|Q_1| (m_1 \tilde{v}_1 d)^{-1} N^{\varepsilon}}{h_1 h_2 h_2' X_1^2 m_1 |q_2 - q_2'|} \ll \frac{1}{M_0^{1/3}} \\ \iff |\xi_2' - \xi_2''| \ll & X N^{\varepsilon} \cdot \frac{Q}{M_0^{1/3} N^{1/3}}. \end{aligned} \tag{5.39}$$

In that range, we can bound $\mathcal{I}_2$ by

$$\mathcal{I}_2 \ll X Y_2 \ll \frac{X X_1 |q_2 - q_2'| m_1}{N_1^{1/3}}.$$

Hence

$$\begin{aligned} \int_{\xi_2' \sim X} |\alpha_2'|^2 \int_{\xi_2'' \sim X} \mathcal{I}_2 \ll & \int_{\xi_2' \sim X} |\alpha_2'|^2 X^2 \cdot \frac{|q_2 - q_2'|}{Q/m_1} \cdot \frac{X_1 Q}{N^{1/3}} \cdot \frac{Q}{M_0^{1/3} N^{1/3}} \\ \ll & \int_{\xi_2' \sim X} |\alpha_2'|^2 \cdot \frac{X^2 X_1 Q^2}{M_0^{1/3} N^{2/3}}. \end{aligned}$$

Then, S_4 is bounded by

$$\begin{aligned} S_4 \ll & \frac{X^2 X_1 Q^2}{M_0^{1/3} N^{2/3}} \sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^{\infty} \\ (\tilde{v}_1, \tilde{v}_2) = 1}} \int_{\xi_2' \sim X} \sum_{h_2' \sim \frac{QT}{N}} \sum_{u_2' \sim \frac{Q}{d \tilde{v}_2 m_1}} |\alpha_2'|^2 \sum_{\hat{h}_2 | h_2'} \\ & \times \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\tilde{h}_2 \sim \frac{QT}{N \hat{h}_2}} \sum_{u_2 \sim \frac{Q}{d \tilde{v}_2 m_1}} \frac{Q}{m_1 \tilde{v}_1 d m h_1 h_2 h_2'} \times \sum_{\tilde{u}_1 \sim Q_1} |\mathcal{C}| \cdot \frac{Q^2 |q_2 - q_2'|^{-1}}{m_1 M_0^{1/3} N^{1/3}}. \end{aligned}$$

Comparing it with S_3 (5.32), we observe that

$$|S_4| \ll \frac{X^2 X_1 Q^2}{M_0^{1/3} N^{2/3}} |S_3|$$

We recall

$$\Delta \ll \sup_{||\alpha||_2=1} \sup_{\frac{N^\varepsilon}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{XM_0} \cdot |S_3|^{1/4} |S_4|^{1/4}$$

and (5.37)

$$S_3 \ll \frac{Q^{15/2} X_1^{5/2} T^{5/2}}{N^{17/6} M_0} + \frac{Q^{11/2} X_1^{1/2} T^{3/2}}{M_0^{2/3} N^{11/6}}.$$

Hence, we have

$$\begin{aligned}
\Delta &\ll \sup_{\frac{N^\varepsilon}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{XM_0} \cdot \frac{X^{1/2} X_1^{1/4} Q^{1/2}}{M_0^{1/12} N^{1/6}} \cdot \left(\frac{Q^{15/4} X_1^{5/4} T^{5/4}}{N^{17/12} M_0^{1/2}} + \frac{Q^{11/4} X_1^{1/4} T^{3/4}}{M_0^{1/3} N^{11/12}} \right) \\
&\ll \sup_{\frac{N^\varepsilon}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} \left(\frac{Q^{21/4} X_1^{3/2} X T^{5/4}}{M_0^{1/12} N^{19/12}} + \frac{Q^{17/4} X_1^{1/2} X T^{3/4} M_0^{1/12}}{N^{13/12}} \right) \\
&\ll \left(\frac{Q^{15/4} X T^{5/4}}{M_0^{1/12} N^{13/12}} + \frac{Q^{15/4} X T^{3/4} M_0^{1/12}}{N^{11/12}} \right). \qquad (5.40)
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathfrak{M} \ll & \frac{N^{4/3+\varepsilon}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \times \Delta \\
\ll & \frac{N^{4/3+\varepsilon}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \times \left(\frac{Q^{15/4} X T^{5/4}}{M_0^{1/12} N^{13/12}} + \frac{Q^{15/4} X T^{3/4} M_0^{1/12}}{N^{11/12}} \right) \\
\ll & \frac{N^{4/3+\varepsilon}}{T} \sup_{Q \ll Q_0} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} \left(\frac{Q^{7/4} X T^{5/4} M_0^{1/4}}{N^{13/12}} + \frac{Q^{7/4} X T^{3/4} M_0^{5/12}}{N^{11/12}} \right) \\
\ll & \frac{N^{4/3+\varepsilon}}{T} \left(\frac{Q_0^{7/4} X T^{5/4} N^{1/2}}{N^{13/12} Q_0^{3/4}} + \frac{Q_0^{7/4} X T^{3/4} N^{5/6}}{Q_0^{5/4} N^{11/12}} \right) \\
\ll & \frac{N^{4/3+\varepsilon}}{T} \left(\frac{Q_0 X T^{5/4}}{N^{7/12}} + \frac{Q_0^{1/2} X T^{3/4}}{N^{1/12}} \right) \\
\ll & \frac{N^{4/3+\varepsilon}}{T} \cdot \left(\frac{X^{3/2} T^{3/4}}{N^{1/12}} + X^{5/4} T^{1/2} N^{1/6} \right) \\
\ll & \frac{N^{4/3+\varepsilon}}{T} \cdot \frac{X^{3/2} T^{3/4}}{N^{1/12}} \left(1 + \frac{N^{1/4}}{X^{1/4} T^{1/4}} \right) \\
\ll & N^{5/4+\varepsilon} T^{-1/4} X^{3/2}.
\end{aligned} \tag{5.41}$$

5.6 Non-generic cases

5.6.1 $X_1 \ll \frac{N^\varepsilon}{M_0^{1/3}}$

When $m \neq 0$ and $X_1 \ll \frac{N^\varepsilon}{M_0^{1/3}}$, from (4.23) we get

$$S_{m,2} \ll \frac{dQ^2 N^\varepsilon}{m_1^2}.$$

And from the dyadic relation $\Xi_2 - \Xi_1 \asymp X_1$, we get

$$|\xi_2 - B\xi_1| \ll \frac{X X_1 Q}{N^{1/3}}$$

for some B depending on q_1, q_2, h_1, h_2 with value bounded between absolute constants ($B \sim 1$). Then, Δ (4.21) is bounded by

$$\begin{aligned}
\Delta = & \sup_{||\alpha||_2=1} \sup_{X_1 \ll \frac{N^\varepsilon}{M_0^{1/3}}} \sum_{m_1 \ll Q} m_1^2 \int_{\xi_1 \sim X} \sum_{q_1 \sim \frac{Q}{m_1}} \sum_{h_1 \sim \frac{QT}{N}} \int_{\xi_2 \sim X} \sum_{q_2 \sim \frac{Q}{m_1}} \sum_{h_2 \sim \frac{QT}{N}} |\alpha_1|^2 S_{\mathcal{M},2} \\
\ll & N^\varepsilon \sup_{X_1 \ll \frac{N^\varepsilon}{M_0^{1/3}}} \sum_{m_1 \ll Q} m_1^{-1} \int_{\xi_1 \sim X} \sum_{q_1 \sim \frac{Q}{m_1}} \sum_{h_1 \sim \frac{QT}{N}} |\alpha_1|^2 \cdot \frac{XX_1Q}{N^{1/3}} \cdot Q \cdot \frac{QT}{N} \cdot Q^2 \\
\ll & \frac{TXQ^5}{N^{4/3}M_0^{1/3}}.
\end{aligned} \tag{5.42}$$

Hence,

$$\begin{aligned}
\mathfrak{M} \ll & \frac{N^{4/3}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \times \Delta \\
\ll & \frac{N^{4/3+\varepsilon}}{T} \frac{TXQ_0^3}{N^{4/3}} \\
\ll & N^\varepsilon X Q_0^3 \sim N^{3/2+\varepsilon} X^{5/2} T^{-3/2}.
\end{aligned} \tag{5.43}$$

5.6.2 $\tilde{u}_1 = 0$

In this case, there is no stationary phase in the $\mathcal{I}$ (5.20). From (5.26), we note that the integral $\mathcal{I}$ is negligibly small unless

$$\frac{Y_1\sqrt{Q}}{\sqrt{m}} \cdot |\Xi_2' - \Xi_2| \sqrt{q_2 m_1} M'^{1/2} \ll N^\varepsilon \implies |\Xi_2 - \Xi_2'| \ll \frac{N^\varepsilon}{Y_1 M_0^{1/3}}.$$

Hence, we get our first condition that

$$|\Xi_2 - \Xi_2'| \ll \frac{N^\varepsilon}{Y_1 M_0^{1/3}} \implies |\xi_2' - B\xi_2| \ll \frac{XQ}{Y_1 N^{1/3} M_0^{1/3}}. \tag{5.44}$$

where B depends on q_2, q_2', h_2, h_2' and is bounded between absolute constants $c_1 < B < c_2$. On the other hand, from this relation and (5.24), we get that

$$|q_2 - q_2'| \ll \frac{QN^\varepsilon}{m_1 Y_1 X_1 M_0^{1/3}} \iff |u_2 - u_2'| \ll \frac{QN^\varepsilon}{m_1 d\tilde{v}_2 Y_1 X_1 M_0^{1/3}}. \tag{5.45}$$

We note that contribution of $\mathcal{C}$ and $\mathcal{I}$ for $\tilde{u}_1 = 0$ as $\mathcal{C}_0$ and $\mathcal{I}_0$. In this range, $\mathcal{I}_0$ can be replaced by a smooth function with absolute value bounded by $N^\varepsilon XY_1^2$. From (5.30), we get that

$$\Delta \ll \sup_{||\alpha||_2=1} \sup_{\frac{N^\varepsilon}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{M_0} \Bigg(\sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\substack{\hat{h}_2 \ll \frac{QT}{N} \\ \hat{h}_2 | (m^* h_1)^\infty}} \int_{\xi_2 \sim X} \sum_{\tilde{h}_2 \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} \alpha_2 \int_{\xi_2' \sim X} \sum_{\tilde{h}_2' \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} \alpha_2' \frac{Q}{m_1 v_1 m h_1 h_2 h_2'} \cdot \mathcal{C}_0 \mathcal{I}_0 \Bigg)^{1/2}. \tag{5.46}$$

By Cauchy's inequality, we can bound

$$\Delta \ll \sup_{||\alpha||_2=1} \sup_{\frac{N^\varepsilon}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{M_0} \Bigg(\sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \int_{\xi_2 \sim X} \sum_{h_2 \sim \frac{QT}{N}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} |\alpha_2|^2 \times \sum_{\hat{h}_2 | h_2} \sum_{\tilde{h}_2' \sim \frac{QT}{N\hat{h}_2}} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \frac{Q}{m_1 v_1 m h_1 h_2 h_2' d} \cdot \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} |\mathcal{C}_0| \int_{\xi_2' \sim X} |\mathcal{I}_0| \Bigg)^{1/2}.$$

Once we replace $\mathcal{C}_0$ by (5.11), we get

$$\begin{aligned} & \frac{Q}{m_1 v_1 m h_1 h_2 h_2' d} \cdot \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} |\mathcal{C}_0| \int_{\xi_2' \sim X} |\mathcal{I}_0| \\ \ll & N^\varepsilon \frac{Q}{m_1 \tilde{v}_1 d m h_1 h_2 h_2' d} \cdot d^3 d_m \sqrt{m^* \hat{h}_1} h_2 h_2' \left(\frac{Q_2}{m_1 d \tilde{v}_2} + \sqrt{m^* \hat{h}_1} \tilde{h}_1 \right) \cdot XY_1^2 \cdot \frac{XQ}{Y_1 N^{1/3} M_0^{1/3}} \\ \ll & \left(\frac{Y_1 Q_2 Q \sqrt{d_m \hat{h}_1}}{m_1^2 \sqrt{d} \tilde{v}_1 \tilde{v}_2 \sqrt{m} h_1} + \frac{Q}{m_1 \tilde{v}_1} \right) \cdot \frac{X^2 Q}{N^{1/3} M_0^{1/3}} \end{aligned}$$

along with the condition $\tilde{h}_2' = \tilde{h}_2 = \frac{h_2}{\hat{h}_2}$. Here, we have used the fact $Y_1 \ll 1$. Putting $Q_2 = \frac{Q}{X_1 Y_1 M_0^{1/3}}$, we get

$$\begin{aligned}
&\sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \frac{Q}{m_1 v_1 m h_1 h_2 h_2' d} \cdot \sum_{u_2' \sim \frac{Q}{d \tilde{v}_2 m_1}} |\mathcal{C}_0| \int_{\xi_2' \sim X} |\mathcal{I}_0| \\
&\ll N^{\varepsilon} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \left(\frac{Q^2 \sqrt{d_m \hat{h}_1}}{m_1^2 \sqrt{d} \tilde{v}_1 \tilde{v}_2 \sqrt{m} h_1 X_1 M_0^{1/3}} + \frac{Q}{m_1 \tilde{v}_1} \right) \cdot \frac{X^2 Q}{N^{1/3} M_0^{1/3}} \\
&\ll \left(\frac{Q^2 \cdot Q \sqrt{X_1}}{m_1^2 d^{3/2} \tilde{v}_1 \tilde{v}_2 X_1 M_0^{2/3}} + \frac{Q^4 X_1 T}{m_1 \tilde{v}_1 d N M_0^{2/3}} \right) \cdot \frac{X^2 Q}{N^{1/3} M_0^{1/3}}.
\end{aligned}$$

Here, we have used (5.36) and (5.35). Then, we get

$$\begin{aligned}
\Delta &\ll \sup_{\frac{N^{\varepsilon}}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q \sqrt{M_0} \left(\frac{Q^3}{\sqrt{X_1} M_0^{2/3}} + \frac{Q^4 X_1 T}{N M_0^{2/3}} \right)^{1/2} \cdot \frac{X Q^{1/2}}{N^{1/6} M_0^{1/6}} \\
&\ll \sup_{\frac{N^{\varepsilon}}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q \sqrt{M_0} \left(\frac{Q^{3/2}}{X_1^{1/4}} + \frac{Q^2 X_1^{1/2} T^{1/2}}{N^{1/2}} \right) \cdot \frac{X Q^{1/2}}{N^{1/6} M_0^{1/2}} \\
&\ll \left(Q^{3/2} M_0^{1/12} + \frac{Q^2 N^{1/6} T^{1/2}}{\sqrt{Q} N^{1/2}} \right) \frac{X Q^{3/2}}{N^{1/6}} \\
&\ll \frac{X Q^3}{N^{1/6}} \left(M_0^{1/12} + \frac{T^{1/2}}{N^{1/3}} \right) \\
&\ll \frac{X Q^3 M_0^{1/12}}{N^{1/6}}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\mathfrak{M} \ll & \frac{N^{4/3}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \times \Delta \\
\ll & \frac{N^{4/3}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \cdot \frac{XQ^3 M_0^{1/12}}{N^{1/6}} \\
\ll & \frac{N^{4/3} Q_0 X}{T N^{1/6}} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{5/12} \\
\ll & \frac{N^{4/3+\varepsilon} Q_0 X}{T N^{1/6}} \cdot \frac{N^{5/6}}{Q_0^{5/4}} \\
\ll & N^{15/8+\varepsilon} X^{7/8} T^{-7/8}
\end{aligned}
\tag{5.47}
$$

5.6.3 $|q_2 - q_2'| \ll \frac{QN^\varepsilon}{m_1 M_0^{1/3} Y_1 X_1}$ and $\tilde{u}_1 \neq 0$

In this case, the phase function of $\mathcal{I}$ does not have a stationary point. But from (5.24), we have

$$|\Xi_2 - \Xi_2'| \ll \frac{N^\varepsilon}{M_0^{1/3} Y_1} \implies |\xi_2' - B\xi_2| \ll \frac{XQ}{N^{1/3} M_0^{1/3} Y_1}.$$

From (5.26), we also get that $\mathcal{I}$ is negligibly small unless

$$\tilde{u}_1 \ll \frac{m h_1 h_2 h_2' m_1 \tilde{v}_1 d}{Q Y_1}.$$

In that range, we can bound $\mathcal{I}$ by $X^2 Y_1^2$. If we denote $u_3 = u_2 - u_2'$ and $Q_3 := \frac{m h_1 h_2 h_2' m_1 \tilde{v}_1 d N^\varepsilon}{Q}$, by (5.14), for a fixed u_2', we get

$$\begin{aligned}
&\frac{Q}{m_1 v_1 m h_1 h_2 h_2'} \cdot \sum_{|u_3| \ll \frac{QN^\varepsilon}{m_1 \tilde{v}_2 d Y_1 M_0^{1/3} X_1}} \sum_{|\tilde{u}_1| \ll Y_1^{-1} Q_3 N^\varepsilon} |\mathcal{C}| \int_{\xi_2} |\mathcal{I}| \\
\ll & \frac{1}{Q_3} \cdot N^\varepsilon d^3 d_m \sqrt{m^* \hat{h}_1} h_2 h_2' \cdot Q_3 Y_1^{-1} \cdot \left(\frac{Q\sqrt{\tilde{h}_1}}{m_1 d \tilde{v}_2 M_0^{1/3} X_1 Y_1} + \sqrt{m^* \hat{h}_1} \right) \cdot \frac{X Y_1^2 \cdot XQ}{N^{1/3} M_0^{1/3} Y_1} \\
\ll & N^\varepsilon d^2 \left(\frac{QT}{N} \right)^2 \left(\frac{Q\sqrt{m h_1 d_m}}{m_1 \sqrt{d} \tilde{v}_2 M_0^{1/3} X_1 Y_1} + m \hat{h}_1 \right) \cdot \frac{X^2 Q}{N^{1/3} M_0^{1/3}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\tilde{h}_2 \sim \frac{QT}{N \hat{h}_2}} \frac{Q}{m_1 v_1 m h_1 h_2 h_2'} \cdot \sum_{|u_3| \ll \frac{QN^\varepsilon}{m_1 \tilde{v}_2 d M_0^{1/3} X_1}} \sum_{|\tilde{u}_1| \ll Q_3 Y_1^{-1} N^\varepsilon} |\mathcal{C}| \int_{\xi_2} |\mathcal{I}| \\
\ll & \frac{d^2 N^\varepsilon X^2 Q}{N^{1/3} M_0^{1/3}} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\tilde{h}_2 \sim \frac{QT}{N \hat{h}_2}} \left(\frac{QT}{N} \right)^2 \left(\frac{Q\sqrt{m h_1 d_m}}{m_1 \sqrt{d} \tilde{v}_2 M_0^{1/3} Y_1 X_1} + m \hat{h}_1 \right) \\
\ll & \frac{d^2 N^\varepsilon X^2 Q}{N^{1/3} M_0^{1/3}} \cdot \left(\frac{QT}{N} \right)^4 \cdot \frac{Q^2 X_1}{M_0^{2/3} d} \cdot \left(\frac{Q^2 \cdot Q^{1/2} X_1^{1/2} T^{1/2}}{m_1 \sqrt{d} \tilde{v}_2 M_0^{2/3} X_1 Y_1 N^{1/2}} + \frac{Q^2 X_1}{M_0^{2/3}} \right) \\
\ll & \frac{N^\varepsilon d Q^5 X^2}{N^{1/3} M_0^{5/3}} \cdot \left(\frac{QT}{N} \right)^4 \left(\frac{Q^{1/2} T^{1/2} \sqrt{X_1}}{m_1 d^{1/2} \tilde{v}_2 Y_1 N^{1/2}} + X_1^2 \right) \\
\ll & \frac{N^\varepsilon d Q^5 X^2}{N^{1/3} M_0^{5/3}} \cdot \left(\frac{QT}{N} \right)^4 \left(\frac{T^{1/2}}{m_1 d^{1/2} \tilde{v}_2 \sqrt{X_1 Q} N^{1/6}} + X_1^2 \right)
\end{aligned}$$

Then,

$$
\begin{aligned}
\Delta \ll & \sup_{||\alpha||_2=1} \sup_{\frac{N^\varepsilon}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{M_0} \Bigg(\sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \int_{\xi_2' \sim X} \sum_{h_2' \sim \frac{QT}{N}} \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} |\alpha_2'|^2 \\
& \sum_{\hat{h}_2 | h_2'} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\tilde{h}_2 \sim \frac{QT}{N\hat{h}_2}} \frac{Q}{m_1 v_1 m h_1 h_2 h_2'} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} \sum_{\tilde{u}_1 \in \mathbb{Z}} |\mathcal{C}| \int_{\xi_2 \sim X} |\mathcal{I}| \Bigg)^{1/2} \\
\ll & N^\varepsilon \sup_{\frac{N^\varepsilon}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{M_0} \frac{Q^{5/2} X}{N^{1/6} M_0^{5/6}} \cdot \left(\frac{QT}{N}\right)^2 \left(\frac{T^{1/4}}{X_1^{1/4} Q^{1/4} N^{1/12}} + X_1 \right) \\
\ll & \frac{Q^{11/2} X T^2}{N^{13/6} M_0^{1/3}} \cdot \left(\frac{T^{1/4} M_0^{1/12}}{Q^{1/4} N^{1/12}} + \frac{N^{1/3}}{Q} \right) \\
\ll & \frac{Q^{11/2-1/4} X T^{9/4} M_0^{1/12}}{N^{13/6+1/12} M_0^{1/3}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathfrak{M} \ll & \frac{N^{4/3}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \times \Delta \\
\ll & \frac{N^{4/3}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \times \frac{Q^{11/2-1/4} X T^{9/4} M_0^{1/12}}{N^{13/6+1/12} M_0^{1/3}} \\
\ll & \frac{N^{4/3+\varepsilon}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \cdot \frac{N^{1/6}}{Q^{1/4}} \cdot \frac{Q^{11/2-1/4} X T^{9/4}}{N^{13/6+1/12}} \\
\ll & N^{4/3+1/6-13/6-1/12+\varepsilon} T^{9/4-1} Q_0^{-2-1/2+11/2} X \\
\ll & N^{-3/4+\varepsilon} Q_0^3 X T^{5/4} \\
\ll & N^{3/4+\varepsilon} X^{5/2} T^{-1/4}.
\end{aligned}
\tag{5.48}
$$

5.6.4 Error term

In this case, we consider the error term of (5.22), i.e.,

$$
\mathcal{I} \ll \frac{XQ^2 |q_2 - q_2'|^{-2}}{M_0^{2/3} X_1^2 m_1^2}.
$$

We also have

$$|\tilde{u}_1| \ll m_1 \tilde{v}_1 d \cdot \frac{m h_1 h_2 h_2' N^{\varepsilon}}{Q} \cdot \frac{m_1 |q_2 - q_2'| X_1^{3/2}}{\sqrt{m}}$$

and

$$\Xi_2' - \Xi_2 \sim \frac{(q_2 - q_2') X_1}{Q/m_1} \implies \xi_2' - B\xi_2 \sim \frac{X X_1 m_1 (q_2 - q_2')}{N^{1/3}}$$

where B depends on q_2, q_2', h_2, h_2' and $B \sim 1$. Then, if we consider the error term, we basically have

$$\int\limits_{\xi_2} |\alpha_2|^2 \int\limits_{\xi_2' - B\xi_2 \sim \frac{X X_1 m_1 (q_2 - q_2')}{N^{1/3}}} \mathcal{I} \ll \int\limits_{\xi_2} |\alpha_2|^2 \cdot \frac{X Q^2 |q_2 - q_2'|^{-2}}{M_0^{2/3} X_1^2 m_1^2} \cdot \frac{X X_1 m_1 (q_2 - q_2')}{N^{1/3}}$$
$$\ll \frac{X^2}{X_1 M_0^{1/3}} \int\limits_{\xi_2} |\alpha_2|^2 \frac{Q^2 |q_2 - q_2'|^{-1}}{m_1 M_0^{1/3} N^{1/3}}.$$

Now, if we apply Cauchy's inequality on all the sums in (5.3), we have

$$\Delta \ll \sup_{||\alpha||_2 = 1} \sup_{\frac{N^{\varepsilon}}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{M_0} \Bigg(\sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^{\infty} \\ (\tilde{v}_1, \tilde{v}_2) = 1}} \sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\substack{\hat{h}_2 \ll \frac{QT}{N} \\ \hat{h}_2 | (m^* h_1)^{\infty}}}$$
$$\int\limits_{\xi_2' \sim X} \sum_{\tilde{h}_2' \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} \alpha_2' \int\limits_{\xi_2 \sim X} \sum_{\tilde{h}_2 \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} \alpha_2 \frac{Q}{m_1 v_1 m h_1 h_2 h_2'} \sum_{\tilde{u}_1 \in \mathbb{Z}} \mathcal{C}\mathcal{I} \Bigg)^{1/2}$$
$$\ll \sup_{||\alpha||_2 = 1} \sup_{\frac{N^{\varepsilon}}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{M_0} \cdot \frac{X}{\sqrt{X_1 M_0^{1/3}}} \cdot S_3^{1/4} \cdot S_3'^{1/4}$$

where S_3 is the same as (5.32)

$$S_3 = \sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^{\infty} \\ (\tilde{v}_1, \tilde{v}_2) = 1}} \int\limits_{\xi_2} \sum_{h_2 \sim \frac{QT}{N}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} |\alpha_2|^2 \sum_{\hat{h}_2 | h_2}$$
$$\sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\tilde{h}_2' \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} \frac{Q}{m_1 v_1 m h_1 h_2 h_2'} \sum_{\tilde{u}_1 \sim Q_1} |\mathcal{C}| \cdot \frac{Q^2 |q_2 - q_2'|^{-1}}{m_1 M_0^{1/3} N^{1/3}}$$

and S_3' is symmetrical to S_3 where the role of q_2, h_2, ξ_2 is reversed with q_2', h_2' and ξ_2':

$$S_3' = \sum_{m_1 \ll Q} \sum_{d \ll \frac{Q^2 X_1}{M_0^{2/3}}} \frac{1}{d} \sum_{\substack{\tilde{v}_1, \tilde{v}_2 | d^\infty \\ (\tilde{v}_1, \tilde{v}_2)=1}} \int_{\xi_2'} \sum_{h_2' \sim \frac{QT}{N}} \sum_{u_2' \sim \frac{Q}{d\tilde{v}_2 m_1}} |\alpha_2|^2 \sum_{\hat{h}_2 | h_2'}$$
$$\sum_{h_1 \sim \frac{QT}{N}} \sum_{\tilde{m} \sim \frac{Q^2 X_1}{M_0^{2/3} d}} \sum_{\tilde{h}_2 \sim \frac{QT}{N\hat{h}_2}} \sum_{u_2 \sim \frac{Q}{d\tilde{v}_2 m_1}} \frac{Q}{m_1 v_1 m h_1 h_2 h_2'} \sum_{\tilde{u}_1 \sim Q_1} |\mathcal{C}| \cdot \frac{Q^2 |q_2 - q_2'|^{-1}}{m_1 M_0^{1/3} N^{1/3}}.$$

Hence, S_3 and S_3' would have the same bound (5.37). i.e.

$$S_3 \ll \frac{Q^{15/2} X_1^{5/2} T^{5/2}}{N^{17/6} M_0} + \frac{Q^{11/2} X_1^{1/2} T^{3/2}}{M_0^{2/3} N^{11/6}}$$

Hence, we can bound Δ by

$$\begin{aligned}
\Delta &\ll \sup_{||\alpha||_2=1} \sup_{\frac{N^\varepsilon}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} Q\sqrt{M_0} \cdot \frac{X}{\sqrt{X_1 M_0^{1/3}}} \left(\frac{Q^{15/2} X_1^{5/2} T^{5/2}}{N^{17/6} M_0} + \frac{Q^{11/2} X_1^{1/2} T^{3/2}}{M_0^{2/3} N^{11/6}} \right)^{1/2} \\
&\ll QX \sup_{\frac{N^\varepsilon}{M_0^{1/3}} \ll X_1 \ll \frac{N^{1/3}}{Q}} \left(\frac{Q^{15/4} X_1^{3/4} T^{5/4}}{M_0^{1/6} N^{17/12}} + \frac{Q^{11/4} T^{3/4}}{X_1^{1/4} N^{11/12}} \right) \\
&\ll QX \left(\frac{Q^3 T^{5/4}}{M_0^{1/6} N^{7/6}} + \frac{Q^{11/4} T^{3/4} M_0^{1/12}}{N^{11/12}} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathfrak{M} &\ll \frac{N^{4/3}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \times \Delta \\
&\ll \frac{N^{4/3}}{T} \sup_{Q \ll Q_0} \frac{1}{Q^2} \sup_{M_0 \ll \frac{N^{2+\varepsilon}}{Q_0^3}} M_0^{1/3} \times QX \left(\frac{Q^3 T^{5/4}}{M_0^{1/6} N^{7/6}} + \frac{Q^{11/4} T^{3/4} M_0^{1/12}}{N^{11/12}} \right) \\
&\ll \left(\frac{N^{4/3} Q_0^2 T^{5/4} X N^{1/3}}{Q_0^{1/2} T N^{7/6}} + \frac{N^{4/3} X T^{3/4}}{N^{1/12} Q^{1/2} T} \right) \\
&\ll N^{5/4} X^{7/4} T^{-1/2} + N X^{3/4}. \qquad (5.49)
\end{aligned}$$

5.7 Final calculation

From (4.8), (5.41), (5.43), (5.47), (5.48) and (5.49), we collect all the bounds on $\mathfrak{M}$ and we get

$$\mathfrak{M} \ll N^{\varepsilon}\left(\frac{N^{3/2}\sqrt{T}}{\sqrt{X}} + \frac{N^{5/4}X^{3/2}}{T^{1/4}} + \frac{N^{3/2}X^{5/2}}{T^{3/2}} + \frac{N^{15/8}X^{7/8}}{T^{7/8}} + \frac{N^{3/4}X^{5/2}}{T^{1/4}} + \frac{N^{5/4}X^{7/4}}{T^{1/2}}\right). \tag{5.50}$$

We equate the first (diagonal) and the second bound (generic off-diagonal) to get

$$\frac{N^{3/2}\sqrt{T}}{\sqrt{X}} = N^{5/4}T^{-1/4}X^{3/2} \iff X = N^{1/8}T^{3/8}.$$

Hence,

$$\mathfrak{M} \ll N^{\varepsilon}\left(N^{23/16}T^{5/16} + N^{29/16}T^{-9/16} + N^{127/64}T^{-35/64} + N^{17/16}T^{11/16} + N^{47/32}T^{5/32}\right). \tag{5.51}$$

As $N \ll T^{3/2+\varepsilon}$, all the other bounds are smaller than the first bound. Hence,

$$\mathfrak{M} \ll N^{23/16+\varepsilon}T^{5/16}. \tag{5.52}$$

We also need to verify our assumption

$$Q_0 > \frac{T}{X} \iff N > \left(\frac{T}{X}\right)^3.$$

With our choice of $X = N^{1/8}T^{3/8}$, this is equivalent to

$$N > \frac{T^{15/8}}{N^{3/8}} \iff N^{11/8} > T^{15/8} \iff N > T^{15/11}.$$

As we have already assumed $N > T^{15/11+\varepsilon}$, this condition is satisfied.

Finally, we have

$$\begin{aligned}
M_F(T) &\ll \sup_{T^{15/11+\varepsilon} \ll N \ll T^{3/2+\varepsilon}} \frac{T^{\varepsilon}}{NX} \cdot X \cdot \frac{T}{X} \cdot \mathfrak{M} + T^{15/11+\varepsilon} \\
&\ll \sup_{T^{15/11+\varepsilon} \ll N \ll T^{3/2+\varepsilon}} \frac{T^{1+\varepsilon}}{NX} \cdot \mathfrak{M} + T^{15/11+\varepsilon} \\
&\ll \sup_{T^{15/11+\varepsilon} \ll N \ll T^{3/2+\varepsilon}} T^{\varepsilon} T^{5/8} N^{23/16-1/8-1} T^{5/16} + T^{15/11+\varepsilon} \\
&\ll \sup_{T^{15/11+\varepsilon} \ll N \ll T^{3/2+\varepsilon}} T^{\varepsilon} T^{15/16} N^{5/16} + T^{15/11+\varepsilon} \\
&\ll T^{45/32+\varepsilon} + T^{15/11+\varepsilon} \\
&\ll T^{3/2-3/32+\varepsilon}.
\end{aligned}$$

This concludes the proof of our Theorem 1.

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