Design of reliability acceptance sampling plans

Thesis submitted to the
Indian Statistical Institute
for the award of the degree

of

Doctor of Philosophy in Quality, Reliability & Operations Research

by

Rathin Das

under the guidance of

Prof. Biswabrata Pradhan

Statistical Quality Control & Operations Research Unit
Indian Statistical Institute, Kolkata

June, 2026

Dedicated To

My Beloved Parents

Shri Rabin Das and Smt. Laxmi Das

Certificate

This is to certify that the thesis entitled **"Design of reliability acceptance sampling plans"** being submitted by **Rathin Das** to the Indian Statistical Institute, Kolkata, India, for the award of a degree of Doctor of Philosophy in Quality, Reliability & Operations Research is a record of bonafide research work carried out by him under my supervision during the last five years. The results embodied in this thesis, to the best of my knowledge, have not been submitted for the award of any other degree/diploma anywhere. In my opinion, the thesis fulfils the requirements for the award of the degree.

(**Prof. Biswabrata Pradhan**)
Statistical Quality Control & Operations Research Unit
Indian Statistical Institute, Kolkata
203, B.T. Road, Kolkata-700108, West Bengal, India.

Acknowledgements

I would like to express my sincere gratitude to all those who have supported and contributed to the successful completion of this research work.

First and foremost, I express my deepest gratitude to my supervisor, Prof. Biswabrata Pradhan, Statistical Quality Control and Operations Research Unit, Indian Statistical Institute, Kolkata, for his invaluable guidance, constant encouragement, and academic support throughout the course of this research. His dedication to academic excellence, insightful suggestions, and constructive criticism have been instrumental in shaping this work. It has been a privilege to work under his supervision.

I would also like to sincerely thank my co-author, Dr. Soumya Roy, for his encouragement, guidance, and constructive suggestions, which have significantly contributed to the outcomes of this research.

I am grateful to the faculty members of the Statistical Quality Control & Operations Research Unit of the Indian Statistical Institute for their cooperation, support, and valuable assistance during the course of my research work.

I acknowledge with thanks the faculty members of the Department of Mathematics at Hooghly Mohsin College and the Indian Institute of Engineering Science and Technology, Shibpur, as well as other teachers who contributed to my academic training, for their guidance and support in helping me develop a strong foundation in Statistics and higher Mathematics. I also gratefully acknowledge my teachers from my early academic years, whose instruction laid the foundation for my academic pursuits.

I would like to thank my collaborators, Dr. Tanmay Sen and Dr. Ritwik Bhattacharya, for their collaboration and shared academic interest in our ongoing and future research work.

I gratefully acknowledge the Indian Statistical Institute for providing the necessary financial assistance for carrying out this research. I also thank the Statistical Quality Control & Operations Research Unit, Indian Statistical Institute, Kolkata, for providing the infrastructure and research facilities essential for this work.

I am deeply indebted to my family for their constant support and encouragement. I am especially thankful to my parents, Mr. Rabin Das and Mrs. Laxmi Das, for their sacrifices and unwavering support. I also thank my sister, Ms. Mowpriya Das, and my brother-in-law, Mr. Sudip Guria, for their encouragement and support throughout this period.

Finally, I would like to thank my friends for their support and encouragement during the course of this research.

(**Rathin Das**)
Indian Statistical Institute, Kolkata

Contents

Chapter 1

Introduction

Sampling plans are important tools in statistical quality control, particularly within the manufacturing industry. They are used to evaluate various characteristics of a lot, such as quality and performance, based on a selected sample. In this method, a sample is drawn from a lot to obtain information about its characteristics. Rather than inspecting every item in a lot, which is often impractical due to time or cost constraints, a representative sample is tested. Based on the data obtained from this sample, appropriate statistical measures are computed to assess the characteristics of the lot under consideration. This information helps to determine whether the lot meets the required standards. Consequently, a decision is made to either accept or reject the lot. This process of making decisions based on sampled data is known as the Acceptance Sampling Plan (ASP). Such methods play a crucial role in manufacturing and related industries, as the acceptance of nonconforming or defective lots can affect the manufacturer's reputation, and future sales.

There are two kinds of ASP, attribute and variable sampling plans.

Attribute sampling plans: The decision to accept or reject the lot is taken based on the number of defective items in the sample. If the number of defective items in the sample is less than or equal to a prespecified number, the lot is accepted; otherwise, it is rejected (see Montgomery [86]).

Variable sampling plans: The decision to accept or reject the lot is taken based on a measurable quality characteristic of interest of the product under consideration. A suitable statistic is computed by using the sample selected from the lot. The lot is accepted or rejected by comparing the statistic with an acceptance limit, π_L, say. The sample size and the acceptance limit must be pre-specified (see Lieberman and Resnikoff [68]).

These traditional ASPs focus on the quality characteristics of the product under consideration at the time of inspection. However, many products, such as military devices, healthcare devices, and electronic appliances, must be evaluated not only based on their initial quality but also on their ability to perform over time. Condra [33] emphasized that "reliability is quality over time". Formally, reliability is defined as *the probability that a product will perform a required function adequately for a given period of time when used under stated operating and environmental conditions.* The reliability and other lifetime characteristics of any product have a great impact on a consumer's decision. The reliability acceptance sampling plan (RASP) is a variant of variable ASP in which the reliability of the item is considered as a quality characteristic to determine lot acceptance or rejection of lot.

In the problem of designing RASP, two fundamental issues must be addressed:

(i) Whether the lot should be accepted or rejected based on the observed data. This involves specifying or determining the decision criterion and the acceptance limit π_L used for lot acceptance or rejection.

(ii) Selecting an appropriate sample size and other design parameters to ensure a reliable decision while maintaining cost effectiveness.

To address these issues, various methods for designing RASP have been developed. For example, producer's and consumer's risk point schemes (see Lieberman and Resnikoff [68]), defense sampling schemes (see Kao [49]), Dodge and Roming's plan (see Dodge and Romig [37]) and decision-theoretic plans (see Wetherill and Chiu [118]).

Among these, in Bayesian approach, decision-theoretic approach is beneficial from an economic point of view. This approach is widely used because it is decided upon by making the best choice based on some economic considerations, such as maximizing the expected utility or minimizing the expected loss, taking into account both prior information and the cost associated with different actions. The principle of Bayesian decision theory is widely discussed by Berger [16]. In the classical (frequentist) approach, the producer's and consumer's risk point scheme is one of the most widely used due to its simplicity and practical appeal. This scheme typically focuses on controlling the producer's risk (probability of rejecting a good lot) and the consumer's risk (probability of accepting a bad lot).

To implement RASP, one must estimate the reliability, which is typically done through life testing. However, due to cost, time, and resource constraints, full inspection of all units may not be feasible. In such situations, censored life-testing is employed, where the test may be stopped before observing all lifetime information of the items in the sample. Designing RASP under censored life data is a challenging task but essential in manufacturing industries.

In recent years, global market competitiveness and rapid technological advancement have pushed manufacturers to produce products with very high reliability. For such products, the mean time to failure under normal operating conditions is often prohibitively long. Consequently, under time constraints, sufficient lifetime information may not be obtained from conventional life testing experiments. In such cases, an accelerated life test (ALT) or a partial accelerated life test (PALT) is used to gather sufficient information within a shorter time. In ALT, items are tested under elevated stress conditions to induce earlier failures, and the results are extrapolated to normal conditions using acceleration models (see Xu et al. [125], Chaloner and Larntz [21], Wu et al. [123]). PALT is a hybrid approach in which items are tested under both normal and elevated stress conditions (see Lim et al. [69], Kumar et al. [54], Zheng and Fang [134]). This approach provides a better balance between cost and precision and has become increasingly important in RASP design.

Furthermore, due to the complexity of modern products, items may fail for multiple causes. For instance, in an automobile, failure may occur due to a surface defect, an interior defect, or both. To investigate the product's failure mechanisms, it is often necessary to record not only the time to failure but also the cause of failure. Such models are known as competing risk models, which are widely used in reliability studies (see Sarhan et al. [109], Azizi et al. [6], Crowder [34], David and Moeschberger [35]). Consequently, the design of RASP for competing risk data has become an important issue in manufacturing industries.

Finally, as products become increasingly complex in structure and highly reliable, failures typically occur only after long periods of operation under normal conditions and may arise from multiple competing causes. In such scenario, ALT or PALT within a competing risks framework is often conducted (see Pascual [94], Zhang et al. [131], Mukhopadhyay and Roy [88]). Thus, the design of RASP for competing risk data under ALT or PALT settings has also emerged as an important issue in manufacturing industries.

The rest of the chapter is organized as follows. Section 1.1 introduces the abbreviations and notation that are commonly used in the thesis. Section 1.2 summarizes the common probability distributions that are generally used in the thesis. Section 1.3 provides a detailed discussion of different censoring schemes. Section 1.4 presents an overview of PALT. Section 1.5 discusses the competing risks model. Section 1.6 discusses warranty analysis. Section 1.7 presents a review of existing works on RASP. Section 1.8 summarizes the remaining chapter. Section 1.8 presents a summary of the chapter. Finally, Section 1.9 lists the publications arising from this thesis.

1.1 Abbreviations and notations

Table 1.1: List of Abbreviations

Abbreviation	**Description**
ALT	Accelerated life test
ASSSPALT	Adaptive simple step-stress partial accelerated life test
ASP	Acceptance sampling plan
BRASP	Bayesian reliability acceptance sampling plan
BSPAA	Bayesian Sampling plan through adaptive simple step-stress partial accelerated life test
CBSP	Conventional Bayesian sampling plan through Non-Accelerated Life Test
CBSPA	Conventional Bayesian sampling plan through simple step-stress partial accelerated life test
CDF	Cumulative distribution function
CEM	Cumulative exposure model
CSH	Cause-specific hazard
CW	Combined warranty
FRW	Free replacement warranty
HCS-I	Type-I hybrid censoring scheme
ICS	Interval censoring scheme

Continued on next page

Table 1.1 – continued from previous page

Abbreviation	Description
IID	Independent and identically distributed
LFT	Latent failure time
MCMC	Markov chain Monte Carlo
MLE	Maximum likelihood estimator
PALT	Partial accelerated life test
PDF	Probability density function
PICS-I	Progressive type-I interval censoring scheme
PMF	probability mass function
PRW	Pro-rata warranty
RASP	Reliability acceptance sampling plan
RDSP	Random decision-making sampling plan
RF	Reliability function
RRS	Relative risk saving
RV	Random variable
SSPALT	Step-stress partial accelerated life test
SSSPALT	Simple step-stress partial accelerated life test
w.r.t.	With respect to

Table 1.2: List of Notations Used Throughout the Thesis

Notation	Description
$\mathcal{A}$	Acceptance decision
$a(\boldsymbol{x} \mid \boldsymbol{\zeta})$	Decision function based on observed data and design parameters
$a^*(\boldsymbol{x} \mid \boldsymbol{\zeta})$	Optimal decision function
$B(\cdot,\cdot)$	Beta function
$\mathcal{C}$	Consumer
C_A	Selling price of the product
C_a	Additional acceleration cost
C_i	Cause of failure of the ith item
C_r	Rejection cost
C_s	Sampling cost per item
C_t	Cost per unit testing time
$C_{CW}(x)$	Warranty cost under combined warranty policy
d	Observed total number of failures
d_i	Observed value of D_i
$d^{(i)}$	Cumulative failures up to τ_i
d_{mj}	Observed number of failures due to cause j during interval m

Continued on next page

Table 1.2 – continued from previous page

Notation	**Description**
D	Total number of failures during life testing
D_i	Number of failures in the interval $(\tau_{i-1}, \tau_i]$
D_{mj}	Number of failures due to cause j during interval m
$E[D]$	Expected number of failures during life testing of the proposed model
$E[D^*]$	Expected number of failures during life testing at optimal design of the proposed model
$E[I]$	Expected number of inspections during life testing of the proposed model
$E[I^*]$	Expected number of inspections during life testing at optimal design of the proposed model
$E[\tau]$	Expected duration of the life test of the proposed model
$E[\tau^*]$	Expected duration of the life test at optimal design of the proposed model
$F_0(\cdot \mid \cdot)$	Lifetime CDF under normal stress level
$F_1(\cdot \mid \cdot)$	Lifetime CDF under accelerated stress level
$G(j, x \mid \cdot)$	Sub-distribution function for cause j
$g(j, x \mid \cdot)$	Sub-density function
$\Gamma(\cdot)$	Gamma function
$\gamma(x, \cdot, \cdot)$	PDF of the gamma distribution
$h(\boldsymbol{\theta})$	Loss function associated with model parameters
I	Number of inspections during life testing
$I_x(\cdot, \cdot)$	CDF of the beta distribution
J	Number of competing causes of failure
k	Number of inspections
$L(\boldsymbol{\theta} \mid \boldsymbol{x})$	Likelihood function
$\mathcal{M}$	Manufacturer
m	Failure threshold used in adaptive testing
n	Sample size
N_i	Number of units at risk at the beginning of interval i
$p(\boldsymbol{\theta})$	Prior distribution of θ
$p(\boldsymbol{\theta} \mid \boldsymbol{x})$	Posterior distribution of θ
$p_{\mathcal{C}}(\boldsymbol{\theta})$	Prior distribution of θ w.r.t consumer
$p_{\mathcal{C}}(\boldsymbol{\theta} \mid \boldsymbol{x})$	Posterior distribution of θ w.r.t consumer
$p_{\mathcal{M}}(\boldsymbol{\theta})$	Prior distribution of θ w.r.t manufacturer
$p_{\mathcal{M}}(\boldsymbol{\theta} \mid \boldsymbol{x})$	Posterior distribution of θ w.r.t manufacturer
$\mathcal{R}$	Rejection decision
$R_B(\boldsymbol{\zeta}, a)$	Bayes risk corresponding to design ζ and decision rule a

Continued on next page

Table 1.2 – continued from previous page

Notation	Description
r	Required number of failures under Type-II censoring
R_i	Number of withdrawn units at inspection time τ_i
r_i	Observed value of R_i
$r^{(i)}$	Cumulative withdrawals up to τ_i
$R_j(x)$	Cause-specific reliability function
s_0	Normal stress level
s_1	Accelerated stress level
v_s	Salvage value per surviving item
w	Warranty period
w_1, w_2	Warranty limits under combined warranty policy
X_i	Lifetime of the ith item
X_{ij}	Latent lifetime of item i due to cause j
$X_{(i)}$	ith order statistic
$\boldsymbol{x}$	Observed life-test data
$x_{(i)}$	Observed value of $X_{(i)}$
$\mathcal{X}$	Acceptance region
Z	Shared frailty variable
ζ	Vector of design parameters of the life-testing plan
$\boldsymbol{\zeta}^*$	Optimal design parameter vector
$\boldsymbol{\theta}$	Vector of model parameters
λ	Rate parameter
λ_0	Rate parameter under normal stress level
λ_1	Rate parameter under accelerated stress level
λ_{0j}	Cause-specific hazard rate for cause j
$\lambda_j(x)$	Cause-specific hazard function
$\Lambda_j(x)$	Cumulative hazard function
ν	Shape parameter
$\Phi(\cdot)$	Standard normal cumulative distribution function
τ	Duration of life testing
τ_0	Starting time of the life test
τ_1	Inspection or stress-change time
τ_i	Inspection time ($i = 1, \ldots, k$)
$\psi_{\mathcal{M}}(\cdot)$	Manufacturer's expected utility

1.2 Summary of common probability distributions

This section summarizes commonly used probability distributions along with their notation, probability density functions (PDFs) and cumulative distribution functions (CDFs).

1.2.1 Normal distribution

Consider a random variable (RV) X that follows a normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$. Its PDF is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}.$$

The corresponding CDF is

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

where $\Phi(\cdot)$ denotes the standard normal CDF and it is defined by

$$\Phi(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.$$

This distribution is denoted by $\mathcal{N}(\mu, \sigma^2)$.

1.2.2 Exponential distribution

Consider RV X that follows an exponential distribution with rate parameter $\lambda > 0$. Its PDF is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

The corresponding CDF is

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

This distribution is denoted by $\mathrm{Exp}(\lambda)$.

1.2.3 Gamma distribution

Consider RV X that follows a gamma distribution with shape parameter $\nu > 0$ and rate parameter $\lambda > 0$. Its PDF is given by

$$f(x) = \gamma(x, \nu, \lambda) = \frac{\lambda^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\lambda x}, \quad x \geq 0,$$

where $\Gamma(\cdot)$ denotes the gamma function. The corresponding CDF is

$$F(x) = \frac{\Gamma_1(\nu, \lambda x)}{\Gamma(\nu)},$$

where $\Gamma(\cdot, \cdot)$ denotes the lower incomplete gamma function, and it is defined by

$$\Gamma_1(a, x) = \int_0^x u^{a-1} e^{-u} du.$$

It is noted that $\Gamma_1(a, \infty) = \Gamma(a)$. This distribution is denoted by $G(\nu, \lambda)$.

1.2.4 Inverse Gamma distribution

Consider RV X that follows an inverse gamma distribution with parameters $\lambda > 0$ and $\nu > 0$. Its PDF is given by

$$f(x) = \frac{\lambda^\nu}{\Gamma(\nu)} x^{-(\nu+1)} \exp\left(-\frac{\lambda}{x}\right), \quad x > 0.$$

The corresponding CDF is

$$F(x) = 1 - \frac{\gamma\left(\nu, \frac{\lambda}{x}\right)}{\Gamma(\nu)}.$$

This distribution is denoted by $IG(\nu, \lambda)$.

1.2.5 Beta distribution

Consider RV X that follows a beta distribution with parameters $\lambda > 0$ and $\nu > 0$. Its PDF is given by

$$f(x) = \frac{1}{\mathrm{B}(\nu,\lambda)} x^{\nu-1}(1-x)^{\lambda-1}, \quad 0 \le x \le 1,$$

where $\mathrm{B}(\nu,\lambda)$ denotes the beta function. The corresponding CDF is

$$F(x) = I_x(\nu,\lambda) = \frac{1}{\mathrm{B}(\nu,\lambda)} B_x(\nu,\lambda),$$

where $B_x(\nu,\lambda)$ denotes the lower incomplete beta function, and it is defined by

$$B_x(\nu,\lambda) = \int_0^x u^{\nu-1}(1-u)^{\lambda-1}$$

It is noted that $B_1(\nu,\lambda) = B(\nu,\lambda)$. This distribution is denoted by $\mathrm{B}(\nu,\lambda)$.

1.2.6 Uniform distribution

Consider RV X that follows a uniform distribution with parameters $a, b \in \mathbb{R}$, where $a < b$. Its PDF is given by

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b.$$

The corresponding CDF is

$$F(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \le x \le b, \\ 1, & x > b. \end{cases}$$

This distribution is denoted by $\mathcal{U}(a,b)$

1.2.7 Weibull distribution

Consider RV X that follows Weibull distribution with shape parameter $\nu > 0$ and rate parameter $\lambda > 0$. Its PDF is given by

$$f(x) = \nu\lambda\,(\lambda x)^{\nu-1} \exp[-(\lambda x)^{\nu}], \quad x \ge 0.$$

The corresponding CDF is

$$F(x) = 1 - \exp[-(\lambda x)^{\nu}].$$

This distribution is denoted by $\mathrm{Wei}(\nu,\lambda)$.

1.3 Different censoring schemes

Suppose n items are put on life test at time $\tau_0 = 0$. The lifetimes of n items $X_1, \ldots, X_n$ are independent and identically distributed (IID) with common CDF $F(\cdot \mid \boldsymbol{\theta})$, PDF $f(\cdot \mid \boldsymbol{\theta})$ and reliability function (RF) $R(\cdot \mid \boldsymbol{\theta}) = 1 - F(\cdot \mid \boldsymbol{\theta})$. Let $X_{(1)}, \ldots, X_{(n)}$ be the ordered lifetime of n items and $x_{(i)}$ is the realization of $X_{(i)}$, for $i = 1, \ldots, n$. Let D and τ denote the number of failures during the life test and the duration of the life test, respectively. Let d be the observed value of D.

The observed data vector of the proposed sampling plan is denoted by $\boldsymbol{x}$. This thesis considers the type-I censoring scheme, the type-II censoring scheme, Type-I hybrid censoring scheme (HCS-I), interval censoring scheme (ICS), and progressive type-I interval censoring scheme (PICS-I). The details of these censoring schemes are discussed in Sections 1.3.1 to 1.3.5, respectively.

1.3.1 Type-I censoring

The life test is terminated at a predetermined time τ_1 in the type-I censoring scheme. The observed data can be written as

$$\boldsymbol{x} = \begin{cases} (d=0) & \text{if } x_{(1)} > \tau_1 \\ (x_{(1)}, x_{(2)}, \ldots, x_{(d)}, d) & \text{if } x_{(d)} < \tau_1 < x_{(d+1)},\ 0 < d < n \\ (x_{(1)}, x_{(2)}, \ldots, x_{(n)}, d = n) & \text{if } x_{(n)} < \tau_1, \end{cases}$$

where d is the observed number of failures up to the time point τ_1. A schematic representation is given in Figure 1.1. If all items fail before the time point τ_1, then the test stops at $X_{(n)} < \tau_1$. Therefore, the duration of the life test is $\tau = \min\{X_{(n)}, \tau_1\}$.

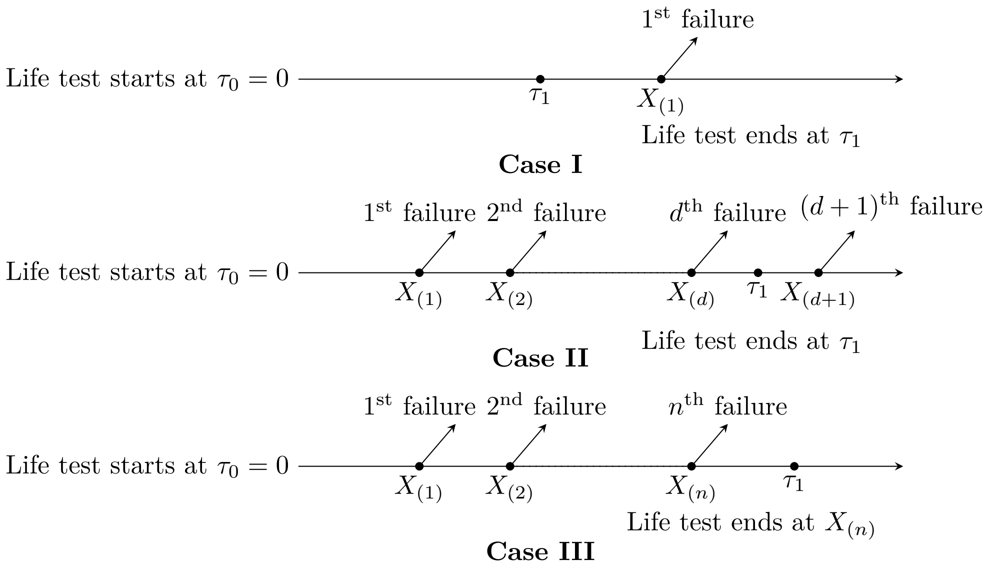


Figure 1.1: Schematic diagram of type-I censoring.

1.3.2 Type-II censoring

The life test is terminated after a fixed number of failures $r < n$. The observed data can be written as $\boldsymbol{x} = (x_{(1)}, \ldots, x_{(r)})$ and the duration of the life test is $\tau = X_{(r)}$. A schematic representation is given in Figure 1.2.

1.3.3 Type-I hybrid censoring

The drawback of the type-I censoring scheme is that very few or no failures could be observed. Therefore, we can obtain a poor estimation of model parameters. Also, the drawback of the type-II censoring scheme is that r^{th} failure can occur after a very long time. Epstein [38] proposed HCS-I,

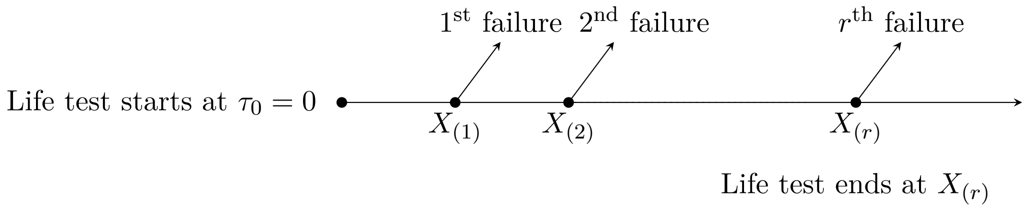


Figure 1.2: Schematic representation of a type-II censored life test.

which is a combination of type-I and type-II censoring schemes. It allows for limiting both the number of failures and testing time. The life test is terminated after the failure of r^{th} or at the time τ_1, whichever occurs earlier. For HCS-I, $\tau = \min\{X_{(r)}, \tau_1\}$. The observed data under HCS-I can be written as

$$\boldsymbol{x} = \begin{cases} (d = 0), & \text{if } x_{(1)} > \tau_1 \\ (x_{(1)}, x_{(2)}, \ldots, x_{(d)}, d), & \text{if } x_{(d)} < \tau_1 < x_{(d+1)} < \ldots < x_{(r)},\ d < r \\ (x_{(1)}, x_{(2)}, \ldots, x_{(r)}, d = r), & \text{if } x_{(r)} < \tau_1. \end{cases} \tag{1.3.1}$$

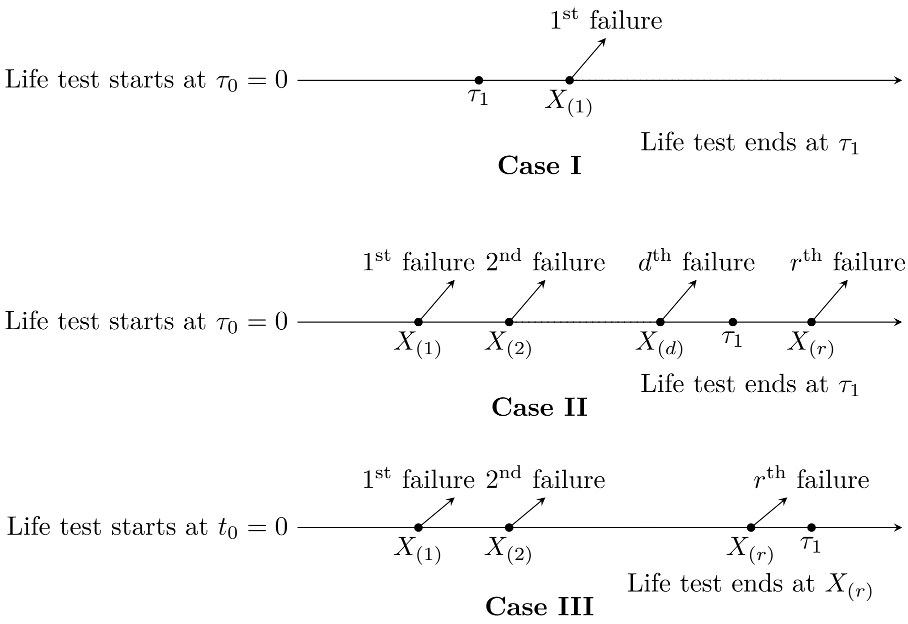


Figure 1.3: Schematic diagram of HCS-I

A schematic representation is given in Figure 1.3. Note that if we take $r = n$, the HCS-I is converted to a type-I censoring scheme and if $\tau_1 = \infty$, the HCS-I is converted to a type-II censoring scheme.

1.3.4 Interval censoring

In many situations, it is not possible to monitor the test continuously, and the items on the test are inspected at some pre-specified inspection times. Here, the failure times are not observed, but only the number of failures is observed at each inspection time. Such a test is called the interval-censored test, and the collected data are called interval-censored data (see Lu and Tsai Lu and Tsai [80]). In ICS, all items are put on test at $\tau_0 = 0$, and the items on the test are inspected at pre-determined k inspection times $\tau_1 < \cdots < \tau_k$. The test stops at τ_k, which leads to censoring all items surviving at τ_k. Let N_i denote the number of units that are at risk of failure at the start of the ith interval $(\tau_{i-1}, \tau_i]$ and d_i be the observed number of failures in the interval $(\tau_{i-1}, \tau_i]$, for $i = 1, \ldots, k$. The observed data can be written as

$$\boldsymbol{x} = \begin{cases} (d_1, \ldots, d_k), & \text{if } d(k-1) < n \;\&\; d(k) \leq n \\ (d_1, \ldots, d_i), & \text{if } d(i) = n, \end{cases}$$

where $d(i) = d_1 + \cdots + d_i$, for $i = 1, \ldots, k$. A schematic representation is given in Figure 1.4.

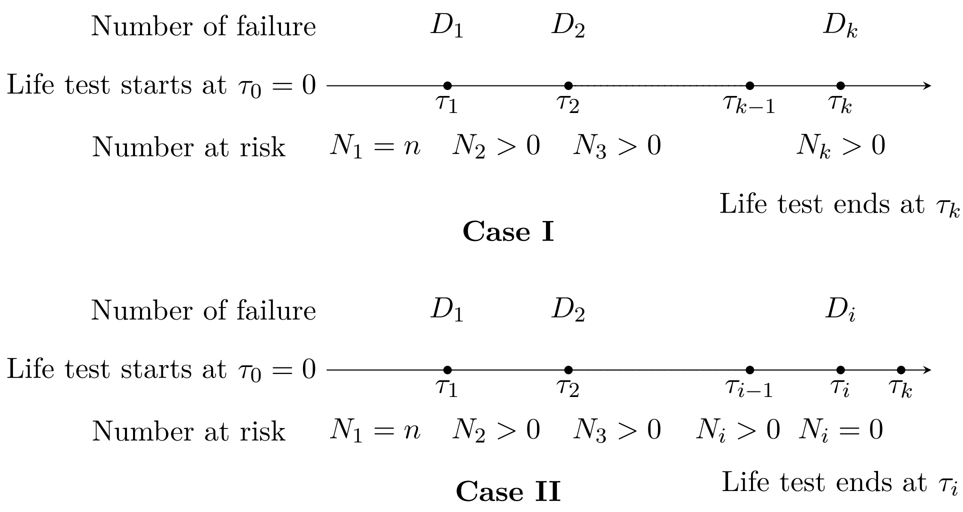


Figure 1.4: Schematic representation of interval censoring scheme.

1.3.5 Progressive type-I interval censoring

One of the drawbacks of this conventional censoring scheme is that no items can be removed at any time point other than the final termination point of the experiment. However, it is also possible that some items need to be removed at any point during life testing due to accidental breakage. This type of loss may occur unintentionally. But sometimes we remove the items from the experiment intentionally to save time and cost of the experiment, or they can be used for another test. For details on the importance of removing items during a life-testing experiment (See Balakrishnan and Aggarwala [8]). Since conventional censoring schemes cannot accommodate these scenarios, progressive censoring schemes were introduced by Cohen [32]. Type I and type II progressive censoring schemes are the most common progressive censoring schemes.

A similar limitation exists in conventional interval censoring schemes. However, such intermediate withdrawals may be unavoidable for various practical reasons [60]. For example, consider the life test performed on battery cells, as discussed by Meeker et al. [82, p. 633]. Some battery cells are withdrawn from the ongoing experiment to assess the extent of physical degradation. To accommodate such intermediate withdrawals of test units from the life test, PICS-I are proposed in the reliability literature [1]. The scheme can be described as follows.

All times are put on the test at $\tau_0 = 0$. The units are inspected at the pre-decided inspection times $\tau_1 < \tau_2 < \ldots < \tau_k$, where τ_k is the pre-fixed termination time of the experiment. Note that, in ICS, only the number of failures is observed at τ_i. Furthermore, in an k point PICS-I, R_i surviving items are removed from the life test at inspection time τ_i, for $i = 1, \ldots, k-1$. Subsequently, all remaining test units, say R_k, are removed from the life test at τ_k. For convenience, let $\boldsymbol{D} = (D_1, \ldots, D_k)$ and $\boldsymbol{R} = (R_1, \ldots, R_k)$ denote the vectors of number of failures and number of withdrawals for the k-point PICS-I, respectively. Let r_i be the observed value of R_i, $d(i) = d_1 + \cdots + d_i$ be the total number of failures up to time τ_i, and $r(i) = r_1 + \cdots + r_i$ be the total number of items withdrawn up to time τ_i. The observed data can be written as

$$\boldsymbol{x} = \begin{cases} (d_1, r_1, d_2, r_2, \ldots, d_k, r_k), & \text{if } d(k-1) + r(k-1) < n \ \& \ d(k) + r(k) = n \\ (d_1, r_1, d_2, r_2, \ldots, d_i, r_i), & \text{if } d(i) + r(i) = n. \end{cases}$$

A schematic representation is given in Figure 1.5.

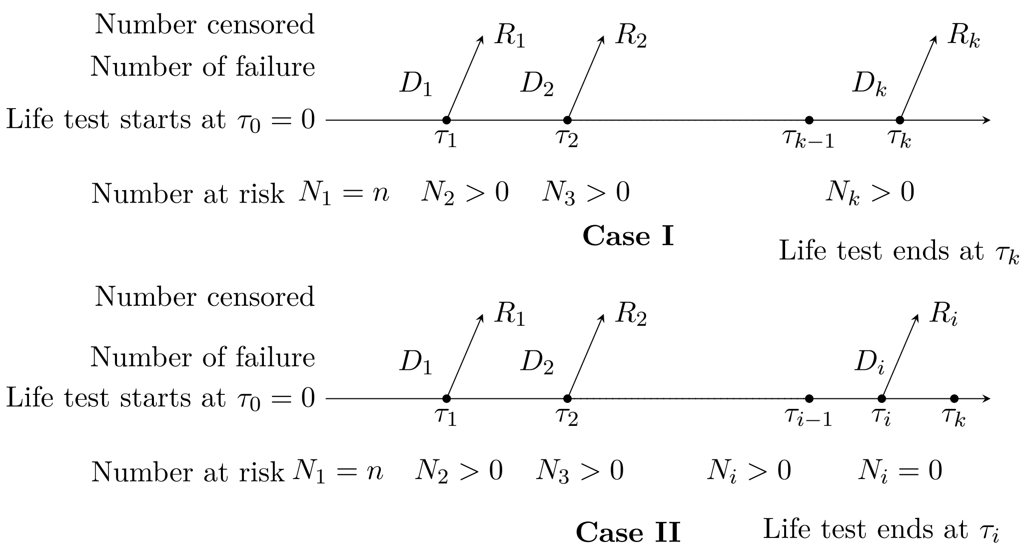


Figure 1.5: Schematic diagram of progressive type-I interval censoring scheme.

1.4 Partial accelerated life testing (PALT)

In reliability engineering, it is often impractical to obtain sufficient lifetime data at normal-use stress levels, since many highly reliable products operate for long periods without failure. Collecting such data under ordinary conditions would require prohibitively long test durations and high costs. To address this issue, ALT are employed, in which products are tested at higher stress levels so that failures occur more quickly and sufficient information can be gathered within a feasible timeframe (see Pham [95]). For comprehensive reviews, see Nelson [91], Bagdonavicius and Nikulin [7] and Kundu and Ganguly [57]. In ALT, inference at normal-use conditions is obtained through extrapolation models that relate lifetime behavior at accelerated conditions to the normal operating condition. However, ALTs depend on extrapolation models to relate accelerated and normal operating conditions, which may not be reliable for new products with limited data at normal stress levels. To address this limitation, the experiment is conducted under both normal operating conditions and higher stress levels in PALTs. One effective PALT strategy is Step-stress partially accelerated life testing (SSPALT),

1.4.1 Step-stress partially accelerated life testing (SSPALT)

In SSPALT, the experimenter changes the level of the stress factors in a sequential manner during the experiment. In SSPALT, all units are put on a life test at normal operating conditions s_0, then shifted to s_1 at time τ_1, to s_2 at τ_2, and so on. Formally,

$$s(x) = \begin{cases} s_0, & \tau_0 \leq x < \tau_1, \\ \vdots \\ s_k, & x \geq \tau_k. \end{cases}$$

The one-step case $k = 1$ is called simple step-stress partial accelerated life test (SSSPALT). Generally, SSSPALT is used in our thesis. In SSSPALT, the life test is conducted under the stress level s_0 at $\tau = 0$. After a time τ_1, the stress level is increased to the stress level s_1.

1.4.2 Cumulative exposure model (CEM)

Under step-stress testing, the stress level varies over time, and an appropriate model is required to describe how accumulated stress affects the lifetime of a product. Among the different accelerated-life models, the most widely used is CEM, first proposed by Sedyakin [111] and later studied extensively by Nelson [91] and Kundu and Ganguly [57]. CEM assumes that the residual life of a unit depends only on the cumulative exposure experienced so far, with no memory of how this exposure was accumulated. At any fixed stress level, units fail according to the CDF at that stress, but the overall lifetime distribution remains continuous across stress changes.
Let $F(\cdot \mid \boldsymbol{\theta})$ denote CDF of the lifetime of items under this SSSPALT and $F_i(\cdot \mid \boldsymbol{\theta}_i)$ be the CDF of the lifetime of the item under the stress level s_i, for $i = 0, 1$. Thus, CEM can be interpreted as time-shifting each subsequent stress-level distribution so that the combined CDF $F(x \mid \boldsymbol{\theta})$ remains continuous across the stress-change points.

- For $0 \leq x \leq \tau_1$, failures occur according to the CDF at the normal operating conditions s_0:

$$F(x \mid \boldsymbol{\theta}) = F_0(x \mid \boldsymbol{\theta}_1).$$

- For $x \geq \tau_1$, when stress changes from s_0 to s_1 at time τ_1, the effect is equivalent to a time shift of the CDF F_1. i.e,

$$F(x \mid \boldsymbol{\theta}) = F_1(x - h_1 \mid \boldsymbol{\theta}_1), \quad x \geq \tau_1,$$

  where h_1 is chosen to preserve continuity at τ_1. Therefore, h_1 can be obtained as the solution of the equation

$$F_1(\tau_1 - h_1 \mid \boldsymbol{\theta}_1) = F_0(\tau_1 \mid \boldsymbol{\theta}_0)$$

Therefore, CDF of the lifetime of the item under SSSPALT can be written as

$$F(x \mid \boldsymbol{\theta}) = \begin{cases} F_0(x \mid \boldsymbol{\theta}_0), & 0 \leq x \leq \tau_1 \\ F_1(x - h_1 \mid \boldsymbol{\theta}_1), & x \geq \tau_1. \end{cases}$$

- **Exponential distribution:** Let hazard rate at the stress level s_i be λ_i, $i = 0, 1$. Therefore, h_1 can be obtained by solving the equation

$$\begin{aligned} 1 - \exp[-\lambda_1(\tau_1 - h_1)] &= 1 - \exp[-\lambda_0 \tau_1] \\ \implies h_1 &= \tau_1 - \frac{\lambda_0}{\lambda_1}\tau_1. \end{aligned}$$

  Therefore, CDF under SSSPALT is given by

$$F(x \mid \boldsymbol{\theta}) = \begin{cases} 1 - \exp[-\lambda_0 x] & \text{for } 0 < x \leq \tau_1 \\ 1 - \exp\left[-\lambda_0 \tau_1 - \lambda_1(x - \tau_1)\right]. \end{cases} \tag{1.4.1}$$

- **Weibull distribution:** Let the lifetime of the item at the stress level s_i follows Weibull distribution with CDF

$$F_i(x \mid \boldsymbol{\theta}_i) = 1 - \exp[-(\lambda_i x)^{\nu}],$$

  for $i = 0, 1$. Similar to exponential distribution we get $h_1 = \tau_1 - \frac{\lambda_0}{\lambda_1}\tau_1$. Therefore, CDF under SSSPALT is given by

$$F(x \mid \boldsymbol{\theta}) = \begin{cases} 1 - \exp[-(\lambda_0 x)^{\nu}] & \text{for } 0 < x \leq \tau_1 \\ 1 - \exp\left[-(\lambda_0 \tau_1 - \lambda_1(x - \tau_1))^{\nu}\right] & \text{for } x > \tau_1. \end{cases} \tag{1.4.2}$$

1.5 Competing risks

Modern industrial products, characterized by ever-increasing structural complexity and operation under diverse environmental conditions, often fail due to multiple distinct causes. For example, a vehicle component might fail due to either a surface defect or an internal flaw. In order to address the presence of multiple causes of failure, reliability engineers employ competing risk models, which consider both the time to failure and the cause of failure (Kalbfleisch and Prentice [48], Lawless [59], Crowder [34], David and Moeschberger [35]). The two primary approaches for modeling competing risks data are the cause-specific hazard approach and the latent failure time approach.

Cause-specific hazard (CSH) approach

Let X denote the overall failure time and $C \in \{1, \ldots, J\}$ the cause of failure. The cause-specific hazard for cause j is defined as

$$\lambda_j(x) = \lim_{\Delta x \to 0} \frac{P(x \leq X < x + \Delta x,\, C = j \mid X \geq x)}{\Delta x},$$

which characterizes the instantaneous risk of failing from cause j at time x, given survival up to x. The joint distribution of (X, C) is determined by the collection $\{\lambda_j(x)\}_{j=1}^{J}$. The overall hazard function is $\lambda(x) = \sum_{j=1}^{J} \lambda_j(x)$. Then the RF of X is

$$R(x \mid \boldsymbol{\theta}) = P(X > x \mid \boldsymbol{\theta}) = \exp\left\{-\int_0^x h(u)du\right\} = \exp\left\{-\int_0^x \sum_{j=1}^{J} \lambda_j(x)(u)du\right\},$$

where θ is a vector of all parameters. This follows from standard hazard–survival theory, since the total instantaneous risk of failure at time x is the sum of the cause-specific risks.

Latent failure time (LFT) approach

In LFT framework, one assumes latent failure times X_{ij} for item i due to cause j, with $i = 1, \ldots, n$ and $j = 1, \ldots, J$. The observed failure time and cause are

$$X_i = \min_{1\leq j\leq J} X_{ij}, \qquad C_i = \arg\min_{1\leq j\leq J} X_{ij},$$

so that $X_i = X_{iC_i}$. For each cause j, let $f_{0j}(x \mid \boldsymbol{\theta}_{0j}), F_{0j}(x \mid \boldsymbol{\theta}_{0j}), R_{0j}(x \mid \boldsymbol{\theta}_{0j}) = 1 - F_{0j}(x \mid \boldsymbol{\theta}_{0j})$ denote PDF, CDF and RF of X_{ij}, respectively, where $\boldsymbol{\theta}_{0j}$ is the parameter vector associated with cause j under normal operating condition s_0.

The joint RF of the latent lifetimes $\overline{\boldsymbol{X}}_i = (X_{i1}, \ldots, X_{iJ})$ is

$$\overline{R}(\boldsymbol{x} \mid \boldsymbol{\theta}) = \Pr[X_{i1} > x_1, \ldots, X_{iJ} > x_J],$$

where $\boldsymbol{x} = (x_1, \ldots, x_J)$ and $\boldsymbol{\theta} = (\theta_{01}, \ldots, \theta_{0l})$ is vector of all parameters.

The subdensity for the observed pair $\{C_i = j, X_i = x\}$ is

$$g(j, x \mid \boldsymbol{\theta}) = \left[-\frac{\partial \overline{R}(\boldsymbol{x} \mid \boldsymbol{\theta}_0)}{\partial x_j}\right]_{x\mathbf{1}_J},$$

and the overall RF is

$$R(x \mid \boldsymbol{\theta}) = \overline{R}(x\mathbf{1}_J \mid \boldsymbol{\theta}),$$

where $\mathbf{1}_J = (1, \ldots, 1)$ denotes the J-dimensional vector of ones and $[\cdot]_{x\mathbf{1}_J}$ indicates evaluation at $\boldsymbol{x} = x\mathbf{1}_J = (x, \ldots, x)$. Equivalently, the distribution of (X, C) may be expressed via the sub-distribution function

$$G(j, x \mid \boldsymbol{\theta}) = P(C = j,\ X \leq x),$$

or through the sub-survivor function

$$\overline{G}(j, x \mid \boldsymbol{\theta}) = P(C = j,\ X > x) = \int_x^\infty g(j, u \mid \boldsymbol{\theta})\, du,$$

see Crowder [34, Chapter 3].

Independent risks

If all risks are independent, the joint survival decomposes as

$$\overline{R}(\boldsymbol{x} \mid \boldsymbol{\theta}) = \prod_{j=1}^{J} R_j(x_j \mid \boldsymbol{\theta}_{0j}).$$

Hence,

$$R(x \mid \boldsymbol{\theta}) = \prod_{j=1}^{J} R_j(x \mid \boldsymbol{\theta}_{0j}),$$

and the subdensity due to cause j simplifies to

$$g(j, x \mid \boldsymbol{\theta}) = f_j(x \mid \boldsymbol{\theta}_{0j}) \prod_{i \neq j} R_i(x \mid \boldsymbol{\theta}_{0i}) = R(x \mid \boldsymbol{\theta})\, \lambda_j(x)(x \mid \boldsymbol{\theta}_{0j}).$$

Thus, the subdensity equals the overall survival multiplied by CSH.

Dependent risks

When risks are dependent, alternative formulations are required. Two widely used approaches are based on copulas (see Nelsen [90]) and frailties (see Wienke [119], Hanagal [42]).

Copula models

By Sklar's theorem [114], the joint RF $\overline{R}(\cdot \mid \boldsymbol{\theta})$ can be expressed in terms of the marginal survivals $R_j(\cdot \mid \boldsymbol{\theta}_{0j})$ and a copula function $\overline{C}(\cdot)$ [92]:

$$\overline{R}(\boldsymbol{x} \mid \boldsymbol{\theta}) = \overline{C}(R_1(x_1 \mid \boldsymbol{\theta}_{01}), \ldots, R_J(x_J \mid \boldsymbol{\theta}_{0J}))\,, \quad \boldsymbol{x} \geq \mathbf{0}.$$

Frailty models

Another common approach is to introduce a shared unobserved random effect, or frailty, denoted Z. Suppose the cumulative hazards $\Lambda_1(x_1), \ldots, \Lambda_J(x_J)$ share a common frailty by replacing $\Lambda_j(x_j)$ with $Z\Lambda_j(x_j)$. Conditional on $Z = z$, the latent failure times are independent and the joint RF is

$$\overline{R}(\boldsymbol{x} \mid z, \boldsymbol{\theta}) = \exp\left(-z \sum_{j=1}^{J} \Lambda_j(x_j)\right).$$

If $\mathcal{F}$ is the distribution of Z over $(0, \infty)$, the unconditional joint RF is

$$\overline{R}(\boldsymbol{x} \mid \boldsymbol{\theta}) = \int_0^\infty \exp\left(-z \sum_{j=1}^{J} \Lambda_j(x_j)\right) d\mathcal{F}(z).$$

1.6 Warranty policy

A warranty is a contractual commitment by the manufacturer that guarantees repair, replacement, or monetary compensation if a product fails within a specified warranty period. For non-repairable products, the repair option is infeasible; therefore, warranties for such products commonly involve reimbursement in the form of rebates. Under rebate-based warranty policies, if a failure occurs within the warranty period, the manufacturer compensates the consumer either fully or proportionally.

There are three types of warranty policies: free replacement warranty (FRW), pro-rata warranty (PRW), and combined FRW and PRW (CW). We consider the warranty defined in the interval

$[0, w]$. Under FRW policy, if a product fails during the warranty period, the manufacturer gives the consumer a full refund (see Murthy and Blischke [89]). Let C_A be the sales price of the product. The cost of reimbursing an item that fails at time x is given by

$$C_{PRW}(x) = C_A, \quad \text{if } 0 \leq x \leq w.$$

In contrast, under PRW policy, the manufacturer provides compensation proportional to the remaining age (or mileage) of the product if it fails during the warranty period (see Menke [84]). A common form of this compensation is linear, so that the cost of reimbursing an item that fails at time x is

$$C_{PRW}(x) = C_A \left(1 - \frac{x}{w}\right), \quad \text{if } 0 \leq x \leq w.$$

Under CW policy, the warranty period is defined over $[0, w_2]$. A point w_1 is specified with $w_1 \leq w_2$, such that the FRW policy applies to failures in $[0, w_1]$, while the PRW policy applies to failures in $(w_1, w_2]$ (see Thomas [115]). The cost of reimbursing an item that fails at time x is

$$C_{CW}(x) = \begin{cases} C_A, & \text{if } 0 \leq x \leq w_1 \\ C_A \left(\dfrac{w_2 - x}{w_2 - w_1}\right), & \text{if } w_1 \leq x \leq w_2 \end{cases}$$

Note that it reduces to FRW policy when $w_1 = w_2$ and reduces to PRW policy when $w_1 = 0$.

1.7 Literature review on RASP

There are numerous works on RASPs under different censoring schemes using the classical (producer's and consumer's risk) approach. Schneider [110] proposed RASP for log-normal and Weibull distributions under type-II censoring. Balasooriya et al. [14], Balasooriya and Saw [13], Singh and Tripathi [112] and Balasooriya and Balakrishnan [12] considered RASPs for the Weibull, two-parameter exponential, and log-normal distributions under PCS-I, respectively. In these sampling plans the authors described the method of determining the sample size n and an acceptability constant, say π_L, given the remaining design parameters, so as to obtain RASP that satisfies the constraints imposed by producer's and consumer's risks. Those remaining parameters are usually set by the experts and therefore may not be optimal in any particular sense. Bhattacharya et al. [18] addressed the determination of optimal design-parameter values for RASPs by minimizing a variance measure subject to a cost constraint using the producer's risk and consumer's risk framework under HCS-I. Huang and Wu [45] presented a cost-function-based approach to obtain optimal RASPs for the exponential lifetime model.

There have been a considerable number of works on determination of BRASPs under various censoring schemes. For example, Yeh [127, 128] and Yeh and Choy [129] studied optimal RASP design using Bayesian decision theory under type-II, type-I censoring and random censoring, respectively. Chen et al. [23] developed RASPs for HCS based on the Kaplan–Meier estimator of reliability as a decision function, while Lin et al. [72, 73] considered HCS plans using the maximum likelihood estimator (MLE) of the mean lifetime. These sampling plans for censored data have not considered the prior belief of the parameter in the decision function.

Papazoglou [93] observed that decisions can be made from existing prior assessments. However, the consumer cannot be certain about acceptance or rejection of the lot using their prior information. Hence, additional information is often obtained through life testing at a cost, and the test results are combined with prior information to form an updated decision. Lin et al. [74] introduced a Bayes decision function that uses existing prior assessments. The Bayes decision framework was also studied by Chen et al. [24] under type-II censoring, Liang and Yang [67] under HCS, Prajapati et al. [98] under generalized HCS-I, and Chen et al. [29] under ICS. For progressive HCS-I and HCS-II, Bayes decision functions were considered by Liang [64, 65].

In these sampling plan procedures, the manufacturer takes the decision of acceptance or rejection of the lot only at the completion of the life test. Given that life testing is often time-consuming and costly, early decision is desirable whenever the accumulated failure information is sufficient before completion of the life test to take a proper decision. In the classical approach, a sequential testing procedure of RASP is studied by many authors (see Aslam et al. [4], Jun et al. [47], Rasay et al. [102], Balamurali et al. [11], Aslam and Jun [3]). In Bayesian approach, Tsai et al. [116] proposed a sequential testing procedure for the problem of designing BRASP under type-I censoring where a decision can be taken before the end of the life test. This BRASP is known as curtailment BRASP (see Kundu [56]). The curtailment BRASP was studied by Chen et al. [30] under type-II censored data and by Chen et al. [25] under HCS-I.

Many of these BRASPs minimize the total expected cost with respect to a common prior for the lifetime parameters and a utility function agreed by both manufacturer and consumer. However, the consumer and manufacturer may be adversarial, and they can agree on a common lifetime distribution but differ on prior distributions and utility functions. In that case, the consumer can take the decision to accept or reject a lot based on the existing information about the product. If the decision is rejection, the manufacturer and consumer agree for a life test to obtain additional information on lifetime. The consumer takes the decision on accepting or rejecting the lot based on the updated lifetime information obtained from the life test.

Lindley and Singpurwalla [75, 76] developed a Bayesian procedure for this adversarial scenario. Lindley and Singpurwalla [75] discussed testing in the attribute acceptance sampling plan for Bernoulli, Poisson, and normal models, and Lindley and Singpurwalla [76] later considered a sequential complete life test. Rufo et al. [107] extended these ideas to the exponential family, and Rufo et al. [108] proposed a Bayesian sequential negotiation model for multiple parties negotiating price and product quality.

The aforementioned BRASP studies under various censoring schemes assumed items are tested under normal-use conditions. In the classical literature, many works focus on RASPs based on ALT. For example, Kim and Yum [50] and Li et al. [63] studied RASPs under type-I censoring for the Weibull and log-normal distributions, respectively, while Lou et al. [79] and Ramyamol and Kumar [101] also addressed RASPs based on ALT. Designing BRASPs based on ALT or PALT with censored samples has received less attention. Recently, Chen et al. [26, 27] and Prajapati and Kundu [97] studied BRASP design under SSSPALT with type-II censored data, and Chen et al. [28] investigated curatailment BRASP design under SSSALT with complete data.

In the sampling plans discussed so far, it has been assumed that products fail due to only one cause. However, in many real situations, a product may fail for more than one reason, meaning that a competing-risks model is needed. Research on RASPs for competing-risk data is more limited. Wu and Huang [122] first studied RASPs when items may fail from different causes and assumed that the lifetimes for each cause follow independent exponential distributions. Later, Prajapati et al. [99] proposed optimal BRASPs for competing-risk data under type-II censoring and HCS-I. More recently, Liang et al. [66] developed optimal curtailed BRASPs for competing-risk data, which allow decisions to be made early before the test ends, helping to reduce testing time and cost.

1.8 The summary of the thesis

First, the relationships among the chapters of the thesis are illustrated in Figure 1.6. Then, a chapter-wise summary of the thesis is provided below.

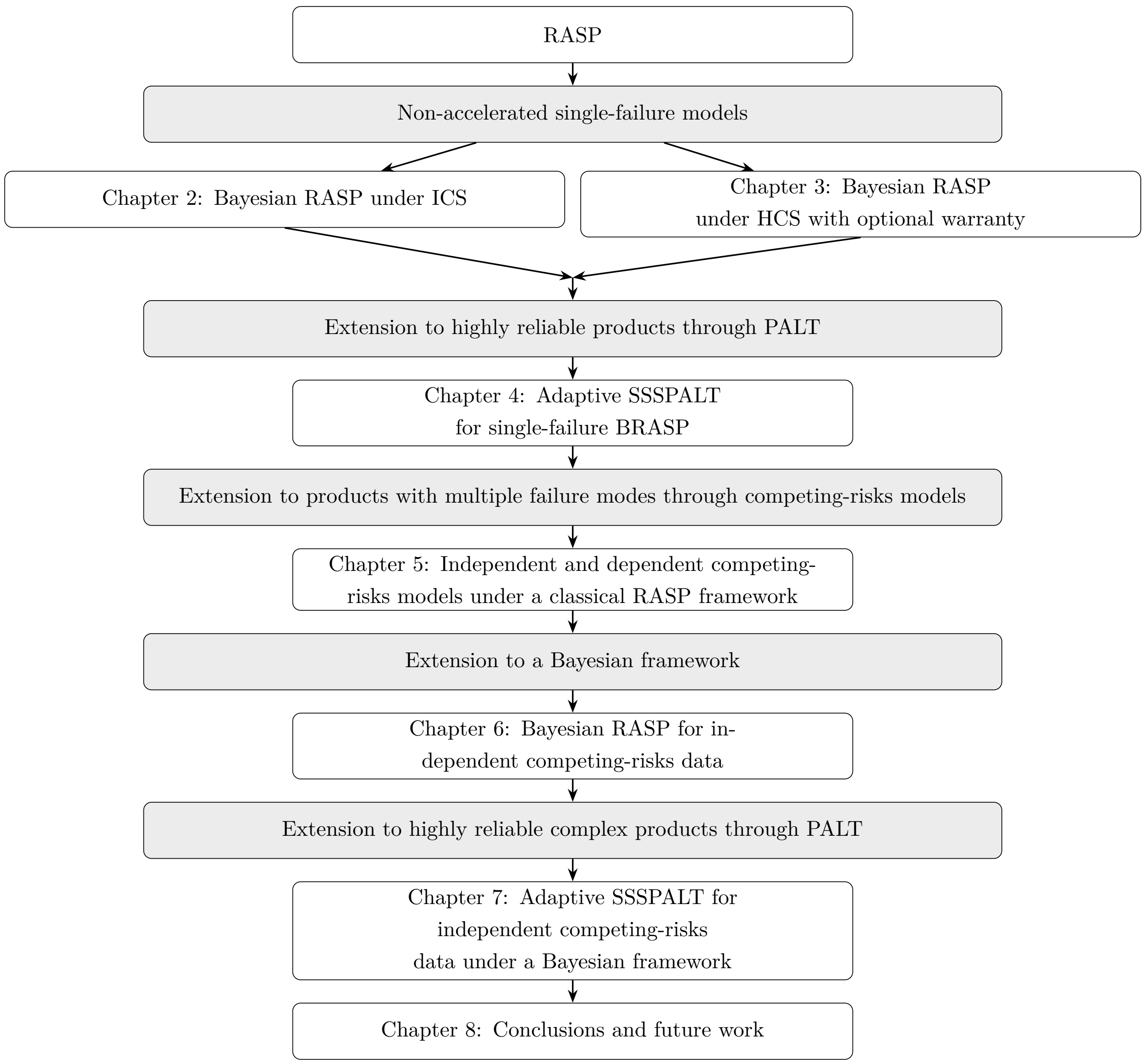


Figure 1.6: Structural relationship and evolution of the research presented in the thesis.

[Chapter 2] RASP under interval censoring by Bayesian approcah

This chapter considers RASP by a Bayesian approach under ICS. It is considered that the consumer and manufacturer agree on a common lifetime distribution of the product. However, they differ in their assessment of the prior distributions and utility functions because of the adversarial nature of the consumer and manufacturer. The consumer accepts or rejects a lot of products based on his/her own prior knowledge about the quality of the product and its utility function. If the consumer rejects the lot, the manufacturer gives additional information on lifetime based on a life test to update the consumer's belief. The consumer decides whether to accept or reject the lot based on updated information. The life test is conducted under ICS, where the decision on acceptance or rejection of the lot can be taken sequentially at pre-specified inspection times. The manufacturer's task is to determine the optimal interval-censoring life testing plan which maximizes his/her utility. The chapter explores two distinct scenarios:

First, a strong common knowledge assumption for prior and utility function is considered. This means that the manufacturer knows the consumer's utility and prior. Under this assumption, a general framework for designing RASP under ICS by Bayesian decision theoretic approach is developed. To illustrate of our methodology, we consider the determination of optimum RASPs for the exponential and Weibull distributions.

Next, it is considered that the consumer's utility and prior are uncertain to the manufacturer. To tackle this situation, a random decision-making sampling plan (RDSP) is introduced where the manufacturer does not require the knowledge of the consumer's utility function and prior distribution for planning the life test. The proposed methodology is illustrated using the exponential distribution.

[Chapter 3] RASP with optional warranty under hybrid censoring by Bayesian approach

This chapter considers the design of RASP by a Bayesian approach under HCS-I for the products sold under an optional warranty. Similar to Chapter 2, it is assumed that the consumer and the manufacturer agree on a common lifetime distribution of the product. However, they differ in the assessment of the prior distributions because of the adversarial nature of the consumer and manufacturer. The consumer takes a decision based on his/her utility and prior beliefs without a warranty. If the decision is rejection, the manufacturer provides a warranty offer to the consumer, which can increase the chance of acceptance of the lot. Due to the warranty, the manufacturer incurs an expense if the product fails during the warranty period. To compensate for expenses due to the warranty offer, the manufacturer can increase the selling price. Nevertheless, the consumer always prefers products with low prices. Therefore, a negotiation occurs between the manufacturer and the consumer, and an optional warranty is worth considering for decision-making. In this context, a CW rebate warranty policy is considered. If the consumer still rejects the lot even with the warranty, the manufacturer conducts a life test under HCS-I and provides lifetime information to the consumer. The consumer updates his/her belief based on lifetime information provided by the manufacturer. The consumer then takes the final decision about the lot based on updated belief. The task of the manufacturer is to determine the optimal life testing plan. Similar to Chapter 2, we explore two distinct scenarios:

First, a strong common knowledge assumption is made on prior and utility. Under this assumption, a general framework for designing RASP under HCS-I with optional warranty by a Bayesian decision theoretic approach is developed. To illustrate our methodology, we consider the determination of optimum RASPs for exponential and Weibull distributions. Next, we consider that the consumer's utility and prior are uncertain to the manufacturer. RDSP is discussed using the exponential distribution.

[Chapter 4] RASP under adaptive SSSPALT under type-I censoring by Bayesian approach

In Chapters 2 and 3, we study BRASP under non-accelerated life test. In recent years, global market competitiveness and rapid technological advancement have pushed manufacturers to produce products with very high reliability. For such products, the mean time to failure under normal operating conditions is often prohibitively long. To address this issue, this chapter proposes a BRASP based on adaptive SSSPALT under type-I censoring. In the conventional SSSPALT, the items are put on normal operating conditions up to a certain time, and after that, the stress is increased to get the failure time information early. However, when the stress increases, an additional cost is incorporated that increases the cost of the life test. In view of this, we consider an adaptive SSSPALT where changing the stress level after time τ_1 depends on the number of failures up to time τ_1. The adaptive test is described as follows. A sample of size n is put on a life test in normal stress s_0 at time $\tau_0 = 0$. If the number of failures up to time τ_1 is less than a pre-specified number m, then the stress s_0 changes to s_1. Otherwise, the stress remains unchanged after τ_1. Next, if the number of failures by time τ_2 is less than n, the test continues until τ_2 under the current stress level. If all items fail before time τ_2, the test is stopped at the time of the last failure.

The proposed methodology provides a general framework for designing BRASP under type-I censoring, covering both accelerated and non-accelerated plans. When $m = 0$, it reduces to BRASP under conventional SSSPALT with type-I censoring and when $m = n$, it reduces to BRASP under conventional non-accelerated life testing with type-I censoring, which was studied by Lin et al. [74]. We derive the Bayes decision rule under both general and quadratic loss functions, evaluate their performances through numerical studies, and demonstrate practical relevance using a real-world example.

[Chapter 5] RASP for competing risks data under PICS-I

In Chapters 2 and 3, it is assumed that the products fail due to single cause. However, for the complex products, there would be multiple causes of failure. This chapter discusses the development of RASPs under PICS-I in the presence of competing causes of failure. A general framework is considered to accommodate the presence of independent or dependent competing causes of failure. The asymptotic properties of MLEs are discussed, which are essential in obtaining the sampling plans. For illustration, frailty model is considered to accommodate the dependence among the potential causes of failure. The frailty model provides an independent competing risks model as a limiting case. We then present the traditional sampling plans for both independent and dependent competing risks models using producer's and consumer's risks methods. A c-optimal design criterion is considered to obtain more useful reliability acceptance sampling plans in the presence of budgetary constraints. A comprehensive numerical experiment is undertaken to examine the impact of the level of dependence among the potential failure times on the resulting sampling plans. An application of the developed methodology is demonstrated using a real-life example and a simulation study is also performed.

This chapter contributes to three important aspects of designing RASPs in the presence of competing risk data. First, a comprehensive framework is performed to model PICS-I datasets while accounting for the presence of competing failure modes. This framework allows us to incorporate both independence and dependence structure among the competing failure modes, which may be useful for modeling ICS data often found in reliability applications. Second, A generic expression of the Fisher information matrix is provided, then the asymptotic properties of MLEs are obtained under a set of regularity conditions. These results may be useful for performing statistical inference based on ICS data. Third, the general framework is developed for design of RASPs under three different setups.

[Chapter 6] RASP for independent competing risks data under ICS by Bayesian approach

This chapter discusses the development of designing BRASP for the competing risk data under ICS. It is assumed that the latent failure times are statistically independent. At first, we obtain RASP where the decision criteria of accepting a lot is pre-fixed. The optimal sampling plan is obtained by minimizing the Bayes risk. For large samples, computing Bayes risk is computationally intensive. Therefore, an approximate Bayes risk is obtained using the asymptotic properties of MLEs. Lastly, we obtain the BRASP, where the decision function is arbitrary and a single prior assessment is used by both the manufacturer and consumer. The manufacturer can derive an optimal decision function that minimizes the Bayes risk among all the decision functions. This optimal decision function is known as the Bayes decision function. The closed form of the Bayes decision function and Bayes risk are obtained for the quadratic loss function. The algorithms are provided for the computation of optimum BRASP. Numerical results are provided and comparisons between RASPs are carried out.

[Chapter 7] RASP under adaptive SSSPALT for competing risks data under type-II censoring by Bayesian approach

In Chapters 5 and 6, we study RASP for competing risk data under non-accelerated condition and Chapter 4 focuses on BRASP for single-failure-mode products under accelerated conditions. As modern products become increasingly complex and highly reliable, failures typically occur only after prolonged use and may arise from multiple causes. To address this setting, this chapter proposes a RASP by Bayesian approach based on an adaptive SSSPALT for competing risks data under type-II censoring. The adaptive testing scheme is introduced in Chapter 4.

Thus, the proposed methodology provides a general framework for designing BRASP under type-II censoring for competing risk data, covering both non-accelerated and accelerated testing plans. When $m = 0$, it reduces to BRASP under conventional SSSPALT with type-II censoring for competing risk data and when $m = r$, it reduces to BRASP under conventional non-accelerated life testing with type-II censoring for competing risk data, which was studied by Prajapati et al. [99].

We derive the Bayes decision rule under both general and quadratic loss functions, evaluate their performances through numerical studies, and demonstrate practical relevance using a real-world example.

[Chapter 8] Conclusions and future work

This chapter provides an overall conclusion of the proposed works considered in the thesis and discusses the scope of future areas.

1.9 Publications from the thesis

1. Das, R. & Pradhan, B. (2025). Bayesian reliability acceptance sampling plans under interval censoring. Communications in Statistics-Theory and Methods, 54(6), 1676-1701. [Out of Chapter 2].
2. Das, R. & Pradhan, B. (2025). Bayesian reliability acceptance sampling plan with optional warranty under hybrid censoring. Quality and Reliability Engineering International, 41(2), 872–896. [Out of Chapter 3].
3. Das, R. & Pradhan, B. (2024). Bayesian reliability acceptance sampling plans under adaptive simple step stress partial accelerated life test. Quality Technology & Quantitative Management. https://doi.org/10.1080/16843703.2026.2678452. [Out of Chapter 4].
4. Das, R., Roy, S. & Pradhan, B. (2024). Reliability acceptance sampling plans under progressive type-I interval censoring schemes in presence of dependent competing risks. arXiv preprint arXiv:2411.01324. Under revision. [Out of Chapter 5].
5. Pradhan, B. & Das, R. (2025). Bayesian reliability acceptance sampling plans for competing risks data under interval censoring. arXiv preprint arXiv:2510.16740. Accepted as a book chapter in Springer Handbook of Reliability. [Out of Chapter 6].
6. Das, R., Roy, S. & Pradhan, B. (2024). Bayesian reliability acceptance sampling plans under adaptive accelerated type-II censored competing risk data. arXiv preprint arXiv:2507.23293. Communicated. [Out of Chapter 7].

Chapter 2

RASP under interval censoring by Bayesian approach [1]

This chapter considers the design of BRASP under ICS. The problem of designing BRASP under ICS is considered by many authors. For example, Chen et al. [29], Lu and Tsai [81], and Aslam et al. [2] considered designing BRASP under the assumption that both the manufacturer ($\mathcal{M}$) and the consumer ($\mathcal{C}$) share a common prior distribution and a common utility or loss function. However, this assumption does not reflect many real-world scenarios where $\mathcal{M}$ and $\mathcal{C}$ differ in prior assessment and utility functions because of their adversarial nature. For example, $\mathcal{M}$ may have access to historical data or internal production information that is unavailable to $\mathcal{C}$. This asymmetry in information naturally leads to different prior beliefs about product reliability, even if both parties agree on the same parametric form of the lifetime distribution. Similarly, their utility or loss functions may also differ.

Motivated by these practical considerations, this chapter develops BRASP under the assumption that $\mathcal{M}$ and $\mathcal{C}$ have different prior beliefs and utility functions. A Bayesian negotiation framework is introduced to model the interaction between the two parties.

The negotiation procedure is as follows. It is assumed that $\mathcal{M}$ offers a lot to $\mathcal{C}$. $\mathcal{C}$ may accept or reject the lot based on prior knowledge about $\boldsymbol{\theta}$ and utility functions for acceptance and rejection, $\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{\theta})$ and $\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid \boldsymbol{\theta})$, respectively. $\mathcal{C}$ accepts the lot using the prior belief if

$$E_{\boldsymbol{\theta}}(\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{\theta})) \geq E_{\boldsymbol{\theta}}(\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid \boldsymbol{\theta})). \tag{2.0.1}$$

If $\mathcal{C}$ rejects the lot based on prior belief, i.e., inequality (2.0.1) is not satisfied, $\mathcal{M}$ may conduct a life test under ICS and provide data to update $\mathcal{C}$'s belief.

During the life-testing procedure, $\mathcal{C}$ updates his/her belief after each inspection based on the data available from the life test. Based on the updated belief, $\mathcal{C}$ can make a decision about the lot. Three types of decisions can be taken after each inspection except after the k^{th} inspection:

[1] One paper based on this chapter has appeared under:

1. Bayesian reliability acceptance sampling plans under interval censoring *Communications in Statistics: Theory & Methods*, 25(6), 1676-1701, 2025.

(i) $\mathcal{C}$ accepts the lot, and $\mathcal{M}$ terminates the life-testing procedure.

(ii) $\mathcal{C}$ rejects the lot, and $\mathcal{M}$ terminates the life-testing procedure.

(iii) $\mathcal{C}$ and $\mathcal{M}$ proceed to the next inspection.

Now, we elaborate on $\mathcal{C}$'s action after each inspection. After the 1^{st} inspection, $\mathcal{C}$ accepts the lot at τ_1 if the observed data satisfy $\mathcal{C}$'s acceptance criteria, which can be derived from inequality (2.0.1) by replacing the prior expectation of $\boldsymbol{\theta}$ with the posterior expectation, i.e.,

$$E_{\boldsymbol{\theta}|\text{observed data}}(\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{\theta})) \geq E_{\boldsymbol{\theta}|\text{observed data}}(\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid \boldsymbol{\theta})). \tag{2.0.2}$$

If the observed data do not satisfy $\mathcal{C}$'s acceptance criteria, $\mathcal{M}$ either stops the life test after the 1^{st} inspection or continues the test for another inspection to update $\mathcal{C}$'s beliefs. Accordingly, $\mathcal{C}$ either rejects the lot or proceeds to the 2^{nd} inspection. The process continues up to the k^{th} inspection. After the k^{th} inspection, $\mathcal{C}$ must accept or reject the lot. The advantages of the proposed sampling plan are that continuous monitoring of the life test is not required. Also, it may not be necessary to observe the test up to the last inspection.

Lindley and Singpurwalla [76], in their work, considered utilities for acceptance or rejection of a lot in terms of cost. However, reliability is an important characteristic that must be adhered to at some specific level. Therefore, we consider $\mathcal{C}$'s utility in terms of the reliability of the product. $\mathcal{M}$'s utility functions for acceptance and rejection are adopted from Lindley and Singpurwalla [76]. To propose BRASP under this scenario, it is considered that $\mathcal{M}$ not only knows his/her own utility and prior but also knows $\mathcal{C}$'s utility, prior, and actions. A general framework is developed for determining BRASP under this common-knowledge assumption. The proposed model is illustrated with the exponential and Weibull distributions.

In real-life scenarios, it may happen that $\mathcal{C}$'s utility, prior and action are unknown to $\mathcal{M}$. Considering this, a random decision-making sampling plan (RDSP) is introduced, where $\mathcal{C}$'s action before the life test, as well as his/her actions at each stage of the life test, are uncertain. In this case, it is assumed that the form of $\mathcal{C}$'s utility function and prior distribution are known to $\mathcal{M}$, but the parameters of the utility function and prior are not known. The exponential distribution is used to illustrate the proposed methodology.

The organization of this chapter is as follows. Section 2.1 presents the proposed model for designing BRASP. Section 2.2 considers the procedures of deriving decision rules for $\mathcal{C}$ and $\mathcal{M}$. Section 2.3 describes the determination of BRASP for the exponential and Weibull distributions. Section 2.4 discusses the RDSP illustrated with the exponential distribution. Section 2.5 provides numerical examples along with the computational procedure for determining the optimum BRASP. Section 2.6 presents the discussion.

2.1 Model

$\mathcal{C}$'s utility functions for acceptance and rejection of an item are taken as

$$\mathcal{U}_{\mathcal{C}}(\mathcal{A} \,|\, \boldsymbol{\theta}) = R(t_0 \,|\, \boldsymbol{\theta}) \quad \text{and} \quad \mathcal{U}_{\mathcal{C}}(\mathcal{R} \,|\, \boldsymbol{\theta}) = R_0, \tag{2.1.1}$$

respectively, where R_0 is the specified value of the reliability and $\boldsymbol{\theta}$ is the vector of model parameters. Using inequality (2.0.1), $\mathcal{C}$ accepts the lot without life testing if

$$E_{\boldsymbol{\theta}}[R(t_0 \,|\, \boldsymbol{\theta})] \geq R_0. \tag{2.1.2}$$

If inequality (2.1.2) is not satisfied, then $\mathcal{C}$ rejects the lot. In that case, $\mathcal{M}$ performs a life test, which is carried out under ICS.

Suppose n items are put on life test at time $\tau_0 = 0$, and the items are inspected at pre-determined inspection times $\tau_1 < \cdots < \tau_k$, where k, the number of inspections, is also pre-determined. At time τ_i, only the number of failures in the interval $(\tau_{i-1}, \tau_i]$ is recorded. The exact failure times of items are unknown. $\mathcal{C}$ updates his/her belief at each inspection time, as described in the introduction to the chapter. Let D_i be the number of failure in $[\tau_{i-1}, \tau_i)$, and $X(i) = (D_1, \ldots, D_i)$ denote the data till i^{th} inspection. In this sampling plan, the observed data and the vector of inspection times are dependent on the inspection number i. Therefore, the observed data vector and inspection times vector up to inspection τ_i are denoted by $\boldsymbol{x}(i) = (d_1, d_2, \ldots, d_i)$ instead of usual notation $\boldsymbol{x}$ and $\boldsymbol{\tau}(i) = (\tau_1, \ldots, \tau_i)$, respectively, where d_i is the observed value of D_i. Then, the likelihood function up to time τ_i is given by

$$L(\boldsymbol{\theta} \,|\, \boldsymbol{x}(i)) \propto \prod_{l=1}^{i} \left[F(\tau_l \,|\, \boldsymbol{\theta}) - F(\tau_{l-1} \,|\, \boldsymbol{\theta})\right]^{d_l} \left[1 - F(\tau_i \,|\, \boldsymbol{\theta})\right]^{n-d(i)}, \tag{2.1.3}$$

where $d(i) = d_1 + \cdots + d_i$ is the number of failures up to time τ_i..
$\mathcal{C}$ finds the posterior distribution of $\boldsymbol{\theta}$ using the prior $p_{\mathcal{C}}(\boldsymbol{\theta})$ as

$$p_{\mathcal{C}}(\boldsymbol{\theta} \,|\, \boldsymbol{x}(i)) = \frac{L(\boldsymbol{\theta} \,|\, \boldsymbol{x}(i))\, p_{\mathcal{C}}(\boldsymbol{\theta})}{\int_{\boldsymbol{\theta}} L(\boldsymbol{\theta} \,|\, \boldsymbol{x}(i))\, p_{\mathcal{C}}(\boldsymbol{\theta}) d\boldsymbol{\theta}}. \tag{2.1.4}$$

Thus, $\mathcal{C}$ accepts the lot after the i^{th} inspection if

$$\begin{aligned} \mathcal{U}_{\mathcal{C}}(\mathcal{A} \,|\, \boldsymbol{x}(i)) &= \int_{\boldsymbol{\theta}} \mathcal{U}_{\mathcal{C}}(\mathcal{A} \,|\, \boldsymbol{\theta})\, p_{\mathcal{C}}(\boldsymbol{\theta} \,|\, \boldsymbol{x}(i)) d\boldsymbol{\theta} \\ &\geq \int_{\boldsymbol{\theta}} \mathcal{U}_{\mathcal{C}}(\mathcal{R} \,|\, \boldsymbol{\theta})\, p_{\mathcal{C}}(\boldsymbol{\theta} \,|\, \boldsymbol{x}(i)) d\boldsymbol{\theta} \\ &= \mathcal{U}_{\mathcal{C}}(\mathcal{R} \,|\, \boldsymbol{x}(i)). \end{aligned} \tag{2.1.5}$$

If inequality (2.1.5) does not hold for the data $\boldsymbol{x}(i)$, then $\mathcal{C}$ rejects the lot after the i^{th} inspection. If the decision is rejection, $\mathcal{M}$ either stops the life test after the i^{th} inspection or continues to the $(i+1)^{\text{st}}$ inspection to update $\mathcal{C}$'s belief.

Now consider the case where $\mathcal{M}$ either stops the life test after the i^{th} inspection or performs another inspection. To find this criterion, a modified data vector $M(\boldsymbol{x}(i))$ with k components for the observed data $\boldsymbol{x}(i)$ is defined as $M(\boldsymbol{x}(i)) = (d_1, d_2, \ldots, d_i, 0, 0, \ldots, 0)$, where $M(\boldsymbol{x}(i))$ has $(k-i)$ zero components. Therefore, $M(\boldsymbol{x}(i))$ represents the observed data when all remaining items survive up to the k^{th} inspection after $\boldsymbol{x}(i)$ is observed. The likelihood function for $M(\boldsymbol{x}(i))$ is

$$L(\boldsymbol{\theta} \,|\, M(\boldsymbol{x}(i))) \propto \prod_{l=1}^{i} \left[F(\tau_l \,|\, \boldsymbol{\theta}) - F(\tau_{l-1} \,|\, \boldsymbol{\theta})\right]^{d_l} \left[1 - F(\tau_k \,|\, \boldsymbol{\theta})\right]^{n-d(i)}. \tag{2.1.6}$$

In likelihood (2.1.6), it is assumed that the remaining items survive up to τ_k after inspection τ_i. This is the difference between likelihood functions (2.1.3) and (2.1.6). $\mathcal{C}$ rejects the lot after the i^{th} inspection based on the data $\boldsymbol{x}(i)$ if

$$\mathcal{U}_{\mathcal{C}}(\mathcal{A} \,|\, M(\boldsymbol{x}(i))) < \mathcal{U}_{\mathcal{C}}(\mathcal{R} \,|\, M(\boldsymbol{x}(i))). \tag{2.1.7}$$

The modified vector $M(\boldsymbol{x}(i))$ is said to be the best output for $\boldsymbol{x}(i)$ if the test continues up to the k^{th} inspection. Therefore, if inequality (2.1.7) holds for the data $\boldsymbol{x}(i)$, $\mathcal{M}$ stops the life test after the i^{th} inspection and $\mathcal{C}$ rejects the lot. Otherwise, both parties proceed to the next inspection. After the k^{th} inspection, if inequality (2.1.5) does not hold, $\mathcal{C}$ rejects the lot.

Let $\mathcal{X}(i)$ be the set of all possible $\boldsymbol{x}(i)$ that satisfy the acceptance criterion (2.1.5), for which $\mathcal{C}$ accepts the lot after the i^{th} inspection and $\mathcal{M}$ terminates the test at τ_i. Similarly, let $\mathcal{Y}(i)$ denotes the set of all possible $\boldsymbol{x}(i)$ that satisfy the rejection criterion (2.1.7), for which $\mathcal{M}$ stops the life test after the i^{th} inspection and $\mathcal{C}$ rejects the lot. Let $\mathcal{Z}(i)$ denotes the set of all possible $\boldsymbol{x}(i)$ that satisfy neither inequality (2.1.5) nor (2.1.7), for which the test proceeds to the next inspection.

Next, we consider $\mathcal{M}$'s task of planning the life test, where optimum values of n, k, and $(\tau_1, \ldots, \tau_k)$ are to be determined. Let $\boldsymbol{\zeta} = (n, \tau_1, \tau_2, \ldots, \tau_k, k)$ denote the vector of decision variables. $\mathcal{M}$ determines the optimum $\boldsymbol{\zeta}$ by maximizing his/her utility. $\mathcal{M}$'s utility functions for acceptance and rejection (see Lindley and Singpurwalla [76]) after the i^{th} inspection are

$$\mathcal{U}_{\mathcal{M}}(\mathcal{A} \,|\, \boldsymbol{x}(i), \boldsymbol{\theta}) = b_1 E_{X \,|\, (\boldsymbol{x}(i), \boldsymbol{\theta})}[X^q] - b_2 - C_s n - C_d d(i) - C_t \tau_i - C_I i, \tag{2.1.8}$$

and

$$\mathcal{U}_{\mathcal{M}}(\mathcal{R} \,|\, \boldsymbol{x}(i), \boldsymbol{\theta}) = b_3 - C_s n - C_d d(i) - C_t \tau_i - C_I i, \tag{2.1.9}$$

respectively. Each cost component in the above expressions is motivated as follows:

- **Cost weight associated with product lifetime (b_1):** $b_1 > 0$ represents the cost weight that $\mathcal{M}$ assigns to the product's lifetime. A higher value of b_1 implies that extending the lifetime of the product yields greater economic benefits by reducing warranty expenses, minimizing replacement costs, and improving brand reliability.
- **Baseline or minimum cost requirement (b_2):** b_2 denotes the baseline cost that must be recovered for the production process to be economically viable. It includes fixed manufacturing costs, target profit margins, and quality assurance expenses that $\mathcal{M}$ considers essential for sustainable operation.
- **Cost associated with product rejection (b_3):** b_3 represents the cost effect of rejecting items that fail to satisfy life-testing requirements. A positive b_3 indicates a cost saving such as avoiding warranty claims or customer dissatisfaction whereas a negative b_3 reflects a loss, arising from wasted testing effort, material disposal, or foregone sales opportunities.
- **Risk attitude parameter (q):** The exponent q regulates the curvature of the power utility function and determines $\mathcal{M}$'s risk preference regarding product lifetime. If $0 < q < 1$, the function is concave, corresponding to a risk-averse attitude. If $q = 1$, the function is linear, reflecting risk neutrality. For $q < 0$ or $q > 1$, the function becomes convex, which corresponds to a risk-seeking behavior (see Happich and Michael [43]).

- **Sampling cost (C_s):** The unit cost incurred for subjecting a test unit to the life test. This includes expenses such as technician labor, test facility usage, measurement equipment, and quality monitoring. Thus, C_s represents the *per-unit sampling cost.*
- **Failure cost (C_d):** When a unit fails during the life test, it is discarded and cannot be reused in production. This generates an additional expense referred to as the failure cost. The *per-unit failure cost*, denoted by C_d, accounts for disposal, replacement, rework, or other direct expenses associated with a failed item.
- **Operating cost (C_t):** The execution of the life test consumes operational resources over time, including power, maintenance, etc.. The operating cost is assumed to be proportional to the duration of the test. Hence, C_t represents the *per-unit time operating cost.*
- **Inspection cost (C_I):** Each inspection of the test units incurs a fixed cost C_I, regardless of the sample size or outcome. This includes expenses such as setup, measurement, record-keeping, and quality assurance procedures at each inspection point.

Note that

$$E_{X\,|\,(\boldsymbol{x}(i),\boldsymbol{\theta})}[X^q] = \int_0^\infty x^q f_{(\boldsymbol{x}(i),\boldsymbol{\theta})}(x)\,dx = \int_0^\infty x^q f_{\boldsymbol{\theta}}(x)\,dx. \tag{2.1.10}$$

The lifetime of an item depends only on the parameters, and hence $E_{X\,|\,(\boldsymbol{x}(i),\boldsymbol{\theta})}[X^q]$ is independent of $\boldsymbol{x}(i)$. $\mathcal{M}$'s utility for the observed data $\boldsymbol{x}(i)$ after the i^{th} inspection is given by

$$\psi_{\mathcal{M}}(\boldsymbol{\zeta}\,|\,\boldsymbol{x}(i),\boldsymbol{\theta}) = \mathbb{1}_{\mathcal{X}(i)}(\boldsymbol{x}(i))\,\mathcal{U}_{\mathcal{M}}(\mathcal{A}\,|\,\boldsymbol{x}(i),\boldsymbol{\theta}) + \mathbb{1}_{\mathcal{Y}(i)}(\boldsymbol{x}(i))\,\mathcal{U}_{\mathcal{M}}(\mathcal{R}\,|\,\boldsymbol{x}(i),\boldsymbol{\theta}),$$

where $\mathbb{1}_{\mathcal{V}}(\boldsymbol{v})$ denotes the indicator function over a set $\mathcal{V}$ which is defined as

$$\mathbb{1}_{\mathcal{V}}(\boldsymbol{v}) = \begin{cases} 1, & \boldsymbol{v} \in \mathcal{V}, \\ 0, & \boldsymbol{v} \notin \mathcal{V}. \end{cases} \tag{2.1.11}$$

Let $p_{\mathcal{M}}(\boldsymbol{\theta})$ be the prior distribution of $\boldsymbol{\theta}$ for $\mathcal{M}$. Then, the expected utility with respect to $\boldsymbol{\theta}$ and the data $\boldsymbol{X}(i)$, for $i = 1, \ldots, k$, is given by

$$\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = E_{\boldsymbol{\theta}}\left[\sum_{i=1}^{k} E_{\boldsymbol{X}(i)}\big(\psi_{\mathcal{M}}(\boldsymbol{\zeta}\,|\,\boldsymbol{X}(i),\boldsymbol{\theta})\big)\right].$$

The optimal decision $\boldsymbol{\zeta}^* = (n^*, \tau_1^*, \tau_2^*, \ldots, \tau_k^*, k^*)$ is obtained by maximizing $\psi_{\mathcal{M}}(\boldsymbol{\zeta})$, i.e.,

$$\boldsymbol{\zeta}^* = \arg\max_{\boldsymbol{\zeta}} \psi_{\mathcal{M}}(\boldsymbol{\zeta}).$$

2.2 The procedure of finding the decision rule

Here we consider the procedure of finding the decision rules for $\mathcal{C}$'s and $\mathcal{M}$'s actions. $\mathcal{C}$ needs to find the decision rule based on the data available from the life test, whereas $\mathcal{M}$ has to consider the optimal design of the life test.

2.2.1 Consumer's decision

A flow chart is provided in Figure 2.1, which describes the procedure when $\mathcal{C}$ accepts or rejects the lot after i^{th} inspection and stops the life test, and when $\mathcal{M}$ and $\mathcal{C}$ proceed to the next inspection.

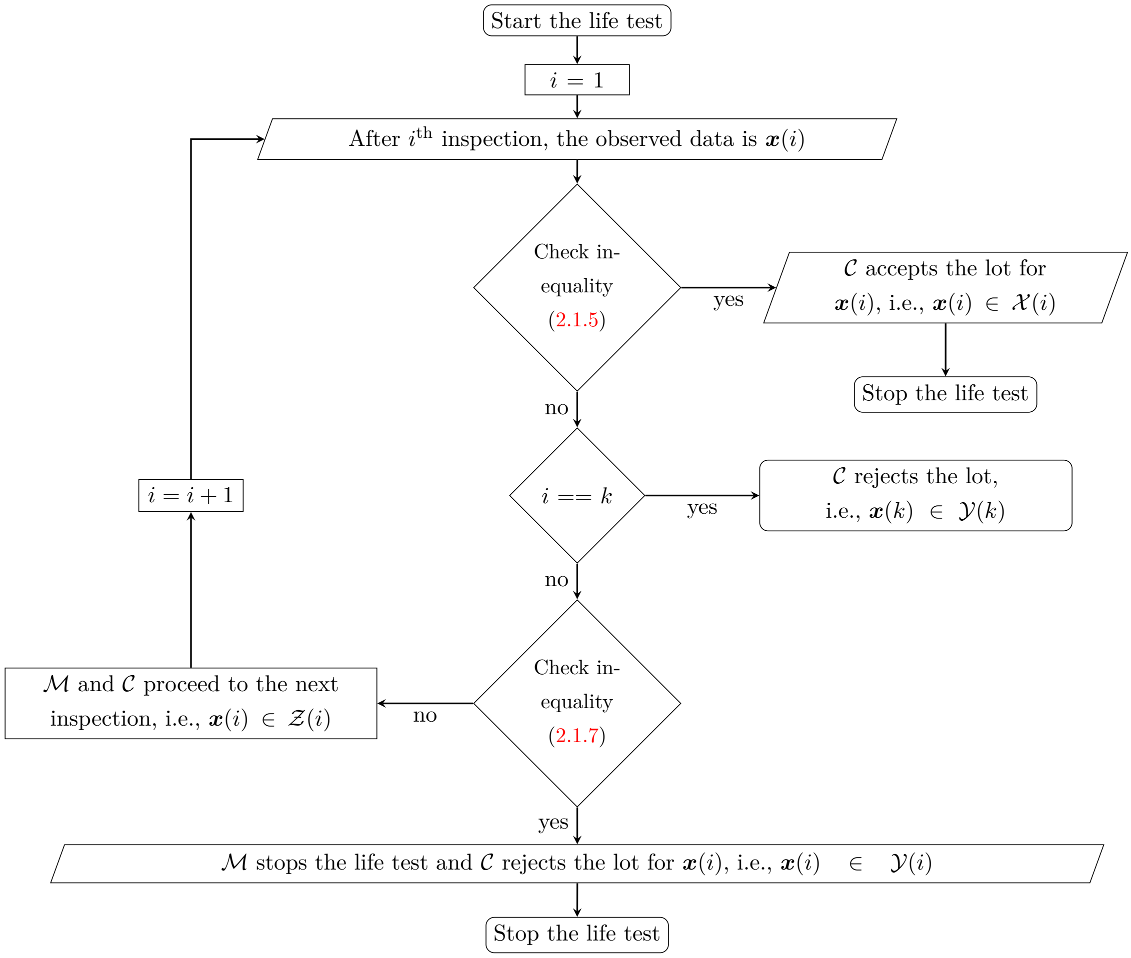


Figure 2.1: The procedure of $\mathcal{C}$'s actions.

We give an example to illustrate the sets $\mathcal{X}(i)$ and $\mathcal{Y}(i)$. Consider $n = 7$ and $k = 3$ for illustration. The sets $\mathcal{X}(i)$ and $\mathcal{Y}(i)$ corresponding to $i = 1, 2, 3$ are provided in Table 2.1. The algorithm to find the sets $\mathcal{X}(i)$ and $\mathcal{Y}(i)$, $i = 1, \ldots, k$, is provided in Algorithm 2.1.

Table 2.1: The data sets for which $\mathcal{C}$ accepts or rejects the lot or goes for the next inspection.

Inspection (i)	$\mathcal{X}(i)$	$\mathcal{Y}(i)$	$\mathcal{Z}(i)$
1	0, 1	4, 5, 6, 7	2, 3
2	(2, 0), (2,1)	(2, 3), (2,4), (2, 5), (3, 2), (3, 3), (3, 4)	(2, 2), (3, 0), (3, 1)
3	(2, 2, 0), (3, 0, 0), (3, 1, 0)	(2, 2,1), (2, 2, 2), (2, 2, 3), (3, 0, 1), (3, 0, 2), (3, 0, 3), (3, 0, 4), (3, 1, 1), (3, 1, 2), (3, 1, 3)	

Algorithm 2.1: Procedure to Determine Sets $\mathcal{X}(i)$ and $\mathcal{Y}(i)$.

Input: $\mathcal{D}(1) = \{\boldsymbol{x}(1) : \boldsymbol{x}(1) = d_1,\ 0 \le d_1 \le n\}$
Output: $\mathcal{X}(i), \mathcal{Y}(i),$ for $i = 1, \ldots, n$
for $i \leftarrow 1$ **to** k **do**
  Initialize: $\boldsymbol{z}(i) \leftarrow \phi$, $\boldsymbol{y}(i) \leftarrow \phi$ where ϕ is the null set; **foreach** $\boldsymbol{x}(i) \in \mathcal{D}^i$ **do**
    1. Compute likelihood function after the i^{th} inspection using (2.1.3);
    2. Compute posterior distribution of $\boldsymbol{\theta}$ and update $\mathcal{C}$'s belief ;
    3. Compute expected utility of $\mathcal{C}$ based on the posterior of $\boldsymbol{\theta}$;
    4. Evaluate acceptance condition using inequality (2.1.5);
    if *Inequality (2.1.5) holds* **then**
      Update: $\mathcal{X}(i) \leftarrow \text{append}(\boldsymbol{z}(i), \boldsymbol{x}(i))$;
    else
      if *$(i < k)$* **then**
        1. Construct modified vector $M(\boldsymbol{x}(i))$ with k components;
        2. Compute likelihood for $M(\boldsymbol{x}(i))$ using (2.1.6);
        3. Compute the posterior of $\boldsymbol{\theta}$;
        4. Compute the expected utility of $\mathcal{C}$ based on the posterior of $\boldsymbol{\theta}$;
        5. Evaluate rejection condition using inequality (2.1.7);
        if *Inequality (2.1.7) holds* **then**
          Update: $\mathcal{Y}(i) \leftarrow \text{append}(\boldsymbol{y}(i), \boldsymbol{x}(i))$;
  if *$(i == k)$* **then**
    $\mathcal{Y}(i) \leftarrow \mathcal{D}(i) - \mathcal{X}(i)$;
  Update set: $\mathcal{D}(i) \leftarrow \mathcal{D}(i) - (\mathcal{X}(i) \cup \mathcal{Y}(i))$;
  Define next-stage set:

$$\mathcal{D}(i+1) = \left\{ \boldsymbol{x}(i+1) = (\boldsymbol{x}(i), d_{i+1}) \ : \ \boldsymbol{x}(i) \in \mathcal{D}(i),\ \sum_{j=1}^{i} x_j + d_{i+1} \le n \right\}.$$

2.2.2 Manufacturer's decision

It is assumed that $\mathcal{C}$'s utility and the prior are known to $\mathcal{M}$. Therefore, $\mathcal{M}$ can use $\mathcal{C}$'s acceptance and rejection criteria to evaluate his/her expected utility. If $\boldsymbol{x}(i) \in \mathcal{X}(i)$, then $\mathcal{D} \equiv \mathcal{A}$; and if $\boldsymbol{x}(i) \in \mathcal{Y}(i)$, then $\mathcal{D} \equiv \mathcal{R}$. Thus, we have

$$\begin{aligned}\psi_{\mathcal{M}}(\boldsymbol{\zeta} \,|\, \boldsymbol{x}(i), \boldsymbol{\theta}) =& \mathbb{1}_{\mathcal{X}(i)}(\boldsymbol{x}(i))\Big(b_1 E_{X\,|\,(\boldsymbol{x}(i),\boldsymbol{\theta})}[X^q] - b_2 - C_s n - C_d d(i) - C_t \tau_i - C_I i\Big) \\ &+ \mathbb{1}_{\mathcal{Y}(i)}(\boldsymbol{x}(i))\Big(b_3 - C_s n - C_d d(i) - C_t \tau_i - C_I i\Big).\end{aligned}$$

Now, the expected utility with respect to $\boldsymbol{x}(i)$ is obtained as

$$\begin{aligned}\psi_{\mathcal{M}}(\boldsymbol{\zeta} \,|\, \boldsymbol{\theta}) = \sum_{i=1}^{k} \sum_{\boldsymbol{x}(i)} \Big[& \mathbb{1}_{\mathcal{X}(i)}(\boldsymbol{x}(i))(b_1 E_{X\,|\,\boldsymbol{\theta}}[X^q] - b_2 - C_s n - C_d d(i) - C_t \tau_i - C_I i) \\ & + \mathbb{1}_{\mathcal{Y}(i)}(\boldsymbol{x}(i))(b_3 - C_s n - C_d d(i) - C_t \tau_i - C_I i)\Big] P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i) \,|\, \boldsymbol{\theta}),\end{aligned} \tag{2.2.1}$$

where $P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i)\,|\,\boldsymbol{\theta})$ is the joint distribution of $X(i)$ and it can be expressed as

$$P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i)\,|\,\boldsymbol{\theta}) = \frac{n!}{d_1!\dots d_i!(n-d(i))!}\prod_{l=1}^{i}\left[F(\tau_l\,|\,\boldsymbol{\theta})-F(\tau_{l-1}\,|\,\boldsymbol{\theta})\right]^{d_l}\left[1-F(\tau_i\,|\,\boldsymbol{\theta})\right]^{n-d(i)}.$$

Let A_i be the event that $\mathcal{C}$ accepts the lot at inspection τ_i. Then

$$P(A_i\,|\,\boldsymbol{\theta}) = \sum_{\boldsymbol{x}(i)\in\mathcal{X}(i)} P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i)\,|\,\boldsymbol{\theta}). \tag{2.2.2}$$

Let R_i be the event that $\mathcal{C}$ rejects the lot at inspection τ_i. Then

$$P(R_i\,|\,\boldsymbol{\theta}) = \sum_{\boldsymbol{x}(i)\in\mathcal{Y}(i)} P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i)\,|\,\boldsymbol{\theta}). \tag{2.2.3}$$

Now we find the expected number of failures $E[D \mid \boldsymbol{\theta}]$. Note that if the test runs up to the inspection time τ_i, then the expected number of failures is $nF(\tau_i \mid \boldsymbol{\theta})$. We have

$$E[D \mid \boldsymbol{\theta}] = \sum_{i=1}^{k} nF(\tau_i \mid \boldsymbol{\theta})\left[P(A_i \mid \boldsymbol{\theta}) + P(R_i \mid \boldsymbol{\theta})\right]. \tag{2.2.4}$$

Note that

$$\sum_{i=1}^{k}\left[P\left(A_i \mid \boldsymbol{\theta}\right) + P\left(R_i \mid \boldsymbol{\theta}\right)\right] = 1. \tag{2.2.5}$$

Proof: Let $p_i = F(\tau_i \mid \boldsymbol{\theta}) - F(\tau_{i-1} \mid \boldsymbol{\theta})$. Now,

$$\begin{aligned}
&\sum_{d_1=0}^{n}\binom{n}{d_1}p_1^{d_1}(1-p_1)^{n-d_1}\\
=&\sum_{\boldsymbol{x}(1)\in\mathcal{X}(1)}\binom{n}{d_1}p_1^{d_1}(1-p_1)^{n-d_1} + \sum_{\boldsymbol{x}(1)\in\mathcal{Y}(1)}\binom{n}{d_1}p_1^{d_1}(1-p_1)^{n-d_1} + \sum_{\boldsymbol{x}(1)\in\mathcal{Z}(1)}\binom{n}{d_1}p_1^{d_1}(1-p_1)^{n-d_1}\\
=&P\left(A_1 \mid \boldsymbol{\theta}\right) + P\left(R_1 \mid \boldsymbol{\theta}\right) + \sum_{\boldsymbol{x}(1)\in\mathcal{Z}(1)}\binom{n}{d_1}p_1^{d_1}\left[p_2 + (1-p_1-p_2)\right]^{n-d_1}\\
=&P\left(A_1 \mid \boldsymbol{\theta}\right) + P\left(R_1 \mid \boldsymbol{\theta}\right) + \sum_{\boldsymbol{x}(1)\in\mathcal{Z}(1)}\binom{n}{d_1}p_1^{d_1}\sum_{d_2=0}^{n-d_1}\binom{n-d_1}{d_2}p_2^{d_2}(1-p_1-p_2)^{n-d_1-d_2}\\
=&P\left(A_1 \mid \boldsymbol{\theta}\right) + P\left(R_1 \mid \boldsymbol{\theta}\right) + \sum_{\boldsymbol{x}(2)\in\mathcal{X}(2)}\frac{n!}{d_1!d_2!(n-d_1-d_2)!}p_2^{d_2}(1-p_1-p_2)^{n-d_1-d_2}\\
&+\sum_{\mathcal{Y}(2)}\frac{n!}{d_1!d_2!(n-d_1-d_2)!}p_2^{d_2}(1-p_1-p_2)^{n-d_1-d_2} + \sum_{\mathcal{Z}(2)}\frac{n!}{d_1!d_2!(n-d_1-d_2)!}p_2^{d_2}(1-p_1-p_2)^{n-d_1-d_2}\\
=&P\left(A_1 \mid \boldsymbol{\theta}\right) + P\left(R_1 \mid \boldsymbol{\theta}\right) + P\left(A_2 \mid \boldsymbol{\theta}\right) + P\left(R_2 \mid \boldsymbol{\theta}\right) + \sum_{\mathcal{Z}(2)}\frac{n!}{d_1!d_2!(n-d_1-d_2)!}p_2^{d_2}(1-p_1-p_2)^{n-d_1-d_2}.
\end{aligned}$$

Proceeding in this way, we get

$$\sum_{i=1}^{k}\left[P\left(A_i \mid \boldsymbol{\theta}\right) + P\left(R_i \mid \boldsymbol{\theta}\right)\right] = \sum_{d_1=0}^{n}\binom{n}{d_1}p_1^{d_1}(1-p_1)^{n-d_1}.$$

Also, it is known that
$$\sum_{d_1=0}^{n}\binom{n}{d_1}p_1^{d_1}(1-p_1)^{n-d_1}=1.$$

Hence, the proof is completed. (Proved)

From (2.2.1), the expected utility given $\boldsymbol{\theta}$ can be written as

$$\begin{aligned}\psi_{\mathcal{M}}(\boldsymbol{\zeta}\mid\boldsymbol{\theta}) =& \sum_{i=1}^{k}\Big[\left\{b_1E_{X\mid\boldsymbol{\theta}}(X^q)-b_2-C_sn-C_dnF(\tau_i\mid\boldsymbol{\theta})-C_t\tau_i-C_Ii\right\}P(A_i\mid\boldsymbol{\theta})\\ &+\{b_3-C_sn-C_dnF(\tau_i\mid\boldsymbol{\theta})-C_t\tau_i-C_Ii\}P(R_i\mid\boldsymbol{\theta})\Big]\\ =&\sum_{i=1}^{k}\left[\left(b_1E_{X\mid\boldsymbol{\theta}}(X^q)-b_2\right)P(A_i\mid\boldsymbol{\theta})+b_3P(R_i\mid\boldsymbol{\theta})-(C_t\tau_i+C_Ii)\right.\\ &\times\{P(A_i\mid\boldsymbol{\theta})+P(R_i\mid\boldsymbol{\theta})\}]-C_sn-C_dE[D\mid\boldsymbol{\theta})].\end{aligned}$$

$\mathcal{M}$'s utility $\psi_{\mathcal{M}}(\boldsymbol{\zeta})$ is then given by

$$\begin{aligned}\psi_{\mathcal{M}}(\boldsymbol{\zeta})=&\sum_{i=1}^{k}\left[\int_{\boldsymbol{\theta}}b_1E_{X\mid\boldsymbol{\theta}}(X^q)P(A_i\mid\boldsymbol{\theta})p_{\mathcal{M}}(\boldsymbol{\theta})d\boldsymbol{\theta}-b_2P(A_i)+b_3P(R_i)-(C_t\tau_i+C_Ii)\right.\\ &\left.\times\{P(A_i)+P(R_i)\}\right]-C_sn-C_dE[D],\end{aligned}\tag{2.2.6}$$

where $P(A_i)=\int_{\boldsymbol{\theta}}P(A_i\,|\,\boldsymbol{\theta})\ p_{\mathcal{M}}(\boldsymbol{\theta})\ d\boldsymbol{\theta},\ \ P(R_i)=\int_{\boldsymbol{\theta}}P(R_i\,|\,\boldsymbol{\theta})\ p_{\mathcal{M}}(\boldsymbol{\theta})\ d\boldsymbol{\theta},\ \ E[D]=\int_{\boldsymbol{\theta}}E[D\,|\,\boldsymbol{\theta}]\ p_{\mathcal{M}}(\boldsymbol{\theta})\ d\boldsymbol{\theta}.$

2.3 Optimal BRASP

Here, we obtain optimal BRASP for the exponential and Weibull distributions.

2.3.1 Exponential distribution case

Here, we illustrate the methodology for the exponential distribution. The lifetimes of the items are IID and follow $\text{Exp}(1/\eta)$, which is defined in Section 1.2.2. It is assumed that for $\mathcal{C}$'s perspective, η follows $IG(\alpha_1,\beta_1)$ with PDF $p_{\mathcal{C}}(\cdot)$, which is defined in Section 1.2.4. Using (2.1.1), the utility functions of $\mathcal{C}$ corresponding to the decisions acceptance and rejection are given by

$$\mathcal{U}_{\mathcal{C}}(\mathcal{A}\mid\eta)=\exp(-t_0/\eta)\quad\text{and}\quad\mathcal{U}_{\mathcal{C}}(\mathcal{R}\mid\eta)=R_0,\tag{2.3.1}$$

respectively. We have

$$E_\eta[\mathcal{U}_{\mathcal{C}}(\mathcal{A}\mid\eta)]=\int_\eta\exp(-t_0/\eta)p_{\mathcal{C}}(\eta)d\eta=\left(\frac{\beta_1}{\beta_1+t_0}\right)^{\alpha_1}.$$

Using (2.1.2), $\mathcal{C}$ accepts the lot without life testing if

$$\left(\frac{\beta_1}{\beta_1+t_0}\right)^{\alpha_1}\geq R_0.\tag{2.3.2}$$

If the inequality (2.3.2) does not hold, then $\mathcal{C}$ rejects the lot. Now, if $\mathcal{C}$'s decision is rejection, $\mathcal{M}$ performs a life test to update $\mathcal{C}$'s belief.

Inspection intervals of equal length:

For the analytical solution, we assume that the inspection times are of equal length, i.e., $\tau_i = ih$, where h is the length of each interval and $i = 1, \ldots, k$. The decision vector is $\boldsymbol{\zeta} = (n, h, k)$.

Result 2.1 *Suppose the lifetime of the product follows Exp$(1/\eta)$ and η follows $IG(\alpha_1, \beta_1)$. If n items are put to a life test and are inspected at equal time intervals of length h, then $\mathcal{C}$ accepts the lot after the i^{th} inspection if the following inequality is satisfied.*

$$\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}(i), h) = \frac{\sum_{j=0}^{d(i)} \binom{d(i)}{j} (-1)^j \frac{1}{(jh+\beta_1+t_0+b(i)h)^{\alpha_1}}}{\sum_{j=0}^{d(i)} \binom{d(i)}{j} (-1)^j \frac{1}{(jh+\beta_1+b(i)h)^{\alpha_1}}} \geq R_0,$$

where $b(i) = ni - \sum_{j=1}^{i} (i - (j-1)) d_j$.

Proof: The items are inspected at intervals of equal length h. Therefore, $\tau_i = ih$ for $i = 1, \ldots, k$. Note that

$$\mathcal{U}_{\mathcal{C}}(\mathcal{D} \mid \boldsymbol{x}(i), h) = \int_0^\infty \mathcal{U}_{\mathcal{C}}(\mathcal{D} \mid \eta) p_{\mathcal{C}}(\eta \mid \boldsymbol{x}(i), h) d\eta, \tag{2.3.3}$$

where $\mathcal{D} \in \{\mathcal{A}, \mathcal{R}\}$, $p_{\mathcal{C}}(\eta \mid \boldsymbol{x}(i), h)$ is the posterior distribution of η and $\mathcal{U}_{\mathcal{C}}(\mathcal{D} \mid \eta)$ is given in (2.3.1). The posterior distribution of η based on observed data $\boldsymbol{x}(i)$ up to time $\tau_i = ih$ is

$$p_{\mathcal{C}}(\eta \mid \boldsymbol{x}(i), h) = \frac{L(\eta \mid \boldsymbol{x}(i), h) p_{\mathcal{C}}(\eta)}{\int_0^\infty L(\eta \mid \boldsymbol{x}(i), h) p_{\mathcal{C}}(\eta) d\eta}.$$

Using (2.1.3), the likelihood function up to time $\tau_i = ih$ can be written as

$$L(\eta \mid \boldsymbol{x}(i), h) \propto (1 - \exp(-h/\eta))^{d_1+d_2+\ldots+d_i} \exp(-b(i)h/\eta),$$

where $b(i) = ni - \sum_{j=1}^{i} (i - (j-1)) d_j$. Therefore, the posterior distribution of η is given by

$$p_{\mathcal{C}}(\eta \mid \boldsymbol{x}(i), h) = \frac{\eta^{-\alpha_1-1} \sum_{j=0}^{d(i)} \binom{d(i)}{j} (-1)^j \exp\left(-\frac{jh+\beta_1+b(i)h}{\eta}\right)}{\sum_{j=0}^{d(i)} \binom{d(i)}{j} (-1)^j \frac{\Gamma(\alpha_1)}{(jh+\beta_1+b(i)h)^{\alpha_1}}}.$$

From (2.3.3), we get the expected acceptance utility for $\mathcal{C}$ after life-testing up to time $\tau_i = ih$ as

$$\begin{aligned}
\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}(i), h) =& \frac{1}{\sum\limits_{j=0}^{d(i)} \frac{\binom{d(i)}{j}(-1)^j \Gamma(\alpha_1)}{(jh+\beta_1+b(i)h)^{\alpha_1}}} \int\limits_0^\infty \exp\left(-\frac{t_0}{\eta}\right) \eta^{-\alpha_1-1} \sum_{j=0}^{d(i)} \binom{d(i)}{j} (-1)^j \exp\left(-\frac{jh+\beta_1+b(i)h}{\eta}\right) d\eta \\
=& \frac{1}{\sum_{j=0}^{d(i)} \binom{d(i)}{j} \frac{(-1)^j}{(jh+\beta_1+b(i)h)^{\alpha_1}}} \sum_{j=0}^{d(i)} \binom{d(i)}{j} \frac{(-1)^j}{(jh+\beta_1+t_0+b(i)h)^{\alpha_1-p}}
\end{aligned}$$

and the expected rejection utility for $\mathcal{C}$ after life-testing up to time $\tau_i = ih$ is given by $\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid \boldsymbol{x}(i), h) = R_0$. Now, using the inequality (2.1.5), we get the desired result. (Proved)

Result 2.2 *Suppose the lifetime of the product follows Exp$(1/\eta)$ and η follows $IG(\alpha_1, \beta_1)$. If n items are put to a life test and are inspected at an equal time intervals of length h, then $\mathcal{C}$ rejects the lot after the i^{th} inspection if the following inequality is satisfied.*

$$\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid M(\boldsymbol{x}(i)), h) = \frac{\sum_{j=0}^{d(i)} \binom{d(i)}{j} (-1)^j \frac{1}{(jh+\beta_1+t_0+b'(i)h)^{\alpha_1}}}{\sum_{j=0}^{d(i)} \binom{d(i)}{j} (-1)^j \frac{1}{(jh+\beta_1+b'(i)h)^{\alpha_1}}} < R_0,$$

where $b'(i) = nk - \sum_{j=1}^{i} (i - (j-1)) d_j$.

Proof: The proof is similar to that of Result 2.1. (Proved)

Using Results 2.1 and 2.2, and Algorithm 2.1, we find the sets $\mathcal{X}(i)$ and $\mathcal{Y}(i)$ for $i = 1, 2, \ldots, k$. Using (2.1.8) and (2.1.9), the expected utility functions of $\mathcal{M}$ corresponding to the decisions $\mathcal{A}$ and $\mathcal{R}$ are given by

$$\mathcal{U}_{\mathcal{M}}(\mathcal{A} \mid \boldsymbol{x}(i), \eta) = b_1\eta^q\Gamma(q+1) - b_2 - C_s n - C_d D - C_t\tau_i - C_I i$$

and

$$\mathcal{U}_{\mathcal{M}}(\mathcal{R} \mid \boldsymbol{x}(i), \eta) = b_3 - C_s n - C_d d(i) - C_t\tau_i - C_I i,$$

respectively. η follows $IG(\alpha_2, \beta_2)$ for $\mathcal{M}$'s prespective with PDF $p_{\mathcal{M}}(\cdot)$.

Result 2.3 *Suppose the lifetime of the product follows $Exp(1/\eta)$ and η follows $IG(\alpha_2, \beta_2)$. If n items are put to a life test and are inspected at equal time intervals of length h up to the maximum number of inspection (k), then $\mathcal{M}$'s expected utility function is given by*

$$\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = \sum_{i=1}^{k}\left[b_1\Gamma(q+1)\frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}\sum_{\boldsymbol{x}(i)\in\mathcal{X}(i)}\frac{n!}{d_1!\ldots d_i!(n-d(i))!}\sum_{j=0}^{d(i)}\binom{d(i)}{j}(-1)^j\frac{\Gamma(\alpha_2-q)}{(jh+b(i)h+\beta_2)^{\alpha_2-q}}\right]$$
$$+\sum_{i=1}^{k}[-(b_2+C_tih+C_Ii)P(A_i)+(b_3-C_tih-C_Ii)P(R_i)] - C_s n - C_d E[D],$$

where

$$P(A_i) = \beta_2^{\alpha_2}\sum_{\boldsymbol{x}(i)\in\mathcal{X}(i)}\frac{n!}{d_1!\ldots d_i!(n-d(i))!}\sum_{j=0}^{d(i)}\binom{d(i)}{j}(-1)^j\frac{1}{(jh+b(i)h+\beta_2)^{\alpha_2}},$$

$$P(R_i) = \beta_2^{\alpha_2}\sum_{\boldsymbol{x}(i)\in\mathcal{Y}(i)}\frac{n!}{d_1!\ldots d_i!(n-d(i))!}\sum_{j=0}^{d(i)}\binom{d(i)}{j}(-1)^j\frac{1}{(jh+b(i)h+\beta_2)^{\alpha_2}},$$

$$E[D] = n - n\left[\beta_2^{\alpha_2}\sum_{i=1}^{k}\sum_{\boldsymbol{x}(i)\in\mathcal{X}(i)\cup\mathcal{Y}(i)}\frac{n!}{d_1!\ldots d_i!(n-d(i))!}\sum_{j=0}^{d(i)}\binom{d(i)}{j}(-1)^j\frac{1}{(jh+ih+b(i)h+\beta_2)^{\alpha_2}}\right]$$

and $\alpha_2 > q$.

Proof: From (2.2.6), we get $\mathcal{M}$'s utility as

$$\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = \sum_{i=1}^{k}\left[\int_0^\infty b_1\Gamma(q+1)\eta^q P(A_i \mid \eta))p_{\mathcal{M}}(\eta)d\eta - b_2P(A_i) + b_3P(R_i) - (C_tih + C_Ii)\right.$$
$$\left.\times(P(A_i)+P(R_i))\right] - C_s n - C_d E[D]. \tag{2.3.4}$$

Using (2.2.2) and (2.2.3), we get

$$P(A_i \mid \eta) = \sum_{\boldsymbol{x}(i)\in\mathcal{X}(i)}\frac{n!}{d_1!\ldots d_i!(n-d(i))!}\sum_{j=0}^{d(i)}\binom{d(i)}{j}(-1)^j\exp(-jh/\eta))\exp(-b(i)h/\eta)$$

and

$$P(R_i \mid \eta) = \sum_{\boldsymbol{x}(i)\in\mathcal{Y}(i)}\frac{n!}{d_1!\ldots d_i!(n-d(i))!}\sum_{j=0}^{d(i)}\binom{d(i)}{j}(-1)^j\exp(-jh/\eta))\exp(-b(i)h/\eta),$$

respectively. Using (2.2.4) and (2.2.5), the expected number of failures is given by

$$\begin{aligned} E[D \mid \eta] &= \sum_{i=1}^{k} n(1-\exp(ih/\eta))[P(A_i \mid \eta) + P(R_i \mid \eta)] \\ &= n - \sum_{i=1}^{k} n \exp(ih/\eta)[P(A_i \mid \eta) + P(R_i \mid \eta)]. \end{aligned}$$

Now, $\int_0^\infty \eta^q P(A_i \mid \eta) p_{\mathcal{M}}(\eta) d\eta$

$$= \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \sum_{\boldsymbol{x}(i)\in\mathcal{X}(i)} \frac{n!}{d_1!\ldots d_i!(n-d(i))!} \sum_{j=0}^{d(i)} \binom{d(i)}{j} (-1)^j \int_0^\infty \eta^{-(\alpha_2-q)-1} \exp\left(-\frac{jh+b(i)h+\beta_2}{\eta}\right) d\eta$$

$$= \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \sum_{\boldsymbol{x}(i)\in\mathcal{X}(i)} \frac{n!}{d_1!\ldots d_i!(n-d(i))!} \sum_{j=0}^{d(i)} \binom{d(i)}{j} (-1)^j \frac{\Gamma(\alpha_2-q)}{(jh+b(i)h+\beta_2)^{\alpha_2-q}}, \tag{2.3.5}$$

where $\alpha_2 > q$. Similarly, we get, the expression of $P(A_i)$, $P(R_i)$ and $E[D]$. Now, using (2.3.5) in (2.3.4), we get the desired result. (Proved)

Inspection intervals of unequal length:

It may be noted that when the inspection intervals are of unequal length, the optimum scheme cannot be obtained analytically. However, it can be obtained by simulation. The likelihood function up to time τ_i is given by

$$L(\eta \mid \boldsymbol{x}(i)) \propto \prod_{l=1}^{i} \left[\exp\left(-\frac{\tau_{i-1}}{\eta}\right) - \exp\left(-\frac{\tau_i}{\eta}\right)\right]^{d_l} \left[\exp\left(-\frac{\tau_i}{\eta}\right)\right]^{s(i)}.$$

Then the posterior distribution of η is given by

$$p_{\mathcal{C}}(\eta \mid \boldsymbol{x}(i)) = \frac{L(\eta \mid \boldsymbol{x}(i)) p_{\mathcal{C}}(\eta)}{\int_0^\infty L(\eta \mid \boldsymbol{x}(i)) p_{\mathcal{C}}(\eta)\, d\eta}.$$

Therefore, $\mathcal{C}$'s acceptance criteria after i^{th} inspection is given by

$$\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}(i)) = \frac{\int_0^\infty \exp\left(-\frac{t_0}{\eta}\right) L(\eta \mid \boldsymbol{x}(i)) p_{\mathcal{C}}(\eta)\, d\eta}{\int_0^\infty L(\eta \mid \boldsymbol{x}(i)) p_{\mathcal{C}}(\eta)\, d\eta} \geq R_0$$

and $\mathcal{C}$'s rejection criteria after i^{th} inspection is given by

$$\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid M(\boldsymbol{x}(i))) = \frac{\int_0^\infty \exp\left(-\frac{t_0}{\eta}\right) L(\eta \mid M(\boldsymbol{x}(i))) p_{\mathcal{C}}(\eta)\, d\eta}{\int_0^\infty L(\eta \mid M(\boldsymbol{x}(i))) p_{\mathcal{C}}(\eta)\, d\eta} < R_0.$$

Note that $\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}(i))$ and $\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid M(\boldsymbol{x}(i)))$ cannot be obtained analytically. We compute these by simulation. We generate a large number S_1 of observations from the prior distributions $p_{\mathcal{C}}(\eta)$. Let $\eta^{(1)}, \ldots, \eta^{(S_1)}$ be the generated observations. Then,

$$\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}(i)) = \frac{\frac{1}{S_1}\sum_{j=1}^{S_1} \exp\left(-\frac{t_0}{\eta^{(j)}}\right) L(\eta^{(j)} \mid \boldsymbol{x}(i))}{\frac{1}{S_1}\sum_{j=1}^{S_1} L(\eta^{(j)} \mid \boldsymbol{x}(i))}$$

and

$$\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid M(\boldsymbol{x}(i))) = \frac{\frac{1}{S_1}\sum_{j=1}^{S_1} \exp\left(-\frac{t_0}{\eta^{(j)}}\right) L(\eta^{(j)} \mid M(\boldsymbol{x}(i)))}{\frac{1}{S_1}\sum_{j=1}^{S_1} L(\eta^{(j)} \mid M(\boldsymbol{x}(i)))}.$$

Next, we consider $\mathcal{M}$'s utility function. From (2.2.6), $\mathcal{M}$'s utility is then obtained as

$$\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = -C_s n + \int_0^\infty \left[\sum_{i=1}^{k}\Big[(b_1\eta^q\Gamma(q+1) - b_2)P(A_i \mid \eta) + b_3P(R_i \mid \eta) - (C_t\tau_i + C_I i + C_d n\left[1-\exp\{-(\eta\tau_i)^\nu\}\right])(P(A_i \mid \eta) + P(R_i \mid \eta))\Big]\right] p_{\mathcal{M}}(\eta)d\eta,$$

where $P(A_i \mid \eta) = \sum_{\boldsymbol{x}(i)\in\mathcal{X}(i)} P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i) \mid \eta)$ and $P(R_i \mid \eta) = \sum_{\boldsymbol{x}(i)\in\mathcal{Y}(i)} P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i) \mid \eta))$. We approximate the expression of $\psi_{\mathcal{M}}(\boldsymbol{\zeta})$ by simulation. We generate a large number S_2 of observations for η from their prior distributions $p_{\mathcal{M}}(\eta)$. Let $\eta^{(1)}, \ldots, \eta^{(S_2)}$ be the generated observations. Then $\mathcal{M}$'s utility can be written as

$$\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = -C_s n + \frac{1}{S_2}\sum_{j=1}^{S_2}\left[\sum_{i=1}^{k}\left[\left(b_1\left(\eta^{(j)}\right)^q\Gamma(q+1) - b_2\right)P\left(A_i \mid \eta^{(j)}\right) + b_3P\left(R_i \mid \eta^{(j)}\right) - \left(C_t\tau_i + C_I i + C_d n\left[1-\exp\left(-\frac{\tau_i}{\eta^{(j)}}\right)\right]\right)\left(P\left(A_i \mid \eta^{(j)}\right) + P\left(R_i \mid \eta^{(j)}\right)\right)\right]\right], \tag{2.3.6}$$

where $P\left(A_i \mid \eta^{(j)}\right) = \sum_{\boldsymbol{x}(i)\in\mathcal{X}(i)} P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i) \mid \eta^{(j)})$ and $P\left(R_i \mid \eta^{(j)}\right) = \sum_{\boldsymbol{x}(i)\in\mathcal{Y}(i)} P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i) \mid \eta^{(j)})$. The optimal life testing plan $\boldsymbol{\zeta}^*$ is obtained by maximizing $\mathcal{M}$'s utility (2.3.6).

2.3.2 Weibull distribution case

Here, we consider that the lifetime follows Wei(ν, λ), which is defined in Section 1.2.7. For $\mathcal{C}$'s perspective, λ follows $G(\alpha_1, \beta_1)$ with PDF $p^1_{\mathcal{C}}(\cdot)$ and ν follows $G(u_1, v_1)$ with PDF $p^2_{\mathcal{C}}(\cdot)$. The distribution $G(a, b)$ is defined in Section 1.2.3. It is assumed that λ and ν are independent. Therefore, $p_{\mathcal{C}}(\boldsymbol{\theta}) = p^1_{\mathcal{C}}(\lambda)\times p^2_{\mathcal{C}}(\nu)$. Using (2.1.1), the utility functions of $\mathcal{C}$ are given by $\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{\theta}) = \exp(-(\lambda t_0)^\nu)$ and $\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid \boldsymbol{\theta}) = R_0$, respectively. $\mathcal{C}$ accepts the lot without life testing if

$$E_{\boldsymbol{\theta}}[\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{\theta})] = \int_0^\infty\int_0^\infty \exp(-(\lambda t_0)^\nu)p^1_{\mathcal{C}}(\lambda)p^2_{\mathcal{C}}(\nu)\, d\lambda\, d\nu \geq R_0. \tag{2.3.7}$$

If the inequality (2.3.7) is not satisfied, then $\mathcal{C}$ rejects the lot. Now, if $\mathcal{C}$'s decision is rejection, $\mathcal{M}$ performs a life test to update $\mathcal{C}$'s belief. It is seen that the analytical solution can not be obtained for the Weibull distribution. Therefore, we consider the general case. The likelihood function up to time τ_i is given by

$$L(\boldsymbol{\theta} \mid \boldsymbol{x}(i)) \propto \prod_{l=1}^{i}\left[\exp\{-(\lambda\tau_{i-1})^\nu\} - \exp\{-(\lambda\tau_i)^\nu\}\right]^{d_l}\left[\exp\{-(\lambda\tau_i)^\nu\}\right]^{n-d(i)}.$$

Then the joint posterior distribution of λ and ν is given by

$$p_{\mathcal{C}}(\boldsymbol{\theta} \mid \boldsymbol{x}(i)) = \frac{L(\boldsymbol{\theta} \mid \boldsymbol{x}(i))p^1_{\mathcal{C}}(\lambda)p^2_{\mathcal{C}}(\nu)}{\int_0^\infty\int_0^\infty L(\boldsymbol{\theta} \mid \boldsymbol{x}(i))p^1_{\mathcal{C}}(\lambda)p^2_{\mathcal{C}}(\nu)\, d\lambda\, d\nu}.$$

Therefore, $\mathcal{C}$'s acceptance criteria after i^{th} inspection is given by

$$\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}(i)) = \frac{\int_0^\infty \int_0^\infty e^{-(\lambda t_0)^\nu} L(\boldsymbol{\theta} \mid \boldsymbol{x}(i)) p_{\mathcal{C}}^1(\lambda) p_{\mathcal{C}}^2(\nu)\, d\lambda\, d\nu}{\int_0^\infty \int_0^\infty L(\boldsymbol{\theta} \mid \boldsymbol{x}(i)) p_{\mathcal{C}}^1(\lambda) p_{\mathcal{C}}^2(\nu)\, d\lambda\, d\nu} \geq R_0 \tag{2.3.8}$$

and $\mathcal{C}$'s rejection criteria after i^{th} inspection is given by

$$\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid M(\boldsymbol{x}(i))) = \frac{\int_0^\infty \int_0^\infty e^{-(\lambda t_0)^\nu} L(\boldsymbol{\theta} \mid M(\boldsymbol{x}(i))) p_{\mathcal{C}}^1(\lambda) p_{\mathcal{C}}^2(\nu)\, d\lambda\, d\nu}{\int_0^\infty \int_0^\infty L(\boldsymbol{\theta} \mid M(\boldsymbol{x}(i))) p_{\mathcal{C}}^1(\lambda) p_{\mathcal{C}}^2(\nu)\, d\lambda\, d\nu} < R_0. \tag{2.3.9}$$

Note that $\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}(i))$ and $\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid M(\boldsymbol{x}(i)))$ cannot be obtained analytically. We compute by simulation. We generate a large number S_1 of observations from the prior distributions $p_{\mathcal{C}}^1(\lambda)$ and $p_{\mathcal{C}}^2(\nu)$. Let $(\lambda^{(1)}, \nu^{(1)}), \ldots, (\lambda^{(S_1)}, \nu^{(S_1)})$ be the generated observations. Then,

$$\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}(i)) = \frac{\frac{1}{S_1} \sum_{j=1}^{S_1} \exp[-(\lambda^{(j)} t_0)^{\nu^{(j)}}] L(\boldsymbol{\theta}^j \mid \boldsymbol{x}(i))}{\frac{1}{S_1} \sum_{j=1}^{S_1} L(\boldsymbol{\theta}^{(j)} \mid \boldsymbol{x}(i))}$$

and

$$\mathcal{U}_{\mathcal{C}}(\mathcal{R} \mid M(\boldsymbol{x}(i))) = \frac{\frac{1}{S_1} \sum_{j=1}^{S_1} \exp[-(\lambda^{(j)} t_0)^{\nu^{(j)}}] L(\boldsymbol{\theta}^j \mid M(\boldsymbol{x}(i)))}{\frac{1}{S_1} \sum_{j=1}^{S_1} L(\boldsymbol{\theta}^{(j)} \mid M(\boldsymbol{x}(i)))}.$$

Using the inequalities (2.3.8) and (2.3.9) and Algorithm 2.1, we find the sets $\mathcal{X}(i)$ and $\mathcal{Y}(i)$.

Next, we consider $\mathcal{M}$'s utility function. Using (2.1.8) and (2.1.9), the expected utility functions of $\mathcal{M}$ corresponding to the decisions $\mathcal{A}$ and $\mathcal{R}$ are given by

$$\mathcal{U}_{\mathcal{M}}(\mathcal{A} \mid \boldsymbol{x}(i), \boldsymbol{\theta}) = b_1 \frac{1}{\lambda^{q/\nu}} \Gamma\left(\frac{q}{\nu} + 1\right) - b_2 - C_s n - C_d d(i) - C_t \tau_i - C_I i$$

and

$$\mathcal{U}_{\mathcal{M}}(\mathcal{R} \mid \boldsymbol{x}(i), \boldsymbol{\theta}) = b_3 - C_s n - C_d d(i) - C_t \tau_i - C_I i,$$

respectively. For $\mathcal{M}$'s perspective, λ and ν follow $G(\alpha_2, \beta_2)$ with PDF $p_{\mathcal{M}}^1(\cdot)$ and $G(u_2, v_2)$ with PDF $p_{\mathcal{M}}^2(\cdot)$, respectively. The parameters λ and ν are independent. Therefore, $p_{\mathcal{M}}(\boldsymbol{\theta}) = p_{\mathcal{M}}^1(\lambda) \times p_{\mathcal{M}}^2(\nu)$. From (2.2.6), $\mathcal{M}$'s utility is then obtained as

$$\begin{aligned}\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = &- C_s n + \int_{\boldsymbol{\theta}} \Bigg[\sum_{i=1}^{k} \Bigg[\left(b_1 \frac{1}{\lambda^{q/\nu}} \Gamma\left(\frac{q}{\nu} + 1\right) - b_2 \right) P(A_i \mid \boldsymbol{\theta}) + b_3 P(R_i \mid \boldsymbol{\theta}) \\ &- \left(C_t \tau_i + C_I i + C_d n \left[1 - \exp\left\{-(\lambda \tau_i)^\nu\right\}\right]\right) \left(P(A_i \mid \boldsymbol{\theta}) + P(R_i \mid \boldsymbol{\theta})\right) \Bigg] \Bigg] p_{\mathcal{M}}(\boldsymbol{\theta}) d\boldsymbol{\theta},\end{aligned}$$

where $P(A_i \mid \boldsymbol{\theta}) = \sum_{\boldsymbol{x}(i) \in \mathcal{X}(i)} P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i) \mid \boldsymbol{\theta})$and $P(R_i \mid \boldsymbol{\theta}) = \sum_{\boldsymbol{x}(i) \in \mathcal{Y}(i)} P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i) \mid \boldsymbol{\theta}))$. We approximate the expression of $\psi_{\mathcal{M}}(\boldsymbol{\zeta})$ by simulation. We generate a large number S_2 of observations of (λ, ν) from their prior distributions $p_{\mathcal{M}}^1(\lambda)$ and $p_{\mathcal{M}}^2(\nu)$. Let $(\lambda^{(1)}, \nu^{(1)}), \ldots, (\lambda^{(S_2)}, \nu^{(S_2)})$be the generated observations. Then $\mathcal{M}$'s utility can be written as

$$\begin{aligned}\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = &- C_s n + \frac{1}{S_2} \sum_{j=1}^{S_2} \Bigg[\sum_{i=1}^{k} \Bigg[\left(b_1 \frac{1}{(\lambda^{(j)})^{q/\nu^{(j)}}} \Gamma\left(\frac{q}{\nu^{(j)}} + 1\right) - b_2 \right) P\left(A_i \mid \boldsymbol{\theta}^{(j)}\right) + b_3 P\left(R_i \mid \boldsymbol{\theta}^{(j)}\right) \\ &- \left(C_t \tau_i + C_I i + C_d n \left[1 - \exp\left\{-(\lambda^{(j)} \tau_i)^{\nu^{(j)}}\right\}\right]\right) \left(P\left(A_i \mid \boldsymbol{\theta}^{(j)}\right) + P\left(R_i \mid \boldsymbol{\theta}^{(j)}\right)\right) \Bigg] \Bigg],\end{aligned} \tag{2.3.10}$$

where $P\left(A_i \mid \boldsymbol{\theta}^{(j)}\right) = \sum_{\boldsymbol{x}(i) \in \mathcal{X}(i)} P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i) \mid \boldsymbol{\theta}^{(j)})$ and $P\left(R_i \mid \boldsymbol{\theta}^{(j)}\right) = \sum_{\boldsymbol{x}(i) \in \mathcal{Y}(i)} P_{\boldsymbol{X}(i)}(\boldsymbol{x}(i) \mid \boldsymbol{\theta}^{(j)})$. The optimal life testing plan $\boldsymbol{\zeta}^*$ is obtained by maximizing $\mathcal{M}$'s utility (2.3.10).

2.4 Random decision-making sampling plan

We now consider an approach where $\mathcal{M}$'s knowledge about $\mathcal{C}$'s utility, prior distribution, and actions are not required. $\mathcal{M}$ knows the form of the utility function, but the parameters of the utility function are unknown. Similarly, the form of $\mathcal{C}$'s prior distribution is known to $\mathcal{M}$, but the hyperparameters are uncertain. $\mathcal{M}$ estimates the probabilities of $\mathcal{C}$'s actions after the life testing. Since $\mathcal{C}$'s exact actions after the life testing are not known, $\mathcal{M}$'s decisions do not depend on them and consequently, $\mathcal{C}$'s decisions do not affect the design.

The utility functions for $\mathcal{C}$ are discussed in (2.1.1). The methodology is discussed for the exponential distribution. We consider that $\alpha_1 \sim \mathcal{U}[\alpha_{11}\alpha_{12}]$, $\beta_1 \sim \mathcal{U}[\beta_{11}, \beta_{12}]$, $t_0 \sim \mathcal{U}[t_{01}, t_{02}]$ and $R_0 \sim \mathcal{U}[R_{01}, R_{02}]$, where $0 < \alpha_{11} < \alpha_{12}$, $0 < \beta_{11} < \beta_{12}$, $0 < t_{01} < t_{02}$ and $0 < R_{01} < R_{02} < 1$. The distribution $\mathcal{U}[a, b]$ is defined in Section 1.2.6. Let $\mathcal{A}_{wl}$ and $\mathcal{R}_{wl}$ be the events that $\mathcal{C}$ accepts or rejects the lot without inspection, respectively. Using the inequality (2.1.2), the probability of acceptance of the lot by $\mathcal{C}$ without inspection can be written as

$$\begin{aligned} P_{\mathcal{C}}(\mathcal{A}_{wl}) &= P\left[\mathcal{U}_{\mathcal{C}}\left(E_{\boldsymbol{\theta}}[\mathcal{A} \mid \boldsymbol{\theta}]\right) \geq R_0\right] \\ &= \int_{\boldsymbol{\phi}_{\mathcal{C}}} P\left[R_0 \leq s_{01}(\boldsymbol{\phi}_{\mathcal{C}}) \mid \boldsymbol{\phi}_{\mathcal{C}}\right] p(\boldsymbol{\phi}_{\mathcal{C}}) d\boldsymbol{\phi}_{\mathcal{C}}, \end{aligned} \tag{2.4.1}$$

where $\boldsymbol{\phi}_{\mathcal{C}} = (\alpha_1, \beta_1, t_0)$, $s_{01}(\boldsymbol{\phi}_{\mathcal{C}}) = E_{\boldsymbol{\theta}}[\mathcal{U}(\mathcal{A} \mid \boldsymbol{\theta})]$ and $p(\boldsymbol{\phi}_{\mathcal{C}})$ is the joint probability density function of $\boldsymbol{\phi}_{\mathcal{C}}$. The probability of the event $\mathcal{R}_{wl}$ is $P_{\mathcal{C}}(\mathcal{R}_{wl}) = 1 - P_{\mathcal{C}}(\mathcal{A}_{wl})$. Now $\mathcal{M}$ conducts life testing to increase the chance of acceptance of the lot by $\mathcal{C}$. We use the inequalities (2.1.5) and (2.1.7) to estimate the probabilities of acceptance and rejection for each $\boldsymbol{x}(i)$. Let $\mathcal{A}_1(\boldsymbol{x}(1))$ be the event of acceptance by $\mathcal{C}$ after the 1^{st} inspection for the observed data $\boldsymbol{x}(1)$. Using the inequality (2.1.5), the probability of acceptance of the lot by $\mathcal{C}$ after the 1^{st} inspection can be written as

$$\begin{aligned} P_{\mathcal{C}}\left(\mathcal{A}_1(\boldsymbol{x}(1)) \mid \mathcal{R}_{wl}\right) &= P\left[\mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid \boldsymbol{x}(1)\right) > R_0 \mid \mathcal{R}_{wl}\right] \\ &= \frac{1}{P_{\mathcal{C}}(\mathcal{R}_{wl})} P\left[\mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid \boldsymbol{x}(1)\right) \geq R_0, \mathcal{U}_{\mathcal{C}}\left(E_{\boldsymbol{\theta}}[\mathcal{A} \mid \boldsymbol{\theta}]\right) < R_0\right] \\ &= \frac{1}{P_{\mathcal{C}}(\mathcal{R}_{wl})} \int_{\boldsymbol{\phi}_{\mathcal{C}}} P\left[s_{01}(\boldsymbol{\phi}_{\mathcal{C}}) < R_0 \leq s_{11}(\boldsymbol{\phi}_{\mathcal{C}}) \mid \boldsymbol{\phi}_{\mathcal{C}}\right] p(\boldsymbol{\phi}_{\mathcal{C}}) d\boldsymbol{\phi}_{\mathcal{C}}, \end{aligned} \tag{2.4.2}$$

where $s_{11}(\boldsymbol{\phi}_{\mathcal{C}}) = \mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}(1))$ and

$$P(x < R_0 < y) = \begin{cases} \frac{y-x}{R_{02}-R_{01}}, & \text{if } R_{01} < x < y < R_{02}, \\ \frac{y-R_{01}}{R_{02}-R_{01}}, & \text{if } x < R_{01} < y < R_{02}, \\ 1 - \frac{x-R_{01}}{R_{02}-R_{01}}, & \text{if } R_{01} < x < R_{02} < y, \\ 1, & \text{if } x < R_{01} < R_{02} < y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{R}_1(\boldsymbol{x}(1))$ be the event of rejection by $\mathcal{C}$ after the 1^{st} inspection for the observed data $\boldsymbol{x}(1)$. Using inequality (2.1.7), the probability of rejection of the lot by $\mathcal{C}$ after the 1^{st} inspection can be written as

$$\begin{aligned} P_{\mathcal{C}}\left(\mathcal{R}_1(\boldsymbol{x}(1)) \mid \mathcal{R}_{wl}\right) &= P\left[\mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid M(\boldsymbol{x}(1))\right) < R_0 \mid \mathcal{R}_{wl}\right] \\ &= \frac{1}{P_{\mathcal{C}}(\mathcal{R}_{wl})} P\left[\mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid M(\boldsymbol{x}(1))\right) < R_0, \mathcal{U}_{\mathcal{C}}\left(E_{\boldsymbol{\theta}}\left[\mathcal{A} \mid \boldsymbol{\theta}\right]\right) < R_0\right] \\ &= \frac{1}{P_{\mathcal{C}}(\mathcal{R}_{wl})} \int_{\boldsymbol{\phi}_{\mathcal{C}}} P\left[R_0 > \max\left\{s_{12}(\boldsymbol{\phi}_{\mathcal{C}}), s_{01}(\boldsymbol{\phi}_{\mathcal{C}})\right\} \mid \boldsymbol{\phi}_{\mathcal{C}}\right] p(\boldsymbol{\phi}_{\mathcal{C}}) d\boldsymbol{\phi}_{\mathcal{C}}, \end{aligned} \tag{2.4.3}$$

where $s_{12}(\boldsymbol{\phi}_{\mathcal{C}}) = \mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid M(\boldsymbol{x}(1)))$. For the sequential sampling plan, $\mathcal{M}$ and $\mathcal{C}$ go for the i^{th} inspection if $\mathcal{C}$ cannot take a decision up to the $(i-1)^{th}$ inspection, i.e., the inequalities (2.1.5) and (2.1.7) do not hold up to the $(i-1)^{th}$ inspection for the observed data $(d_1, d_2, \ldots, d_{i-1})$.

Let $\mathcal{A}_i(\boldsymbol{x}(i))$ be the event of acceptance by $\mathcal{C}$ after the i^{th} inspection for the data $\boldsymbol{x}(i)$ when $\mathcal{C}$ cannot take a decision for the lot up to the $(i-1)^{th}$ inspection, for $i = 2, \ldots, k$. Therefore, the inequalities (2.1.5) and (2.1.7) do not hold for the data $\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots, \boldsymbol{x}(i-1)$. Then the probability of acceptance of the lot by $\mathcal{C}$ after the i^{th} inspection can be written as

$$
\begin{aligned}
& P_{\mathcal{C}}\left(\mathcal{A}_i(\boldsymbol{x}(i)) \mid \mathcal{R}_{wl}\right) \\
=& \frac{1}{P_{\mathcal{C}}(\mathcal{R}_{wl})} P\left[\mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid \boldsymbol{x}(i)\right) \geq R_0, \bigcap_{j=1}^{i-1}\left\{\mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid \boldsymbol{x}(j)\right) < R_0\right\}, \bigcap_{j=1}^{i-1}\left\{\mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid M(\boldsymbol{x}(j))\right) \geq R_0\right\}, \mathcal{R}_{wl}\right] \\
=& \frac{1}{P_{\mathcal{C}}(\mathcal{R}_{wl})} \int_{\boldsymbol{\phi}_{\mathcal{C}}} P\left[R_0 \leq s_{i1}(\boldsymbol{\phi}_{\mathcal{C}}), \bigcap_{j=0}^{i-1}\left\{R_0 > s_{j1}(\boldsymbol{\phi}_{\mathcal{C}})\right\}, \bigcap_{j=1}^{i-1}\left\{R_0 \leq s_{j2}(\boldsymbol{\phi}_{\mathcal{C}})\right\} \;\middle|\; \boldsymbol{\phi}_{\mathcal{C}}\right] p(\boldsymbol{\phi}_{\mathcal{C}}) d\boldsymbol{\phi}_{\mathcal{C}} \\
=& \frac{1}{P(\mathcal{R}_{wl})} \int_{\boldsymbol{\phi}_{\mathcal{C}}} P\left[S_{i1}(\boldsymbol{\phi}_{\mathcal{C}}) < R_0 \leq \min\left\{s_{i1}(\boldsymbol{\phi}_{\mathcal{C}}), S_2^i(\boldsymbol{\phi}_{\mathcal{C}})\right\} \;\middle|\; \boldsymbol{\phi}_{\mathcal{C}}\right] p(\boldsymbol{\phi}_{\mathcal{C}}) d\boldsymbol{\phi}_{\mathcal{C}},
\end{aligned} \tag{2.4.4}
$$

where $s_{j1}(\boldsymbol{\phi}_{\mathcal{C}}) = \mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid \boldsymbol{x}(j)\right)$, $s_{j2}(\boldsymbol{\phi}_{\mathcal{C}}) = \mathcal{U}_{\mathcal{C}}\left(\mathcal{R} \mid M(\boldsymbol{x}(j))\right)$ for $j = 1, 2, \ldots, (i-1)$, $S_{i1}(\boldsymbol{\phi}_{\mathcal{C}}) = \max_{0 \leq j \leq i-1}\{s_{j1}(\boldsymbol{\phi}_{\mathcal{C}})\}$ and $S_{i2}(\boldsymbol{\phi}_{\mathcal{C}}) = \min_{1 \leq j \leq i-1}\{s_{j2}(\boldsymbol{\phi}_{\mathcal{C}})\}$.

Let $\mathcal{R}_i(\boldsymbol{x}(i))$ be the event of rejection by $\mathcal{C}$ after the i^{th} inspection for the data $\boldsymbol{x}(i)$ when $\mathcal{C}$ cannot accept or reject the lot up to the $(i-1)^{th}$ inspection, for $i = 2, \ldots, k$. Therefore, the inequalities (2.1.5) and (2.1.7) do not hold for the data $\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots, \boldsymbol{x}(i-1)$. Then the probability of rejection of the lot by $\mathcal{C}$ after the i^{th} inspection can be written as

$$
\begin{aligned}
& P_{\mathcal{C}}\left(\mathcal{R}_i(\boldsymbol{x}(i)) \mid \mathcal{R}_{wl}\right) \\
=& \frac{1}{P_{\mathcal{C}}(\mathcal{R}_{wl})} P\left[\mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid M(\boldsymbol{x}(i))\right) < R_0, \bigcap_{j=1}^{i-1}\left\{\mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid \boldsymbol{x}(j)\right) < R_0\right\}, \bigcap_{j=1}^{i-1}\left\{\mathcal{U}_{\mathcal{C}}\left(\mathcal{A} \mid M(\boldsymbol{x}(j))\right) \geq R_0\right\}, \mathcal{R}_{wl}\right] \\
=& \frac{1}{P_{\mathcal{C}}(\mathcal{R}_{wl})} \int_{\boldsymbol{\phi}_{\mathcal{C}}} P\left[R_0 > s_{i2}(\boldsymbol{\phi}_{\mathcal{C}}), \bigcap_{j=0}^{i-1}\left\{R_0 > s_{j1}(\boldsymbol{\phi}_{\mathcal{C}})\right\}, \bigcap_{j=1}^{i-1}\left\{R_0 \leq s_{j2}(\boldsymbol{\phi}_{\mathcal{C}})\right\} \;\middle|\; \boldsymbol{\phi}_{\mathcal{C}}\right] p(\boldsymbol{\phi}_{\mathcal{C}})\, d\boldsymbol{\phi}_{\mathcal{C}} \\
=& \frac{1}{P_{\mathcal{C}}(\mathcal{R}_{wl})} \int_{\boldsymbol{\phi}_{\mathcal{C}}} P\left[\max\left\{S_{i1}(\boldsymbol{\phi}_{\mathcal{C}}), s_{i2}(\boldsymbol{\phi}_{\mathcal{C}})\right\} < R_0 \leq S_{i2}(\boldsymbol{\phi}_{\mathcal{C}}) \;\middle|\; \boldsymbol{\phi}_{\mathcal{C}}\right] p(\boldsymbol{\phi}_{\mathcal{C}})\, d\boldsymbol{\phi}_{\mathcal{C}}.
\end{aligned} \tag{2.4.5}
$$

Next, using (2.4.2), (2.4.3), (2.4.4) and (2.4.5), $\mathcal{M}$'s expected utility for a given data $\boldsymbol{x}(i)$ if the test is stopped at τ_i can be written as

$$
\begin{aligned}
\psi_{\mathcal{M}}(\boldsymbol{x}(i) \mid \boldsymbol{\theta}) =& P_{\mathcal{C}}(\mathcal{A}_i(\boldsymbol{x}(i)) \mid \mathcal{R}_{wl})\left(b_1 E_{X \mid (\boldsymbol{x}(i), \boldsymbol{\theta})}(X^q) - b_2 - C_s n - C_d d(i) - C_t \tau_i - C_I i\right) \\
& + P_{\mathcal{C}}(\mathcal{R}_i(\boldsymbol{x}(i)) \mid \mathcal{R}_{wl})\left(b_3 - C_s n - C_d d(i) - C_t \tau_i - C_I i\right).
\end{aligned}
$$

The expected utility is obtained as

$$
\begin{aligned}
\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = \sum_{i=1}^{k} \sum_{\boldsymbol{x}(i)} \int_{\boldsymbol{\theta}} \Big[& P_{\mathcal{C}}(\mathcal{A}_i(\boldsymbol{x}(i)) \mid \mathcal{R}_{wl})\left(b_1 E_{X|(\boldsymbol{x}(i), \boldsymbol{\theta})}(X^q) - b_2 - C_s n - C_d n F(\tau_i \mid \boldsymbol{\theta}) - C_t \tau_i - C_I i\right) \\
& + P_{\mathcal{C}}(\mathcal{R}_i(\boldsymbol{x}(i)) \mid \mathcal{R}_{wl})\left(b_3 - C_s n - C_d n F(\tau_i \mid \boldsymbol{\theta}) - C_t \tau_i - C_I i\right)\Big] P_{\mathbf{X}(i)}(\boldsymbol{x}(i) \mid \boldsymbol{\theta})\; p_{\mathcal{M}}(\boldsymbol{\theta})\; d\boldsymbol{\theta}.
\end{aligned} \tag{2.4.6}
$$

The optimal decision $\boldsymbol{\zeta}^* = (n^*, \tau_1^*, \tau_2^*, \ldots, \tau_k^*, k^*)$ is obtained by maximizing (2.4.6) , i.e.,

$$
\boldsymbol{\zeta}^* = \arg\max_{\boldsymbol{\zeta}} \psi_{\mathcal{M}}(\boldsymbol{\zeta}).
$$

2.5 Numerical example

Here we illustrate the proposed method with examples and compute the optimum RASPs.

Example 2.1 *The lifetime is assumed to follow Exp*$(1/\eta)$. *To choose the hyperparameters of* $p_{\mathcal{M}}(\eta)$*, the prior distribution of* η *for* $\mathcal{M}$*, we assume that* $\mathcal{M}$*'s belief about the product reliability is* 0.98 *at time* $t_0 = 1$*, i.e.,* $E_\eta(R(1 \mid \eta)) = 0.98$*. Thus,*

$$\int_0^\infty P(X > 1 \mid \eta) p_{\mathcal{M}}(\eta) d\eta = 0.98 \quad \Longrightarrow \quad \left(\frac{\beta_2}{\beta_2 + 1}\right)^{\alpha_2} = 0.98. \tag{2.5.1}$$

Due to the randomness of η*, the probability that reliability at* $t_0 = 1$ *is less than* 0.98 *is* 0.6*. That is,* $P(R(1 \mid \eta) < 0.98) = 0.6$*. This implies*

$$P(\exp(-1/\eta) < 0.98) = 0.6 \quad \Longrightarrow \quad P(\eta < \ln(1/0.98)) = 0.6. \tag{2.5.2}$$

Solving (2.5.1) and (2.5.2), we obtain $\alpha_2 = 1.66$ *and* $\beta_2 = 81.84$*. The cost components and risk parameter of* $\mathcal{M}$*'s utility function are chosen as* $b_1 = 200$*,* $b_2 = 5000$*,* $b_3 = 250$*,* $C_s = 15$*,* $C_d = 4$*,* $C_t = 30$*,* $C_I = 5$*,* $q = 0.8$*. Next, we choose the hyperparameters of the prior distribution of* η *for* $\mathcal{C}$*. Two different priors are considered for illustration.*
Prior 1: $\mathcal{C}$*'s belief about the product reliability is* 0.85 *at time* $t_0 = 1$ *and* $P(R(1 \mid \eta) < 0.7) = 0.1$*. Solving this, we obtain* $\alpha_1 = 1.5$ *and* $\beta_1 = 8.82$*.*
Prior 2: $\mathcal{C}$*'s belief about the product reliability is* 0.75 *at time* $t_0 = 1$ *and* $P(R(1 \mid \eta) < 0.6) = 0.1$*. Solving this, we obtain* $\alpha_1 = 3.61$ *and* $\beta_1 = 12.05$*.*
For both priors, $\mathcal{C}$ *accepts the lot if the mean reliability of the product achieves* 0.95 *at time* $t_0 = 1$*.*

Using Prior 1, $P(A_i)$ for $i = 1, \ldots, k$ under different scenarios are shown in Figures 2.2 and 2.3.

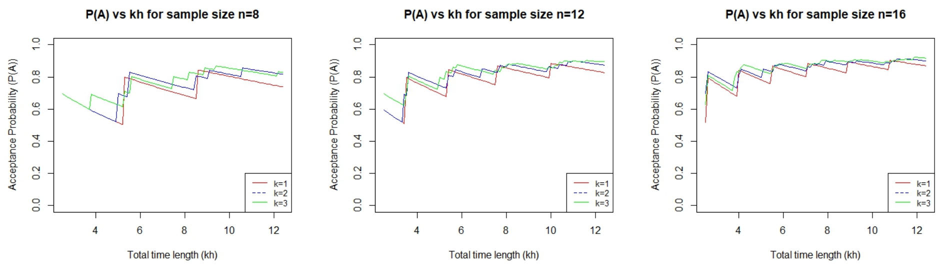


Figure 2.2: Probability of acceptance versus inspection time (kh) for different values of n and k.

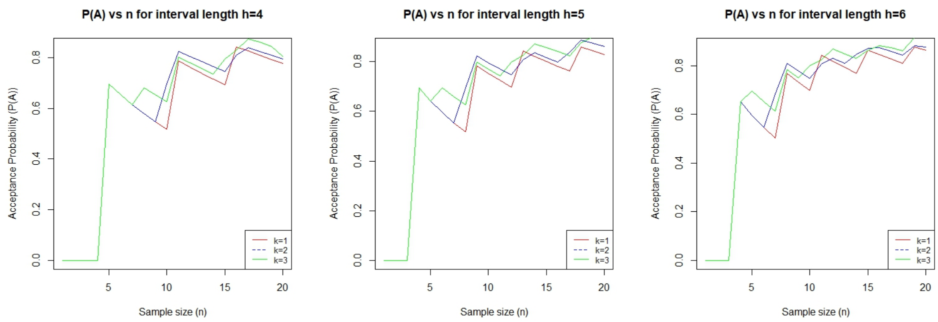


Figure 2.3: Probability of acceptance versus sample size (n) for different values of h and k.

It is assumed that inspection intervals are of equal length h, so that $\tau_i = ih$, $i = 1, \ldots, k$. The probabilities of acceptance for different values of n, k, and inspection times (kh) are calculated. In Figure 2.2, the probability of acceptance is plotted versus inspection time (kh) for $k = 1, 2, 3$, and for fixed sample sizes $n = 8, 12, 16$. It is observed that the probability of acceptance decreases up to a certain time and then fluctuates upwards. This is possibly due to the fact that the number of elements in the set $\mathcal{X}(i)$ increases after that time. It is also observed that as n and k increase, the time length between two fluctuations decreases. Consequently, $\mathcal{C}$ can take decisions more quickly. In addition, the probability of acceptance increases when n and k increase. In Figure 2.3, the probability of acceptance is plotted versus sample size (n) for $k = 1, 2, 3$ and for fixed inspection intervals $h = 4, 5, 6$. Similar behavior is observed as in Figure 2.2. The probability of acceptance decreases up to a certain sample size and then fluctuates upwards. This is again due to the fact that the number of elements in the set $\mathcal{X}(i)$ increases after that sample size.

Example 2.2 *The lifetime is assumed to follow Wei*(ν, λ). *The hyperparameters for the priors* $p^1_{\mathcal{C}}(\lambda)$ *and* $p^2_{\mathcal{C}}(\nu)$ *are taken as* $(\alpha_1, \beta_1) = (10, 20)$ *and* $(u_1, v_1) = (100, 100)$. *With these values,* $\mathcal{C}$*'s initial belief about the product reliability is* 0.78 *at time* $t_0 = 0.5$*, i.e.,*

$$\int_0^\infty \int_0^\infty \exp(-(\lambda t_0)^\nu) p^1_{\mathcal{C}}(\lambda) p^2_{\mathcal{C}}(\nu)\, d\lambda\, d\nu = 0.78.$$

$\mathcal{C}$ *accepts the lot when the product reliability reaches* 0.9 *at time* $t_0 = 0.5$. *The hyperparameters for* $\mathcal{M}$*'s priors are taken as* $(\alpha_1, \beta_1) = (4, 100)$ *and* $(u_1, v_1) = (50, 50)$. *The cost parameters of* $\mathcal{M}$*'s utility are the same as in Example 2.1 and the risk parameter is* $q = 1$.

Example 2.3 *Here we consider the RDSP approach and assume that the lifetime follows Exp*$(1/\eta)$. *The parameters of* $\mathcal{C}$*'s utility and prior are assumed random. We take* $\alpha_1 \sim \mathcal{U}(1.1, 4)$, $\beta_1 \sim \mathcal{U}(8, 20)$, $t_0 \sim \mathcal{U}(0.5, 1.5)$ *and* $R_0 \sim \mathcal{U}(0.85, 0.95)$. *The cost parameters of* $\mathcal{M}$*'s utility are the same as in Example 2.1 and the risk parameter is* $q = 0.8$.

2.5.1 Computation of optimum BRASP

We now consider the computational procedures of the optimal BRASP, the acceptance set $\mathcal{X}(i)$, the rejection set $\mathcal{Y}(i)$, and the acceptance probability $P(A_i)$ for each inspection $i = 1, \ldots, k$.

Inspection intervals of equal length:

Here the optimal values of n, h, and k are obtained simultaneously using Algorithm 2.2.

Algorithm 2.2: Procedure for determining of ζ^* for equal inspection intervals.

Input : Upper bounds n_0, k_0, h_0 for n, k, and h, respectively
Output : Optimal values (n^*, h^*, k^*) maximizing $\mathcal{M}$'s utility

```
Fix n_0, k_0, and h_0 as sufficiently large upper bounds for n, k, and h;
for n ← 1 to n_0 do
    for k ← 1 to k_0 do
        Maximize ψ_M(n, h, k) from (2.2.6) with respect to h, where 0 < h ≤ h_0;
        Let h*(n, k) be the value of h that maximizes ψ_M(n, h, k) for given (n, k);
Choose (n*, h*, k*) such that (n*, h*, k*) = arg max_{1≤n≤n_0, 1≤k≤k_0} ψ_M(n, h*(n, k), k).
```

The results for the exponential and Weibull distributions with equal interval lengths are presented in Tables 2.2 and 2.3, respectively, using the parameter values specified in Examples 2.1 and 2.2.

Table 2.2: Optimal value (n^*, h^*, k^*), acceptance set $\mathcal{X}(i)$, rejection set $\mathcal{Y}(i)$ and acceptance probability $P(A_i)$ after each inspection i for the exponential distribution with equal inspection intervals.

	(n^*, h^*, k^*)	i	$P(A_i)$	$\mathcal{X}(i)$	$\mathcal{Y}(i)$
Prior 1	(11, 2.9, 2)	1	0.58	{0}	{2-12}
		2	0.15	{(1, 0)}	{(1, 1-10)}
Prior 2	(13, 1.5, 5)	1	0	{}	{2-13}
		2	0	{}	{(0, 2-13), (1, 1-12)}
		3	0.41	{(0, 0, 0)}	{(0, 0, 2-13), (0, 1, 1-13),(1, 0, 1-13)}
		4	0	{}	{(0, 0, 1, 1-12), (0, 1, 0, 1-12),(1, 0, 0, 1-12)}
		5	0.16	{(0, 0, 1, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 0, 0)}	{(0, 0, 1, 0, 1-12), (0, 1, 0, 0, 1-12), (1, 0, 0, 0, 1-12)}

Table 2.3: Optimal value of (n^*, h^*, k^*), acceptance set $\mathcal{X}(i)$, rejection set $\mathcal{Y}(i)$ and acceptance probability $P(A_i)$ after each inspection i for the Weibull distribution with equal inspection intervals.

(n^*, h^*, k^*)	i	$P(A_i)$	$\mathcal{X}(i)$	$\mathcal{Y}(i)$
(7, 4.9, 2)	1	0.31	{0}	{3-7}
	2	0.39	{(1, 0-2), (2, 0)}	{(1, 3-6), (2, 1-5)}

Using the values of the parameters given in Example 2.3, the optimal values of (n, h, k) under the RDSP approach are obtained and the results are reported in Table 2.4.

Table 2.4: Optimal value (n^*, h^*, k^*) and $P(\boldsymbol{x}(i))$ for $d(i) = 0, 1, 2, 3$ after each inspection i for the exponential distribution under the RDSP.

(n^*, h^*, k^*)	i	$d(i)=0$		$d(i)=1$		$d(i)=2$		$d(i)=3$	
		$\boldsymbol{x}(i)$	$P(\boldsymbol{x}(i))$	$\boldsymbol{x}(i)$	$P(\boldsymbol{x}(i))$	$\boldsymbol{x}(i)$	$P(\boldsymbol{x}(i))$	$\boldsymbol{x}(i)$	$P(\boldsymbol{x}(i))$
(9, 3.4, 3)	1	(0)	0.84	(1)	0.61	(2)	0.34	(3)	0.12
	2	(0, 0)	0.13	(0, 1)	0.08	(0, 2)	0.01	(0, 3)	0
				(1, 0)	0.30	(1, 1)	0.17	(1, 2)	0.03
						(2, 0)	0.41	(2, 1)	0.22
								(3, 0)	0.38
	3	(0, 0, 0)	0.02	(0, 0, 1)	0.01	(0, 0, 2)	0	(0, 0, 3)	0
				(0, 1, 0)	0.06	(0, 1, 1)	0.02	(0, 1, 2)	0
				(1, 0, 0)	0.07	(1, 0, 1)	0.03	(1, 0, 2)	0
						(0, 2, 0)	0.08	(0, 2, 1)	0.03
						(1, 1, 0)	0.14	(1, 1, 1)	0.05
						(2, 0, 0)	0.16	(2, 0, 1)	0.06
								(1, 2, 0)	0.05
								(2, 1, 0)	0.22
								(3, 0, 0)	0.25

Using (2.4.1h), we get that the probability of acceptance by $\mathcal{C}$ without life testing is $P(\mathcal{A}_{wl}) = 0.1$. Let $P(\boldsymbol{x}(i))$ denote the probability of acceptance after the i^{th} inspection for the observed data $\boldsymbol{x}(i) = (d_1, \ldots, d_i)$, given that $\mathcal{C}$ rejects the lot without life testing. The values of $P(\boldsymbol{x}(i))$ for $d(i) = 0, 1, 2, 3$ are reported in Table 2.4.

Inspection intervals of unequal length:

Here the optimal values of $(n, \tau_1, \ldots, \tau_k, k)$ are obtained simultaneously using Algorithm 2.3.

Algorithm 2.3: Procedure for determining $\boldsymbol{\zeta}^*$ with different inspection intervals.

Input : Upper bounds n_0 and k_0 for n and k; maximum interval width h_0

Output : Optimal values $(n^*, \tau_1^*, \ldots, \tau_{k^*}^*, k^*)$ that maximizing $\mathcal{M}$'s utility

Choose sufficiently large values n_0 and k_0 as upper bounds for n and k;

for $n \leftarrow 1$ **to** n_0 **do**

  for $k \leftarrow 1$ **to** k_0 **do**

    1. Set initial inspection times $(\tau_1, \ldots, \tau_k) = (h, 2h, \ldots, kh)$;
    2. Maximize $\psi_{\mathcal{M}}(n, \boldsymbol{\tau}, k)$ with respect to h, for $0 < h \leq h_0$;
    3. Let $h^*(n, k)$ be the optimal value, and set $\tau_i = i \cdot h^*(n, k)$ for $i = 1, \ldots, k$;
    4. Use $(\tau_1, \ldots, \tau_k)$ as the initial values and further maximize $\psi_{\mathcal{M}}(n, \boldsymbol{\tau}, k)$ w.r.t. each τ_i, allowing unequal spacing;

Select the $(n^*, \boldsymbol{\tau}^*, k^*)$ that maximizes $\psi_{\mathcal{M}}(n, \boldsymbol{\tau}, k)$ over all combinations;

The results for the exponential using the value of the parameters given in Example 2.1 and Weibull distributions using the value of the parameters given in Example 2.2 with different inspection intervals are reported in Tables 2.5 and 2.6, respectively.

Table 2.5: Optimal value $(n^*, \tau_1^*, \ldots, \tau_k^*, k^*)$, acceptance set $\mathcal{X}(i)$, rejection set $\mathcal{Y}(i)$ and acceptance probability $P(A_i)$ for each inspection i for the exponential distribution with different inspection intervals.

	$(n^*, \tau_1^*, \ldots, \tau_k^*, k^*)$	i	$P(A_i)$	$\mathcal{X}(i)$	$\mathcal{Y}(i)$
Prior 1	(11, 3.6, 5.8, 2)	1	0.52	{0}	{2-12}
		2	0.19	{(1, 0)}	{(1, 1-10)}
Prior 2	(14, 4.8, 9.9, 2)	1	0.38	{0}	{3-14}
		2	0.25	{(1, 0-1), (2, 0)}	{(1, 2-13), (2, 1-12)}

Table 2.6: Optimal value $(n^*, \tau_1^*, \ldots, \tau_k^*, k^*)$, acceptance set $\mathcal{X}(i)$, rejection set $\mathcal{Y}(i)$ and acceptance probability $P(A_i)$ after each inspection i for the Weibull distribution with different inspection intervals.

$(n^*, \tau_1^*, \ldots, \tau_k^*, k^*)$	i	$P(A_i)$	$\mathcal{X}(i)$	$\mathcal{Y}(i)$
(8, 4.9, 11.8, 2)	1	0.60	{0-1}	{3-8}
	2	0.18	{(2, 0-2)}	{(2, 3-6)}

The optimal value of $\mathcal{M}$'s utility with different inspection intervals is larger than that with equal inspection intervals. For example, the simulated optimal value of $\mathcal{M}$'s utility for the exponential distribution under Prior 1 with equal inspection intervals is 2958, whereas with different inspection intervals it is 2969.

Remark 2.1 *If, in any of the scenarios, $\mathcal{M}$'s maximum utility is less than b_3, the optimal sampling plan is $n = 0$. This means that no sample size can convince $\mathcal{C}$ about product reliability.*

Remark 2.2 *There are some computational limitations in the procedure. For large n, the procedure becomes computationally intensive. We have taken $S_1 = S_2 = 10000$ for generating observations. Better results may be obtained if the number of generated observations S_1 and S_2 is increased, but this will also increase computation time. Moreover, under the RDSP approach, the probability of acceptance is obtained for each observed dataset, so the same computational limitations arise.*

2.5.2 Illustration of the manufacturer's and consumer's action after each inspection

In all cases, after determining the value of ζ^*, $\mathcal{M}$ conducts the life testing to update $\mathcal{C}$'s belief. In Example 2.1, we obtain the optimal design $(13, 1.5, 5)$ for equal inspection intervals when $\mathcal{C}$'s hyperparameters are $\alpha_1 = 3.61$ and $\beta_1 = 12.05$. Therefore, in that case, $\mathcal{M}$ conducts a life test with a sample size of 13 to update $\mathcal{C}$'s beliefs, and after a time length of 1.5, the number of failures is observed. Based on the observed number of failures, $\mathcal{C}$ makes the decision.

Next, we generate data and the corresponding decisions of $\mathcal{M}$ and $\mathcal{C}$ using the flow chart given in Figure 2.1 and Table 2.2, as discussed below:

Data 1: After the 1st inspection, 3 failures are observed, i.e., $\boldsymbol{x}(1) = 3$. Since $\boldsymbol{x}(1) \in \mathcal{Y}(1)$, $\mathcal{M}$ stops the life testing after the 1st inspection, and $\mathcal{C}$ rejects the lot.

Data 2: After the 1st inspection, no failures are observed, i.e., $\boldsymbol{x}(1) = 0$. Since $\boldsymbol{x}(1) \notin \mathcal{X}(1)$ and $\boldsymbol{x}(1) \notin \mathcal{Y}(1)$, $\mathcal{M}$ and $\mathcal{C}$ proceed to the 2nd inspection. After the 2nd inspection, 1 failure is observed, so $\boldsymbol{x}(2) = (0, 1)$. Since $\boldsymbol{x}(2) \notin \mathcal{X}(2)$ and $\boldsymbol{x}(2) \notin \mathcal{Y}(2)$, they proceed to the 3rd inspection. After the 3rd inspection, 1 failure is observed, so $\boldsymbol{x}(3) = (0, 1, 1)$. Since $\boldsymbol{x}(3) \in \mathcal{Y}(3)$, $\mathcal{M}$ stops the life testing after the 3rd inspection, and $\mathcal{C}$ rejects the lot.

Data 3: After the 1st inspection, no failures are observed, i.e., $\boldsymbol{x}(1) = 0$. Since $\boldsymbol{x}(1) \notin \mathcal{X}(1)$ and $\boldsymbol{x}(1) \notin \mathcal{Y}(1)$, $\mathcal{M}$ and $\mathcal{C}$ proceed to the 2nd inspection. After the 2nd inspection, no failures are observed, so $\boldsymbol{x}(2) = (0, 0)$. Since $\boldsymbol{x}(2) \notin \mathcal{X}(2)$ and $\boldsymbol{x}(2) \notin \mathcal{Y}(2)$, they proceed to the 3rd inspection. After the 3rd inspection, no failures are observed, so $\boldsymbol{x}(3) \in \mathcal{X}(3)$. Hence, $\mathcal{C}$ accepts the lot after the 3rd inspection.

Data 4: After the 1st inspection, 1 failure is observed, i.e., $\boldsymbol{x}(1) = 1$. Since $\boldsymbol{x}(1) \notin \mathcal{X}(1)$ and $\boldsymbol{x}(1) \notin \mathcal{Y}(1)$, $\mathcal{M}$ and $\mathcal{C}$ proceed to the 2nd inspection. After the 2nd inspection, 1 failure is observed, so $\boldsymbol{x}(2) = (1, 1)$. Since $\boldsymbol{x}(2) \in \mathcal{Y}(2)$, $\mathcal{M}$ stops the life testing after the 2nd inspection, and $\mathcal{C}$ rejects the lot.

Data 5: After the 1st inspection, no failures are observed, i.e., $\boldsymbol{x}(1) = 0$. Since $\boldsymbol{x}(1) \notin \mathcal{X}(1)$ and $\boldsymbol{x}(1) \notin \mathcal{Y}(1)$, $\mathcal{M}$ and $\mathcal{C}$ proceed to the 2nd inspection. After the 2nd inspection, no failures are observed, so $\boldsymbol{x}(2) = (0, 0)$. Since $\boldsymbol{x}(2) \notin \mathcal{X}(2)$ and $\boldsymbol{x}(2) \notin \mathcal{Y}(2)$, they proceed to the 3rd inspection. After the 3rd inspection, 1 failure is observed, so $\boldsymbol{x}(3) = (0, 0, 1)$. Since $\boldsymbol{x}(3) \notin \mathcal{X}(3)$ and $\boldsymbol{x}(3) \notin \mathcal{Y}(3)$, they proceed to the 4th inspection. After the 4th inspection, no failures are observed, i.e., $\boldsymbol{x}(4) = (0, 0, 1, 0)$. Since $\boldsymbol{x}(4) \notin \mathcal{X}(4)$ and $\boldsymbol{x}(4) \notin \mathcal{Y}(4)$, they proceed to the 5th

inspection. After the 5th inspection, no failures are observed, so $\boldsymbol{x}(5) = (0, 0, 1, 0, 0) \in \mathcal{X}(5)$. Hence, $\mathcal{C}$ accepts the lot after the 5th inspection.
Next, we illustrate the no-sampling cases.

Case 1: In Example 2.1, $\mathcal{C}$ accepts the lot if the mean reliability of the product achieves 0.95 at time $t_0 = 1$. Assume that the hyperparameters of the prior distribution of η for $\mathcal{C}$ are $\alpha_1 = 1.1$ and $\beta_1 = 25$. Therefore, $\mathcal{C}$'s initial belief about the product mean reliability at $t_0 = 1$ is 0.96, which meets the acceptance criteria without life testing. In this scenario, $\mathcal{C}$ accepts the lot without life testing, since the initial belief about $\mathcal{M}$'s product is satisfactory.

Case 2: In Example 2.1, for $b_1 = 100$ and $\mathcal{C}$'s hyperparameter values $\alpha_1 = 3.61$ and $\beta_1 = 12.05$, we obtain the optimal design $(1, 0, 1)$ using Algorithm 2.2. However, it is seen that $\mathcal{M}$'s utility under this design is 230, which is less than b_3. Therefore, the optimal design is $(0, 0, 0)$. This means that $\mathcal{M}$ does not conduct life testing, and $\mathcal{C}$ makes the decision based on the initial beliefs and rejects the lot.

2.6 Discussion

In this chapter, we have considered the determination of optimum BRASP under ICS. The advantage of using this proposed methodology is that the decision regarding lot acceptance/rejection can be taken sequentially. We have considered exponential and Weibull distributions for illustration when the $\mathcal{M}$ not only knows his/her own utility and prior but also knows $\mathcal{C}$'s utility, prior and actions. We have considered RDSP, when the forms of $\mathcal{C}$'s utility function and prior distribution are known to $\mathcal{M}$, but $\mathcal{M}$ does not know the parameters of utility function and prior.

Chapter 3

RASP with optional warranty under HCS by Bayesian Approach [1]

This chapter considers BRASP under HCS-I. The problem of designing BRASP under HCS-I has been addressed by several authors, such as Liang and Yang [67], Lin et al. [72], and Prajapat et al. [96]. These studies assumed that $\mathcal{M}$ and $\mathcal{C}$ share a common prior distribution and a common utility (loss) function. However, such assumptions may not hold in many real-world applications, where prior beliefs and utility assessments of the two parties differ due to their adversarial nature. $\mathcal{C}$ can take a decision to accept or reject a lot based on the existing information about the product. In the event of rejection of a lot by $\mathcal{C}$, $\mathcal{M}$ offers a warranty to $\mathcal{C}$. When a lot is sold under a warranty policy, it increases the chance of acceptance of the lot. However, $\mathcal{M}$ incurs an expense if the product fails during the warranty period. To compensate for expenses due to the warranty offer, $\mathcal{M}$ can increase the selling price. Nevertheless, $\mathcal{C}$ always prefers products with lower prices. Therefore, negotiations arise between $\mathcal{M}$ and $\mathcal{C}$, making an optional warranty worth considering in the decision-making process. CW rebate warranty policy, as introduced by Thomas [115], is considered. This combined policy allows a more practical and flexible mechanism for risk-sharing and cost management under BRASP framework with HCS-I.

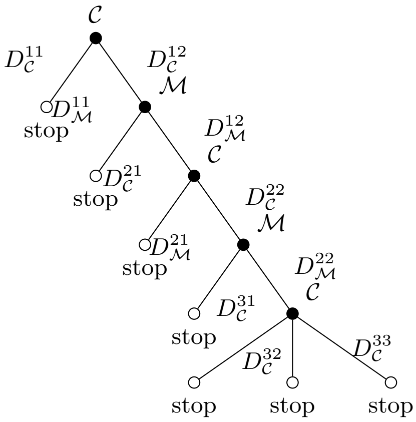
In the proposed framework, $\mathcal{M}$ initially offers the lot without any warranty. Based on their prior belief, $\mathcal{C}$ either accepts or rejects the lot. If the lot is rejected, $\mathcal{M}$ offers an optional warranty to enhance $\mathcal{C}$'s utility and encourage acceptance. If $\mathcal{C}$ still rejects the lot, $\mathcal{M}$ may provide a sample drawn from a life test to demonstrate the quality of the lot. $\mathcal{C}$ then updates their belief based on the test results and makes a final decision on lot acceptance or rejection.

$\mathcal{M}$'s goal is to determine the optimal life testing plan. Following the setup in Chapter 2, two cases of information sharing between $\mathcal{M}$ and $\mathcal{C}$ are considered in this chapter. In the first case, the common knowledge assumption is used, where both parties are aware of each other's prior and utility functions. In the second case, RDSP is introduced to model scenarios where $\mathcal{C}$'s action

[1] One paper based on this chapter has appeared under:

1. Bayesian reliability acceptance sampling plan with optional warranty under hybrid censoring *Quality and Reliability Engineering International.*

before the life test, as well as their actions at each stage of the life test, are uncertain. In both cases, $\mathcal{M}$ computes the expected utility based on $\mathcal{C}$'s acceptance–rejection behavior and makes a decision to maximize it. This highlights the interdependence of $\mathcal{M}$'s and $\mathcal{C}$'s actions. The schematic representation of the proposed approach is shown in Figure 3.1.



$D_{\mathcal{C}}^{11}$	Lot is accepted without warranty and life testing
$D_{\mathcal{C}}^{12}$	Lot is rejected
$D_{\mathcal{M}}^{11}$	Warranty is not offered to $\mathcal{C}$
$D_{\mathcal{M}}^{12}$	Warranty is offered to $\mathcal{C}$
$D_{\mathcal{C}}^{21}$	Lot is accepted with a warranty and without life testing
$D_{\mathcal{C}}^{22}$	Lot is rejected
$D_{\mathcal{M}}^{21}$	Sample is not given
$D_{\mathcal{M}}^{22}$	Sample is given from a life-testing
$D_{\mathcal{C}}^{31}$	Lot is accepted without warranty
$D_{\mathcal{C}}^{32}$	Lot is accepted with warranty
$D_{\mathcal{C}}^{33}$	Lot is rejected

Figure 3.1: Schematic representation of the proposed work.

Lindley and Singpurwalla [76] considered utility (profit) functions for modeling the decisions of accepting or rejecting a lot. In contrast, the present work adopts $\mathcal{C}$'s loss function framework for acceptance or rejection decisions. As a result, the inequalities (2.0.1) and (2.0.2) are reversed to reflect the loss-based criterion. The utility functions of $\mathcal{M}$ associated with lot acceptance and rejection are adopted from Lindley and Singpurwalla [76]. A general decision-making framework is formulated under this setup. To illustrate the proposed BRASP model under the common knowledge assumption, both exponential and Weibull lifetime distributions are used. In the case of RDSP, the methodology is demonstrated using the exponential distribution.

The organization of the paper is as follows. Section 3.1 discusses the proposed model. Section 3.2 considers determination of optimum BRASPs for the exponential and Weibull distributions. Section 3.3 discusses RDSP and illustrates with the exponential distribution. Section 3.4 provides numerical examples under different scenarios. Section 3.5 considers analysis of a real data set to illustrate the proposed method. Section 3.6 provides discussion.

3.1 Development of the model with an optional warranty

Suppose that $\mathcal{M}$ is negotiating with $\mathcal{C}$ for the sale of a product under a warranty policy. In the negotiation process, $\mathcal{C}$ and $\mathcal{M}$ take their decision based on their utility functions. Sections 3.1.1 and 3.1.2 discuss $\mathcal{C}$'s utility function and action, respectively. Sections 3.1.3 and 3.1.4 present $\mathcal{M}$'s utility function and action, respectively.

3.1.1 Consumer's utility function

$\mathcal{C}$ has a requirement of minimum lifetime L for the product. If the product fails before the lifetime L, $\mathcal{C}$ has a certain amount of loss. It is assumed that the loss function is linearly decreasing with lifetime. Therefore, the loss function for acceptance without warranty and rejection in terms of lifetime can be taken as

$$\mathcal{L}_{\mathcal{C}}(\mathcal{A}_{wo} \mid X) = \begin{cases} a_1\left(1 - \frac{X}{L}\right) + a_2, & \text{if } X \leq L, \\ a_2, & \text{if } X > L \end{cases} \quad \text{and} \quad \mathcal{L}_{\mathcal{C}}(\mathcal{R} \mid X) = a_3, \tag{3.1.1}$$

where a_1 is the proportional loss with the lifetime of the product if it fails before the time L, a_2 is the fixed cost which is paid by $\mathcal{C}$ when the lot is accepted and a_3 is the loss due to rejection. The $\mathcal{C}$ accepts the lot without warranty and life-testing if

$$E_{\boldsymbol{\theta}}\left[E_{X \mid \boldsymbol{\theta}}\left\{\mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid X)\right\}\right] \leq a_3. \tag{3.1.2}$$

If the inequality (3.1.2) does not hold, $\mathcal{M}$ offers a warranty to $\mathcal{C}$. $\mathcal{C}$ updates his/her utility function using warranty offer. Here $\mathcal{M}$ sells his/her products under the combined FRW-PRW rebate warranty policy as described in Introduction. Let C_p be the selling price of the item without any warranty. If the lot is sold with a warranty, $\mathcal{C}$ needs to pay the additional cost C_w per item for the lot due to warranty offer. Therefore, $(C_p + C_w)$ is the selling price of an item with the warranty. Let $C_{CW}(X)$ be the amount of rebate of an item with lifetime X. Then, for rebate warranty from Murthy and Blischke [89], the expected cost for an item to $\mathcal{C}$ is $(C_p + C_w) - E_{X \mid \boldsymbol{\theta}}\left[C_{CW}(X)\right]$ and to $\mathcal{M}$ it is $C_m + E_{X \mid \boldsymbol{\theta}}\left[C_{CW}(X)\right]$, where C_m is the cost of supplying an item. The cost $C_{CW}(X)$ under combined FRW-PRW rebate warranty policy (see Thomas [115]) is given by

$$C_{CW}(X) = \begin{cases} (C_p + C_w), & \text{if } 0 \leq X < w_1, \\ (C_p + C_w)\frac{w_1 - X}{w_1 - w_2}, & \text{if } w_1 \leq X < w_2, \\ 0, & \text{if } X > w_2. \end{cases}$$

So the expected cost of accepting the lot with warranty is

$$E_{X \mid \boldsymbol{\theta}}\left\{\mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid X)\right\} - E_{X \mid \boldsymbol{\theta}}\left\{C_{CW}(X)\right\} + C_w. \tag{3.1.3}$$

Therefore, $\mathcal{C}$ accepts the lot with a warranty and without life-testing if

$$E_{\boldsymbol{\theta}}\left[E_{X \mid \boldsymbol{\theta}}\left\{\mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid X)\right\}\right] - E_{\boldsymbol{\theta}}\left[E_{X \mid \boldsymbol{\theta}}\left\{C_{CW}(X)\right\}\right] + C_w \leq a_3. \tag{3.1.4}$$

We have

$$E_{X \mid \boldsymbol{\theta}}[\mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid X)] = \frac{a_1}{L}\int_0^L F(x \mid \boldsymbol{\theta})\, dx + a_2. \tag{3.1.5}$$

From Budhiraja & Pradhan [20] we get,

$$E_{X \mid \boldsymbol{\theta}}[C_{CW}(X)] = \frac{C_p + C_w}{w_2 - w_1}\int_{w_1}^{w_2} F(x \mid \boldsymbol{\theta})\, dx. \tag{3.1.6}$$

If the inequality (3.1.4) does not hold, $\mathcal{C}$'s decision is rejection with a warranty offer. In the event of rejection of lot with warranty offer, $\mathcal{M}$ performs a life test. $\mathcal{C}$ updates his/her belief based on lifetime information obtained from the life test. Now based on the updated belief, $\mathcal{C}$ takes the decision of acceptance or rejection of the lot.

3.1.2 Consumer's decision

Suppose n items are put on a life test under HCS-I under the design $\boldsymbol{\zeta} = (n, r, \tau_1)$. The lifetimes of n items $X_1, \ldots, X_n$ are IID with common CDF $F(\cdot \mid \boldsymbol{\theta})$ and PDF $f(\cdot \mid \boldsymbol{\theta})$. The observed data $\boldsymbol{x}$

under HCS-I is given in (1.3.1). The likelihood function is given by

$$L(\boldsymbol{\theta} \mid \boldsymbol{x}) = \begin{cases} (F(\tau_1 \mid \boldsymbol{\theta}))^n, & \text{if } d = 0, \\ \frac{n!}{(n-d)!} \prod_{i=0}^{d} f(x_{(i)} \mid \boldsymbol{\theta})(1 - F(\tau_1 \mid \boldsymbol{\theta}))^{n-d}, & \text{if } d = 1, \cdots, r-1, \\ \frac{n!}{(n-r)!} \prod_{i=1}^{d} f(x_{(i)} \mid \boldsymbol{\theta})(1 - F(x_{(r)} \mid \boldsymbol{\theta}))^{n-r}, & \text{if } d = r. \end{cases}$$

Therefore, the posterior distribution of $\boldsymbol{\theta}$ given data $\boldsymbol{x}$ under the design $\boldsymbol{\zeta} = (n, r, \tau_1)$ is given in (2.1.4). $\mathcal{C}$ accepts the lot without warranty after life testing based on the observed data $\boldsymbol{x}$ if

$$\mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}) = \int_{\theta} \mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{\theta}) p_{\mathcal{C}}(\boldsymbol{\theta} \mid \boldsymbol{x}) d\boldsymbol{\theta} \leq a_3. \tag{3.1.7}$$

If the inequality (3.1.7) is not satisfied, $\mathcal{M}$ offers a warranty to $\mathcal{C}$. Now, $\mathcal{C}$ accepts the lot after life testing with warranty if

$$\mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}) - E_{\boldsymbol{\theta} \mid \boldsymbol{x}} \left[E_{X \mid \boldsymbol{\theta}}[C_{CW}(X)] \right] + C_w \leq a_3. \tag{3.1.8}$$

If the inequality is not satisfied after life-testing with a warranty offer, $\mathcal{M}$ fails to convince $\mathcal{C}$ and $\mathcal{C}$ rejects the lot. The flowchart of $\mathcal{C}$'s decision at every step is shown in Figure 3.2.

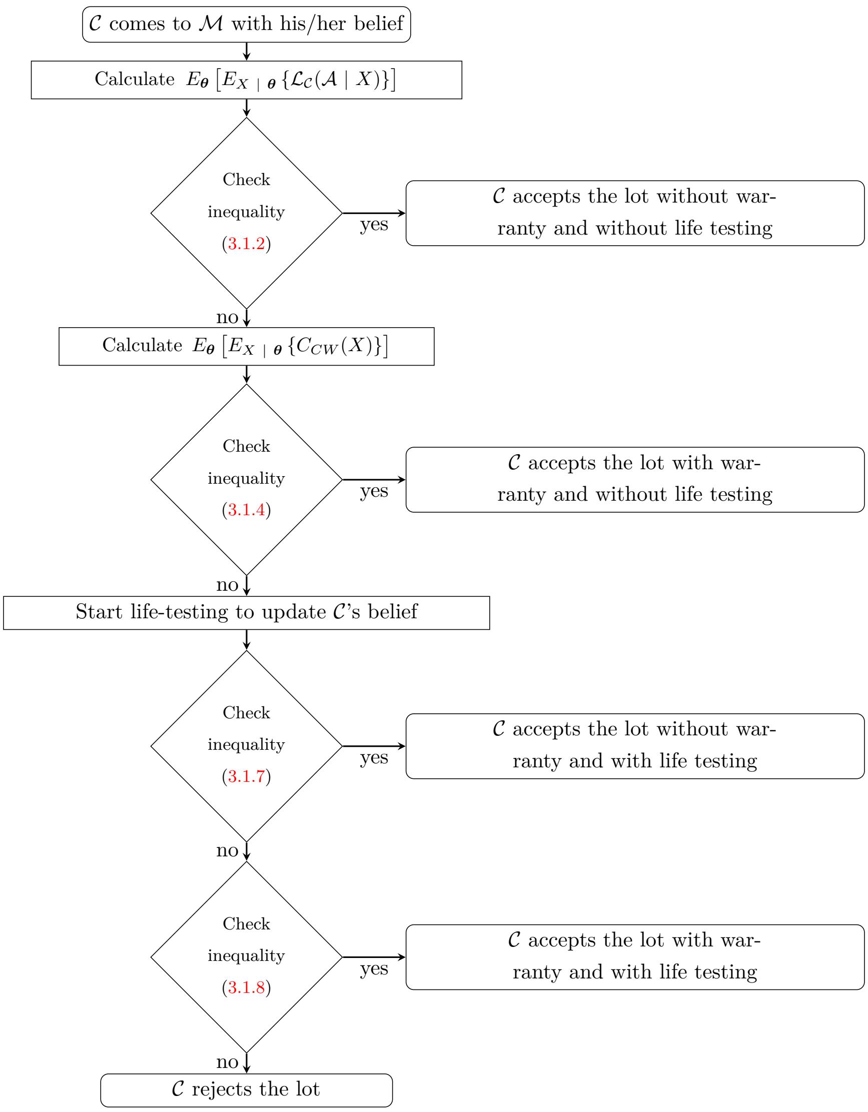


Figure 3.2: The procedure of $\mathcal{C}$'s actions.

We now define three sets of data for $\boldsymbol{x}$ values as follows: $\mathcal{X}_{wo}$ is the set of all possible values of $\boldsymbol{x}$ which satisfies the inequality (3.1.7), $\mathcal{X}_{ww}$ is the set of all possible values of $\boldsymbol{x}$ which does not satisfy the inequality (3.1.7) and satisfies the inequality (3.1.8) and $\mathcal{Y}$ is the set of all possible values of $\boldsymbol{x}$ which do not satisfy both inequalities (3.1.7) and (3.1.8). Therefore, $\mathcal{X}_{wo}$ and $\mathcal{X}_{ww}$ contain the values of $\boldsymbol{x}$ for which $\mathcal{C}$ accepts the lot without warranty and with warranty, respectively and $\mathcal{Y}$ contains the values of $\boldsymbol{x}$ for which $\mathcal{C}$ rejects the lot. Next, we consider $\mathcal{M}$'s decision of optimal planning of the life test. $\mathcal{M}$ needs to determine optimum values of $\boldsymbol{\zeta} = (n, r, \tau_1)$ by maximizing his/her utility function.

3.1.3 Manufacturer's utility function

The utility functions of $\mathcal{M}$ corresponding to accepting the lot without warranty (see Lindley and Singpurwalla [76]) and with a warranty after life testing for a given data $\boldsymbol{x}$ are

$$\mathcal{U}_{\mathcal{M}}(\mathcal{A}_{wo} \mid \boldsymbol{x}, \boldsymbol{\theta}) = b_1 E_{X \mid (\boldsymbol{x},\boldsymbol{\theta})}[X^q] - b_2 - C_s n - C_d d - C_t \tau, \tag{3.1.9}$$

and

$$\begin{aligned}\mathcal{U}_{\mathcal{M}}(\mathcal{A}_{ww} \mid \boldsymbol{x}, \boldsymbol{\theta}) &= \mathcal{U}_{\mathcal{M}}(\mathcal{A}_{wo} \mid \boldsymbol{x}, \boldsymbol{\theta}) + C_w - E_{X \mid (\boldsymbol{x},\boldsymbol{\theta})}[C_{CW}(X)] \\ &= b_1 E_{X \mid (\boldsymbol{x},\boldsymbol{\theta})}[X^q] + C_w - b_2 - E_{X \mid (\boldsymbol{x},\boldsymbol{\theta})}[C_{CW}(X)] - C_s n - C_d d - C_t \tau, \end{aligned} \tag{3.1.10}$$

respectively. The utility corresponding to rejection (see Lindley and Singpurwalla [76]) is taken as

$$\mathcal{U}_{\mathcal{M}}(\mathcal{R} \mid \boldsymbol{x}, \boldsymbol{\theta}) = b_3 - C_s n - C_d d - C_t \tau, \tag{3.1.11}$$

τ is the duration of the test and D is the number of failures up to time τ. $E_{X \mid (\boldsymbol{x},\boldsymbol{\theta})}[X^q]$ is defined in (2.1.10). Here, it is assumed that $\mathcal{M}$ knows $\mathcal{C}$'s utility and the prior. Therefore, $\mathcal{M}$ can use $\mathcal{C}$'s acceptance and rejection criteria to determine his/her expected utility and find the optimum life-testing plan.

3.1.4 Manufacturer's expected utility

$\mathcal{M}$'s utility function for the given data $\boldsymbol{x}$ is given by

$$\psi_{\mathcal{M}}(\boldsymbol{\zeta} \mid \boldsymbol{x}, \boldsymbol{\theta}) = \mathbb{1}_{\mathcal{X}_{wo}}(\boldsymbol{x}) \mathcal{U}_{\mathcal{M}}(\mathcal{A}_{wo} \mid \boldsymbol{x}, \boldsymbol{\theta}) + \mathbb{1}_{\mathcal{X}_{ww}}(\boldsymbol{x}) \mathcal{U}_{\mathcal{M}}(\mathcal{A}_{ww} \mid \boldsymbol{x}, \boldsymbol{\theta}) + \mathbb{1}_{\mathcal{Y}}(\boldsymbol{x}) \mathcal{U}_{\mathcal{M}}(\mathcal{R} \mid \boldsymbol{x}, \boldsymbol{\theta}),$$

where $\mathbb{1}_{\mathcal{V}}(\boldsymbol{v})$ denotes the indicator function over a set $\mathcal{V}$ defined in (2.1.11). Let $p_{\mathcal{M}}(\boldsymbol{\theta})$ is the prior distribution of $\boldsymbol{\theta}$ for $\mathcal{M}$ and $\boldsymbol{X} = (X_{(1)}, \ldots, X_{(D)}, D)$ be the random variable of the observed data $\boldsymbol{x}$. The expected utility w.r.t. the data $\boldsymbol{X}$ and parameter $\boldsymbol{\theta}$ is given by

$$\begin{aligned}\psi_{\mathcal{M}}(\boldsymbol{\zeta}) &= E_{\boldsymbol{\theta}}\Big[E_{\boldsymbol{X}|\boldsymbol{\theta}}\{\psi_{\mathcal{M}}(\boldsymbol{\zeta} \mid \boldsymbol{X}, \boldsymbol{\theta})\}\Big] \\ &= \int_{\boldsymbol{\theta}} \sum_{d} \int_{\boldsymbol{y}} \Big[\mathbb{1}_{\mathcal{X}_{wo} \cup \mathcal{X}_{ww}}(\boldsymbol{x})\{b_1 E_{X|(\boldsymbol{x},\boldsymbol{\theta})}(X^q) - b_2\} - \mathbb{1}_{\mathcal{X}_{ww}}(\boldsymbol{x}) E_{X|(\boldsymbol{x},\boldsymbol{\theta})}[C_{CW}(X)] \\ &\qquad + \mathbb{1}_{\mathcal{Y}}(\boldsymbol{x}) b_3\Big]\, g_{\boldsymbol{X}}(\boldsymbol{y}, d \mid \boldsymbol{\theta})\, d\boldsymbol{y}\, p_{\mathcal{M}}(\boldsymbol{\theta})\, d\boldsymbol{\theta} - C_s n - C_d E[D \mid \boldsymbol{\theta}] - C_t E[\tau \mid \boldsymbol{\theta}],\end{aligned} \tag{3.1.12}$$

where $\boldsymbol{x} = (\boldsymbol{y}, d)$ with $\boldsymbol{y} = (x_{(1)}, \ldots, x_{(d)})$, and $g_{\boldsymbol{X}}(\boldsymbol{y}, d \mid \boldsymbol{\theta})$ is the conditional joint PDF defined as

$$g_{\boldsymbol{X}}(\boldsymbol{y}, d \mid \boldsymbol{\theta}) = g_{\boldsymbol{Y}|D,\boldsymbol{\theta}}(\boldsymbol{y} \mid d, \boldsymbol{\theta})\, P(D = d \mid \boldsymbol{\theta}),$$

where $\boldsymbol{Y}$ is the random variable corresponding to the observed value $\boldsymbol{y}$. Here, $g_{\boldsymbol{Y}|D,\boldsymbol{\theta}}(\boldsymbol{y} \mid d, \boldsymbol{\theta})$ denotes the conditional joint PDF of $\boldsymbol{y}$ given $(D, \boldsymbol{\theta})$, and $P(D = d \mid \boldsymbol{\theta})$ is the probability mass function (PMF) of D.

Let A_{wo} and A_{ww} be the events that $\mathcal{C}$ accepts the lot without warranty and with warranty, respectively. Then,

$$P(A_{wo} \mid \theta) = \sum_d \int_{\boldsymbol{y}} \mathbb{1}_{\mathcal{X}_{wo}}(x) g_{\boldsymbol{X}}(\boldsymbol{y}, d \mid \boldsymbol{\theta})\ d\boldsymbol{y} \quad \text{and} \quad P(A_{ww} \mid \theta) \ = \sum_d \int_{\boldsymbol{y}} \mathbb{1}_{\mathcal{X}_{ww}}(x) g_{\boldsymbol{X}}(\boldsymbol{y}, d \mid \boldsymbol{\theta})\ d\boldsymbol{y}.$$

Let R be the event that $\mathcal{C}$ rejects the lot. Then,

$$P(R \mid \boldsymbol{\theta}) = \sum_d \int_{\boldsymbol{y}} \mathbb{1}_{\mathcal{Y}}(x) g_{\boldsymbol{X}}(\boldsymbol{y}, d \mid \boldsymbol{\theta})\ d\boldsymbol{y}.$$

Let $L_w(\boldsymbol{\theta})$ be the expected loss of $\mathcal{M}$ due to the warranty policy. Then $L_w(\boldsymbol{\theta})$ can be written as

$$L_w(\boldsymbol{\theta}) = \int_{\boldsymbol{y}} \sum_d \Big[E_{X \mid (\boldsymbol{x},\boldsymbol{\theta})}(C_{CW}(X)) - C_w\Big]\, \mathbb{1}_{\mathcal{X}_{ww}}(x) g_{\boldsymbol{X}}(\boldsymbol{y}, d \mid \boldsymbol{\theta})\ d\boldsymbol{y}.$$

The expected number of failures is given by (see Bhattacharya et al. [17])

$$E[D \mid \boldsymbol{\theta}] = \sum_{j=0}^{r-1} j \binom{n}{j} (1 - F(\tau_1 \mid \boldsymbol{\theta}))^j (F(\tau_1 \mid \boldsymbol{\theta}))^{n-j} + r \sum_{j=r}^{n} \binom{n}{j} (1 - F(\tau_1 \mid \boldsymbol{\theta}))^j (F(\tau_1 \mid \boldsymbol{\theta}))^{n-j}.$$

The expected duration of the test is given by (see Bhattacharya et al. [17])

$$E[\tau \mid \boldsymbol{\theta}] = \tau_1 \left(1 - \sum_{j=r}^{n} \binom{n}{j} (F(\tau_1 \mid \boldsymbol{\theta})^j (1 - F(\tau_1 \mid \boldsymbol{\theta}))^{n-j}\right) + r \binom{n}{r} \int_0^{\tau_1} x (F(x \mid \boldsymbol{\theta}))^{r-1} (1 - F(x \mid \boldsymbol{\theta}))^{n-r} f(x \mid \boldsymbol{\theta}) dx.$$

The expected utility of $\mathcal{M}$, which is given in (3.1.12), can be written as

$$\begin{aligned} \psi_{\mathcal{M}}(\boldsymbol{\zeta}) = &\int_{\boldsymbol{\theta}} \sum_{\boldsymbol{x}} \Big[b_1 E_{X \mid (\boldsymbol{x},\boldsymbol{\theta})}(X^q) [P(A_{wo} \mid \boldsymbol{\theta}) + P(A_{ww} \mid \boldsymbol{\theta})\Big]\, p_{\mathcal{M}}(\boldsymbol{\theta}) d\boldsymbol{\theta} \\ &- b_2 [P(A_{wo}) + P(A_{ww})] - L_w + b_3 P(R) - C_s n - C_d E[D] - C_t E[\tau], \end{aligned} \tag{3.1.13}$$

where $P(A_{wo}) = \int_{\boldsymbol{\theta}} P(A_{wo} \mid \boldsymbol{\theta})\ p_{\mathcal{M}}(\boldsymbol{\theta})\ d\boldsymbol{\theta}$, $P(A_{ww}) = \int_{\boldsymbol{\theta}} P(A_{ww} \mid \boldsymbol{\theta})\ p_{\mathcal{M}}(\boldsymbol{\theta})\ d\boldsymbol{\theta}$, $P(R) = \int_{\boldsymbol{\theta}} P(R \mid \boldsymbol{\theta})$ $p_{\mathcal{M}}(\boldsymbol{\theta})\ d\boldsymbol{\theta}$, $L_w = \int_{\boldsymbol{\theta}} L_w(\boldsymbol{\theta}) p_{\mathcal{M}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$ $E[D] = \int_{\boldsymbol{\theta}} E[D \mid \boldsymbol{\theta}] p_{\mathcal{M}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$ and $E[\tau] = \int_{\boldsymbol{\theta}} E[\tau \mid \boldsymbol{\theta}] p_{\mathcal{M}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$. The optimal design $\boldsymbol{\zeta}^*$ is obtained by

$$\boldsymbol{\zeta}^* = (n^*, r^*, {\tau_1}^*) = \arg\max_{\boldsymbol{\zeta}} \psi_{\mathcal{M}}(\boldsymbol{\zeta}).$$

Remark 3.1 *$\mathcal{M}$'s utility of accepting the lot without warranty and life test is*

$$\psi_{\mathcal{M}}((0,0,0)) = \int_{\boldsymbol{\theta}} b_1 E_{X \mid \boldsymbol{\theta}}[X^q]\ p_{\mathcal{M}}(\boldsymbol{\theta})\ d\boldsymbol{\theta} - b_2.$$

Remark 3.2 *$\mathcal{M}$'s utility of accepting the lot with warranty and without life test is*

$$\psi_{\mathcal{M}}((0,0,0)) = \int_{\boldsymbol{\theta}} b_1 E_{X \mid \boldsymbol{\theta}}[X^q - C_{CW}(X)]\ p_{\mathcal{M}}(\boldsymbol{\theta})\ d\boldsymbol{\theta} - b_2 + C_w.$$

Remark 3.3 *$\mathcal{M}$'s utility of rejecting the lot without life-testing is*

$$\psi_{\mathcal{M}}((0,0,0)) = b_3.$$

Remark 3.4 *After life testing, $\mathcal{M}$'s utility must be greater than the above three values. The conditions under which the lot is accepted without life testing are given in (3.1.7) and (3.1.8). If $\mathcal{M}$'s utility is less than b_3 among all sampling plans, then the lot is rejected without life testing.*

3.2 Optimal RASP

Here, we obtain optimal RASP for the exponential and Weibull distributions.

3.2.1 Exponential distribution case

We obtain optimal RASP when lifetime distribution follows exponential distribution. The lifetimes of the items are IID and follow $\text{Exp}(1/\eta)$, which is defined in Section 1.2.2.

Consumer's decision

It is assumed that for $\mathcal{C}$'s perspective, η follows $IG(\alpha, \beta)$ with PDF $p_{\mathcal{C}}(\cdot)$, which is defined in Section 1.2.4. For the $\text{Exp}(1/\eta)$, the utility functions corresponding to acceptance and rejection of $\mathcal{C}$ without warranty using (3.1.1) and (3.1.5) are given by

$$\mathcal{L}_{\mathcal{C}}(\mathcal{A}_{wo} \mid \eta) = E_{X \mid \eta}[\mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid X)] = \frac{a_1}{L}\left[L + \eta \exp(-\eta/L) - \eta\right] + a_2 \quad \text{and} \quad \mathcal{L}_{\mathcal{C}}(\mathcal{R} \mid \eta) = a_3,$$

respectively. Using (3.1.2), $\mathcal{C}$ accepts the lot without life-testing and warranty if

$$\frac{a_1}{L}\left[L + \frac{\beta_1^{\alpha_1}}{(\alpha_1 - 1)}\left(\frac{1}{(\beta_1 + L)}\right)^{\alpha_1 - 1} - \frac{\beta_1}{\alpha_1 - 1}\right] + a_2 \leq a_3. \tag{3.2.1}$$

If the inequality 93.2.1) is not satisfied, $\mathcal{M}$ offers a warranty to $\mathcal{C}$. Using (3.1.6), the expected warranty cost is obtained as

$$E_{X \mid \eta}[C_{CW}(X)] = \frac{C_p + C_w}{w_2 - w_1}\left[(w_2 - w_1) + \eta\left\{\exp(-w_2/\eta) - \exp(w_1/\eta)\right\}\right].$$

Taking expectation w.r.t. η, we get

$$E_\eta[E_{X \mid \eta}[C_{CW}(X)]] = (C_p + C_w) - \frac{C_p + C_w}{w_2 - w_1}\frac{\beta_1^{\alpha_1}}{(\alpha_1 - 1)}\left[\frac{1}{(\beta_1 + w_1)^{\alpha_1 - 1}} - \frac{1}{(\beta_1 + w_2)^{\alpha_1 - 1}}\right].$$

Using (3.1.4), $\mathcal{C}$ accepts the lot without life-testing and with a warranty, if

$$\begin{aligned} &\frac{a_1}{L}\left[L + \frac{\beta_1^{\alpha_1}}{(\alpha_1 - 1)}\left(\frac{1}{(\beta_1 + L)}\right)^{\alpha_1 - 1} - \frac{\beta_1}{\alpha_1 - 1}\right] + a_2 \\ &+ \frac{C_p + C_w}{w_2 - w_1}\frac{\beta_1^{\alpha_1}}{(\alpha_1 - 1)}\left[\frac{1}{(\beta_1 + w_1)^{\alpha_1 - 1}} - \frac{1}{(\beta_1 + w_2)^{\alpha_1 - 1}}\right] \geq a_3 + C_p. \end{aligned} \tag{3.2.2}$$

If both inequalities (3.2.1) and (3.2.2) are not satisfied, $\mathcal{M}$ conducts a life test and provides data to $\mathcal{C}$. Suppose n items are put on a life test under HCS-I. The likelihood function is given by

$$L(\eta \mid \boldsymbol{x}) \propto \eta^{-d} \exp\left(-\frac{v(\boldsymbol{x})}{\eta}\right), \tag{3.2.3}$$

where

$$v(\boldsymbol{x}) = \begin{cases} n\tau_1, & \text{if } d = 0, \\ \sum_{i=1}^{d} x_{(i)} + (n-d)\tau_1, & \text{if } 1 \le d < r, \\ \sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}, & \text{if } d = r. \end{cases}$$

is the observed value of the random variable

$$V(\boldsymbol{X}) = \begin{cases} n\tau_1, & \text{if } D = 0, \\ \sum_{i=1}^{D} X_{(i)} + (n-D)\tau_1, & \text{if } 1 \le D < r, \\ \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}, & \text{if } D = r. \end{cases}$$

The upper bound of $v(\boldsymbol{x})$ is $n\tau_1$ for all d. The posterior distribution of η is given by

$$p(\eta \mid \boldsymbol{x}) \propto \eta^{-d-\alpha_1-1} \exp\left(-\frac{v(\boldsymbol{x})+\beta_1}{\eta}\right).$$

This shows that $\eta \mid \boldsymbol{x}$ follows $IG(\alpha_1+d, \beta_1+v(\boldsymbol{x}))$ with PDF $p(\cdot \mid \boldsymbol{x})$. Using (3.1.7), we get,

$$\begin{aligned} &\int_0^\infty \left[\frac{a_1}{L}\left\{L + \eta \exp\left(-\frac{\eta}{L}\right) - \eta\right\} + a_2\right] \frac{(v(\boldsymbol{x})+\beta_1)^{\alpha_1+d}}{\Gamma(\alpha_1+d)} \eta^{-\alpha_1+d-1} \exp\left(-\frac{v(\boldsymbol{x})+\beta_1}{\eta}\right) d\eta \le a_3 \\ \implies &\frac{a_1}{L}\left[L + \frac{(\beta_1+v(\boldsymbol{x}))^{\alpha_1+d}}{(\alpha_1+d-1)} \left(\frac{1}{(\beta_1+v(\boldsymbol{x})+L)}\right)^{\alpha_1+d-1} - \frac{\beta_1+v(\boldsymbol{x})}{\alpha_1+d-1}\right] + a_2 \le a_3 \\ \implies &A_1((v(\boldsymbol{x}),d) \mid \boldsymbol{\zeta}) \le \frac{L(-a_2+a_3)}{a_1}, \end{aligned} \tag{3.2.4}$$

where

$$A_1((v(\boldsymbol{x}),d) \mid \boldsymbol{\zeta}) = \left[L + \frac{(\beta_1+v(\boldsymbol{x}))^{\alpha_1+d}}{(\alpha_1+d-1)} \left(\frac{1}{(\beta_1+v(\boldsymbol{x})+L)}\right)^{\alpha_1+d-1} - \frac{\beta_1+v(\boldsymbol{x})}{\alpha_1+d-1}\right].$$

If the inequality (3.2.4) is satisfied, $\mathcal{C}$ accepts the lot without warranty. If the inequality (3.2.4) is not satisfied, $\mathcal{M}$ offers a warranty to $\mathcal{C}$. Now, $\mathcal{C}$ will accept the lot with a warranty, using (3.1.8), if

$$\begin{aligned} &\int_0^\infty \left[\mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid \eta) - \frac{C_p + C_w}{w_2 - w_1} \int_{w_1}^{w_2} \left[1 - \exp\left(-\frac{x}{\eta}\right)\right] dx - C_w\right] \frac{(v(\boldsymbol{x})+\beta_1)^{\alpha_1+d}}{\Gamma(\alpha_1+d)} \\ &\qquad \times \eta^{-\alpha_1+d-1} \exp\left(-\frac{v(\boldsymbol{x})+\beta_1}{\eta}\right) d\eta \le a_3 \\ &\implies \frac{a_1}{L} A_1((v(\boldsymbol{x}),d) \mid \boldsymbol{\zeta}) - A_2((v(\boldsymbol{x}),d) \mid \boldsymbol{\zeta}) \le -a_2 + a_3, \end{aligned}$$

where

$$\begin{aligned} A_2((v(\boldsymbol{x}),d) \mid \boldsymbol{\zeta}) =& C_p - \frac{C_p + C_w}{w_2 - w_1} \frac{(\beta_1+v(\boldsymbol{x}))^{\alpha_1+d}}{(\alpha_1+d-1)} \\ &\times \left[\frac{1}{(\beta_1+w_1+v(\boldsymbol{x}))^{\alpha_1+d-1}} - \frac{1}{(\beta_1+w_2+v(\boldsymbol{x}))^{\alpha_1+d-1}}\right]. \end{aligned}$$

Next, we provide an alternative form of $\mathcal{C}$´s decision function. The alternative form of the decision function is useful to calculate $\mathcal{M}$´s expected utility. For developing an alternative form of $\mathcal{C}$'s decision function, we consider the following results.

Result 3.1 *$A_1((v(\boldsymbol{x}),d) \mid \boldsymbol{\zeta})$ is decreasing in $v(\boldsymbol{x})$ for fixed d.*

Proof: We present the proof in Appendix 3.A.1. (Proved)

Result 3.2 $\frac{a_1}{L}A_1((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})-A_2((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})$ *is decreasing in* $v(\boldsymbol{x})$ *for fixed* d.

Proof: We provide the proof in Appendix 3.A.1. (Proved)

If $A_1(v(\boldsymbol{x})=0,d\mid\boldsymbol{\zeta})>\frac{L(a_3-a_2)}{a_1}$, then using Result 3.1, there exists a threshold $c(d)$ such that

$$\begin{aligned} A_1((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})&<\frac{L(a_3-a_2)}{a_1}, &&\text{for } v(\boldsymbol{x})>c(d)\\ A_1((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})&>\frac{L(a_3-a_2)}{a_1}, &&\text{for } v(\boldsymbol{x})<c(d). \end{aligned}$$

If $A_1(v(\boldsymbol{x})=0,d\mid\boldsymbol{\zeta})\le\frac{L(a_3-a_2)}{a_1}$. In that scenario, $c(\boldsymbol{d})$ is taken as 0. If $\frac{a_1}{L}A_1((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})-A_2((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})>(a_3-a_2)$, then using Result 3.2, there exists a threshold $c'(d)$ such that

$$\begin{aligned} \frac{a_1}{L}A_1((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})-A_2((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})&\le(a_3-a_2), &&\text{for } v(\boldsymbol{x})>c'(d)\\ \frac{a_1}{L}A_1((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})-A_2((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})&\le(a_3-a_2), &&\text{for } v(\boldsymbol{x})<c'(d). \end{aligned}$$

If $\frac{a_1}{L}A_1((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})-A_2((v(\boldsymbol{x}),d)\mid\boldsymbol{\zeta})\le(a_3-a_2)$. In that scenario, $c'(\boldsymbol{d})$ is taken as 0. Note that $0\le v(\boldsymbol{x})\le n\tau_1$, we have

$$\mathcal{X}_{wo}=\left\{(v(\boldsymbol{x}),d):\ v(\boldsymbol{x})>c(d)\ \&\ 0<v(\boldsymbol{x})<n\tau_1\ \&\ 0\le d\le r\right\},$$

$$\mathcal{X}_{ww}=\{(v(\boldsymbol{x}),d):\ v(\boldsymbol{x})<c(d)\ \&\ v(\boldsymbol{x})>c'(d)\ \&\ 0<v(\boldsymbol{x})<n\tau_1\ \&\ 0\le d\le r\}$$

and

$$\mathcal{Y}=\{(v(\boldsymbol{x}),d):\ v(\boldsymbol{x})>c'(d)\ \&\ 0<v(\boldsymbol{x})<n\tau_1\ \&\ 0\le d\le r\}.$$

Let $c_1(d)=\min\{\max\{0,c(d)\},n\tau_1\}$ and $c_2(d)=\min\{\max\{0,c'(d)\},c(d)\}$. Then $\mathcal{X}_{wo}$, $\mathcal{X}_{ww}$ and $\mathcal{Y}$ can be written as

$$\mathcal{X}_{wo}=\left\{(v(\boldsymbol{x}),d):\ c_1(d)<v(\boldsymbol{x})<n\tau_1\ \&\ 0\le d\le r\right\},$$

$$\mathcal{X}_{ww}=\{(v(\boldsymbol{x}),d):\ c_2(d)<v(\boldsymbol{x})<c_1(d)\ \&\ 0\le d\le r\}$$

and

$$\mathcal{Y}=\{(v(\boldsymbol{x}),d):\ 0<v(\boldsymbol{x})<c_2(d)\ \&\ 0\le d\le r\}.$$

Manufacturer's decision

Using (3.1.9), (3.1.10) (3.1.11) and (2.1.10), the expected utility functions of $\mathcal{M}$ corresponding to the decisions $\mathcal{A}_{wo}$, $\mathcal{A}_w$ and $\mathcal{R}$ are obtained as

$$\begin{aligned} \mathcal{U}_{\mathcal{M}}(\mathcal{A}_{wo}\mid\boldsymbol{x},\eta)&=b_1\eta^q\Gamma(q+1)-b_2-C_sn-C_dd-C_t\tau,\\ \mathcal{U}_{\mathcal{M}}(\mathcal{A}_{ww}\mid\boldsymbol{x},\eta)&=\mathcal{U}_{\mathcal{M}}(\mathcal{A}_{wo}\mid\boldsymbol{x},\eta)+C_w-C_p+\frac{C_p\eta}{w_2-w_1}\left[\exp\left(-\frac{w_1}{\eta}\right)-\exp\left(-\frac{w_2}{\eta}\right)\right] \end{aligned}$$

and

$$\mathcal{U}_{\mathcal{M}}(\mathcal{R}\mid\boldsymbol{x},\eta)=b_3-C_sn-C_dd-C_t\tau,$$

respectively. It is assumed that for $\mathcal{M}$'s perspective, η follows an $IG(\alpha_2,\beta_2)$ with PDF $p_{\mathcal{M}}(\cdot)$. Using the sets $\mathcal{X}_{wo},\mathcal{X}_{ww}$ and $\mathcal{Y}$, $\mathcal{M}$'s expected utility which is given in (3.1.13) can be written as

$$\psi_{\mathcal{M}}(\boldsymbol{\zeta})=-C_sn+b_3P(R)-C_dE[D]-C_tE[\tau]-b_2[P(A_{wo})+P(A_{ww})]-L_w$$

$$+ b_1\Gamma(q+1)\sum_{d=0}^{r}\int_0^\infty\int_{c_2(d)}^{n\tau_1}\eta^q g_{(V(\boldsymbol{X}),D)}(y,d)p_{\mathcal{M}}(\eta)\ dy\ d\eta,$$

where

$$P(A_{wo}) = \sum_{d=0}^{r}\int_0^\infty\int_{c_1(d)}^{n\tau_1} g_{(V(\boldsymbol{X}),D)}(y,d)p_{\mathcal{M}}(\eta)\ dy\ d\eta, \tag{3.2.5}$$

$$P(A_{ww}) = \sum_{d=0}^{r}\int_0^\infty\int_{c_2(d)}^{c_1(d)} g_{(V(\boldsymbol{X}),D)}(y,d)p_{\mathcal{M}}(\eta)\ dy\ d\eta, \tag{3.2.6}$$

$$P(R) = \sum_{d=0}^{r}\int_0^\infty\int_{0}^{c_2(d)} g_{(V(\boldsymbol{X}),D)}(y,d)p_{\mathcal{M}}(\eta)\ dy\ d\eta, \tag{3.2.7}$$

$$L_w = \sum_{d=0}^{r}\int_0^\infty\int_{c_2(d)}^{c_1(d)}\left[C_p - \frac{C_p\eta}{w_2-w_1}\left\{\exp\left(-\frac{w_1}{\eta}\right)-\exp\left(-\frac{w_2}{\eta}\right)\right\}-C_w\right] g_{(V(\boldsymbol{X}),D)}(y,d)p_{\mathcal{M}}(\eta)\ dy\ d\eta, \tag{3.2.8}$$

and $g_{(V(\boldsymbol{X}),D)}$ is the joint distribution of $V(\boldsymbol{X})$ and D.

Theorem 3.1 *[Childs et al. [31]] The expression of $g_{(V(\boldsymbol{X}),D)}(y,d)$ is given by*

$$g_{(V(\boldsymbol{X}),D)}(y,d) = \begin{cases} \exp\left(-\frac{n\tau_1}{\eta}\right)\mathbb{1}_{\{n\tau_1\}}(y), & \text{if } d=0,\\ \binom{n}{d}\sum_{i=0}^{d}\binom{d}{i}\exp\left[-\frac{(n-d+i)\tau_1}{\eta}\right](-1)^i\gamma\left(y-\tau_1(n-d+i),\frac{1}{\eta},d\right), & \text{if } 1\le d\le r-1,\\ g(y,\frac{1}{\eta},r)+r\binom{n}{r}\sum_{k=1}^{r}\frac{(-1)^k\exp\left[-(n-r+k)\tau_1/\eta\right]}{n-r+k}\binom{r-1}{k-1}\gamma\left(y-(n-r+k)\tau_1,\frac{1}{\eta},r\right), & \text{if } d=r, \end{cases}$$

where the function $\gamma(y,a,b)$ is PDF of the gamma distribution defined in 1.2.3. and $\mathbb{1}_{\{x\}}(y)$ is PMF of a degenerate distribution at the point x defined in (2.1.11).

Now consider the result which is required for the computation of $\mathcal{M}$'s expected utility.

Result 3.3 *Let p_1, p_2, p and b be the positive real numbers. Also, let x_1 and x_2 be the non-negative real numbers such that $x_2 > p_2$ and $x_2 > x_1$ and $\rho_i = (h_i - p_2)/(h_i + p_1)$, $i = 2, 3$, $h_1 = \max\{x_1, p_2\}$ and $h_2 = x_2$. Then*

$$\int_0^\infty\int_{x_1}^{x_2}\eta^{-b-1}\exp(-(p_1+p_2)/\eta)\gamma(y-p_2,1/\eta,p)\ dy\ d\eta = \frac{\Gamma(b)}{(p_1+p_2)^b}\left[I_{\rho_1}(p,b)-I_{\rho_2}(p,b)\right],$$

where $I_\rho(p,b)$ is the CDF of the beta distribution defined in Section 1.2.5.

Proof:

$$\begin{aligned}
&\int_0^\infty\int_{x_1}^{x_2}\eta^{-b-1}\exp(-(p_1+p_2)/\eta)g(y-p_2,1/\eta,p)\ dy\ d\eta\\
=&\int_0^\infty\int_{h_1}^{x_2}\eta^{-b-1}\exp\left(-\frac{p_1+p_2}{\eta}\right)\frac{\eta^{-p}}{\Gamma(p)}(y-p_2)^{p-1}\exp\left(-\frac{y-p_2}{\eta}\right)\ dy\ d\eta \qquad [h_1=\max(p_2,x_1)]\\
=&\frac{1}{\Gamma(p)}\int_{h_1}^{h_2}\int_0^\infty\eta^{-(b+p)-1}\exp\left(-\frac{p_1+y}{\eta}\right)(y-p_2)^{p-1}\ d\eta\ dy \qquad [h_2=x_2]\\
=&\frac{1}{\Gamma(p)}\int_{h_1}^{h_2}(y-p_2)^{p-1}\frac{\Gamma(b+p)}{(p_1+y)^{b+p}}\ dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(b+p)}{\Gamma(p)} \int_{h_1-p_2}^{h_2-p_2} \frac{z^{p-1}}{(p_1+z+p_2)^{b+p}}\, dz && [z = y-c] \\
&= \frac{\Gamma(b+p)}{\Gamma(p)(p_1+p_2)^{b+p}} \int_{h_1-p_2}^{h_2-p_2} \frac{z^{p-1}}{\left(1+\frac{z}{p_1+p_2}\right)^{b+p}}\, dz \\
&= \frac{\Gamma(b+p)}{\Gamma(p)(p_1+p_2)^{b}} \int_{(h_1-p_2)/(p_1+p_2)}^{(h_2-p_2)/(p_1+p_2)} \frac{z_1^{p-1}}{(1+z_1)^{b+p}}\, dz_1 && [z_1 = z/(a+c)] \\
&= \frac{\Gamma(b)}{(p_1+p_2)^{b}} \int_{\rho_1}^{\rho_2} z_2^{p-1}(1-z_2)^{b-1}\, dz_2 && [z_2 = z_1/(1+z_1)] \\
&= \frac{\Gamma(b)}{(p_1+p_2)^{b}} \frac{B_{\rho_2}(p,b)-B_{\rho_1}(p,b)}{B(p,b)} \\
&= \frac{\Gamma(b)}{(p_1+p_2)^{b}} \left[I_{\rho_2}(p,b)-I_{\rho_1}(p,b)\right],
\end{aligned}
$$

where $\rho_i = \frac{(h_i-p_2)/(p_1+p_2)}{1+(h_i-p_2)/(p_1+p_2)} = \frac{h_i-p_2}{h_i+p_1}$, for $i = 1, 2$. (Proved)

If $p_1 = \beta_2 + w$, $p_2 = (n-d+j)\tau_1$, $b = \alpha_2 + l$, $p = d$ and $x_2 > (n-d+j)\tau_1$, then using Result 3.3, we get,

$$
\begin{aligned}
&\int_0^\infty \int_{x_1}^{x_2} \eta^{(-(\alpha_2-l)-1)} \exp(-(\beta_2+w+(n+j-d)\tau_1)/\eta)\gamma\left(x-\tau_1(n-d+j), \frac{1}{\eta}, d\right)\, dy\, d\eta \\
&= \frac{\Gamma(\alpha_2-l)}{(\beta_2+w+(n+j-d)T)^{\alpha_2-l}}(I_{\rho_2}(d,\alpha_2-q)-I_{\rho_1}(d,\alpha_2-q)) = H_{(w,l,j,d)}(\rho_1,\rho_2), \text{ say,} \qquad (3.2.9)
\end{aligned}
$$

where $\rho_i = (h_i-(n-d+j)\tau_1)/(h_i+\beta_2+w)$, for $i = 1, 2$, $h_1 = \max\{x_1, (n-d+j)\tau_1\}$ and $h_2 = x_2$. Note that in this case, the points ρ_1 and ρ_2 varies with w, j and d.

Result 3.4 *Suppose the lifetime of the product follows Exp$(1/\eta)$ and η follows inverse gamma with parameters (α_2, β_2). If n items are put on a life test under HCS-I, then $\mathcal{M}$'s expected utility function is given by*

$$
\begin{aligned}
\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = b_1 &\left(\frac{\beta_2^{\alpha_2}\Gamma(\alpha_2-q)}{\Gamma(\alpha_2)(\beta_2+n\tau_1)^{\alpha_2-q}} \mathbb{1}_{\mathcal{S}_1}(\boldsymbol{\zeta}) + \sum_{i=0}^{d}\binom{d}{i}\binom{n}{d}(-1)^i \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} H_{(0,q,i,d)}(\zeta_2,\zeta_4) \right. \\
&\left. + \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} H_{(0,q,r-n,r)}(\zeta_2,\zeta_4) + \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} r\binom{n}{r} \sum_{k=1}^{r} \frac{(-1)^k}{n-r+k}\binom{r-1}{k-1} H_{(0,q,k,r)}(\zeta_2,\zeta_4) \right)\Gamma(q+1) \\
&- b_2[P(A_{wo})+P(A_{ww})] - L_w + b_3P(R) - C_s n - C_d E[D] - C_t E[\tau],
\end{aligned}
$$

where $\alpha_2 > q$, $\mathcal{S}_1 = \{\boldsymbol{\zeta} : n\tau_1 > c'(0)\}$, $\rho_i = (h_i-(n+i-d)\tau_1)/(h_i+\beta_2+w)$ for $i = 1, \cdots, 4$, $h_1 = (n-d+i)\tau_1$, $h_2 = \max\{c_2(d), (n-d+i)\tau_1\}$, $h_3 = \max\{c_1(d), (n-d+i)\tau_1\}$ and $h_4 = n\tau_1$.

The expressions of $P(A_{wo})$, $P(A_{ww})$, $P(R)$, L_w, $E[D]$, $E[\tau]$ are given in Appendix 3.A.2.

3.2.2 Weibull distribution case

Here, we derive the optimal sampling plan for the Weibull distribution. Let X follows Wei$(\nu, (\lambda^*)^{1/\nu})$, which is defined in Section 1.2.7. From Section 1.2.7, the CDF of X can be written as

$$F(x \mid \boldsymbol{\theta}) = 1 - \exp(-\lambda^* x^\nu).$$

The priors of ν and λ^* for $\mathcal{C}$ follow $G(u_1, v_1)$ and $G(\alpha_1, \beta_1)$, respectively. The distribution $G(a, b)$ is defined in Section 1.2.3. It is assumed that ν and λ are independent. Here $\boldsymbol{\theta} = (\nu, \lambda^*)$. Using (3.1.1) and (3.1.5), the expected utility functions of $\mathcal{C}$ corresponding to acceptance and rejection without warranty are obtained as

$$\mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{\theta}) = a_1 \left[1 - \frac{\Gamma\left(\frac{1}{\nu}\right)\left(\gamma\left(\lambda^* L^\nu, \frac{1}{\nu}, 1\right)\right.}{\nu \lambda^{*\frac{1}{\nu}} L}\right] + a_2 \quad \text{and} \quad \mathcal{L}_{\mathcal{C}}(\mathcal{R} \mid \boldsymbol{\theta}) = a_3,$$

respectively. The expected acceptance utility functions of $\mathcal{C}$ w.r.t. prior with warranty are $\mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{\theta})$ $-E_{X \mid \boldsymbol{\theta}}(C_{CW}(X))$, where

$$\begin{aligned} E_{X \mid \boldsymbol{\theta}}(C_{CW}(X)) =& \frac{C_p + C_w}{w_2 - w_1} \int_{w_2}^{w_1} [1 - \exp(-\lambda^* x^\nu)] dx \\ =& \frac{C_p + C_w}{w_2 - w_1} \left[1 - \frac{\Gamma\left(\frac{1}{\nu}\right)}{\nu \lambda^{*\frac{1}{\nu}}} \left\{\gamma\left(w_2^*, \frac{1}{\nu}, 1\right) - \gamma\left(w_1^*, \frac{1}{\nu}, 1\right)\right\}\right] = Q(\boldsymbol{\theta}), \quad \text{say,} \end{aligned}$$

where $w_i^* = \lambda^* w_i^\nu$ for $i = 1, 2$. If the inequalities (3.1.2) and (3.1.4) do not hold, we go for life testing under HCS-I. The joint posterior distribution of $\boldsymbol{\theta} = (\nu, \lambda^*)$ is given by

$$p(\boldsymbol{\theta} \mid \boldsymbol{x}) \propto \lambda^{*\alpha_1 + d - 1} \nu^{u_1 + d - 1} \prod_{i=0}^{d} x_{(i)}^{\nu - 1} \exp\left[-\lambda^*(u(\boldsymbol{x}) + \beta_1)\right] \exp(-v_1 \nu), \tag{3.2.10}$$

where $x_{(0)} = 1$ and

$$u(\boldsymbol{x}) = \begin{cases} n t^\nu, & \text{if } d = 0, \\ \sum_{i=1}^{d} x_{(i)}^\nu + (n - d) t^\nu, & \text{if } 1 \le d < r, \\ \sum_{i=1}^{r} x_{(i)}^\nu + (n - r) x_{(r)}^\nu, & \text{if } d = r. \end{cases}$$

$\mathcal{C}$'s decision cannot be obtained analytically as in the case of Weibull distribution. We calculate (3.1.7) and (3.1.8) by simulation, for which we draw the sample from the joint posterior distribution given in (3.2.10). The procedure for drawing a sample is given in Algorithm 3.1. To use Algorithm 3.1, the following properties of the joint posterior distribution, as given in (3.2.10), are needed.

Properties:

1. The conditional posterior distribution of λ^* given ν follows the gamma distribution with parameters $(\alpha_1 + d, u(\boldsymbol{x}) + \beta_1)$.
2. The marginal posterior distribution of ν is log-concave if $r \ge 1$.

The proofs of the properties are given in the paper of Kundu [55]. We now provide the Algorithm 3.1 to generate (ν, λ^*) from $p(\nu, \lambda^* \mid \boldsymbol{x})$.

Algorithm 3.1: Procedure to generate $(\nu^{(i)}, \lambda^{*(i)})$ from $p(\nu, \lambda^* \mid \boldsymbol{x})$.

Input: Observed data $\boldsymbol{x}$, number of iterations S

Output: Samples $\{(\nu^{(i)}, \lambda^{*(i)})\}_{i=1}^{S}$ from $p(\nu, \lambda^* \mid \boldsymbol{x})$

for $i = 1$ **to** S **do**

  Generate $\nu^{(i)}$ from the log-concave marginal density $p(\nu \mid \boldsymbol{x})$ using the method of Devroye [36];

  Generate $\lambda^{*(i)}$ from $G(\alpha_1 + d,\ u(\boldsymbol{x}) + \beta_1)$ conditional on $\nu^{(i)}$;

Next, we consider $\mathcal{M}$'s utility function. Using the (3.1.9), (3.1.10), (3.1.11) and (2.1.10), the expected utility functions of $\mathcal{M}$ corresponding to the decisions $\mathcal{A}_{wo}$, $\mathcal{A}_w$ and $\mathcal{R}$ are obtained as

$$\mathcal{U}_{\mathcal{M}}(\mathcal{A}_{wo} \mid \boldsymbol{x}, \boldsymbol{\theta}) = b_1 \lambda^{*-q/\nu} \Gamma(q/\nu + 1) - b_2 - C_s n - C_d d - C_t \tau,$$

$$\mathcal{U}_{\mathcal{M}}(\mathcal{A}_{ww} \mid \boldsymbol{x}, \boldsymbol{\theta}) = \mathcal{U}_{\mathcal{M}}(\mathcal{A}_{wo} \mid \boldsymbol{x}, \boldsymbol{\theta}) + C_w - (C_p + C_w) \left[1 - \frac{\gamma\left(\frac{1}{\nu}, w_2^*\right) - \gamma\left(\frac{1}{\nu}, w_1^*\right)}{\nu \lambda^{*\frac{1}{\nu}} (w_2 - w_1)}\right]$$

and

$$\mathcal{U}_{\mathcal{M}}(\mathcal{R} \mid \boldsymbol{x}, \boldsymbol{\theta}) = b_3 - C_s n - C_d d - C_t \tau,$$

respectively. The prior of ν and λ^* for $\mathcal{M}$ follow $G(u_2, v_2)$ and $G(\alpha_2, \beta_2)$, respectively. It is assumed that ν and λ are independent. Here, an analytical solution cannot be obtained as in the case of the Weibull distribution. $\mathcal{M}$'s expected utility function can be obtained using Algorithm 3.2.

Algorithm 3.2: Computation of $\mathcal{M}$'s Expected Utility.

Input: Design parameter $\boldsymbol{\zeta} = (n, \boldsymbol{\tau_1}, k)$, number of simulations S_1, S_2

Output: Expected utility $\psi_{\mathcal{M}}(\boldsymbol{\zeta})$

Initialize $\psi_0 \leftarrow 0$;

for $u = 1$ **to** S_1 **do**

Generate $x_{(1)}^{(u)}, \ldots, x_{(d^{(u)})}^{(u)}$, $d^{(u)}$, and $\eta^{(u)}$ using the algorithm in Appendix 3.A.3;

Generate $\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(S_2)} \sim p_{\mathcal{C}}(\boldsymbol{\theta} \mid \boldsymbol{x}^{(u)})$, where $\boldsymbol{x}^{(u)} = (x_{(1)}^{(u)}, \ldots, x_{(d^{(u)})}^{(u)}, d^{(u)})$;

Compute:

$$\mathcal{L}_{\mathcal{C}}(\mathcal{D} \mid \boldsymbol{x}^{(u)}) = \frac{1}{S_2} \sum_{s=1}^{S_2} \mathcal{L}_{\mathcal{C}}(\mathcal{D} \mid \boldsymbol{\theta}^{(s)}), \quad E_{\boldsymbol{\theta} \mid \boldsymbol{x}^{(u)}}[Q(\boldsymbol{\theta})] = \frac{1}{S_2} \sum_{s=1}^{S_2} Q(\boldsymbol{\theta}^{(s)})$$

if *Inequality (3.1.7) holds for* $\boldsymbol{x}^{(u)}$ **then**

$r_1 \leftarrow 1,\ r_2 \leftarrow 0,\ r_3 \leftarrow 0;$

else if *Inequality (3.1.8) holds for* $\boldsymbol{x}^{(u)}$ **then**

$r_1 \leftarrow 0,\ r_2 \leftarrow 1,\ r_3 \leftarrow 0;$

else

$r_1 \leftarrow 0,\ r_2 \leftarrow 0,\ r_3 \leftarrow 1;$

Compute:

$$\psi_1 = (r_1 + r_2)\left(b_1 E_{X \mid \boldsymbol{x}^{(u)}, \boldsymbol{\theta}}[X^q] - b_2\right) - r_2 E_{X \mid \boldsymbol{x}^{(u)}, \boldsymbol{\theta}}[C_{CW}(X)] + r_3 b_3 - C_d d^{(u)} - C_t \eta_0^{(u)}$$

Update: $\psi_0 \leftarrow \psi_0 + \psi_1$;

Return:

$$\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = -C_s n + \frac{\psi_0}{S_1}.$$

3.3 Random decision-making sampling plan

Similar to Chapter 2, we consider the RDSP approach where $\mathcal{M}$'s knowledge about $\mathcal{C}$'s utility, prior and actions are not required. Here, $\mathcal{M}$ knows the form of the utility function and the prior distribution of $\mathcal{C}$. The parameters of the utility function and the hyperparameters are uncertain. $\mathcal{M}$ estimates the probabilities of $\mathcal{C}$'s action after the life test. $\mathcal{C}$'s exact action after the life testing is not known to $\mathcal{M}$. Therefore, $\mathcal{M}$'s action does not depend on $\mathcal{C}$'s action and hence does not affect the optimal value of the sampling parameter.

The utility functions for $\mathcal{C}$ are discussed in (3.1.1) and (3.1.3). The methodology is discussed for the exponential distribution. We consider that $\alpha_1 \sim \mathcal{U}[\alpha_1^1, \alpha_1^2]$, $\beta_1 \sim \mathcal{U}[\beta_1^1, \beta_1^2]$, $a_1 \sim \mathcal{U}[a_1^1, a_1^2]$ and $a_2 \sim \mathcal{U}[a_2^1, a_2^2]$, $a_3 \sim \mathcal{U}[a_3^1, a_3^2]$, $L \sim \mathcal{U}[L^1, L^2]$ where $0 < \alpha_1^1 < \alpha_1^2$, $0 < \beta_1^1 < \beta_1^2$, $0 < a_1^1 < a_1^2$, $a_2^1 < a_2^1$, $a_3^1 < a_3^2$, $L^1 < L^2$. The distribution $\mathcal{U}[a, b]$ is defined in (1.2.6).

Let $\mathcal{A}_{wo}^0$, $\mathcal{A}_{ww}^0$ and $\mathcal{R}^0$ be the events that $\mathcal{C}$ accepts the lot without warranty, with warranty, and rejects the lot without inspection, respectively. Also, let $\mathcal{A}_{wo}^1(\boldsymbol{x})$, $\mathcal{A}_{ww}^1(\boldsymbol{x})$ and $\mathcal{R}^1(\boldsymbol{x})$ be the events that $\mathcal{C}$ accepts the lot without warranty, with warranty, and rejects the lot based on the data $\boldsymbol{x}$ after life testing. The probabilities of these events are estimated using Algorithm 3.3.

Algorithm 3.3: Compute $P(\mathcal{A}_{wo}^1(\boldsymbol{x}) \mid \mathcal{R}_0)$, $P(\mathcal{A}_{ww}^1(\boldsymbol{x}) \mid \mathcal{R}_0)$, and $P(\mathcal{R}^1(\boldsymbol{x}) \mid \mathcal{R}_0)$.

Input: Sample $\boldsymbol{x} = (x_{(1)}, x_{(2)}, \ldots, x_{(d)}, d)$, number of simulations S

Output: Estimated conditional probabilities of $\mathcal{C}$ action given initially $\mathcal{C}$ rejects the lot

Initialize: $r_{01} \leftarrow 0$, $r_{02} \leftarrow 0$, $r_{03} \leftarrow 0$, $r_{11} \leftarrow 0$, $r_{12} \leftarrow 0$, $r_{13} \leftarrow 0$;

```
for i = 1 to S do
    Generate parameters:;
        α1^(i) ~ U[α1^1, α1^2],  β1^(i) ~ U[β1^1, β1^2],  a1^(i) ~ U[a1^1, a1^2],  a2^(i) ~ U[a2^1, a2^2],  a3^(i) ~ U[a3^1, a3^2],
        L^(i) ~ U[L^1, L^2];
    Compute: Compute the expressions (3.1.1), (3.1.3), U^0(A | x) and E_{θ|x}[E_{X|θ}[C_CW(X)]]; if
     Inequality (3.1.2) holds then
        r01 ← r01 + 1;
    else
        if Inequality (3.1.4) holds then
            r02 ← r02 + 1;
        else
            r03 ← r03 + 1;
            if Inequality (3.1.7) holds for x then
                r11 ← r11 + 1;
            else
                if Inequality (3.1.8) holds for x then
                    r12 ← r12 + 1;
                else
                    r13 ← r13 + 1;
```

Compute final probabilities:;

$P(\mathcal{A}_{wo}^0) = \frac{r_{01}}{S}$, $P(\mathcal{A}_{ww}^0) = \frac{r_{02}}{S}$, $P(\mathcal{R}^0) = \frac{r_{03}}{S}$;

$P(\mathcal{A}_{wo}^1(\boldsymbol{x}) \mid \mathcal{R}_0) = \frac{r_{11}}{r_{03}}$, $P(\mathcal{A}_{ww}^1(\boldsymbol{x}) \mid \mathcal{R}_0) = \frac{r_{12}}{r_{03}}$, $P(\mathcal{R}^1(\boldsymbol{x}) \mid \mathcal{R}_0) = \frac{r_{13}}{r_{03}}$;

Then, using these estimated probabilities, the expected utility of $\mathcal{M}$ is computed as

$$\begin{aligned}\psi_{\mathcal{M}}(\boldsymbol{\zeta}) = \int_\theta \int_{\boldsymbol{y}} \sum_d \Big[& P(\mathcal{A}_{wo}^1(\boldsymbol{x}) \mid \mathcal{R}_0) \left\{ b_1 E_{X \mid (\boldsymbol{x}, \boldsymbol{\theta})}(X^q) - b_2 - C_s n - C_d d - C_t \tau \right\} \\ & + P(\mathcal{A}_{ww}^1(\boldsymbol{x}) \mid \mathcal{R}_0) \left\{ b_1 E_{X|(\boldsymbol{x}, \boldsymbol{\theta})}(X^q) - b_2 - C_s n - C_d d - C_t \tau \right\} \\ & + P(\mathcal{R}^1(\boldsymbol{x}) \mid \mathcal{R}_0) \left\{ b_3 - C_s n - C_d d - C_t \tau \right\} \Big] \, g_{\boldsymbol{X}}(\boldsymbol{y}, d \mid \boldsymbol{\theta}) \, p_{\boldsymbol{M}}(\theta) \, d\boldsymbol{y} \, d\theta. \end{aligned} \tag{3.3.1}$$

The optimal decision $\boldsymbol{\zeta}^* = (n^*, r^*, {\tau_1}^*)$ is obtained by maximizing (3.3.1), i.e,

$$\boldsymbol{\zeta}^* = \arg\max_{\boldsymbol{\zeta}} \psi_{\mathcal{M}}(\boldsymbol{\zeta}).$$

3.4 Numerical example

Here, we consider the computation of optimum BRASPs under different scenarios.

3.4.1 Optimal BRASP under exponential distribution

Here we assume that the lifetime of the item follows $\text{Exp}(1/\eta)$, which is defined in Section 1.2.2. The hyperparameters of the prior distribution of η for $\mathcal{M}$ are taken as $\alpha_2 = 1.8$ and $\beta_2 = 18$. The cost components and risk parameter of $\mathcal{M}$'s utility function are chosen as $b_1 = 10$, $b_2 = 5$, $b_3 = 25$, $C_s = 1$, $C_d = 0.5$, $C_t = 0.5$ and $q = 0.8$. The warranty time points are $w_1 = 5$, $w_2 = 10$, warranty price $C_w = 0.5$ and the selling price of the product is $C_p = 2$. $\mathcal{C}$'s required lifetime is $L = 15$. The cost parameters of $\mathcal{C}$'s utility are $a_1 = 10$, $a_2 = 5$, $a_3 = 9$ and the hyperparameters of the prior distribution of η for $\mathcal{C}$ are $\alpha_1 = 2$ and $\beta_1 = 3$. The optimal design $\boldsymbol{\zeta}^*$, $\psi_{\mathcal{M}}(\boldsymbol{\zeta}^*)$, the probability of acceptance, rejection after life testing, $E[D^*]$ and $E[I^*]$ at $\boldsymbol{\zeta}^*$ are given in Table 3.1.

Table 3.1: Optimal values $(n^*, r^*, {\tau_1}^*)$ for the exponential distribution.

$\boldsymbol{\zeta}^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\boldsymbol{\zeta}^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
(5, 2, 5.75)	70.79	[0.18, 0.24, 0.58]	1.40	3.96

Next, we consider the effect of parameters on optimum solution. The effect of the parameters a_3, b_1, b_3, (α_2, β_2) and C_t are given in Tables 3.2, 3.3, 3.5, 3.4 and 3.6, respectively.

Table 3.2: Optimal values $(n^*, r^*, {\tau_1}^*)$ for different values of a_3 for the exponential distribution.

a_3	$\boldsymbol{\zeta}^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\boldsymbol{\zeta}^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
8	(5, 2, 10.63)	57.59	[0.10, 0.13, 0.77]	1.69	5.46
10	(4, 2, 4.03)	81.29	[0.32, 0.32, 0.36]	1.05	3.35
11	(1, 1, 1.71)	89.37	[0, 0.85, 0.15]	0.15	1.58
12	(0, 0, 0)	92.03	[0, 1, 0]	0	0

It is observed in Table 3.2 that when a_3 increases for fixed values of other parameters, the probability of acceptance without warranty increases. This is due to the fact that the lot is accepted without warranty, which is the first priority to $\mathcal{C}$ because of the low price of the product. When $a_3 = 12$, the optimal design is $(0, 0, 0)$ and $P(A_{wo}) = 1$. This means that the lot is accepted with a warranty and without life-testing. Also, it is observed that when a_3 increases, the sample size and time point decrease. This happens because as a_3 increases, the value of $E_\eta[\mathcal{U}_\mathcal{C}(\mathcal{A} \mid \eta)]$ gets closer to a_3.

Table 3.3: Optimal values of $(n^*, r^*, {\tau_1}^*)$ for different values of b_1 for the exponential distribution.

b_1	$\boldsymbol{\zeta}^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\boldsymbol{\zeta}^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
3	(5, 1, 3.30)	29.31	[0, 0.31, 0.69]	0.52	3.12
5	(2, 1, 6.55)	38.59	[0, 0.37, 0.63]	0.63	3.99
15	(8, 5, 10.21)	106.85	[0.34, 0.13, 0.53]	3.75	7.89
20	(10, 7, 11.76)	144.26	[0.36, 0.14, 0.50]	5.28	9.47

In Table 3.3, it is observed that when b_1 increases, the probability of rejection of the lot decreases. This is due to the fact that if b_1 increases, $\mathcal{M}$'s utility of acceptance increases. Also, we see that sample size increases with b_1.

Table 3.4: Optimal values $(n^*, r^*, {\tau_1}^*)$ for different values of b_3 for the exponential distribution.

b_3	$\boldsymbol{\zeta}^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\boldsymbol{\zeta}^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
15	(8, 4, 7.46)	65.22	[0.32, 0.14, 0.54]	2.94	5.65
35	(5, 2, 5.75)	76.62	[0.18, 0.24, 0.58]	1.40	3.95
55	(4, 2, 7.67)	88.37	[0.17, 0.24, 0.69]	1.42	5.21
110	(2, 1, 11.50)	123.10	[0.23, 0, 0.77]	0.77	5.43

In Table 3.4, when b_3 increases, the probability of rejection of the lot increases. This is due to the fact that if b_3 increases, $\mathcal{M}$'s utility of rejection increases. Also, it is seen that the sample size decreases with b_3.

Table 3.5: Optimal values $(n^*, r^*, {\tau_1}^*)$ for different values of (α_2, β_2) for the exponential distribution.

(α_2, β_2)	$\boldsymbol{\zeta}^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\boldsymbol{\zeta}^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
(2.8, 18)	(2, 1, 6.55)	33.21	[0, 0.22, 0.78]	0.78	3.13
(2.8, 28)	(5, 2, 5.75)	48.42	[0.14, 0.24, 0.62]	1.48	3.88
(1.8, 28)	(8, 5, 10.13)	112.96	[0.51, 0.15, 0.34]	3.10	8.93
(18,180)	(2, 1, 6.55)	30.40	[0, 0.28, 0.72]	0.72	3.69

Table 3.6: Optimal values $(n^*, r^*, {\tau_1}^*)$ for different values of C_t for the exponential distribution.

C_t	$\boldsymbol{\zeta}^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\boldsymbol{\zeta}^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
0	(5, 5, 33.29)	76.41	[0.33, 0.13, 0.54]	4.24	23.06
0.1	(6, 5, 17.37)	74.36	[0.33, 0.13, 0.54]	3.91	13.18
1	(6, 2, 4.60)	68.95	[0.19, 0.24, 0.57]	1.39	3.19
3	(8, 2, 3.29)	63.67	[0.20, 0.23, 0.57]	1.37	2.30

In Table 3.5, when α_2 is fixed and β_2 increases from 18 to 28, it is observed that $\mathcal{M}$'s utility and probability of acceptance increases. This is due to the fact that when β_2 increases, the mean lifetime of the product increases. In Table 3.6, it is observed that when C_t increases, $E[\tau^*]$ decreases as expected.

3.4.2 Optimal BRASP under Weibull distribution

Here, we assume that the lifetime of the item follows Wei$(\nu, (\lambda^*)^{1/\nu})$. The hyperparameters of the priors of ν and λ^* for $\mathcal{C}$ are taken as $(u_1, v_1) = (10, 10)$ and $(\alpha_1, \beta_1) = (2, 3)$, respectively. The hyperparameters of the priors of ν and λ^* for $\mathcal{M}$ are $(u_2, v_2) = (100, 100)$ and $(\alpha_2, \beta_2) = (1.8, 18)$, respectively. The cost components and risk parameter of $\mathcal{M}$'s utility function are chosen as $b_1 = 10$, $b_2 = 5$, $b_3 = 25$, $C_s = 1$, $C_d = 0.5$, $C_t = 0.5$ and $q = 0.8$. The warranty time points are $w_1 = 5$, $w_2 = 10$, warranty price $C_w = 0.5$ and selling price of the product is $C_p = 2$. $\mathcal{C}$'s required lifetime is $L = 15$. The cost parameters of $\mathcal{C}$'s utility are $a_1 = 10$, $a_2 = 5$, $a_3 = 9$. The optimal design, $\mathcal{M}$'s utility, the probability of acceptance, rejection after life testing, the expected number of failures and the expected time duration of the test are provided in Table 3.7.

Table 3.7: Optimal values $(n^*, r^*, {\tau_1}^*)$ for the Weibull distribution.

$\zeta^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\zeta^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
(6, 3, 7.51)	141.45	[0.31, 0.15, 0.54]	2.08	5.42

Next, we consider the effect of parameters on the optimum solution. The effect of the parameters a_3, b_1 and (α_1, β_1) are given in the Tables 3.8, 3.9 and 3.10, respectively.

Table 3.8: Optimal values $(n^*, r^*, {\tau_1}^*)$ for different values of a_1 for the Weibull distribution.

a_1	$\zeta^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\zeta^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
6	(0, 0, 0)	161.55	[0, 1, 0]	0	0
8	(3, 2, 4.31)	153.19	[0.36, 0.35, 0.29]	0.91	3.75
9	(6, 3, 4.90)	147.25	[0.41, 0.20, 0.39]	1.77	4.06
11	(9, 4, 7.53)	135.67	[0.28, 0.12, 0.60]	3.03	5.30

Table 3.9: Optimal values $(n^*, r^*, {\tau_1}^*)$ for different values of b_1 for the Weibull distribution.

b_1	$\zeta^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\zeta^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
1	(0, 0, 0)	25	[0, 0, 1]	0	0
3	(3, 1, 5.21)	45.90	[0.11, 0.21, 0.68]	0.68	2.86
5	(4, 2, 8.10)	71.67	[0.23, 0.17, 0.60]	1.41	5.26
15	(10, 5, 5.04)	213.48	[0.34, 0.13, 0.53]	3.75	7.89

Table 3.10: Optimal values $(n^*, r^*, {\tau_1}^*)$ for different values of (α_1, β_1) for the Weibull distribution.

(α_1, β_1)	$\zeta^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\zeta^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
(3, 3)	(9, 5, 8.88)	130.10	[0.23, 0.16, 0.61]	3.68	6.62
(3, 9)	(6, 3, 7.12)	142.02	[0.31, 0.17, 0.52]	2.05	5.24
(2, 9)	(3, 2, 6.45)	148.89	[0.35, 0.19 0.46]	1.93	4.63
(20, 30)	(0 ,0, 0)	25	[0, 0, 1]	0	0

It is observed in Table 3.8 that when a_3 increases, the probability of acceptance decreases. This is due to the fact that when a_1 increases the expected acceptance loss for $\mathcal{C}$, $E_\eta[\mathcal{U}_{\mathcal{C}}(\mathcal{A} \mid \eta)]$ increases. When $a_1 = 6$, the optimal design is $(0, 0, 0)$ and $P(A_{ww}) = 1$. This means that the lot is rejected by $\mathcal{C}$ without warranty. However after a warranty is given by $\mathcal{M}$, $\mathcal{C}$ is convinced and accepted the lot with warranty. In Table 3.9, if b_1 increases, $\mathcal{M}$'s utility of acceptance increases, which is similar to the exponential distribution case given in Table 3.3. In Table 3.10, when α_1 is fixed and β_1 increases from 3 to 9, it is observed that the n^* and $E[\tau_1]$ decrease. This is due to the fact that when β_1 increases, the prior belief about the mean lifetime of the product for $\mathcal{C}$ increases. Therefore, $\mathcal{C}$ is easily convinced by $\mathcal{M}$. Also, it is seen that for $(\alpha_1, \beta_1) = (20, 30)$, the probability of rejection equals to 1. We can say that $\mathcal{C}$'s initial belief about the product is very poor. Therefore, $\mathcal{M}$ does not agree to conduct a life test to convince $\mathcal{C}$ and the lot is rejected.

3.4.3 Optimal BRASP under RDSP approach

Here we consider RDSP and assume that the lifetime follows Exp$(1/\eta)$. The parameters of $\mathcal{C}$'s utility and prior are random. We consider that $a_1 \sim \mathcal{U}(10, 20)$, $a_2 \sim \mathcal{U}(1, 7)$, $a_3 \sim \mathcal{U}(3, 12)$ $\alpha_1 \sim \mathcal{U}(1, 8)$, $\beta_1 \sim \mathcal{U}(1.5, 3.5)$ and $L \sim \mathcal{U}(12, 18)$. The cost components and risk parameter of $\mathcal{M}$'s utility function are chosen as $b_1 = 10$, $b_2 = 5$, $b_3 = 25$, $C_s = 1$, $C_d = 0.5$, $C_t = 0.5$, $C_w = 0.5$, $C_p = 2$ and $q = 0.8$. The warranty time points are $w_1 = 5$, $w_2 = 10$. The optimal design is given in Table 3.11 and the effect of the parameters b_1 and C_t are given in Tables 3.12 and 3.13, respectively.

Table 3.11: Optimal values $(n^*, r^*, {\tau_1}^*)$ under RDSP approach.

$\zeta^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\zeta^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
(3, 3, 4.73)	85.08	[0.50, 0.37, 0.13]	1.03	4.59

Table 3.12: Optimal values $(n^*, r^*, {\tau_1}^*)$ for different values of b_1 under RDSP approach.

b_1	$\zeta^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\zeta^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
5	(2, 1, 5.45)	39.37	[0.46, 0.27, 0.27]	0.57	3.56
10	(3, 3, 4.73)	85.08	[0.50, 0.37, 0.13]	1.03	4.59
15	(5,4,4.93)	132.38	[0.68, 0.23, 0.09]	1.62	4.40
20	(5, 5, 4.94)	180.13	[0.69, 0.22, 0.09]	1.74	4.90

Table 3.13: Optimal values $(n^*, r^*, {\tau_1}^*)$ for different values of C_t under RDSP approach.

C_t	$\zeta^* = (n^*, r^*, {\tau_1}^*)$	$\psi_{\mathcal{M}}(\zeta^*)$	$[P(A_{wo}), P(A_{ww}), P(R)]$	$E[D^*]$	$E[\tau^*]$
0	(2, 2, 25.49)	94.52	[0.43, 0.23, 0.34]	1.98	23.06
0.1	(4, 3, 17.37)	90.36	[0.43, 0.23, 0.34]	2.91	13.18
1	(6, 2, 4.60)	88.95	[0.39, 0.24, 0.37]	1.39	3.19
3	(8, 2, 3.29)	83.67	[0.30, 0.23, 0.37]	1.37	2.30

In Table 3.12, it is observed that when b_1 increases, the probability of rejection of the lot decreases. This is due to the fact that if b_1 increases, $\mathcal{M}$'s utility of acceptance increases, which is similar to the non-random case for the exponential distribution given in Section 3.4.1. Also, it is seen that sample size increases with b_1.

3.5 Data analysis

The proposed methodology of determining optimum RASP is illustrated by using data on failure times (in hours) of the air-conditioning system of plane "7913" taken from Proschan [100]. The data set is given below:

1, 4, 11, 16, 18, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216.

Mondal and Kundu [85] analyzed this data dividing by 100 and fitted Wei$(\nu, (\lambda^*)^{1/\nu})$. The ML estimates of ν and λ^* obtained in Mondal and Kundu [85] for the complete data are $\hat{\nu} = 1.123$ and $\hat{\lambda^*} = 1.286$, respectively. Here $\mathcal{M}$ of the airconditioning system is treated as $\mathcal{M}$ and the aircraft $\mathcal{M}$ is considered as $\mathcal{C}$. Suppose that $\mathcal{M}$ is negotiating with $\mathcal{C}$ for the sale of air conditioning systems. Both $\mathcal{M}$ and $\mathcal{C}$ agree that the lifetime follows Weibull distribution. We assume that $\mathcal{M}$ considered gamma priors for ν and λ^* with respective means 1.123 and 1.286. $\mathcal{C}$ also considered gamma priors for ν and λ^*. The mean values of ν and λ^* considered by $\mathcal{C}$ are 1.123 and 2.572, respectively. The hyper-parameters of gamma priors of parameters ν and λ^* for $\mathcal{C}$ are $u_1 = 11.23$, $v_1 = 10$ and $\alpha_1 = 25.72$, $\beta_1 = 10$, respectively and for $\mathcal{M}$ $u_2 = 112.3$ $v_2 = 100$ and $\alpha_2 = 128.6$, $\beta_2 = 100$, respectively. $\mathcal{C}$ believes that the mean lifetime of the product is 0.373 hour and $\mathcal{M}$ believes that the mean lifetime of the product is 0.745 hour.

Next, we consider the cost components and risk parameter of $\mathcal{M}$, which are taken as $b_1 = 3000$, $b_2 = 1000$, $b_3 = 250$, $C_s = 30$, $C_d = 5$, $C_t = 30$ and $q = 1$. The warranty time points are taken as $w_1 = 0.2$ hour and $w_2 = 0.3$ hour, warranty price $C_w = 200$ and selling price of the product is $C_p = 2000$. $\mathcal{C}$'s requirement of a minimum lifetime is $L = 0.5$ hour. The cost parameters of $\mathcal{C}$'s utility are $a_1 = 12000$, $a_2 = 5000$ and $a_3 = 8500$. For these values, the optimum RASP is $(n^*, r^*, {\tau_1}^*) = (6, 4, 0.517)$. The probability of acceptance of the lot without a warranty after life testing is 0.35; the probability of acceptance with a warranty after life testing is 0.24 and the probability rejection of the lot after life testing is 0.41.

Next, we have to carry out a life test under the optimum RASP. For illustration, we generate HCS-I data based on the optimum life testing plan $\boldsymbol{\zeta} = (6, 4, 0.517)$ from the Wei$(1.123, 1.114)$. Then we calculate $e_1 = \mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}) - a_3$ and $e_2 = \mathcal{L}_{\mathcal{C}}(\mathcal{A} \mid \boldsymbol{x}) - E_{\boldsymbol{\theta} \mid \boldsymbol{x}}[E_{X \mid \boldsymbol{\theta}}(C_{CW}(X)] + C_w - a_3$. If $e_1 < 0$, $\mathcal{C}$ accepts the lot without warranty; if $e_1 > 0$ and $e_2 < 0$, $\mathcal{C}$ accepts the lot with warranty; and if $e_1, e_2 > 0$, $\mathcal{C}$ rejects the lot. Some data sets and corresponding decisions are given in Table 3.14 for illustration purposes.

Table 3.14: The data sets and $\mathcal{C}$'s acceptance utility and his/her decision.

i	$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	e_1	e_2	Decision
1	0.360	0.375	-	-	-296.091	-676.824	The lot is accepted without warranty.
2	0.042	0.112	0.390	0.477	1095.776	404.860	The lot is rejected.
3	0.221	0.243	0.370	0.433	355.132	-166.762	The lot is accepted with warranty.
4	0.136	0.268	-	-	228.634	-270.746	The lot is accepted with warranty.
5	0.301	-	-	-	-404.696	-764.607	The lot is accepted without warranty.
6	0.054	0.059	0.506	-	1095.612	402.880	The lot is rejected.
7	-	-	-	-	-675.769	-977.915	The lot is accepted without warranty.

3.6 Discussion

In this work, we have considered determination of optimum Bayesian RASP with optional warranty under hybrid censoring. The work can be extended to other censoring schemes. We have considered exponential and Weibull distributions for illustration. However, the proposed methodology can be extended to other lifetime distributions with appropriate prior distributions. Here we have considered that only one $\mathcal{C}$ negotiated with one $\mathcal{M}$. However, in many situations, $\mathcal{C}$ negotiates with more than one $\mathcal{M}$ for taking a decision to buy a product. This will be considered in future studies. Also, for simplicity, we have assumed that both parties consider common lifetime distribution. However, it can be extended to the case of different lifetime distributions for $\mathcal{M}$ and $\mathcal{C}$.

3.A Appendix

3.A.1 Proof of results

To prove Results 3.1 and 3.2, we need to prove the following lemma.

Lemma 3.1 *Suppose n identical units are put on a life test under the HCS-I. The lifetimes of the units are IID with CDF $F(x \mid \eta) = 1 - \exp(-x/\eta)$. Let $p_{\mathcal{C}}(\eta)$ be the prior of $\mathcal{C}$. Then for any positive decreasing function $h(\eta)$, the posterior expectation of $h(\eta)$ is decreasing in $v(\boldsymbol{x})$ for fixed d.*

Proof: From (3.2.3), the likelihood function is given by

$$L(\eta \mid \boldsymbol{x}) \propto \eta^{-d} \exp\left(-\frac{v(\boldsymbol{x})}{\eta}\right).$$

The posterior expectation of $h(\eta)$, denoted by $A((v(\boldsymbol{x}), d) \mid m)$, is given by

$$A((v(\boldsymbol{x}), d) \mid \boldsymbol{\zeta}) = \frac{\int_0^\infty h(\eta)\eta^{-d} \exp\left(-\frac{v(\boldsymbol{x})}{\eta}\right) p_{\mathcal{C}}(\eta)\, d\eta}{\int_0^\infty \eta^{-d} \exp\left(-\frac{v(\boldsymbol{x})}{\eta}\right) p_{\mathcal{C}}(\eta)\, d\eta}. \tag{3.A.1}$$

Consider two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ such that $v(\boldsymbol{x}_1) < v(\boldsymbol{x}_2)$, it suffices to show that $A((v(\boldsymbol{x}_1), d) \mid \boldsymbol{\zeta})$ $< (>) A((v(\boldsymbol{x}_2), d) \mid \boldsymbol{\zeta})$ when $h(\eta)$ is increasing (decreasing) function. Let $f_1(\eta) = \eta^{-d} \exp\left(-\frac{v(\boldsymbol{x}_1)}{\eta}\right)$, $f_2(\eta) = \eta^{-d} \exp\left(-\frac{v(\boldsymbol{x}_2)}{\eta}\right)$, $g_1(\eta) = h(\eta)p_{\mathcal{C}}(\eta)$ and $g_2(\eta) = p_{\mathcal{C}}(\eta)$. Thus from (3.A.1), we get

$$A((v(\boldsymbol{x}_1), d) \mid \boldsymbol{\zeta}) = \frac{\int_0^\infty f_1(\eta)g_1(\eta)\, d\eta}{\int_0^\infty f_1(\eta)g_2(\eta)\, d\eta}$$

and

$$A((v(\boldsymbol{x}_2), d) \mid \boldsymbol{\zeta}) = \frac{\int_0^\infty f_2(\eta)g_1(\eta)\, d\eta}{\int_0^\infty f_2(\eta)g_2(\eta)\, d\eta}.$$

We assume all integrals are finite. Clearly, $f_2(\eta)$ and $g_2(\eta)$ are non negative functions of η. Note that

$$\frac{f_1(\eta)}{f_2(\eta)} = \exp\left[\frac{(v(\boldsymbol{x}_2) - v(\boldsymbol{x}_1))}{\eta}\right]$$

is decreasing in η when $v(\boldsymbol{x}_1) < v(\boldsymbol{x}_2)$ and $g_1(\eta)/g_2(\eta) = h(\eta)$ is decreasing function in η for $\eta > 0$. By Theorem 2 in Wijsman [120], we get $A((v(\boldsymbol{x}_1), d) \mid \boldsymbol{\zeta}) > A((v(\boldsymbol{x}_2), d) \mid \boldsymbol{\zeta})$ when $v(\boldsymbol{x}_1) < v(\boldsymbol{x}_2)$. Therefore, $A((v(\boldsymbol{x}), d) \mid \boldsymbol{\zeta})$ is decreasing in $v(\boldsymbol{x})$ for fixed d. (Proved)

Proof of Result 3.1:

Proof: We have

$$\begin{aligned} A_1((v(\boldsymbol{x}), d) \mid \boldsymbol{\zeta}) &= \int_0^\infty \left(\int_0^L [1 - \exp(-x/\eta)] dx \right) p_C(\eta \mid \boldsymbol{x}) d\eta \\ &= \int_0^\infty h(\eta) p_C(\eta \mid \boldsymbol{x}) d\eta, \end{aligned}$$

where $h(\eta) = \int_0^L [1 - \exp(-x/\eta)]dx$. Note that $h(\eta)$ is decreasing in η. By Lemma 3.1, it follows that $A_1((v(\boldsymbol{x}, d) \mid \boldsymbol{\zeta})$ is decreasing in $v(\boldsymbol{x})$ for fixed d. (Proved)

Proof of Result 3.2:

Proof: We have

$$\begin{aligned} &\frac{a_1}{L} A_1((v(\boldsymbol{x}), d) \mid \boldsymbol{\zeta}) - A_2((v(\boldsymbol{x}), d) \mid \boldsymbol{\zeta}) \\ &= \int_0^\infty \left[\int_0^L \frac{a_1}{L} [1 - \exp(-x/\eta)] dx - \frac{C_p}{w_2 - w_1} \int_{w_1}^{w_2} [1 - \exp(-x/\eta)] dx \right] p_C(\eta \mid \boldsymbol{x}) d\eta \\ &= \int_0^\infty h(\eta) p_C(\eta \mid \boldsymbol{x}) d\eta, \end{aligned}$$

where $h(\eta) = \int_0^L \frac{a_1}{L} [1 - \exp(-x/\eta)]dx - \frac{C_p}{w_2 - w_1} \int_{w_1}^{w_2} [1 - \exp(-x/\eta)]dx$. Note that $h(\eta)$ is decreasing in η. By Lemma 3.1, it follows that $A_1((v(\boldsymbol{x}, d) \mid \boldsymbol{\zeta})$ is decreasing in $v(\boldsymbol{x})$ for fixed d. (Proved)

3.A.2 Expressions of $P(A_{wo}), P(A_{ww}), P(R), L_w, E[D]$ and $E[\tau]$:

The expressions of $E[D]$ and $E[\tau]$ are given in Liang and Yang [67]. For the expressions of other terms of the result, we put the value of $f_{(v(\boldsymbol{x}),D)}(y, d)$ in (3.2.5), (3.2.6), (3.2.7) and (3.2.8) and simplify by using the (3.2.9). The expressions of $P(A_{wo})$, $P(A_{ww})$, $P(R)$, L_w, $E[D]$and $E[\tau]$ are given below:

$$\begin{aligned} P(A_{wo}) = &\frac{\beta_2^{\alpha_2}}{(\beta_2 + n\tau_1)^{\alpha_2}} \mathbb{1}_{S_1}(\boldsymbol{\zeta}) + \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \left[\sum_{d=1}^{r} \sum_{i=0}^{d} \binom{d}{i} \binom{n}{d} (-1)^i H_{(0,0,i,d)}(\zeta_3, \zeta_4) \right. \\ &\left. + H_{(0,0,r-n,r)}(\zeta_3, \zeta_4) + r \binom{n}{r} \sum_{k=1}^{r} \frac{(-1)^k}{n - r + k} \binom{r-1}{k-1} H_{(0,0,k,r)}(\zeta_3, \zeta_4) \right], \end{aligned}$$

$$\begin{aligned} P(A_{ww}) = &\frac{\beta_2^{\alpha_2}}{(\beta_2 + n\tau_1)^{\alpha_2}} \mathbb{1}_{S_2}(\boldsymbol{\zeta}) + \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \left[\sum_{d=1}^{r} \sum_{i=0}^{d} \binom{d}{i} \binom{n}{d} (-1)^i H_{(0,0,i,d)}(\zeta_2, \zeta_3) \right. \\ &\left. + H_{(0,0,r-n,r)}(\zeta_2, \zeta_3) + r \binom{n}{r} \sum_{k=1}^{r} \frac{(-1)^k}{n - r + k} \binom{r-1}{k-1} H_{(0,0,k,r)}(\zeta_2, \zeta_3) \right], \end{aligned}$$

$$\begin{aligned} P(R) = &\frac{\beta_2^{\alpha_2}}{(\beta_2 + n\tau_1)^{\alpha_2}} \mathbb{1}_{S_3}(\boldsymbol{\zeta}) + \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \left[\sum_{d=1}^{r} \sum_{i=0}^{d} \binom{d}{i} \binom{n}{d} (-1)^i H_{(0,0,i,d)}(\zeta_1, \zeta_2) \right. \\ &\left. + H_{(0,0,r-n,r)}(\zeta_1, \zeta_2) + r \binom{n}{r} \sum_{k=1}^{r} \frac{(-1)^k}{n - r + k} \binom{r-1}{k-1} H_{(0,0,k,r)}(\zeta_1, \zeta_2) \right], \end{aligned}$$

$$\begin{aligned}L_w =& C_pP(A_w) - C_wP(A_w) + \frac{C_p}{w_2 - w_1}\left(\sum_{j=1}^{2}(-1)^{j-1}\left[\frac{\beta_2^{\alpha_2}}{\alpha_2(\beta_2 + w_j + nt)^{\alpha_2-1}}\mathbb{1}_{\mathcal{S}_2}(\boldsymbol{\zeta})\right.\right.\\ &+\sum_{i=0}^{d}\binom{d}{i}\binom{n}{d}(-1)^i\frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}\sum_{d=1}^{r}H_{(w_1,1,i,d)}(\zeta_2,\zeta_3) + \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}H_{(w_1,1,r-n,r)}(\zeta_2,\zeta_3)\\ &\left.\left.+\frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}r\binom{n}{r}\sum_{k=1}^{r}\frac{(-1)^k}{n-r+k}\binom{r-1}{k-1}H_{(w_1,1,k,r)}(\zeta_2,\zeta_3\right]\right),\end{aligned}$$

$$E[D] = \sum_{j=0}^{r-1} j\binom{n}{j}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}\frac{\beta_2^{\alpha_2}}{(\tau_1(n-i)+\beta_2)^{\alpha_2}} + r\sum_{j=r}^{n}\binom{n}{j}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}\frac{\beta_2^{\alpha_2}}{(\tau_1(n-i)+\beta_2)^{\alpha_2}}$$

and

$$\begin{aligned}E[\tau] =& \tau_1\left(1 - \sum_{j=r}^{n}\binom{n}{j}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}\frac{\beta_2^{\alpha_2}}{(\tau_1(n-i)+\beta_2)^{\alpha_2}}\right) + r\binom{n}{r}\sum_{i=0}^{r-1}(-1)^{r-1-i}\binom{r-1}{j}\\ &\times\left[\frac{\beta_2}{(\alpha_2-1)} - \frac{\alpha_2\tau_1(n-i)+\beta_1}{(\alpha_2-1)}\frac{\beta_2^{\alpha_2}}{(\tau_1(n-i)+\beta_2)^{\alpha_2}}\right]\frac{1}{(n-i)^2},\end{aligned}$$

where $\mathcal{S}_2 = \{\boldsymbol{\zeta} : c'(0) < n\tau_1 \le c(0)\}$ and $\mathcal{S}_3 = \{\boldsymbol{\zeta} : n\tau_1 \le c(0)\}$.

3.A.3 Alogorithm for generating HCS-I sample:

The algorithm generates a pseudo-random HCS-I sample with n units and r failures and truncation time τ_1 (see Meeker [82]). Define $U_{(0)}^{(u)} = 0$, start with $i = 1$, and generate the sequence as follows:

1. Generate $\boldsymbol{\theta}_i^{(u)}$ from $p_{\mathcal{M}}(\boldsymbol{\theta})$.
2. A pseudo-random observation $U_i^{(u)}$ is generated form a U(0,1) . Compute $U_{(i)}^{(u)} = 1 - [1 - U_{(i-1)}^{(u)}] \times (1 - U_i^{(u)})^{1/(n-i+1)}$ and $x_{(i)}^{(u)} = F^{-1}[U_{(i)}^{(u)} \mid \boldsymbol{\theta}_i^{(u)}]$.
3. If $i > r$ or $x_{(i)}^{(u)} > \tau_1$, stop; and take $d^{(u)} = (i-1)$ and $\tau^{(u)} = x_{(i-1)}^{(u)}$ the sample data is represented by $\boldsymbol{x}^{(u)} = (x_{(1)}^{(u)}, \cdots, x_{(i-1)}^{(u)}, d^{(u)})$.
4. If $i \le r$ and $x_i^{(u)} \le \tau_1$, increase the value of i by 1 and return to Step 1.

Chapter 4

RASP through adaptive SSSPALT under type-I censoring by Bayesian approach

This chapter considers the determination of BRASP under type-I censoring, with the aim of minimizing the total expected cost w.r.t. a common prior distribution for the lifetime parameters and a shared utility function between the manufacturer and the consumer. As discussed in Section 1.7, most of the existing literature on BRASP focuses on life testing conducted under non-accelerated conditions. In recent years, global market competitiveness and rapid technological advancement have pushed manufacturers to produce products with very high reliability. For such products, the mean time to failure under normal operating conditions is often prohibitively long. In such cases, conventional life testing often fails to provide sufficient lifetime information to support decision-making due to time and resource constraints.

To address this limitation, ALT or PALT is commonly employed to generate adequate failure data within a shorter duration. Several works have considered inferences and design problems in ALT or PALT settings under various censoring schemes. For example, Wu et al. [123] studied statistical inference for the two-parameter exponential distribution under type-II censoring, while Kim and Sung [51] examined optimal ALT planning illustrated with log-normal distribution. Similarly, Lim et al. [69] and Jin et al. [46] explored ALT or PALT designs under degradation models.

The problem of designing BRASP based on ALT or PALT with censored samples has received less attention. Recently, Chen et al. [26, 27] and Prajapati et al. [98] investigated BRASP under SSSPALT with type-II censored data, while Chen et al. [28] studied BRASP under SSSALT with complete data. In the case of BRASP under SSSPALT, as studied by Chen et al. [26, 27], it has been noted that increasing stress levels introduces an additional cost. Motivated by this, we consider an adaptive simple step-stress partial accelerated life test (ASSSPALT) in which the decision to change the stress level after time τ_1 depends on the number of failures observed up to τ_1. Xiang et al. [124], Wang and Wu [117] have considered ALT design under adaptive settings.

In this chapter, we extend this idea by developing BRASP through ASSSPALT under type-I censoring, referred to here as BSPAA. The adaptive test can be described as follows: A sample of size n is placed on a life test at the normal stress level s_0 at time $\tau_0 = 0$. If the number of failures up to time τ_1 is less than a pre-specified threshold m, the stress level s_0 is increased to s_1; otherwise, the stress remains unchanged after τ_1. Under type-I censoring, the life test continues until time τ_2 at the current stress level, unless all items fail earlier, in which case the test is terminated at the time of the last failure. Two important special cases are noted:

(i) If $m = 0$, then the stress is unchanged after τ_1 irrespective of the number of failures, and the test is continued under initial stress s_0. In this case, the life test becomes a conventional non-accelerated under type-I censoring.

(ii) If $m = n$ and the number of failures up to time τ_1 is less than n, then the stress is always changed to the stress level s_1 after τ_1 and the test is continued under the stress level s_1. If the number of failures up to time τ_1 is equal to n, the life test is stopped before τ_1. As a result, ASSSPALT becomes a conventional SSSPALT under type-I censoring.

Thus, the adaptive design allows both accelerated and non-accelerated life tests to be studied within a unified framework. Moreover, in the adaptive case $(0 < m < n)$, the Bayes risk is minimized compared to the non-accelerated and accelerated cases.

In industrial applications, it is essential to make cost-effective decisions within limited time. The adaptive accelerated life test achieves both objectives: it reduces testing time and provides flexibility to adjust stress levels based on interim test results. Furthermore, decision makers often possess prior knowledge of lifetime parameters, which can be incorporated into the Bayesian framework. Accordingly, we develop a methodology for determining BRASP using data from adaptive accelerated life tests and prior information.

From a managerial perspective, two aspects are of primary importance. First, the implementation of ASSSPALT under type-I censoring and the collection of life test data generated from it. Second, takes decisions of acceptance or rejection for a lot based on the decision criteria developed. The proposed methodology provides a general framework for designing BRASP under type-I censoring. As special cases, it also accommodates BRASP under conventional SSSPALT with type-I censoring, referred to as CBSPA in this chapter, and BRASP under conventional non-accelerated life testing with type-I censoring, referred to as CBSP, which was studied by Lin et al. [74].

The organization of the chapter is as follows. Section 4.1 presents the framework of the BSPAA for the exponential distribution through SSSPALT. Section 4.2 derives the Bayes decision function and Bayes risk under a general loss function and prior distributions. Section 4.3 specifies these results for the quadratic loss function. Section 4.4 addresses the determination of the optimal BSPAA. Section 4.5 provides numerical examples under different scenarios, including comparisons between BSPAA, CBSP, and CBSPA. Section 4.6 presents a real data analysis to demonstrate the proposed method. Finally, Section 4.8 concludes with a discussion.

4.1 Model & assumptions

Suppose n identical items are selected from the lot and put on life test in normal stress s_0 at time $\tau_0 = 0$. Let $\tau_1 > 0$ and $\tau_2 > \tau_1$ denote the pre-fixed time points. Let D_1 be the number of failures that occur by time τ_1 under the stress level s_0. If D_1 is less than a pre-assigned number m $(\leq n)$, the stress level s_0 is changed to stress level s_1. Otherwise, the stress level remains unchanged and the life test continues with stress s_0. For simplicity, stress levels s_0 and s_1 are represented as follows.

$$s = \begin{cases} s_0, & \text{if } \ 0 < x < \tau_1, \\ s_0, & \text{if } \ x \geq \tau_1, \ D_1 \geq m, \\ s_1, & \text{if } \ x \geq \tau_1, \ D_1 < m. \end{cases} \tag{4.1.1}$$

Suppose X denotes the lifetime of an item with CDF $F(\cdot \mid \boldsymbol{\theta})$ and PDF $f(\cdot \mid \boldsymbol{\theta})$ under ASSSPALT. Let the hazard rate at the stress level s_i be λ_i, $i = 0, 1$. Using (1.4.2), the CDF of X is given by

$$F(x \mid \boldsymbol{\theta}) = \begin{cases} 1 - \exp(-\lambda_0 x), & \text{if } \ x < \tau_1, \\ [1 - \exp(-\lambda_0 x)]^{1-\delta}[1 - \exp[-\lambda_0\tau_1 - \lambda_1(x - \tau_1)]]^{\delta}, & \text{if } \ x \geq \tau_1, \end{cases}$$

where δ is the indicator function defined as

$$\delta = \begin{cases} 1, & \text{if } D_1 < m, \\ 0, & \text{if } D_1 \geq m. \end{cases} \tag{4.1.2}$$

The PDF of X is given by

$$f(x \mid \boldsymbol{\theta}) = \begin{cases} \lambda_0 \exp(-\lambda_0 x), & \text{if } \ x < \tau_1, \\ [\lambda_0 \exp(-\lambda_0 x)]^{1-\delta}[\lambda_1 \exp[-\lambda_0\tau_1 - \lambda_1(x - \tau_1)]]^{\delta}, & \text{if } \ x \geq \tau_1. \end{cases}$$

Suppose $X_1, X_2, \ldots, X_n$ be the lifetimes of n items with PDF $f(\cdot \mid \boldsymbol{\theta})$ and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics corresponding to $X_1, X_2, \ldots, X_n$. The test is carried out under type-I censoring as described in Introduction. Therefore, the life test continues up to time τ_2. Let D_2 be the random variable denoting the number of failures in the interval $(\tau_1, \tau_2]$. The observed data is given by

$$\boldsymbol{x} = \begin{cases} (d_1 = 0, d_2 = 0), & \text{if } d = 0, \\ (\boldsymbol{y}, d_1, d_2), & \text{if } d > 0, \end{cases}$$

where $\boldsymbol{y} = (x_{(1)}, \ldots, x_{(d)})$ and $d = d_1 + d_2$. We assume that $\lambda_0 = \lambda$ and $\lambda_1 = \phi\lambda$, with $\phi > 1$, since $s_0 < s_1$. Let $\boldsymbol{\theta} = (\lambda, \phi)$ denote the parameter vector. Given observed data $\boldsymbol{x}$, the likelihood function can be written as

$$L(\boldsymbol{\theta} \mid \boldsymbol{x}) \propto \begin{cases} \lambda^d \exp\left[-\lambda\left(\displaystyle\sum_{j=0}^{d_1} x_{(j)} + (n - d_1)\tau_1 + \phi^{\delta}(n - d)(\tau_2 - \tau_1)\right)\right], & \text{if } d_1 \geq 0 \ \& \ d_2 = 0, \\ \phi^{\delta d_2}\lambda^d \exp\left[-\lambda\left(\displaystyle\sum_{j=0}^{d_1} x_{(j)} + (n - d_1)\tau_1 + \right.\right. & \\ \left.\left.\phi^{\delta}\left(\displaystyle\sum_{j=d_1+1}^{d} (x_{(j)} - \tau_1) + (n - d)(\tau_2 - \tau_1)\right)\right)\right], & \text{if } d_1 \geq 0 \ \& \ d_2 > 0, \end{cases}$$

Here, the sufficient statistics w_1 and w_2 can be written as $w_1(\boldsymbol{x}) = \sum_{j=0}^{d_1} x_{(j)} + (n - d_1)\tau_1$, where $x_{(0)} = 0$ and

$$w_2(\boldsymbol{x}) = \begin{cases} \sum_{j=d_1+1}^{d}(x_{(j)} - \tau_1) + (n - d)(\tau_2 - \tau_1), & \text{if } d_2 > 0, \\ (n - d)(\tau_2 - \tau_1), & \text{if } d_2 = 0. \end{cases}$$

Then the likelihood simplifies to

$$L(\boldsymbol{\theta} \mid (w_1(\boldsymbol{x}), w_2(\boldsymbol{x}), d_1, d_2)) \propto \lambda^d \phi^{d_2 \delta} \exp\left[-\lambda \left(w_1(\boldsymbol{x}) + \phi^\delta w_2(\boldsymbol{x})\right)\right]. \tag{4.1.3}$$

For convenience, we write $w_1 \equiv w_1(\boldsymbol{x})$ and $w_2 \equiv w_2(\boldsymbol{x})$. The observed data can be written as $\boldsymbol{x} = (w_1, w_2, d_1, d_2)$. The vector of the decision variables of the life testing plan under this setup is denoted by $\boldsymbol{\zeta} = (n, \tau_1, \tau_2, m)$.

Let $p(\boldsymbol{\theta})$ be the prior distribution of $\boldsymbol{\theta} = (\lambda, \phi)$. It is assumed that λ and ϕ are independent and $p_1(\cdot)$ and $p_2(\cdot)$ be their PDFs, respectively. Therefore, $p(\boldsymbol{\theta})$ can be written as $p(\boldsymbol{\theta}) = p_1(\lambda)p_2(\phi)$. The posterior distribution of $\boldsymbol{\theta}$ given $\boldsymbol{x}$ is

$$p(\boldsymbol{\theta} \mid \boldsymbol{x}) = \frac{L(\boldsymbol{\theta} \mid \boldsymbol{x})p(\boldsymbol{\theta})}{\int_{\boldsymbol{\theta}} L(\boldsymbol{\theta} \mid \boldsymbol{x})\ p(\boldsymbol{\theta})\ d\boldsymbol{\theta}}. \tag{4.1.4}$$

Let $a(\boldsymbol{x} \mid \boldsymbol{\zeta})$ denotes the action of acceptance sampling and it is defined by

$$a(\boldsymbol{x} \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{if the lot is accepted based on observed data } \boldsymbol{x}, \\ 0, & \text{if the lot is rejected based on observed data } \boldsymbol{x}. \end{cases} \tag{4.1.5}$$

We consider the loss functions for acceptance and rejection after the life testing as

$$\mathcal{L}\left(\mathcal{A} \mid \boldsymbol{x}, \boldsymbol{\theta}\right) = h(\lambda) + nC_s - (n-d)v_s + \tau C_t + \delta(n - d_1)C_a \tag{4.1.6}$$

and

$$\mathcal{L}\left(\mathcal{R} \mid \boldsymbol{x}, \boldsymbol{\theta}\right) = C_r + nC_s - (n-d)v_s + \tau C_t + \delta(n - d_1)C_a, \tag{4.1.7}$$

respectively. The cost components C_s and C_t are defined in Chapter 2. The other cost components in the above expressions are motivated as follows:

- **Loss function ($h(\lambda)$):** Represents the cost associated with accepting a lot. Items are assumed to operate under normal conditions, so the loss depends solely on the failure rate λ. Since higher λ indicates earlier failure, $h(\lambda)$ is taken as a positive, increasing function of λ. For example, if the loss arises from items failing before a specified lifetime L, it can be modeled as $h(\lambda) = NCP(X < L) = NC[1 - \exp(-\lambda L)]$, where N is the lot size, C is the cost per failed item, and X the item lifetime under normal conditions.
- **Salvage value (v_s):** Units that survive the life test may often be reused in production or sold as second-grade products. This recovery generates a cost offset, referred to as the salvage value. Here v_s represents the per-unit salvage value which is always less than C_s.
- **Accelerated stress cost (C_a):** It represents the per-unit accelerated stress cost which arises due to applying elevated stress (e.g., increased temperature, voltage, or pressure) increases marginal resource usage or risk of induced damage. This entails additional expenses.
- **Rejection cost (C_r):** It captures the fixed penalty associated with rejecting a lot, encompassing material disposal, rework, and potential contractual penalties.

4.2 Bayes risk and Bayes decision function

Here, we obtain the Bayes Risk using the loss function given in (4.1.6). Then we determine the life testing plan $\boldsymbol{\zeta}^* = (n^*, \tau_1^*, \tau_2^*, m^*)$ and the optimal decision function $a^*(\boldsymbol{x} \mid \boldsymbol{\zeta}^*) \equiv a^*$ by minimizing the Bayes risk over all such sampling plans.

4.2.1 Bayes risk

The expected loss for a given sampling plan $\boldsymbol{\zeta}$ and observed data $\boldsymbol{x}$ is

$$\begin{aligned}\psi(\boldsymbol{\zeta}, a \mid \boldsymbol{x}, \boldsymbol{\theta}) =& \mathcal{L}(\mathcal{A} \mid \boldsymbol{x}, \boldsymbol{\theta}) a(\boldsymbol{x} \mid \boldsymbol{\zeta}) + \mathcal{L}(\mathcal{R} \mid \boldsymbol{x}, \boldsymbol{\theta}) \left[1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta})\right] \\ =& a(\boldsymbol{x} \mid \boldsymbol{\zeta}) h(\lambda) + (1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta})) C_r + n C_s + \delta(n - d_1) C_a - (n - d) v_s + \tau C_t.\end{aligned}$$

The Bayes risk of a sampling plan $\boldsymbol{\zeta} = (n, \tau_1, \tau_2, m)$ is defined as the expected loss over the joint distribution of the parameters and data:

$$\begin{aligned}R_B(\boldsymbol{\zeta}, a) &= E_{\boldsymbol{\theta}} E_{\boldsymbol{X} \mid \boldsymbol{\theta}} \left[\psi(\boldsymbol{\zeta}, a \mid \boldsymbol{x}, \boldsymbol{\theta})\right] \\ &= n(C_s - v_s) + C_a n_{as} + v_s E[D] + C_t E[\tau] + R_A(\boldsymbol{\zeta}, a),\end{aligned} \tag{4.2.1}$$

where

$$R_A(\boldsymbol{\zeta}, a) = E_{\boldsymbol{\theta}} E_{\boldsymbol{X} \mid \boldsymbol{\theta}} [a(\boldsymbol{X} \mid \boldsymbol{\zeta}) h(\lambda) + (1 - a(\boldsymbol{X} \mid \boldsymbol{\zeta})) C_r], \tag{4.2.2}$$

$n_{as} = E_{\boldsymbol{\theta}}[E_{D_1 \mid \boldsymbol{\theta}}[\delta(n - D_1) \mid \boldsymbol{\theta}]]$, $E[D] = E_{\boldsymbol{\theta}}[E_{D \mid \boldsymbol{\theta}}[D \mid \boldsymbol{\theta}]]$ and $E[\tau] = E_{\boldsymbol{\theta}}[E_{\tau \mid \boldsymbol{\theta}}[\tau \mid \boldsymbol{\theta}]]$. The expressions of $E[D]$, n_{as} and $E[\tau]$ are provided in Appendix 3.A.2.

4.2.2 Bayes decision function

We obtain the Bayes decision function that minimizes the Bayes risk $R_B(\boldsymbol{\zeta}, a)$ among the class of all decision functions. The Bayes risk $R_B(\boldsymbol{\zeta}, a)$ in (4.2.1) consists of the decision function $a(\cdot \mid \boldsymbol{\zeta})$ only in the term $R_A(\boldsymbol{\zeta}, a)$. Note that $R_A(\boldsymbol{\zeta}, a)$ in (4.2.2) can be written as

$$\begin{aligned}R_A(\boldsymbol{\zeta}, a) &= E_{\boldsymbol{\theta}}[h(\lambda)] + E_{\boldsymbol{\theta}} E_{\boldsymbol{X} \mid \boldsymbol{\theta}}[(1 - a(\boldsymbol{X} \mid \boldsymbol{\zeta}))(C_r - h(\lambda))] \\ &= E_{\boldsymbol{\theta}}[h(\lambda)] + E_{\boldsymbol{X}} E_{\boldsymbol{\theta} \mid \boldsymbol{X}}[(1 - a(\boldsymbol{X} \mid \boldsymbol{\zeta}))(C_r - h(\lambda))].\end{aligned} \tag{4.2.3}$$

We have

$$\begin{aligned}& E_{\boldsymbol{X}} E_{\boldsymbol{\theta} \mid \boldsymbol{X}} \left[(1 - a(\boldsymbol{X} \mid \boldsymbol{\zeta}))(C_r - h(\lambda))\right] \\ &= \sum_{d_1=0}^{n} \sum_{d_2=0}^{n-d_1} \int_{w_1} \int_{w_2} \left[1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta})\right] E_{\boldsymbol{\theta} \mid \boldsymbol{x}}[C_r - h(\lambda)] \; g_{\boldsymbol{X}}(\boldsymbol{x}) \; dw_2 \; dw_1 \\ &= \sum_{d_1=0}^{n} \sum_{d_2=0}^{n-d_1} \int_{w_1} \int_{w_2} \left[1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta})\right] \left\{C_r - \int_{\boldsymbol{\theta}} h(\lambda) \, p(\boldsymbol{\theta} \mid \boldsymbol{x}) \; d\boldsymbol{\theta}\right\} g_{\boldsymbol{X}}(\boldsymbol{x}) \; dw_2 \; dw_1 \\ &= \sum_{d_1=0}^{n} \sum_{d_2=0}^{n-d_1} \int_{w_1} \int_{w_2} \left[1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta})\right] \left[C_r - \varphi(\boldsymbol{x})\right] \; g_{\boldsymbol{X}}(\boldsymbol{x}) \; dw_2 \; dw_1,\end{aligned}$$

where

$$\varphi(w_1, w_2, d_1, d_2) = \int_{\boldsymbol{\theta}} h(\lambda)\, p(\boldsymbol{\theta} \mid w_1, w_2, d_1, d_2)\; d\boldsymbol{\theta} \tag{4.2.4}$$

and $g_{\boldsymbol{X}}(\boldsymbol{x})$ is the marginal PDF of $\boldsymbol{X}$, defined as

$$g_{\boldsymbol{X}}(\boldsymbol{x}) = \int_{\boldsymbol{\theta}} g_{\boldsymbol{X} \mid \boldsymbol{\theta}}(\boldsymbol{x} \mid \boldsymbol{\theta})\; p(\boldsymbol{\theta})\; d\boldsymbol{\theta},$$

with conditional joint PDF

$$g_{\boldsymbol{X} \mid \boldsymbol{\theta}}(\boldsymbol{x} \mid \boldsymbol{\theta}) = g_{(W_1,W_2) \mid D_1,D_2,\boldsymbol{\theta}}(w_1, w_2 \mid d_1, d_2, \boldsymbol{\theta}) P(D_1 = d_1, D_2 = d_2 \mid \boldsymbol{\theta}).$$

Here $g_{(W_1,W_2) \mid D_1,D_2,\boldsymbol{\theta}}(w_1, w_2 \mid d_1, d_2, \boldsymbol{\theta})$ denotes the conditional joint PDF of (W_1, W_2) given $(D_1, D_2, \boldsymbol{\theta})$ and $P(D_1 = d_1, D_2 = d_2 \mid \boldsymbol{\theta})$ is the joint PMF of (D_1, D_2).

To obtain Bayes decision, we need to minimize $R_A(\boldsymbol{\zeta}, a)$ w.r.t. a which is equivalent to minimization of $E_{\boldsymbol{X}} E_{\boldsymbol{\theta} \mid \boldsymbol{X}}[(1 - a(\boldsymbol{X} \mid \boldsymbol{\zeta}))(C_r - h(\lambda))]$ w.r.t. $a(\cdot \mid \boldsymbol{\zeta})$. Now we consider two cases:

Case 1: when $\varphi(w_1, w_2, d_1, d_2) \leq C_r$,

$$E_{\boldsymbol{\theta} \mid \boldsymbol{x}}[(1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta}))(C_r - h(\lambda))] = \begin{cases} 0, & \text{if } a(\boldsymbol{x} \mid \boldsymbol{\zeta}) = 1, \\ \geq 0, & \text{if } a(\boldsymbol{x} \mid \boldsymbol{\zeta}) = 0. \end{cases}$$

Case 2: when $\varphi(w_1, w_2, d_1, d_2) > C_r$,

$$E_{\boldsymbol{\theta} \mid \boldsymbol{x}}[(1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta}))(C_r - h(\lambda))] = \begin{cases} 0, & \text{if } a(\boldsymbol{x} \mid \boldsymbol{\zeta}) = 1, \\ \leq 0, & \text{if } a(\boldsymbol{x} \mid \boldsymbol{\zeta}) = 0. \end{cases}$$

Therefore, for each fixed $\boldsymbol{\zeta}$, if we take $a(\boldsymbol{x} \mid \boldsymbol{\zeta}) = 1$ when $C_r - \varphi(w_1, w_2, d_1, d_2) \geq 0$ and $a(\boldsymbol{x} \mid \boldsymbol{\zeta}) = 0$ when $C_r - \varphi(w_1, w_2, d_1, d_2) \leq 0$, then $E_{\boldsymbol{X}} E_{\boldsymbol{\theta} \mid \boldsymbol{X}}[(1 - a(\boldsymbol{X} \mid \boldsymbol{\zeta}))(C_r - h(\lambda))]$ is minimized w.r.t. $a(\cdot \mid \boldsymbol{\zeta})$. So for fixed $\boldsymbol{\zeta}$, the Bayes decision function $a^*(\boldsymbol{x} \mid \boldsymbol{\zeta})$ is given by,

$$a^*(\boldsymbol{x} \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{if } C_r - \varphi(w_1, w_2, d_1, d_2) \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

Next, we provide an alternative form of the Bayes decision function that is useful to obtain a simplified form of (4.2.3).

4.2.3 Alternative form of Bayes decision function

For developing an alternative form of the Bayes decision function, we consider the following theorem.

Theorem 4.1 *If λ follows gamma distribution and $h(\lambda)$ is increasing in λ, then the posterior expectation of $h(\lambda)$ which is given in (4.2.4), satisfies the following monotonicity properties:*

1. *For fixed (w_2, d_1, d_2), $\varphi(w_1, w_2, d_1, d_2)$ is decreasing in w_1.*
2. *For fixed (w_1, d_1, d_2), $\varphi(w_1, w_2, d_1, d_2)$ is decreasing in w_2.*

Proof: The proof is given in Appendix 4.A.2. (Proved)

For fixed $(w_1 = 0, d_1, d_2)$, $\varphi(0, w_2, d_1, d_2)$ is a decreasing function in w_2. If $\varphi(0, 0, d_1, d_2) > C_r$ and $\lim_{w_2\to\infty} \varphi(0, w_2, d_1, d_2) < C_r$, then there exists a unique point $c_1(d_1, d_2)$ such that

$$\begin{aligned} \varphi(0, w_2, d_1, d_2) &> C_r, \quad \text{for } w_2 < c_1(d_1, d_2) \\ \varphi(0, w_2, d_1, d_2) &< C_r, \quad \text{for } w_2 > c_1(d_1, d_2). \end{aligned}$$

If $\varphi(0, 0, d_1, d_2) < C_r$, then $\varphi(0, w_2, d_1, d_2) < C_r$ for all $w_2 > 0$. In that scenario, $c_1(d_1, d_2)$ can be taken as 0. Now, for fixed $(w_2 = c_1(d_1, d_2), d_1, d_2)$, $\varphi(w_1, c_1(d_1, d_2), d_1, d_2)$ is a decreasing function in w_1. Therefore,

$$\varphi(w_1, w_2, d_1, d_2) < C_r, \quad \text{for } w_1 > 0 \text{ and } w_2 > c_1(d_1, d_2).$$

If $\lim_{w_2\to\infty} \varphi(0, w_2, d_1, d_2) > C_r$, then $c_1(d_1, d_2)$ can be taken as ∞. When $0 < w_2 < c_1(d_1, d_2)$, we get $\varphi(0, w_2, d_1, d_2) > C_r$, since for fixed (d_1, d_2, w_2), $\varphi(w_1, w_2, d_1, d_2)$ is decreasing function in w_1. Therefore, when $0 < w_2 < c_1(d_1, d_2)$ say, there exists a unique point $C_{11}(d_1, d_2, w_2)$, such that

$$\begin{aligned} \varphi(w_1, w_2, d_1, d_2) &< C_r, \quad \text{for } w_1 > C_{11}(d_1, d_2, w_2) \text{ and } 0 < w_2 < c_1(d_1, d_2) \\ \varphi(w_1, w_2, d_1, d_2) &> C_r, \quad \text{for } 0 < w_1 < C_{11}(d_1, d_2, w_2) \text{ and } 0 < w_2 < c_1(d_1, d_2). \end{aligned}$$

If $\lim_{w_1\to\infty} \varphi(w_1, w_2, d_1, d_2) > C_r$, then $C_{11}(d_1, d_2, w_2)$ can be taken as ∞. Note that the upper bounds of w_1 and w_2 are $n\tau_1$ and $(n - d_1)(\tau_2 - \tau_1)$, respectively. Define $c_1'(d_1, d_2) = \min\{c_1(d_1, d_2), n\tau_1\}$ and $c_2'(d_1, d_2, w_2) = \min\{C_{11}(d_1, d_2, w_2), (n - d_1)(\tau_2 - \tau_1)\}\}$. Therefore, the Bayes decision function can be written as

$$a^*(\boldsymbol{x} \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{for } w_2 > c_1'(d_1, d_2) \text{ or } 0 < w_2 < c_1'(d_1, d_2) \text{ and } w_1 > c_2'(d_1, d_2, w_2), \\ 0, & \text{for } 0 < w_2 < c_1'(d_1, d_2) \text{ and } 0 < w_1 < c_2'(d_1, d_2, w_2). \end{cases} \tag{4.2.5}$$

4.2.4 Alternative form of Bayes Risk

Now, using (4.2.5), $R_A(\boldsymbol{\zeta}, a^*)$ in (4.2.2) can be written as

$$\begin{aligned} & R_A(\boldsymbol{\zeta}, a^*) \\ =& E_{\boldsymbol{\theta}}[h(\lambda)] + \sum_{d_1=0}^{n} \sum_{d_2=0}^{n-d_1} \int_{\boldsymbol{\theta}} [C_r - h(\lambda)]\ P(0 < W_1 < c_2'(d_1, d_2, w_2), 0 < W_2 < c_1'(d_1, d_2), D_1 = d_1, D_2 = d_2 \mid \boldsymbol{\theta})\ p(\boldsymbol{\theta})\ d\boldsymbol{\theta} \\ =& E_{\boldsymbol{\theta}}[h(\lambda)] + \sum_{d_1=0}^{n} \sum_{d_2=0}^{n-d_1} \int_{w_2=0}^{c_1'(d_1,d_2)} \int_{w_1=0}^{c_2'(d_1,d_2,w_2)} \int_{\boldsymbol{\theta}} [C_r - h(\lambda)]\ g_{\boldsymbol{X}\ |\boldsymbol{\theta}}(w_1, w_2, d_1, d_2 \mid \boldsymbol{\theta}) p(\boldsymbol{\theta})\ dw_1\ dw_2\ d\boldsymbol{\theta} \\ =& E_{\boldsymbol{\theta}}[h(\lambda)] + \sum_{d_1=0}^{n} \sum_{d_2=0}^{n-d_1} H(d_1, d_2), \end{aligned} \tag{4.2.6}$$

where

$$H(d_1, d_2) = \int_{w_2=0}^{c_1'(d_1,d_2)} \int_{w_1=0}^{c_2'(d_1,d_2,w_2)} \int_{\boldsymbol{\theta}} [C_r - h(\lambda)] g_{\boldsymbol{X}}(w_1, w_2, d_1, d_2 \mid \boldsymbol{\theta}) p(\boldsymbol{\theta})\ dw_1\ dw_2\ d\boldsymbol{\theta}.$$

The joint distribution function of (W_1, W_2, D_1, D_2) is given in Appendix 4.A.1.

4.3 Bayes decision function and Bayes risk for quadratic loss function

Here, the loss function is taken as a quadratic loss function $h(\lambda) = C_0 + C_1\lambda + C_{11}\lambda^2$ as considered by Yeh [128]. The priors of λ and ϕ are taken as λ follows a $G(\alpha_1, \beta_1)$ with PDF $p_1(\cdot)$ and ϕ follows $\mathcal{U}(1, l)$ with PDF $p_2(\cdot)$. Now, for given $\boldsymbol{\zeta} = (n, \tau_1, \tau_2, m)$, we derive the Bayes decision function $a^*(\cdot \mid \boldsymbol{\zeta})$. First, we compute the joint posterior distribution of (λ, ϕ) for given $\boldsymbol{x}$. Using the (4.1.4), we get

$$p(\lambda, \phi \mid \boldsymbol{x}) = \frac{\phi^{\delta d_2}\lambda^{d+\alpha_1-1}\exp[-\lambda(w_1 + \phi^\delta w_2 + \beta_1)]}{\int_0^\infty \int_1^l \phi^{\delta d_2}\lambda^{d+\alpha_1-1}\exp[-\lambda(w_1 + \phi^\delta w_2 + \beta_1)]\, d\phi\, d\lambda}.$$

To obtain Bayes decision, we need the following result:

Result 4.1 *If κ is a non-negative real number, then we get*

$$\begin{aligned}\int_1^l \int_0^\infty \phi^{\delta d_2}\lambda^\kappa \exp(-\lambda(w_1 + \phi^\delta w_2))\lambda^{d_1+d_2}\lambda^{\alpha_1-1}\exp(-\beta_1\lambda)\, d\lambda\, d\phi &= \int_1^l \phi^{\delta d_2}\frac{\Gamma(\kappa + d_1 + d_2 + \alpha_1)}{(w_1 + \phi^\delta w_2 + \beta_1)^{\kappa+d_1+d_2+\alpha_1}}\, d\phi \\ &= H_1(w_1, w_2, d_1, d_2, \kappa, \delta), \text{ say.}\end{aligned}$$

If $\delta = 0$,

$$H_1(w_1, w_2, d_1, d_2, \kappa, 0) = \frac{\Gamma(\kappa + d_1 + d_2 + \alpha_1)(l-1)}{(w_1 + w_2 + \beta_1)^{\kappa+d_1+d_2+\alpha_1}}$$

and if $\delta = 1$,

$$\begin{aligned}H_1(w_1, w_2, d_1, d_2, \kappa, 1) &= \int_1^l \phi^{d_2}\frac{\Gamma(\kappa + d + \alpha_1)}{(w_1 + \phi w_2 + \beta_1)^{\kappa+d+\alpha_1}}\, d\phi \\ &= \int_1^l \phi^{d_2}\frac{\Gamma(\kappa + d + \alpha_1)}{(w_1 + \beta_1)^{\kappa+d+\alpha_1}(1 + \phi w_2/(w_1 + \beta_1))^{\kappa+d+\alpha_1}}\, d\phi \\ &= \int_{w_2/(w_1+\beta_1)}^{w_2 l/(w_1+\beta_1)} \frac{\Gamma(\kappa + d + \alpha_1)}{(w_1 + \beta_1)^{\kappa+d_1+\alpha_1+1}w_2^{d_2+1}}\frac{z^{d_2}}{(1+z)^{\kappa+d+\alpha_1}}\, dz \qquad [z = \phi z/(w_1 + \beta_1)] \\ &= \frac{\Gamma(d_2+1)\Gamma(\kappa + d_1 + \alpha_1 - 1)}{(w_1 + \beta_1)^{\kappa+d_1+\alpha_1+1}w_2^{d_2+1}}[I_{\rho_1}(d_2 + 1, \kappa + d_1 + \alpha_1 - 1) - I_{\rho_2}(d_2 + 1, \kappa + d_1 + \alpha_1 - 1)],\end{aligned}$$

where $\rho_i = h_i/(1 + h_i)$, for $i = 1, 2$, $h_1 = w_2 l/(w_1 + \beta_1)$, $h_2 = w_2/(w_1 + \beta_1)$ and $I_\rho(a, b)$ is the CDF of beta distribution defined in Section 1.2.5. Using Result 4.1, $\varphi(\boldsymbol{x})$ can be written as

$$\varphi(\boldsymbol{x}) = C_0 + C_1\frac{H_1(w_1, w_2, d_1, d_2, 1, \delta)}{H_1(w_1, w_2, d_1, d_2, 0, \delta)} + C_{11}\frac{H_1(w_1, w_2, d_1, d_2, 2, \delta)}{H_1(w_1, w_2, d_1, d_2, 0, \delta)}.$$

Therefore, the Bayes decision can be obtained as follows

$$a^*(w_1, w_2, d_1, d_2 \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{if } C_0 + C_1\frac{H_1(w_1,w_2,d_1,d_2,1,\delta)}{H_1(w_1,w_2,d_1,d_2,0,\delta)} + C_{11}\frac{H_1(w_1,w_2,d_1,d_2,2,\delta)}{H_1(w_1,w_2,d_1,d_2,0,\delta)} \le C_r, \\ 0, & \text{otherwise.}\end{cases} \tag{4.3.1}$$

Note that the prior expectation of the loss $h(\lambda)$ is

$$E_{(\lambda,\phi)}[h(\lambda)] = \int_1^l \int_0^\infty h(\lambda)p_1(\lambda)p_2(\phi)\, d\lambda\, d\phi = C_0 + \frac{C_1\alpha_1}{\beta_1} + C_{11}\frac{C_{11}\alpha_1(\alpha_1+1)}{{\beta_1}^2}.$$

Let $\mathbf{0} = (0, 0, 0, 0)$ denote the no sampling case, i.e, the decision is taken without life-testing. For no sampling case, the Bayes decision function is given by

$$a^*(\mathbf{0} \mid \mathbf{0}) = \begin{cases} 1, & \text{if } C_0 + \frac{C_1\alpha_1}{\beta_1} + C_{11}\frac{C_{11}\alpha_1(\alpha_1+1)}{{\beta_1}^2} \leq C_r, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3.2}$$

Using the decision function $a(\cdot \mid \boldsymbol{\zeta})$, the explicit form of Bayes risk is given in the Appendix 4.A.3.

4.4 Optimal BSPAA

Here we find optimal BSPAA that minimizes the Bayes risk among all the possible sampling plans. Therefore, the optimal BSPAA $(\boldsymbol{\zeta}^*, a^*)$ is defined by

$$(\boldsymbol{\zeta}^*, a^*) = \arg\min_{(\boldsymbol{\zeta},a)} R_B(\boldsymbol{\zeta}, a).$$

The Bayes risk is a function of two discrete variables n, m and two continuous variables τ_1, τ_2. The solution cannot be obtained analytically. We provide contour plot of Bayes risk w.r.t. τ_1, τ_2 for $n = 3, 4$ and $m = 0, 2, n$ in Figure 4.1 for hyperparameters $\alpha = 3$, $\beta = 1$ and $l = 10$ and cost coefficients $C_0 = 2$, $C_1 = 3$, $C_{11} = 2$, $C_a = 0.1$, $v_s = 0.2$, $C_s = 0.5$, $C_t = 5$ and $C_r = 30$.

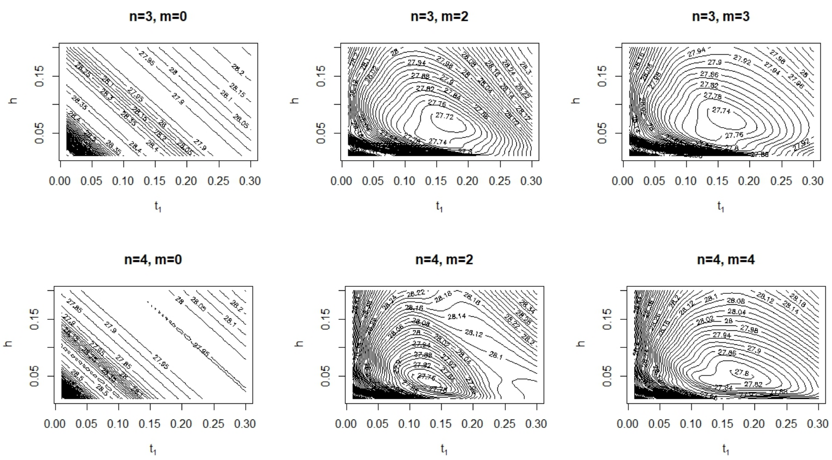


Figure 4.1: Contour Plots for different value of m and for $n = 3, 4$

In Figure 4.1, it is noted that for $m = 0$, the lines are parallel. This is due to the fact that for $m = 0$, BSPAA is converted to CBSP. In CBSP, the life test proceeds entirely under the normal stress level until the termination time τ_2. Although τ_1 is defined within the general adaptive framework, it has no operational effect in BSPAA when $m = 0$. When $m = 0$, one may simply take $\tau_1 = \tau_2$, since the test effectively reduces to a CBSP with a single termination time. Consequently, the Bayes risk and the optimal design depend only on τ_2. For $m > 0$, in Figure 4.1, it is seen that for fixed n, m, the Bayes risk has a unique minimum. Note that by definition, the optimal value m^* is bounded.

Next, we provide an upper bound for n^*. Let $\boldsymbol{\zeta}^* = (n^*, \tau_1^*, \tau_2^*, m^*)$ be the optimal BSPAA. Since all costs are positive and $C_s > v_s \geq 0$, all terms in $R_B(\boldsymbol{\zeta}, a)$ are positive. So we get,

$$R_B(\boldsymbol{\zeta}^*, a^*) \geq n^*(C_s - v_s). \tag{4.4.1}$$

In no sampling case, if the lot is rejected, the Bayes risk is $R(\mathbf{0}, 0) = C_r$ and if the lot is accepted, the Bayes risk is $R(\mathbf{0}, 1) = E_\lambda[h(\lambda)]$. Therefore $R(\mathbf{0}, a^*) = \min\{E_\lambda[h(\lambda)], C_r\}$. Now,

$$R_B(\boldsymbol{\zeta}^*, a^*) \leq \min\{E_\lambda[h(\lambda)], C_r\}. \tag{4.4.2}$$

From (4.4.1) and (4.4.2), we get

$$n_0 = \left\lfloor \frac{\min\{E_\lambda[h(\lambda)], C_r\}}{C_s - v_s} \right\rfloor. \tag{4.4.3}$$

Therefore, we can say that we get a finite value of $\boldsymbol{\zeta}^* = (n^*, \tau_1^*, \tau_2^*, m^*)$. It is noted that for $m = 0$, $\tau_1^* = \tau_2^*$. We now provide Algorithm 4.1 to obtain optimal BSPAA.

Algorithm 4.1: Procedure for Finding Optimal BSPAA

Input: Possible design parameters $\boldsymbol{\zeta} = (n, \tau_1, \tau_2, m)$ with $n, m \in \mathbb{Z}_{\geq 0}$ and $\tau_1, \tau_2 \geq 0$, maximum n_0
Output: Optimal design parameters $\boldsymbol{\zeta}^*$ and Bayes decision function $a^*(\cdot \mid \boldsymbol{q})$ minimizing R_B
for $n = 0, 1, \ldots, n_0$ **do**
  for $m = 0, 1, \ldots, n$ **do**
    for τ_1, τ_2 *in domain* **do**
      Derive Bayes decision function $a^*(\cdot \mid \boldsymbol{\zeta})$ minimizing $R_B(\boldsymbol{\zeta}, a)$ (See Sections 4.2.2 and 4.2.3);
    if $m > 0$ **then**
      Set $\tau_2 = \tau_1 + h$ Let $h^*(n, m)$ be optimal h;
      Find $\tau_1^*(n, m)$ and $\tau_2^*(n, m) = \tau_1^*(n, m) + h^*(n, m)$ by minimizing (4.2.1) over $\tau_1 \geq 0$, $h \geq 0$. i.e,

$$(\tau_1^*(n, m), h^*(n, m)) = \arg \min_{\tau_1 \geq 0, h \geq 0} R_B(n, \tau_1, \tau_1 + h, m, a^*(\cdot \mid (n, \tau_1, \tau_1 + h, m)))$$

    else
      Set $\tau_2 = \tau_1$ Find $\tau_1^*(n, m) = \tau_2^*(n, m)$ by minimizing (4.2.1) over $\tau_1 \geq 0$

$$\tau_1^*(n, m) = \arg \min_{\tau_1 \geq 0} R_B(n, \tau_1, \tau_1, 0, a^*(\cdot \mid (n, \tau_1, \tau_1, 0)))$$

    Find $m^*(n)$ by minimizing (4.2.1) over $0 \leq m \leq n$:

$$m^*(n) = \arg \min_{0 \leq m \leq n} R_B(n, \tau_1^*(n, m), \tau_2^*(n, m), m, a^*(\cdot \mid (n, \tau_1^*(n, m), \tau_2^*(n, m), m)))$$

  Find n^* by minimizing (4.2.1) over $1 \leq n \leq n_0$:

$$n^* = \arg \min_{1 \leq n \leq n_0} R_B(n, \tau_1^*(n, m^*(n)), \tau_2^*(n, m^*(n)), m^*(n), a(\cdot \mid (n, \tau_1^*(n, m^*(n)), \tau_2^*(n, m^*(n)), m^*(n))))$$

Now, we write $a(\cdot \mid (n^*, \tau_1^*(n^*, m^*(n^*), \tau_2^*(n^*, m^*(n^*)), m^*))) = a^*$, $\tau_1^*(n^*, m^*(n^*)) = \tau_1^*$, $\tau_2^*(n^*, m^*(n^*)) = \tau_2^*$ and $m^*(n^*) = m^*$. Therefore, $(\boldsymbol{\zeta}^*, a^*) = (n^*, \tau_1^*, \tau_2^*, m^*, a^*)$ is the optimal BSPAA.

Theorem 4.2 *The sampling plan $(n^*, \tau_1^*, \tau_2^*, m^*, a^*)$ is the optimal BSPAA.*

Proof: It is enough to prove that for any sampling plan $(n, \tau_1, \tau_2, m, a)$, the following inequality holds:

$$R_B(n, \tau_1, \tau_2, m, a) \geq R_B(n^*, \tau_1^*, \tau_2^*, m^*, a^*).$$

Now,

$$\begin{aligned}
R_B(n,\tau_1,\tau_2,m,a) - R_B(n^*,\tau_1^*,\tau_2^*,m^*,a^*) =& [R_B(n,\tau_1,\tau_2,m,a) - R_B(n,\tau_1,\tau_2,m,a^*)] \\
&+ [R_B(n,\tau_1,\tau_2,m,a^*) - R_B(n,\tau_1^*,\tau_2^*,m,a^*)] \\
&+ [R_B(n,\tau_1^*,\tau_2^*,m,a^*) - R_B(n,\tau_1^*,\tau_2^*,m^*,a^*)] \\
&+ [R_B(n,\tau_1^*,\tau_2^*,m^*,a^*) - R_B(n^*,\tau_1^*,\tau_2^*,m^*,a^*)].
\end{aligned}$$

From Algorithm 4.1, it follows that $[R_B(n,\tau_1,\tau_2,m,a) - R_B(n,\tau_1,\tau_2,m,a^*)] \geq 0$, $[R_B(n,\tau_1,\tau_2,m,a^*) - R_B(n,\tau_1^*,\tau_2^*,m,a^*)] \geq 0$, $[R_B(n,\tau_1^*,\tau_2^*,m,a^*) - R_B(n,\tau_1^*,\tau_2^*,m^*,a^*)] \geq 0$, $[R_B(n,\tau_1^*,\tau_2^*,m^*,a^*) - R_B(n^*,\tau_1^*,\tau_2^*,m^*,a^*)] \geq 0$. Therefore, $[R_B(n,\tau_1,\tau_2,m,a) - R_B(n^*,\tau_1^*,\tau_2^*,m^*,a^*)] \geq 0$. Hence, the proof is completed. (Proved)

4.5 Numerical study and comparisons

Here we illustrate the proposed method using an example and compute the optimum BSPAA. Also, we study the effect of cost components and hyperparameters on the optimum solution of BSPAA. The optimum solution is compared with CBSP and CBSPA.

4.5.1 An Illustrated Example

The hyperparameters of the prior are taken as $\alpha_1 = 3$, $\beta_1 = 1$ and $l = 10$. The cost coefficients of the loss function are taken as $C_0 = 2$, $C_1 = 3$ and $C_{11} = 2$. The values of other cost components are $C_a = 0.1$, $v_s = 0.2$, $C_s = 0.5$, $C_t = 5$ and $C_r = 30$. The optimal design $\boldsymbol{\zeta}^* = (n^*,\tau_1^*,\tau_2^*,m^*)$, and $E[\tau^*]$, $E[D^*]$ and the optimal Bayes risk R_B^* at $\boldsymbol{\zeta}^*$ of the proposed plan are given in Table 4.1.

Table 4.1: Optimal design for BSPAA

$(n^*,\tau_1^*,\tau_2^*,m^*)$	$E[\tau^*]$	$E[D^*]$	R_B^*
(3, 0.169, 0.238, 2)	0.220	2.013	27.704

For comparing the CBSP and CBSPA with BSPAA, we calculate the percentage of relative risk savings (RRS) of a BSPAA over CBSP and CBSPA, which are measured by RRS_1 and RRS_2, respectively and provided by

$$RRS_1 = 100 \times \frac{R_1^* - R_B^*}{R_1^*}\% \quad \text{and} \quad RRS_2 = 100 \times \frac{R_2^* - R_B^*}{R_2^*}\%, \tag{4.5.1}$$

where R_1^* and R_2^* are the optimal Bayes risk of CBSP and CBSPA, respectively. The optimal sampling plan (n_C^*,τ_{2C}^*), the expected time duration $E[\tau_C^*]$, the expected number of failures $E[D_C^*]$ and the optimal Bayes risk R_1^* corresponding to CBSP are provided in Table 4.2. The optimal sampling plan $(n_A^*,\tau_{1A}^*,\tau_{2A}^*)$, the expected time duration $E[\tau_A^*]$, the expected number of failures $E[D_A^*]$ and the optimal Bayes risk R_2^* of CBSPA, and for comparison RRS_1 and RRS_2 are given in Table 4.2.

Table 4.2: Optimal design for CBSP and CBSPA, and RRS of BSPAA over CBSP and CBSPA

CBSP				CBSPA				RRS	
(n_C^*, τ_{2C}^*)	$E[\tau_C^*]$	$E[D_C^*]$	R_1^*	$(n_A^*, \tau_{1A}^*, \tau_{2A}^*)$	$E[\tau_A^*]$	$E[D_A^*]$	R_2^*	RRS_1	RRS_2
(4, 0.193)	0.190	1.644	27.837	(3, 0.162, 0.238)	0.213	2.163	27.723	0.48%	0.07%

In Table 4.1, we see that the Bayes risk of our proposed model is 27.702. However, in Table 4.2, we see that the Bayes risk of CBSP and CBSPA are 27.837 and 27.723, respectively. This indicates that adaptive test may have a better impact than conventional ALT and non-accelerated life tests.

4.5.2 Effect of the parameters

Here we study the effect of cost components and hyperparameters on the optimum solution of BSPAA over CBSP and CBSPA. Optimal solutions of BSPAA, CBSP and CBPSA for different values of cost components and hyperparameters for $C_a = 0, 0.1, 0.2$ are provided in Tables 4.3-4.9. Also, for the comparison of BSPAA, CBSP and CBSPA, RRS_1 and RRS_2 are tabulated. The other values of the cost components and hyperparameters, which are not mentioned in the tables, are kept fixed.

In Table 4.3, the hyperparameters α_1 and β_1 vary while other values of the cost components and hyperparameters are fixed. For $(\alpha_1, \beta_1) = (2, 1.2), (2.5, 0.6), (2.5, 1.2), (3, 0.8), (3.5, 1)$, we observe that the optimal BRASPs represent no sampling cases and in these cases, the decision is taken based (4.3.2). It is observed that for fixed $\alpha_1 = 2$ and C_a, when β_1 increases from 0.6 to 1, the Bayes risk and expected time duration $E[\tau^*]$ of BSPAA decrease. This is due to the fact that when β_1 increases, the prior mean $\beta_1/(\alpha_1 - 1)$ increases. Also, it is seen that for fixed $\beta_1 = 1$ and $C_a = 0, 0.1, 0.2$, when α_1 increases from 2 to 3, the Bayes risk and expected time duration $E[\tau^*]$ of BSPAA increase. It is seen that when C_a increases, RRS_1 decreases. Therefore, for the higher values of c_a, CBSP is better than CBSPA and BSPAA is equivalent to CBSP.

In Table 4.4, we provide the effect of C_t when $c_a = 0, 0.1, 0.2$, $v_s = 0$ for fixed values of other parameters and coefficients. In Table 4.5, we provide the effect of C_t when $v_s = 0.2$. It is seen that when c_t increases, the expected time duration decreases as expected. When $C_a = 0$ and $v_s = 0$, it is seen that in the optimal sampling plan of BSPAA, $m^* = n^*$. This means that the optimal sampling plan of BSPAA is equivalent to the optimal sampling plan of CBSPA. Therefore $RRS_2 = 0\%$. This is due to the fact that when C_a and v_s are not incorporated into the Bayes risk, the number of failures in the life test does not depend on the sampling plan. In CBSPA, we get more information in a shorter time duration. Therefore, BSPAA is equivalent to CBSPA.

In Table 4.6, we provide the effect of C_r. For $C_r = 10, 20, 60$, the optimal BRASPs represent no sampling cases. For $C_r = 10, 20$, the Bayes risk is $R_B = C_r$, i.e, lot is rejected without life testing. For $C_r = 60$, $R_B = C_0 + C_1\alpha_1/\beta_1 + C_{11}\alpha_1(\alpha_1 + 1)/\beta_1{}^2$ and the lot is accepted without life testing.

In Table 4.7, we provide the effect of l. When l increases, the Bayes risk decreases for fixed C_a. This is due to the fact that when l increases, the mean accelerated factor ϕ increases. We get more information in a shorter period of time. Due to similar facts, the expected time duration decreases when l increases for fixed C_a. Since the CBSP does not depend on l, the optimal design parameters

Table 4.3: Optimal design parameters of BSPAA, CBSP, and CBSPA and the RRS of BSPAA over CBSP and CBSPA for different values of C_a, α and β.

			BSPAA				CBSP				CBSPA				RRS	
α	β	C_a	$(n^*, \tau_1^*, \tau_2^*, m^*)$	$E[\tau^*]$	$E[D^*]$	R_B^*	(n_C^*, τ_{2C}^*)	$E[\tau_C^*]$	$E[D_C^*]$	R_1^*	$(n_A^*, \tau_{1A}^*, \tau_{2A}^*)$	$E[\tau_A^*]$	$E[D_A^*]$	R_2^*	RRS_1	RRS_2
2	0.6	0	(4, 0.140, 0.185, 2)	0.180	2.214	27.167					(4, 0.142, 0.188)	0.179	2.480	27.196	1.26%	0.11%
2	0.6	0.1	(3, 0.202, 0.273, 2)	0.245	2.059	27.359	(4,0.217)	0.210	1.843	27.514	(3, 0.198, 0.278)	0.240	2.223	27.377	0.56%	0.07%
2	0.6	0.2	(3, 0.213, 0.280, 2)	0.251	2.048	27.497					(3, 0.211, 0.286)	0.248	2.219	27.544	0.06%	0.17%
2	0.8	0	(4, 0.090, 0.132, 2)	0.131	1.781	23.577					(4, 0.093, 0.136)	0.133	1.933	23.593	1.28%	0.07%
2	0.8	0.1	(3, 0.129, 0.196, 2)	0.186	1.727	23.811	(5, 0.220)	0.218	1.781	23.882	(3, 0.125, 0.195)	0.181	1.813	23.824	0.30%	0.05%
2	0.8	0.2	(5, 0.220, 0.220, 0)	0.218	1.781	23.882					(3, 0.136, 0.202)	0.188	1.799	24.046	0.00%	0.68%
2	1	0	(2, 0, 0.045, 1-2)	0.042	0.680	19.662					(2, 0, 0.045)	0.042	0.680	19.662	1.69%	0.00%
2	1	0.1	(2, 0, 0.045, 1-2)	0.042	0.680	19.862	(0, 0)	0	0	20	(2, 0, 0.045)	0.042	0.680	19.862	0.69%	0.00%
2	1	0.2	(0, 0, 0, 0)	0	0	20					(0, 0, 0)	0	0	20	0.00%	0.00%
2	1.2	0	(0,0,0,0)	0	0	15.333	(0,0)	0	0	15.333	(0, 0,0)	0	0	15.333	0.00%	0.00%
2.5	0.6	0	(1,0,0.212,1)	0	0	29.844	(0,0)	0	0	30	(0, 0,0)	0	0	20.40	0.00%	0.00%
2.5	0.8	0	(4, 0.130, 0.174, 2)	0.170	2.166	27.369					(4, 0.130, 0.176)	0.167	2.304	27.399	1.21%	0.11%
2.5	0.8	0.1	(3, 0.186, 0.256, 2)	0.234	2.037	27.572	(5, 0.263)	0.256	2.543	27.703	(3, 0.180, 0.258)	0.227	2.200	27.591	0.47%	0.07%
2.5	0.8	0.2	(5, 0.263, 0.263,, 0)	0.256	2.543	27.703					(3, 0.194, 0.267)	0.236	2.189	27.768	0.00%	0.23%
2.5	1	0	(3, 0.097, 0.169, 3)	0.157	1.818	24.156					(3, 0.097, 0.169)	0.157	1.818	24.156	1.38%	0.00%
2.5	1	0.1	(3, 0.113, 0.178, 2)	0.169	1.728	24.379	(3, 0.204)	0.198	1.114	24.493	(3, 0.108, 0.176)	0.164	1.804	24.391	0.47%	0.05%
2.5	1	0.2	(5, 0.208, 0.208, 0)	0.198	1.114	24.493					(2, 0.099, 0.166)	0.148	1.184	24.579	0.00%	0.35%
2.5	1.2	0	(0,0,0,0)	0	0	20.403	(0,0)	0	0	20.40	(0, 0,0)	0	0	20.40	0.00%	0.00%
3	0.8	0	(0,0,0,0)	0	0	30	(0,0)	0	0	30	(0, 0,0)	0	0	30	0.00%	0.00%
3	1	0	(4, 0.119, 0.162, 2)	0.159	2.121	27.497					(4, 0.118, 0.163)	0.156	2.578	27.526	1.22%	0.11%
3	1	0.1	(3, 0.169, 0.238, 2)	0.220	2.013	27.704	(4, 0.193)	0.190	1.644	27.837	(3, 0.162, 0.238)	0.213	2.163	27.723	0.48%	0.07%
3	1	0.2	(4, 0.193, 0.193, 0)	0.190	1.644	27.837					(2, 0.176 0.266)	0.215	1.516	27.891	0.00%	0.19%
3	1.2	0	(2, 0, 0.067, 2)	0.057	1.034	24.486					(2, 0, 0.067)	0.057	1.034	24.486	1.54%	0.00%
3	1.2	0.1	(2, 0, 0.067, 2)	0.057	1.034	24.686	(3, 0.186)	0.182	1.053	24.870	(2, 0, 0.067)	0.057	1.034	24.686	0.74%	0.00%
3	1.2	0.2	(3, 0.186, 0.186, 0)	0.182	1.053	24.870					(2, 0.043, 0.107)	0.095	1.090	24.879	0.00%	0.04%
3.5	1	0	(0, 0, 0, 0)	0	0	30	(0,0)	0	0	30	(0, 0, 0)	0	0	30	0.00%	0.00%
3.5	1.2	0	(4, 0.108, 0.151, 2)	0.149	2.092	27.569					(3, 0.099, 0.150)	0.142	1.787	27.588	1.19%	0.00%
3.5	1.2	0.1	(3, 0.152, 0.220, 2)	0.205	1.987	27.775	(4, 0.181)	0.179	1.554	27.900	(3, 0.145 0.219)	0.198	2.121	27.792	0.45%	0.06%
3.5	1.2	0.2	(4, 0.181, 0.181, 0)	0.179	1.554	27.900					(2, 0.145, 0.232)	0.191	1.478	27.929	0.00%	0.10%

Table 4.4: Optimal design parameters of BSPAA, CBSP, and CBSPA and the RRS of BSPAA over CBSP and CBSPA for different values of C_a and C_t when $v_s = 0.2$.

		BSPAA				CBSP				CBSPA				RRS	
C_t	C_a	$(n^*, \tau_1^*, \tau_2^*, m^*)$	$E[\tau^*]$	$E[D^*]$	R_B^*	(n_C^*, τ_{2C}^*)	$E[\tau_C^*]$	$E[D_C^*]$	R_1^*	$(n_A^*, \tau_{1A}^*, \tau_{2A}^*)$	$E[\tau_A^*]$	$E[D_A^*]$	R_2^*	RRS_1	RRS_2
1	0	(3, 0.525, 0.579, 3)	0.449	2.486	26.409	(3, 0.752)	0.532	2.442	26.457	(3, 0.525, 0.579)	0.449	2.486	26.409	0.18%	0.00%
1	0.1	(3, 0.752, 0.752, 0)	0.532	2.442	26.457					(3, 0.601, 0.639)	0.481	2.485	26.488	0.00%	0.12%
2	0	(4, 0.266, 0.321, 3)	0.300	2.808	26.805					(3, 0.339, 0.424)	0.351	2.442	26.811	0.53%	0.02%
2	0.1	(3, 0.413, 0.484, 3)	0.393	2.455	26.926	(4, 0.394)	0.360	2.523	26.948	(3, 0.413, 0.484)	0.393	2.455	26.926	0.08%	0.00%
2	0.2	(4, 0.394, 0.394, 0)	0.360	2.523	26.948					(3, 0.483, 0.539)	0.428	2.456	27.025	0.00%	0.28%
5	0	(4, 0.119, 0.162, 2)	0.159	2.121	27.497					(4, 0.118, 0.163)	0.156	2.578	27.526	1.22%	0.11%
5	0.1	(3, 0.169, 0.238, 2)	0.220	2.013	27.704	(4, 0.193)	0.190	1.644	27.837	(3, 0.162, 0.238)	0.213	2.163	27.723	0.48%	0.07%
5	0.2	(4, 0.193, 0.193, 0)	0.190	1.644	27.837					(2, 0.176 0.266)	0.215	1.516	27.891	0.00%	0.19%
7	0	(4, 0.106, 0.152, 2)	0.149	2.138	27.806					(4, 0.103, 0.150)	0.145	2.336	27.827	1.43%	0.08%
7	0.1	(3, 0.103, 0.153, 2)	0.146	1.726	28.049	(4, 0.185)	0.182	1.596	28.209	(3, 0.097, 0.151)	0.141	1.822	28.061	0.57%	0.04%
7	0.2	(4, 0.185, 0.185, 0)	0.182	1.596	28.209					(4, 0.155, 0.186)	0.181	2.243	28.493	0.00%	1.00%

Table 4.5: Optimal design parameters of BSPAA, CBSP, and CBSPA and the RRS of BSPAA over CBSP and CBSPA for different values of C_a and C_t when $v_s = 0$.

		BSPAA				CBSP				CBSPA				RRS	
C_t	C_a	$(n^*, \tau_1^*, \tau_2^*, m^*)$	$E[\tau^*]$	$E[D^*]$	R_B^*	(n_C^*, τ_{2C}^*)	$E[\tau_C^*]$	$E[D_C^*]$	R_1^*	$(n_A^*, \tau_{1A}^*, \tau_{2A}^*)$	$E[\tau_A^*]$	$E[D_A^*]$	R_2^*	RRS_1	RRS_2
1	0	(3, 0.518, 0.592, 3)	0.453	2.553	26.506					(3, 0.518, 0.592)	0.453	2.553	26.506	0.22%	0.00%
1	0.1	(3, 0.602, 0.656, 3)	0.487	2.547	26.585	(3, 0.790)	0.546	2.477	26.565	(3, 0.602, 0.656)	0.487	2.547	26.585	0.08%	0.00%
1	0.2	(3, 0.790, 0.790, 0)	0.546	2.477	26.565					(3, 0.691, 0.727)	0.520	2.548	26.653	0.00%	0.33%
2	0	(3, 0.340, 0.453, 3)	0.362	2.543	26.912					(3, 0.340, 0.453)	0.362	2.543	26.912	0.68%	0.00%
2	0.1	(3, 0.412, 0.503, 3)	0.399	2.529	27.028	(3, 0.713)	0.516	2.403	27.096	(3, 0.412, 0.503)	0.399	2.529	27.028	0.25%	0.00%
2	0.2	(3, 0.713, 0.713,0)	0.516	2.403	27.096					(3, 0.481, 0.553)	0.440	2.199	27.127	0.00%	0.11%
5	0	(3, 0.142, 0.239, 3)	0.206	2.266	27.683					(3, 0.142, 0.239)	0.206	2.266	27.683	1.71%	0.00%
5	0.1	(2, 0.149, 0.255, 2)	0.200	1.549	27.857	(3, 0.279)	0.259	1.566	28.165	(2, 0.149, 0.255)	0.200	1.549	27.857	1.09%	0.00%
5	0.2	(2, 0.172, 0.270, 2)	0.215	1.542	27.986					(2, 0.172, 0.270)	0.215	1.542	27.986	0.63%	0.00%
7	0	(2, 0.027, 0.129, 2)	0.098	1.405	28.009					(2, 0.027, 0.129)	0.098	1.405	28.009	2.36%	0.00%
7	0.1	(2, 0.034, 0.132, 2)	0.102	1.395	28.192	(4, 0.190)	0.187	1.626	28.686	(2, 0.034, 0.132)	0.102	1.395	28.192	1.172%	0.00%
7	0.2	(1, 0, 0.083, 1)	0.050	0.629	28.297					(1, 0, 0.083)	0.050	0.629	28.297	1.36%	0.00%

Table 4.6: Optimal design parameters of BSPAA, CBSP, and CBSPA and the RRS of BSPAA over CBSP and CBSPA for different values of C_a and C_r.

		BSPAA				CBSP				CBSPA				RRS	
C_r	C_a	$(n^*, \tau_1^*, \tau_2^*, m^*)$	$E[\tau^*]$	$E[D^*]$	R_B^*	(n_C^*, τ_{2C}^*)	$E[\tau_C^*]$	$E[D_C^*]$	R_1^*	$(n_A^*, \tau_{1A}^*, \tau_{2A}^*)$	$E[\tau_A^*]$	$E[D_A^*]$	R_2^*	RRS_1	RRS_2
10	0	(0,0,0,0)	0	0	10	(0, 0)	0	0	10	(0, 0, 0)	0	0	10	0.00%	0.00%
20	0	(0,0,0,0)	0	0	20	(0, 0)	0	0	20	(0, 0, 0)	0	0	20	0.00%	0.00%
30	0	(3, 0.200, 0.263, 3)	0.237	2.132	27.621					(3, 0.200,0.263)	0.237	2.132	27.621	0.91%	0.00%
30	0.1	(3,0.204,0.260,2)	0.241	1.965	27.700	(3, 0.272)	0.252	1.542	27.875	(3,0.207,0.269)	0.242	2.136	27.793	0.63%	0.33%
30	0.2	(3, 0.227, 0.272, 0)	0.252	1.542	27.875					(3, 0.241,0.327)	0.281	2.338	27.950	0.00%	0.27%
40	0	(5, 0.119, 0.154, 3)	0.153	2.604	31.537					(4, 0.119, 0.163)	0.158	2.344	31.550	0.92%	0.04%
40	0.1	(4, 0.148, 0.186, 3)	0.181	2.298	31.812	(5, 0.179)	0.178	1.949	31.830	(4, 0.141, 0.180)	0.175	2.339	31.828	0.06%	0.05%
40	0.2	(5, 0.179, 0.179, 0)	0.178	1.949	31.830					(4, 0.175, 0.219)	0.210	2.538	32.084	0.00%	0.79%
50	0	(5, 0.114, 0.148, 4)	0.146	2.649	34.043					(5, 0.114, 0.147)	0.145	2.639	34.045	0.83%	0.01%
50	0.1	(4, 0.115, 0.148, 3)	0.146	2.084	34.342	(4, 0.169)	0.167	1.496	34.329	(4,0.119, 0.157)	0.153	2.234	34.349	0.03%	0.02%
50	0.2	(4, 0.169, 0.169, 0)	0.167	1.496	34.329					(4, 0.113, 0.152)	0.148	2.229	34.344	0.00%	0.04%
60	0	(0,0,0,0)	0	0	35	(0, 0)	0	0	35	(0, 0, 0)	0	0	35	0.00%	0.00%

Table 4.7: Optimal design parameters of BSPAA, CBSP, and CBSPA and the RRS of BSPAA over CBSP and CBSPA for different values of C_a and l.

		BSPAA				CBSP				CBSPA				RRS	
l	C_a	$(n^*, \tau_1^*, \tau_2^*, m^*)$	$E[\tau^*]$	$E[D^*]$	R_B^*	(n_C^*, τ_{2C}^*)	$E[\tau_C^*]$	$E[D_C^*]$	R_1^*	$(n_A^*, \tau_{1A}^*, \tau_{2A}^*)$	$E[\tau_A^*]$	$E[D_A^*]$	R_2^*	RRS_1	RRS_2
10	0	(3, 0.200, 0.263, 3)	0.237	2.132	27.621					(3, 0.200,0.263)	0.237	2.132	27.621	0.91%	0.00%
10	0.1	(3,0.204,0.260,2)	0.241	1.965	27.700	(3, 0.272)	0.252	1.542	27.875	(3,0.207,0.269)	0.242	2.136	27.793	0.63%	0.33%
10	0.2	(3, 0.227, 0.272, 0)	0.252	1.542	27.875					(3, 0.241,0.327)	0.281	2.338	27.950	0.00%	0.27%
20	0	(3, 0.161, 0.203, 2)	0.193	1.755	27.462					(3, 0.158, 0.202)	0.190	1.862	27.469	1.48%	0.02%
20	0.1	(3, 0.174, 0.213, 2)	0.200	2.017	27.626	(3,0.272)	0.252	1.542	27.875	(3, 0.168, 0.210)	0.193	2.182	27.659	0.89%	0.12%
20	0.2	(3, 0.185, 0.221, 2)	0.208	1.994	27.783					(3, 0.179, 0.219)	0.202	2.176	27.845	0.33%	0.22%
50	0	(3, 0.164, 0.182, 2)	0.176	1.654	27.404					(3, 0.162, 0.181)	0.174	1.768	27.424	0.33	0.22%
50	0.1	(3, 0.174, 0.192, 2)	0.185	1.678	27.566	(3, 0.272)	0.252	1.542	27.875	(3, 0.171, 0.189)	0.181	1.769	27.613	1.11%	0.17%
50	0.2	(3, 0.185, 0.221, 2)	0.208	1.994	27.783					(3, 0.179, 0.219)	0.202	2.176	27.845	0.33%	0.22%
100	0	(3, 0.162, 0.172, 3)	0.167	1.508	27.407					(3, 0.162, 0.172)	0.167	1.508	27.407	1.68%	0.00%
100	0.1	(2, 0.117, 0.342, 2)	0.179	1.877	27.536	(3, 0.272)	0.252	1.542	27.875	(2, 0.117, 0.342)	0.179	1.877	27.536	1.22%	0.00%
100	0	(2, 0.131, 0.357, 2)	0.190	1.880	27.678					(2, 0.131, 0.357)	0.190	1.880	27.678	0.70%	0.00%

Table 4.8: Optimal design parameters of BSPAA, CBSP, and CBSPA and the RRS of BSPAA over CBSP and CBSPA for different values of C_a and C_0.

		BSPAA				CBSP				CBSPA				RRS	
C_0	C_a	$(n^*, \tau_1^*, \tau_2^*, m^*)$	$E[\tau^*]$	$E[D^*]$	R_B^*	(n_C^*, τ_{2C}^*)	$E[\tau_C^*]$	$E[D_C^*]$	R_1^*	$(n_A^*, \tau_{1A}^*, \tau_{2A}^*)$	$E[\tau_A^*]$	$E[D_A^*]$	R_2^*	RRS_1	RRS_2
0	0	(3, 0.154, 0.221, 2)	0.205	1.978	26.685					(3, 0.150, 0.222)	0.200	2.112	26.704	0.33%	0.07%
0	0.1	(4, 0.108, 0.149, 2)	0.147	2.060	26.480	(5, 0.235)	0.231	2.346	26.775	(3, 0.138, 0.214)	0.192	2.121	26.503	1.10%	0.09%
0	0.2	(5, 0.235, 0.235, 0)	0.231	2.346	26.775					(3, 0.163, 0.230)	0.208	2.096	26.898	0.00%	0.46%
1	0	(4, 0.113, 0.155, 2)	0.152	2.090	26.998					(4, 0.115, 0.157)	2.293	0.152	27.025	1.18%	0.10%
1	0.1	(3, 0.162, 0.230, 2)	0.213	1.996	27.203	(5, 0.243)	0.239	2.397	27.321	(3, 0.156, 0.230)	0.206	2.138	27.223	0.43%	0.07%
1	0.2	(5, 0.243, 0.243, 0)	0.239	2.397	27.321					(3, 0.170, 0.239)	0.215	2.125	27.414	0.00%	0.34%
2	0	(3, 0.200, 0.263, 3)	0.237	2.132	27.621					(3, 0.200,0.263)	0.237	2.132	27.621	0.91%	0.00%
2	0.1	(3,0.204,0.260,2)	0.241	1.965	27.700	(3, 0.272)	0.252	1.542	27.875	(3,0.207,0.269)	0.242	2.136	27.793	0.63%	0.33%
2	0.2	(3, 0.227, 0.272, 0)	0.252	1.542	27.875					(3, 0.241,0.327)	0.281	2.338	27.950	0.00%	0.27%
3	0	(4, 0.125, 0.170, 2)	0.167	2.165	27.978					(3, 0.116, 0.172)	0.160	1.892	28.004	1.15%	0.09%
3	0.1	(3, 0.175, 0.246, 2)	0.226	2.035	28.185	(4, 0.201)	0.197	1.691	28.306	(3, 0.167, 0.246)	0.218	2.191	28.203	0.43%	0.06%
3	0.2	(4, 0.201, 0.201, 0)	0.197	1.691	28.306					(3, 0.182, 0.256)	0.228	2.182	28.388	0.00%	0.29%
5	0	(3, 0.246, 0.338, 3)	0.288	2.374	29.547					(3,0.246, 0.338)	0.288	2.374	29.547	1.19%	0.00%
5	0.1	(2, 0.341, 0.439, 2)	0.323	1.663	29.656	(3, 0.354)	0.316	1.791	29.902	(2, 0.341, 0.439)	0.323	1.663	29.656	0.82%	0.00%
5	0.2	(2, 0.350, 0.445, 2)	0.327	1.661	29.738					(2, 0.350, 0.445)	0.327	1.661	29.738	0.55%	0.00%
10	0	(0,0,0,0)	0	0	30	(0,0)	0	0	30	(0,0,0)	0	0	30	0.00%	0.00%

Table 4.9: Optimal design parameters of BSPAA, CBSP, and CBSPA and the RRS of BSPAA over CBSP and CBSPA for different values of C_a and C_1.

		BSPAA				CBSP				CBSPA				RRS	
C_1	C_a	$(n^*, \tau_1^*, \tau_2^*, m^*)$	$E[\tau^*]$	$E[D^*]$	R_B^*	(n_C^*, τ_{2C}^*)	$E[\tau_C^*]$	$E[D_C^*]$	R_1^*	$(n_A^*, \tau_{1A}^*, \tau_{2A}^*)$	$E[\tau_A^*]$	$E[D_A^*]$	R_2^*	RRS_1	RRS_2
0	0	(4, 0.109, 0.150, 3)	0.147	2.219	23.462					(4, 0.108, 0.149)	0.145	2.245	23.464	0.02%	1.21%
0	0.1	(3, 0.096, 0.147, 2)	0.140	1.724	23.704	(5, 0.172)	0.223	1.407	23.750	(3, 0.094, 0.148)	0.139	1.814	23.718	0.19%	0.06%
0	0.2	(3, 0.172, 0.172 0)	0.223	1.407	23.750					(3, 0.105, 0.155)	0.146	1.795	23.944	0.00%	0.81%
1	0	(4, 0.127, 0.174, 4)	0.168	2.423	25.025					(4, 0.127, 0.174)	0.168	2.423	25.025	1.12%	0.00%
1	0.1	(3, 0.120, 0.178, 2)	0.168	1.846	25.250	(5, 0.198)	0.196	2.092	25.311	(3, 0.118, 0.178)	0.165	1.939	25.267	0.24%	0.07%
1	0.2	(5, 0.198, 0.198, 0)	0.196	2.092	25.311					(3, 0.129, 0.186)	0.173	1.934	25.478	0.00%	0.65%
2	0.1	(4, 0.101, 0.141, 2)	0.139	2.025	26.375					(3, 0.141, 0.209)	0.190	2.062	26.593	1.15%	0.08%
2	0.1	(3, 0.145, 0.209, 2)	0.195	1.940	26.575	(3, 0.272)	0.252	1.542	26.775	(3, 0.141, 0.209)	0.190	2.062	26.593	1.21%	0.09%
2	0.2	(3, 0.272, 0.272, 0)	0.252	1.542	26.775					(3, 0.163, 0.230)	0.208	2.122	26.898	0.00%	0.36%
3	0	(3, 0.200, 0.263, 3)	0.237	2.132	27.621					(3, 0.200,0.263)	0.237	2.132	27.621	0.91%	0.00%
3	0.1	(3,0.204,0.260,2)	0.241	1.965	27.700	(3, 0.272)	0.252	1.542	27.875	(3,0.207,0.269)	0.242	2.136	27.793	0.63%	0.33%
3	0.2	(3, 0.227, 0.272, 0)	0.252	1.542	27.875					(3, 0.241,0.327)	0.281	2.338	27.950	0.00%	0.27%
5	0	(0,0,0,0)	0	0	30	(0,0)	0	0	30	(0,0,0)	0	0	30	0.00%	0.00%

remain fixed for different values of l.

The effects of C_0 and C_1 are provided Tables 4.8 and 4.9, respectively. It is seen that when C_i, $i = 0, 1$ increases, the Bayes risk increases. For high values of C_i, $i = 0, 1$, the optimal BRASPs are no sampling cases, i.e, lot is rejected without life testing.

From Tables 4.3-4.9, it is observed that for sampling cases, if $n^* = n_C^* = n_A$, then $E[\tau_C^*] \leq E[\tau^*] \leq E[\tau_A]$ and also $E[D_C^*] \geq E[D^*] \geq E[D_A]$. This demonstrates that the ordering of the duration is the opposite of the ordering of the expected number of failures. Therefore, when C_t, v_s and C_a are incorporated into Bayes risk, BSPAA is better than the other two sampling plans, which can be observed from the values of RRS_1 and RRS_2. It is observed that, when $0 < m^* < n$, RRS_1 is always greater than 1.2% whereas RRS_2 is greater than 0.15% in some cases. As CBSP does not depend on C_a, only one value is provided in Tables 4.3-4.9 when C_a varies.

4.6 Data analysis

The proposed methodology of determining optimum RASP by Bayesian approach is illustrated by the data set of oil breakdown times of insulating fluid subjected to different constant levels of high voltage reported in Lawless [59]. For the sake of illustration, 30kV is assumed to be the normal use voltage and 36kV is used as the accelerated voltage. The data is given in Table 4.10. Zheng and Fang [134] analyzed the data and showed that the exponential distribution fits well.

Table 4.10: The data set of oil breakdown times

Normal conditions (30kV):
7.74, 17.05, 20.46, 21.02, 22.66, 43.40, 47.30, 139.07, 144.12, 175.88, 194.90
Accelerated stress condition (36kV):
1.97, 0.59, 2.58, 1.69, 2.71, 25.50, 0.35, 0.99, 3.99, 3.67, 2.07, 0.96, 5.35, 2.90, 13.77

The mean oil breakdown time of insulating fluid at normal conditions and accelerated conditions are 75.78 and 4.61, respectively. The MLEs of λ and ϕ are $\hat{\lambda} = 0.013$ and $\hat{\phi} = 16.45$, respectively. For computational purposes, we assume that λ has gamma prior with mean 0.013. The hyperparameters of gamma prior of λ are taken as $\alpha_1 = 1.3$ and $\beta_1 = 100$, and the hyperparameter of uniform prior of ϕ is taken as $l = 30$. The cost components of the manufacturer are taken as $C_a = 0.1$, $v_s = 0.2$, $c_s = 0.4$, $C_t = 0.05$, $C_0 = 2$, $C_1 = 700$, $C_{11} = 80000$ and $C_r = 30$. The optimum RASP is obtained as $(n^*, \tau_1^*, \tau_2^*, m^*) = (4, 18.29, 28.29, 2)$. Next, we have to carry out a life test under the optimum RASP. For illustration, we generate type-I adaptive step-stress data based on the optimum life testing plan $(n^*, \tau_1^*, \tau_2^*, m^*) = (4, 18.29, 28.29, 2)$ from the exponential distribution with $\lambda = 0.013$ and $\phi = 16.45$. The data sets $(\boldsymbol{y}, d_1, d_2) = (x_{(1)}, \cdots, x_{(d)}, d_1, d_2)$ and the corresponding decisions about the lot acceptance or rejection a^* are given in Table 4.11 for illustration purposes. Also, in Table 4.11, the decision of changing the stress at τ_1^*, w_1 and w_2 are provided.

The BRASP can be illustrated as follows. Four items are put on life test at the stress level 30kV. After time $\tau_1 = 18.29$, the number of failures is observed. If the number of failures is less

Table 4.11: Simulated data sets and corresponding decision about the lot

i	$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	d_1	Change the stress	d_2	w_1	w_2	e	a^*
1	18.76	19.58	20.00	23.56	0	Yes	4	73.16	8.75	4.33	0
2	6.83	7.97	24.72	-	2	No	1	51.39	16.43	54.67	0
3	10.20	19.44	20.02	-	1	Yes	2	65.07	12.88	-2.13	1
4	5.97	7.90	-	-	2	No	0	50.45	20.00	24.62	0
5	15.62	18.98	19.74	21.78	1	Yes	3	70.49	5.64	19.37	0
6	20.09	20.58	21.81	23.49	0	Yes	4	73.16	12.81	-3.31	1
7	14.84	21.55	21.75	28.11	1	Yes	3	69.71	16.54	-1.57	1

than 2, the stress is increased to 36kV and the test continues up to $\tau_2 = 28.29$. If the number of failures is greater than or equal to 2, the stress is unchanged and the test continues up to $\tau_2 = 28.29$ at the stress level 30kV. Then we calculate $e = \varphi(\boldsymbol{y}, \boldsymbol{d}) - C_r$, for $i = 1, \ldots, 7$. Using (4.3.1), we can say that if $e < 0$, then $a^* = 1$ and the lot is accepted. If $e \geq 0$, then $a^* = 0$ and the lot is rejected. Therefore, Table 4.11 summarizes the resulting acceptance or rejection decisions for each simulated data set, which demonstrates the practical operation of the proposed BRASP.

4.7 Extension to Weibull lifetime distribution

Here, consider that the proposed method can be extended to other lifetime distributions, such as the Weibull distribution. Let T_i be the lifetime of a unit at stress level s_i with PDF $f(\cdot \mid \boldsymbol{\theta}_i)$ and CDF $F(\cdot \mid \boldsymbol{\theta}_i)$, for $i = 0, 1$. We develop BRASP for ASSSPALT under the assumption that product lifetimes follow Weibull distribution at any stress level. Then $T_i \sim \text{Wei}(\nu, \lambda_i)$, for $i = 0, 1$. The scale parameters λ_0 and λ_1 are connected by the relation $\lambda_1 = \phi\lambda_0$, where $\phi > 1$ is the acceleration constant. We now consider $\lambda_0 = \lambda$. The vector of parameters is denoted by vector $\boldsymbol{\theta} = (\lambda, \phi, \nu)$. Now using CEM, CDF of the lifetime of an item under ASSSPALT X is given by

$$F(t \mid \boldsymbol{\theta}) = \begin{cases} F_0(t \mid \boldsymbol{\theta}_0), & \text{if } \ t < t_1, \\ [F_0(t \mid \boldsymbol{\theta}_0)]^{1-\delta}[F_1(t - h_1 \mid \boldsymbol{\theta}_1]^{\delta}, & \text{if } \ t \geq t_1, \end{cases}$$

where h_1 is the solution of the equation $F_1(\tau_1 - h_1 \mid \boldsymbol{\theta}_1) = F_0(\tau_1 \mid \boldsymbol{\theta_0})$. PDF of X is given by

$$f(t \mid \boldsymbol{\theta}) = \begin{cases} f_0(t \mid \boldsymbol{\theta}_0), & \text{if } \ t < t_1, \\ [f_0(t \mid \boldsymbol{\theta}_0)]^{1-\delta}[f_1(t - h_1 \mid \boldsymbol{\theta}_1]^{\delta}, & \text{if } \ t \geq t_1, \end{cases}$$

Therefore

$$F(t \mid \boldsymbol{\theta}) = \begin{cases} 1 - \exp\left[-\left(\lambda_0 t\right)^{\nu}\right], & \text{if } \ t < t_1, \\ \left[1 - \exp\left\{-\left(\lambda_0 t\right)^{\nu}\right\}\right]^{1-\delta}\left[1 - \exp\left\{-\left(\lambda_0 t_1 + \lambda_1 (t - t_1)\right)^{\nu}\right\}\right]^{\delta}, & \text{if } \ t \geq t_1, \end{cases}$$

Based on observed data $(\boldsymbol{y}, d_1, d_2)$, the likelihood functions under two cases are obtained as follows.
Case 1: The stress does not change. i.e, $D_1 \geq m$,

$$L_0(\boldsymbol{\theta} \mid (\boldsymbol{y}, d_1, d_2)) = \prod_{i=1}^{d} f_0(x_{(i)} \mid \boldsymbol{\theta}_0)[1 - F_0(\tau_2 \mid \boldsymbol{\theta}_0)]^{n-d}$$

Case 2: The stress increases at t_1. i.e, $D_1 < m$,

$$L_1(\boldsymbol{\theta} \mid (\boldsymbol{y}, d_1, d_2)) = \prod_{i=1}^{d_1} f_0(x_{(i)} \mid \boldsymbol{\theta}_0) \prod_{i=1}^{d_2} f_1(x_{(i+d_1)} - h_1) \mid \boldsymbol{\theta}_1) \times [1 - F_0(\tau_1 \mid \nu, \lambda_0)]^{n-d_1} [1 - F_1(\tau_2 - h_1 \mid \boldsymbol{\theta}_1)]^{n-d}$$

The full likelihood function under ASSSPALT is

$$L(\boldsymbol{\theta} \mid (\boldsymbol{y}, d_1, d_2)) = [L_0(\boldsymbol{\theta} \mid (\boldsymbol{y}, d_1, d_2)]^{1-\delta} [L_1(\boldsymbol{\theta} \mid (\boldsymbol{y}, d_1, d_2))]^{\delta} \tag{4.7.1}$$

Let $p(\boldsymbol{\theta})$ be the prior distribution of $\boldsymbol{\theta}$. It is assumed that λ, ϕ and ν are independent and $p_1(\cdot)$, $p_2(\cdot)$ and $p_3(\cdot)$ be their PDFs, respectively. Let λ and ν follow $G(\alpha_1, \beta_1)$ and $G(\alpha_2, \beta_2)$ respectively and ϕ follows $\mathcal{U}(1, l]$. The posterior distribution of $\boldsymbol{\theta}$ can be derived using (4.1.4) and (4.7.1). The loss function for the Weibull distribution can be taken as $h(\lambda, \nu) = NC[1 - \exp\{-(\lambda t)^{\nu}\}]$. Therefore, the Bayes decision function can be written as

$$a^*(\boldsymbol{y}, d_1, d_2 \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{if } \varphi(\boldsymbol{y}, d_1, d_2) \le C_r, \\ 0, & \text{otherwise}, \end{cases} \tag{4.7.2}$$

where $\varphi(\boldsymbol{y}, d_1, d_2) = \int_{\boldsymbol{\theta}} h(\lambda, \nu)\ p(\boldsymbol{\theta} \mid \boldsymbol{y}, d_1, d_2)\ d\boldsymbol{\theta}$. It may be noted that $\varphi(\boldsymbol{y}, d_1, d_2)$ cannot be obtained analytically. However, it can be computed numerically through either grid-based integration or Markov chain Monte Carlo (MCMC) sampling.

To determine an optimal BRASP, we need to evaluate the Bayes risk for the designing parameter $\boldsymbol{\zeta} = (n, \tau_1, \tau_2, m)$ for the Weibull distribution. The Bayes risk cannot be solved analytically. To compute the Bayes risk, we use the Monte Carlo simulation technique. Therefore, we need to generate the lifetime data under ASSSPALT for Weibull distribution. Using the data, the posterior distribution is computed and the Bayes decision function given in (4.7.2) is executed. Using the Bayes decision function, the total loss is calculated for this data. Repeat this procedure for a large number of simulated generated data and taking the average over these repetitions, we get the Bayes risk. The optimal plan $\boldsymbol{\zeta}^* = (n^*, \tau_1^*, \tau_2^*, m^*)$ is the design that minimizes the Bayes risk.

4.8 Discussion

This chapter considered designing BRASP approach based on ASSSPALT for type-I censored data. It is observed that BSPAA generally leads to lower Bayes risk and higher efficiency than CBSP and CBSPA. BSPAA is a generalized model that encompasses both CBSP and CBSPA, thereby allowing direct comparison between non-accelerated and accelerated Bayesian reliability acceptance plans. The analysis reveals that when the additional cost for increasing the stress level is relatively high and the time-related cost is relatively low, CBSP becomes more advantageous than CBSPA. In contrast, when the acceleration cost $C_a = 0$ and the salvage value $v_s = 0$, CBSPA consistently outperforms the non-accelerated plan for sufficiently large values of the time-cost parameter C_t. This clearly demonstrates that the value of acceleration cost, salvage cost, and time cost plays a crucial role in determining which plan offers the minimum Bayes risk.

In Section 4.7, the extension demonstrates that ASSSPALT framework is not limited to exponential distribution. It can accommodate more general lifetime models such as the Weibull distribution. Therefore, it enhances the applicability of the proposed BRASP in real-world reliability studies. A more detailed investigation of lifetime distributions other than the exponential distribution within this adaptive framework is left for future work.

4.A Appendix

4.A.1 Expression of term of Bayes risk

Expected number of failures ($E[D]$):

The total number of failures during the life test is defined as $D = D_1 + D_2$. Expected number of failures is given by

$$E[D] = \int_{\boldsymbol{\theta}} E_{D|\boldsymbol{\theta}}[D_1 + D_2 \mid \boldsymbol{\theta}]\ p(\boldsymbol{\theta})\ d\boldsymbol{\theta},$$

where

$$\begin{aligned} E_{D|\boldsymbol{\theta}}[D_1 + D_2 \mid \boldsymbol{\theta}] &= E[D_1 \mid \boldsymbol{\theta}] + E[D_2 \mid \boldsymbol{\theta}] \\ &= E_{D_1|\boldsymbol{\theta}}[D_1 \ |\boldsymbol{\theta}] + \sum_{d_1=0}^{n} E_{D_2|D_1,\boldsymbol{\theta}}[D_2 \mid d_1, \boldsymbol{\theta}] P(D_1 = d_1 \mid \boldsymbol{\theta}) \end{aligned} \tag{4.A.1}$$

Now, we know, the distribution of D_1 follows binomial distribution with parameter $(n, F(\tau_1 \mid \boldsymbol{\theta}))$. Then $E_{D_1|\boldsymbol{\theta}}[D_1 \mid \boldsymbol{\theta}] = nF(\tau_1 \mid \boldsymbol{\theta})$. Also, we know that the conditional distribution of $D_2 \mid D_1 = d_1, \boldsymbol{\theta}$ follows binomial distribution with parameter $\left((n-d_1), \frac{F(\tau_2 \mid \boldsymbol{\theta}) - F(\tau_1 \mid \boldsymbol{\theta})}{1 - F(\tau_1 \mid \boldsymbol{\theta})}\right)$. Then

$$E_{D_2|D_1,\boldsymbol{\theta}}[D_2 \mid d_1, \boldsymbol{\theta}] = (n - d_1)\frac{F(\tau_2 \mid \boldsymbol{\theta}) - F(\tau_1 \mid \boldsymbol{\theta})}{1 - F(\tau_1 \mid \boldsymbol{\theta})}$$

. Therefore, (4.A.1) can be written as

$$\begin{aligned} & E_{D_1|\boldsymbol{\theta}}[D_1 \ |\boldsymbol{\theta}] + \sum_{d_1=0}^{n} E_{D_2|D_1,\boldsymbol{\theta}}[D_2 \mid d_1, \boldsymbol{\theta}] P(D_1 = d_1 \mid \boldsymbol{\theta}) \\ &= n[1 - \exp(-\lambda\tau_1)] + \sum_{d_1=0}^{n-1} (n - d_1)\frac{F(\tau_2 \mid \boldsymbol{\theta}) - F(\tau_1 \mid \boldsymbol{\theta})}{1 - F(\tau_1 \mid \boldsymbol{\theta})}\binom{n}{d_1}(F(\tau_1 \mid \boldsymbol{\theta}))^{d_1}(1 - F(\tau_1 \mid \boldsymbol{\theta}))^{n-d_1} \\ &= n[1 - \exp(-\lambda\tau_1)] + n\exp(-\lambda\tau_1) - \left[\sum_{d_1=0}^{m-1}(n - d_1)\exp[-\lambda(\tau_1 + \phi(\tau_2 - \tau_1)]\right. \\ &\qquad \left. + \sum_{d_1=m}^{n-1}(n - d_1)\exp(-\lambda\tau_2)\right]\binom{n}{d_1}(1 - \exp(-\lambda\tau_1))^{d_1}\exp[-\lambda(n - d_1 - 1)\tau_1] \\ &= n - \sum_{d_1=0}^{m-1}\sum_{i=0}^{d_1}(-1)^{d_1-i}\binom{d_1}{i}(n - d_1)\exp[-\lambda((n-i)\tau_1 + \phi(\tau_2 - \tau_1)] \\ &\qquad - \sum_{d_1=m}^{n-1}\sum_{i=0}^{d_1}(-1)^{d_1-i}\binom{d_1}{i}(n - d_1)\exp[-\lambda((n-i)\tau_1 + (\tau_2 - \tau_1))]. \end{aligned}$$

Expected number of items continued to higher stress levels after τ_1 (n_{as}):

Let n_{as} be the expected number of items continued to higher stress levels after τ_1. Then

$$\begin{aligned}
n_{as} &= E_{\boldsymbol{\theta}}[E_{D_1 \mid \boldsymbol{\theta}}[\delta(n-D_1) \mid \boldsymbol{\theta}]] \\
&= \int_{\lambda=0}^{\infty} \sum_{d_1=0}^{m-1} (n-d_1)\binom{n}{d_1}(1-\exp(-\lambda\tau_1))^{d_1} \exp(-(n-d_1)\tau_1)\; p_1(\lambda) d\lambda \\
&= \sum_{d_1=0}^{m-1} \sum_{i=0}^{d_1} (n-d_1)\binom{n}{d_1}\binom{d_1}{i}(-1)^{d_1-i} \int_{\lambda=0}^{\infty} \exp(-(n-i)\tau_1)\; p_1(\lambda)\; d\lambda
\end{aligned}$$

Expected test duration ($E[\tau]$):

The duration of the test under type-I censoring is defined as $\tau = \min\{X_{(n)}, \tau_2\}$.

$$\begin{aligned}
E_{\tau \mid \boldsymbol{\theta}}[\tau \mid \boldsymbol{\theta}] =& E_{X_{(n)}}[\min\{X_{(n)}, \tau_2\} \mid \boldsymbol{\theta}] \\
=& E_{X_{(n)} \mid X_{(n)} \geq \tau_2, \boldsymbol{\theta}}[\tau_2 \mid X_{(n)} \geq \tau_2, \boldsymbol{\theta}] P(X_{(n)} > \tau_2 \mid \boldsymbol{\theta}) \\
&+ E_{X_{(n)} \mid X_{(n)} < \tau_2, \boldsymbol{\theta}}[X_{(n)} \mid X_{(n)} < \tau_2, \boldsymbol{\theta}][1 - P(X_{(n)} > \tau_2 \mid \boldsymbol{\theta})] \\
=& \tau_2(1 - F_{X_{(n)}}(\tau_2 \mid \boldsymbol{\theta}))] - \int_{t=0}^{\tau_2} t \frac{\partial}{\partial t}(1 - F_{X_{(n)}}(t \mid \boldsymbol{\theta}))]\; dt \\
=& \tau_2(1 - F_{X_{(n)}}(\tau_2 \mid \boldsymbol{\theta})) - \tau_2(1 - F_{X_{(n)}}(\tau_2 \mid \boldsymbol{\theta})) + \int_{t=0}^{\tau_2}(1 - F_{X_{(n)}}(t \mid \boldsymbol{\theta}))\; dt \\
=& \int_{t=0}^{\tau_2}(1 - F_{X_{(n)}}(t \mid \boldsymbol{\theta}))\; dt
\end{aligned}$$

When $m > 0$, for $0 < t < \tau_1$,

$$\begin{aligned}
(1 - F_{X_{(n)}}(t \mid \boldsymbol{\theta})) &= 1 - P(X_{(n)} < t \mid \boldsymbol{\theta}) \\
&= 1 - P(\max\{Y_1, \ldots, Y_n\} \leq t \mid \boldsymbol{\theta}) \\
&= 1 - [F(t \mid \boldsymbol{\theta})]^n \\
&= \sum_{i=1}^{n} \binom{n}{i}(-1)^{i+1} \exp(-\lambda i t)
\end{aligned}$$

and for $\tau_1 < t < \tau_2$, the distribution $X_{(n)}$ depends on D_1.

$$\begin{aligned}
&(1 - F_{X_{(n)}}(t \mid \boldsymbol{\theta})) \\
&= \sum_{d_1=0}^{n-1} P(X_{(n)} > t \mid D_1 = d_1, \boldsymbol{\theta}) P(D_1 = d_1 \mid \boldsymbol{\theta}) \\
&= \sum_{d_1=0}^{n-1} \left[1 - P(X_{(n)} < t \mid D_1 = d_1, \boldsymbol{\theta})\right][P(D_1 = d_1 \mid \boldsymbol{\theta})] \\
&= \sum_{d_1=0}^{n-1} [1 - P((n-d_1) \text{ items fail in the interval } (\tau_1, t) \mid \boldsymbol{\theta})] \binom{n}{d_1} [F(\tau_1 \mid \boldsymbol{\theta})]^{d_1} [1 - F(\tau_1 \mid \boldsymbol{\theta})]^{n-d_1} \\
&= \sum_{d_1=0}^{n-1} \left[1 - \left(1 - \frac{1 - F(t \mid \boldsymbol{\theta})}{1 - F(\tau_1 \mid \boldsymbol{\theta})}\right)^{n-d_1}\right] \binom{n}{d_1} \sum_{j=0}^{d_1} \binom{d_1}{j} (-1)^j (1 - F(\tau_1 \mid \boldsymbol{\theta}))^{n-d_1+j}
\end{aligned}$$

$$= -\sum_{d_1=0}^{n-1}\sum_{j=0}^{d_1}\sum_{k=1}^{n-d_1}\binom{n}{d_1}\binom{d_1}{j}\binom{n-d_1}{k}(-1)^{j+k}\left(\frac{1-F(t\mid\boldsymbol{\theta})}{1-F(\tau_1\mid\boldsymbol{\theta})}\right)^k(1-F(\tau_1\mid\boldsymbol{\theta}))^{n-d_1+j}$$

$$= -\sum_{d_1=0}^{m-1}\sum_{j=0}^{d_1}\sum_{k=1}^{n-d_1}\binom{n}{d_1}\binom{d_1}{j}\binom{n-d_1}{k}(-1)^{j+k}\exp[-\lambda((n-d_1+j)\tau_1+\phi k(t-\tau_1))]$$

$$-\sum_{d_1=m}^{n-1}\sum_{j=0}^{d_1}\sum_{k=1}^{n-d_1}\binom{n}{d_1}\binom{d_1}{j}\binom{n-d_1}{k}(-1)^{j+k}\exp[-\lambda((n-d_1+j)\tau_1+k(t-\tau_1))].$$

When $m=0$, the stress does not change after τ_1. Therefore, the hazard rate is unchanged up to τ_2. So, when $m=0$,

$$(1-F_{X_{(n)}}(t\mid\boldsymbol{\theta}))=\sum_{i=1}^{n}\binom{n}{i}(-1)^{i+1}\exp(-\lambda i t)\qquad \text{for } 0<t<\tau_2.$$

Therefore

$$E[\tau]=\int_{\boldsymbol{\theta}}E_{\tau\mid\boldsymbol{\theta}}[\tau\mid\boldsymbol{\theta}]\;p(\boldsymbol{\theta})\;d\boldsymbol{\theta}$$
$$=\int_{\boldsymbol{\theta}}\int_{t=0}^{\tau_1}(1-F_{X_{(n)}}(t\mid\boldsymbol{\theta}))\;p(\boldsymbol{\theta})\;dt\;d\boldsymbol{\theta}+\int_{\boldsymbol{\theta}}\int_{t=\tau_1}^{\tau_2}(1-F_{X_{(n)}}(t\mid\boldsymbol{\theta}))\;p(\boldsymbol{\theta})\;dt\;d\boldsymbol{\theta}.$$

Expression of $g_{\boldsymbol{X}\mid\boldsymbol{\theta}}(\boldsymbol{x}\mid\boldsymbol{\theta})$

Theorem 4.3 *(Balakrishnan et al. [10]) The conditional joint PDF of (W_1,W_2,D_1,D_2) given parameters (λ,ϕ) is expressed as*

$$g_{(W_1,W_2,D_1,D_2)\mid\lambda,\phi)}(w_1,w_2,d_1,d_2\mid\lambda,\phi)$$
$$=\sum_{j=0}^{d_1}\sum_{k=0}^{d_2}A_{d_1,d_2,j,k}q_1(\lambda,j)q_2(\lambda,\phi,j,k)\begin{cases}\mathbb{1}_{\{n\tau_1\}}(w_1)\mathbb{1}_{\{n(\tau_2-\tau_1)\}}(w_2), & \text{if } d_1=0,d_2=0,\\ \mathbb{1}_{\{n\tau_1\}}(w_1)\;\gamma\left(w_2-\tau_{d_1,d_2,k},d_2,\phi^\delta\lambda\right), & \text{if } d_1=0,d_2>0,\\ \gamma\left(w_1-\tau_{d_1,j},d_1,\lambda\right)\mathbb{1}_{\{n(\tau_2-\tau_1)\}}(w_2), & \text{if } d_1>0,d_2=0,\\ \gamma\left(w_1-\tau_{d_1,j},d_1,\lambda\right)\gamma\left(w_2-\tau_{d_1,d_2,k},d_2,\phi^\delta\lambda\right), & \text{if } d_1>0,d_2>0,\end{cases}$$

where $\tau_{d_1,j}=(n-d_1+j)\tau_1$, $\tau_{d_1,d_2,k}=(n-d-k)(\tau_2-\tau_1)$, $A_{d_1,d_2,j,k}=\frac{n!}{d_1!d_2!(n-d)!}\binom{d_1}{j}\binom{d_2}{k}(-1)^{(j+k)}$, $q_1(\lambda,j)=\exp(-\lambda(n-d_1+j)\tau_1)$ and $q_2(\lambda,\phi,j,k)=\exp(-\phi^\delta\lambda(n-d+k)(\tau_2-\tau_1))$, the functions $\gamma(x,\alpha,\beta)$ and $\mathbb{1}y(x)$ are defined in Result 3.1.

4.A.2 Proof of the monotonicity Property of Bayes decision function :

Proof of Theorem 4.1: From (4.1.3), the likelihood function is given by

$$L(\lambda\mid(w_1,w_2,d_1,d_2),\mathbf{q})\propto\lambda^d\phi^{d_2\delta}\exp\left[-\lambda(w_1+\phi^\delta w_2)\right].$$

Note that

$$\varphi_1(w_1,w_2,d_1,d_2,\phi)=\int_\lambda h(\lambda)p_1(\lambda\mid\phi,(w_1,w_2,d_1,d_2))d\lambda$$
$$=\frac{\int_0^\infty h(\lambda)\lambda^{d_1+d_2}\phi^{d_2}\exp(-\lambda(w_1+\phi w_2))p_1(\lambda)p_2(\phi)d\lambda}{\int_0^\infty \lambda^{d_1+d_2}\phi^{d_2}\exp(-\lambda(w_1+\phi w_2))p_1(\lambda)p_2(\phi)d\lambda}\qquad(4.A.2)$$

1. For fixed (w_2, d_1, d_2), consider two points w_{11} and w_{12} such that $w_{11} < w_{12}$. Then it is sufficient to prove that $\varphi(w_{11}, w_2, d_1, d_2) > \varphi(w_{12}, w_2, d_1, d_2)$ when $h(\lambda)$ is increasing function in λ. Let $h_{11}(\lambda) = \lambda^d \exp(-\lambda w_{11})$ and $h_{12}(\lambda) = \lambda^d \exp(-\lambda w_{12})$. Note that $h_{21}(\lambda) = h(\lambda)\int_\phi \phi^{d_2} \exp(-\lambda\phi w_2)p(\boldsymbol{\theta})d\phi$ and $h_{22}(\lambda) = \int_\phi \phi^{d_2} \exp(-\lambda\phi w_2)p(\boldsymbol{\theta})d\phi$ if $\delta = 1$ and $h_{21}(\lambda) = h(\lambda)\exp(-\lambda w_2)$ and $h_{22}(\lambda) = \exp(-\lambda w_2)$ if $\delta = 0$. Thus from (4.A.2), we get
$$\varphi(w_{11}, w_2, d_1, d_2) = \frac{\int_\lambda h_{11}(\lambda)h_{21}(\lambda)d\lambda}{\int_\lambda h_{11}(\lambda)h_{22}(\lambda)d\lambda}$$
and
$$\varphi(w_{12}, w_2, d_1, d_2) = \frac{\int_\lambda h_{12}(\lambda)h_{21}(\lambda)d\lambda}{\int_\lambda h_{12}(\lambda)h_{22}(\lambda)d\lambda}.$$
Assume all integrals are finite. Clearly, $h_{12}(\lambda)$ and $h_{22}(\lambda)$ are non-negative functions of λ. Now, $h_{11}(\lambda)/h_{12}(\lambda) = \exp[\lambda(w_{12} - w_{11})]$ and $h_{21}(\lambda)/h_{22}(\lambda) = h(\lambda)$ which is an increasing function in λ for $\lambda > 0$. By Theorem 2 in Wijsman [120], we get $\varphi(w_{11}, w_2, d_1, d_2) > \varphi(w_{12}, w_2, d_1, d_2)$. Therefore, $\varphi(w_{11}, w_2, d_1, d_2)$ is decreasing in w_1 for fixed (w_2, d_1, d_2).

2. For $\delta = 0$, the proof that $\phi(w_1, w_2, d_1, d_2)$ is decreasing in w_2 for fixed (w_1, d_1, d_2) is similar to that in part (1). Hence, we omit the proof.
For $\delta = 1$, the joint posterior distribution of λ and ϕ is given by
$$p(\lambda, \phi \mid (w_1, w_2, d_1, d_2)) = \frac{\lambda^d \phi^{d_2} \exp[-\lambda(w_1 + \phi w_2)]\ p_1(\lambda)\ p_2(\phi)}{K(w_1, w_2, d_1, d_2)},$$
where
$$\begin{aligned} K(w_1, w_2, d_1, d_2) =& \int_\lambda \int_\phi \lambda^d \phi^{d_2} \exp[-\lambda(w_1 + \phi w_2)]\ p_1(\lambda)\ p_2(\phi)\ d\phi\ d\lambda \\ =& E_{(\lambda,\phi)}[\lambda^d \phi^{d_2} \exp[-\lambda(w_1 + \phi w_2)]]. \end{aligned}$$
The marginal posterior distribution of ϕ is
$$p_2(\phi \mid (w_1, w_2, d_1, d_2)) = \frac{\int_\lambda \lambda^d \phi^{d_2} \exp[-\lambda(w_1 + \phi w_2)]\ p_1(\lambda)\ p_2(\phi)\ d\lambda}{K(w_1, w_2, d_1, d_2)}.$$
Therefore, the posterior conditional distribution of $\lambda|\ \phi$ is given by
$$p_1(\lambda \mid \phi, (w_1, w_2, d_1, d_2)) = \frac{\lambda^d \phi^{d_2} \exp[-\lambda(w_1 + \phi w_2)]\ p_1(\lambda)\ p_2(\phi)}{\int_\lambda \lambda^d \phi^{d_2} \exp[-\lambda(w_1 + \phi w_2)]\ p_1(\lambda)\ p_2(\phi)\ d\lambda}.$$
The posterior expectation of $h(\lambda)$ can be written as
$$\varphi(w_1, w_2, d_1, d_2) = \int_\phi \int_\lambda h(\lambda)\ p_1(\lambda \mid \phi, (w_1, w_2, d_1, d_2))\ p_2(\phi \mid (w_1, w_2, d_1, d_2))\ d\lambda\ d\phi.$$
To prove the decreasing property of $\varphi(w_1, w_2, d_1, d_2)$ w.r.t. w_2 for fixed (w_1, d_1, d_2), we need the following two lemmas:

Lemma 4.1 *If λ follows gamma distribution, then for fixed (d_1, d_2, w_1) and for w_{21}, w_{22} with $w_{21} < w_{22}$, the likelihood ratio*
$$\frac{p_2(\phi \mid w_1, w_{21}, d_1, d_2)}{p_2(\phi \mid w_1, w_{22}, d_1, d_2)}$$
is increasing in ϕ.

Proof: If λ follows gamma distribution with parameter (α, β), then the marginal posterior of ϕ can be written as

$$p_2(\phi \mid (w_1, w_2, d_1, d_2)) = \frac{\beta^\alpha \Gamma(\alpha + d)\phi^{d_2}\ p_2(\phi)\ d\lambda}{(w_1 + \phi w_2)^\beta \Gamma(\alpha) K(w_1, w_2, d_1, d_2)}.$$

$$\frac{p_2(\phi \mid (w_1, w_{21}, d_1, d_2)}{p_2(\phi \mid (w_1, w_{22}, d_1, d_2)} = \frac{K(w_1, w_{22}, d_1, d_2)}{K(w_1, w_{21}, d_1, d_2)} \frac{(w_1 + \phi w_{22})^\beta}{(w_1 + \phi w_{21})^\beta}$$

It is enough to prove that

$$h_3(\phi, w_{21}, w_{22}) = \frac{w_1 + \phi w_{22}}{w_1 + \phi w_{21}}$$

is increasing in ϕ. Now,

$$\frac{\partial h_3(\phi, w_{21}, w_{22})}{\partial \phi} = \frac{w_1(w_{22} - w_{21})}{(w_1 + \phi w_{21})^2} \geq 0$$

This completes the proof. (Proved)

Lemma 4.2 *If $h(\lambda)$ is increasing in λ, then*

$$\varphi_1(w_1, w_2, d_1, d_2, \phi) = \int_\lambda h(\lambda) p_1(\lambda \mid \phi, (w_1, w_2, d_1, d_2)) d\lambda$$

(a) is decreasing in w_2 for fixed (w_1, d_1, d_2, ϕ).

(b) is decreasing in ϕ for fixed (w_1, w_2, d_1, d_2).

Proof: Consider two points w_{21} and w_{21} such that $w_{21} < w_{22}$. Let $h_{21}(\lambda) = h(\lambda)\lambda^{d_1+d_2}\phi^{d_2} p_1(\lambda)$, $h_{22}(\lambda) = \lambda^{d_1+d_2}\phi^{d_2} p_1(\lambda)$, $h_{11}(\lambda) = \exp(-\lambda(w_1 + \phi w_{21}))$ and $h_{12}(\lambda) = \exp(-\lambda(w_1 + \phi w_{22}))$. Clearly, $h_{12}(\lambda), h_{22}(\lambda)$ are non negative functions. Note that $h_{11}(\lambda)/h_{12}(\lambda) = \exp[\lambda\phi(w_{22} - w_{21})]$ is increasing in λ. and $h_{21}(\lambda)/h_{22}(\lambda) = h(\lambda)$ which is also increasing function in λ for $\lambda > 0$. By Theorem 2 in Wijsman [120], we get $\varphi_1(w_1, w_{21}, d_1, d_2, \phi) > \varphi_1(w_1, w_{22}, d_1^2, d_2, \phi)$. Therefore, $\varphi_1(w_1, w_2, d_1, d_2, \phi)$ is decreasing in w_2 for fixed (w_1, d_1, d_2, ϕ).

Similarly, let $\phi_1 < \phi_2$. Let $h_{21}(\lambda) = h(\lambda)\lambda^d p_1(\lambda)$, $h_{22}(\lambda) = \lambda^d p_1(\lambda)$, $h_{11}(\lambda) = \phi_1^{d_2} \exp[-\lambda(w_1 + \phi_1 w_2)] p_2(\phi_1)$ and $h_{12}(\lambda) = \phi_2^{d_2} \exp[-\lambda(w_1 + \phi_2 w_2)] p_2(\phi)$ Then $h_{21}(\lambda)/h_{22}(\lambda) = h(\lambda)$ is increasing in λ and $h_{11}(\lambda)/h_{12}(\lambda) = (\phi_1^{d_2} p_2(\phi_1)/\phi_2^{d_2} p_2(\phi_2)) \exp[\lambda w_2(\phi_2 - \phi_1)]$ is also increasing in λ. By Theorem 2 in Wijsman [120], we get $\varphi_1(w_1, w_2, d_1, d_2, \phi_1) > \varphi_1(w_1, w_2, d_1, d_2, \phi_2)$. Therefore, $\varphi_1(w_1, w_2, d_1, d_2, \phi)$ is decreasing in ϕ for fixed (w_1, w_2, d_1, d_2). (Proved)

Now, we prove that $\varphi(w_1, w_{21}, d_1, d_2) - \varphi(w_1, w_{22}, d_1, d_2) \geq 0$ for $w_{21} < w_{22}$. The proof is in a similar manner as given in Lemma 3.4.2 of Lehmann and Romano [62].

$$\begin{aligned}
&\varphi(w_1, w_{21}, d_1, d_2) - \varphi(w_1, w_{22}, d_1, d_2) \\
&= \int_\phi \varphi_1(w_1, w_{21}, d_1, d_2, \phi) p_2(\phi \mid (w_1, w_{21}, d_1, d_2) d\phi - \int_\phi \varphi_1(w_1, w_{22}, d_1, d_2, \phi) p_2(\phi \mid (w_1, w_{22}, d_1, d_2) d\phi \\
&\geq \int_\phi \varphi_1(w_1, w_{22}, d_1, d_2, \phi) p_2(\phi \mid (w_1, w_{21}, d_1, d_2) d\phi - \int_\phi \varphi_1(w_1, w_{22}, d_1, d_2, \phi) p_2(\phi \mid (w_1, w_{22}, d_1, d_2) d\phi \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{using Lemma 4.2(i)}]
\end{aligned}$$

$$= \int_{\phi} \varphi_1(w_1, w_{22}, d_1, d_2, \phi)[p_2(\phi \mid (w_1, w_{21}, d_1, d_2)) - p_2(\phi \mid (w_1, w_{22}, d_1, d_2)]\ d\phi. \tag{4.A.3}$$

Define the sets U and V as

$$U = \{\phi \mid p_2(\phi \mid (w_1, w_{21}, d_1, d_2)) > p_2(\phi \mid (w_1, w_{22}, d_1, d_2))\}$$

and

$$V = \{\phi \mid p_2(\phi|(w_1, w_{21}, d_1, d_2) \leq p_2(\phi \mid (w_1, w_{22}, d_1, d_2))\}.$$

Let $u = \inf\limits_{U}\{\varphi_1(w_1, w_{22}, d_1, d_2, \phi)\}$and $v = \sup\limits_{V}\{\varphi_1(w_1, w_{22}, d_1, d_2, \phi)\}$. From (4.A.3), we have

$$\begin{aligned}
&\varphi(w_1, w_{21}, d_1, d_2) - \varphi(w_1, w_{22}, d_1, d_2)\\
\geq &\int\limits_{U} \varphi_1(w_1, w_{22}, d_1, d_2, \phi)[p_2(\phi \mid (w_1, w_{21}, d_1, d_2)) - p_2(\phi \mid (w_1, w_{22}, d_1, d_2)]\ d\phi\\
&+ \int\limits_{V} \varphi_1(w_1, w_{22}, d_1, d_2, \phi)[p_2(\phi \mid (w_1, w_{21}, d_1, d_2)) - p_2(\phi \mid (w_1, w_{22}, d_1, d_2)]\ d\phi\\
\geq &u \int\limits_{U} [p_2(\phi \mid (w_1, w_{21}, d_1, d_2)) - p_2(\phi \mid (w_1, w_{22}, d_1, d_2)]\ d\phi\\
&+ v \int\limits_{V} [p_2(\phi \mid (w_1, w_{21}, d_1, d_2)) - p_2(\phi \mid (w_1, w_{22}, d_1, d_2)]\ d\phi\\
&= (u - v) \int\limits_{U} [p_2(\phi \mid (w_1, w_{21}, d_1, d_2)) - p_2(\phi \mid (w_1, w_{22}, d_1, d_2)]\ d\phi.
\end{aligned} \tag{4.A.4}$$

The sets U and V can be written as

$$U = \left\{\phi \mid \frac{p_2(\phi \mid (w_1, w_{21}, d_1, d_2))}{p_2(\phi \mid (w_1, w_{22}, d_1, d_2))} > 1\right\}$$

and

$$V = \left\{\phi \mid \frac{p_2(\phi \mid (w_1, w_{21}, d_1, d_2))}{p_2(\phi \mid (w_1, w_{22}, d_1, d_2))} < 1\right\}.$$

For all $\phi_1 \in U$ and $\phi_2 \in V$, we have

$$\frac{p_2(\phi_1 \mid (w_1, w_{21}, d_1, d_2))}{p_2(\phi_1 \mid (w_1, w_{22}, d_1, d_2))} > 1 > \frac{p_2(\phi_2 \mid (w_1, w_{21}, d_1, d_2))}{p_2(\phi_2 \mid (w_1, w_{22}, d_1, d_2))}.$$

From Lemma 4.1,

$$\frac{p_2(\phi \mid (w_1, w_{21}, d_1, d_2)}{p_2(\phi \mid (w_1, w_{22}, d_1, d_2)}$$

is increasing in ϕ. Therefore,

$$\frac{p_2(\phi_1 \mid (w_1, w_{21}, d_1, d_2))}{p_2(\phi_1 \mid (w_1, w_{22}, d_1, d_2))} > \frac{p_2(\phi_2 \mid (w_1, w_{21}, d_1, d_2))}{p_2(\phi_2 \mid (w_1, w_{22}, d_1, d_2))} \implies \phi_1 > \phi_2.$$

Now, from Lemma 4.2 (ii), $\varphi_1(w_1, w_2, d_1, d_2, \phi)$ is a decreasing function in ϕ. Therefore we get $\varphi_1(w_1, w_2, d_1, d_2, \phi_1) < \varphi_1(w_1, w_2, d_1, d_2, \phi_2)$, $\forall\ \phi_1 \in U$ and $\phi_2 \in V$. This implies that $\inf\limits_{U}\{\varphi_1(w_1, w_2, d_1, d_2, \phi)\} < \sup\limits_{V}\{\varphi_1(w_1, w_2, d_1, d_2, \phi)\} \implies u > v$. From (4.A.4) we get $\varphi(w_1, w_{21}, d_1, d_2) - \varphi(w_1, w_{22}, d_1, d_2) \geq 0$. Therefore $\varphi(w_1, w_2, d_1, d_2)$ is decreasing in w_2 for fixed (w_1, d_1, d_2).

4.A.3 Expression of Bayes risk for quadratic loss function

Consider the quadratic loss function $h(\lambda) = C_0 + C_1\lambda + C_{11}\lambda^2$. The joint distribution of λ and ϕ is given by

$$p(\lambda, \phi) = \frac{\beta^\alpha}{\Gamma(\alpha)(l-1)}\lambda^{\alpha-1}\exp(-\beta\lambda).$$

Using (4.2.6), the Bayes risk is then obtained as

$$R_B(\boldsymbol{\zeta}, a) = n(C_s - v_s) + C_a n_{as} + E[D] + C_t E[\tau] + E_\lambda[h(\lambda)] + \sum_{d_1=0}^{n}\sum_{d_2=0}^{n-d_1} H(d_1, d_2),$$

where $E_\lambda[h(\lambda]$, $H(d_1, d_2)$, $E[D]$, n_{as} and $E[\tau]$ are given Appendix 4.A.3, Appendix 4.A.3, Appendix 4.A.3, Appendix 4.A.3 and Appendix 4.A.3, respectively.

Expression of $E_\lambda[h(\lambda)]$

$$E_\lambda[h(\lambda)] = C_0 + C_1\frac{\alpha}{\beta} + C_{11}\frac{\alpha(\alpha+1)}{\beta^2}.$$

Expression of $H(d_1, d_2)$

For the expression of $H(d_1, d_2)$, we need to consider four cases:

Case 1: If $d_1 = 0$, $d_2 = 0$, then

$$\begin{aligned} H(d_1, d_2) =& A_{0,0,0,0}\frac{\beta^\alpha}{\Gamma(\alpha)(l-1)}\sum_{p=0}^{2}\Bigg[C_p\ \mathbb{1}_{\mathcal{S}_4}(\boldsymbol{\zeta})\ \&\ n(\tau_2-\tau_1) \le c(0,0)) \\ & H_1(n\tau_1, n(\tau_2-\tau_1), 0, 0, p)\Bigg], \end{aligned}$$

where $\mathcal{S}_4 = \{\boldsymbol{\zeta} : n\tau_1 \le c(0,0,0,n(\tau_2-\tau_1))\}$.
Case 2: If $d_1 = 0$, $d_2 > 0$, then

$$\begin{aligned} H(d_1, d_2) =& \frac{\beta^\alpha}{\Gamma(\alpha)(l-1)}\sum_{k=0}^{d_2}\int_{w_2=\tau_{0,d_2,k}}^{\xi(0,d_2,k)} A_{0,d_2,0,k}\sum_{p=0}^{2}\Bigg[C_p \mathbb{1}_{\mathcal{S}_5}(\boldsymbol{\zeta}, d_2, w_2) \\ & \frac{(w_2-\tau_{0,d_2,k})^{d_2-1}}{\Gamma(d_2)}H_1(n\tau_1, w_2, 0, d_2, p)\Bigg]\ dw_2, \end{aligned}$$

where $\mathcal{S}_5 = \{(\boldsymbol{\zeta}, d_2, w_2) : n\tau_1 \le c(0, d_2, w_2)\}$
Case 3: If $d_1 > 0$, $d_2 = 0$, then

$$\begin{aligned} H(d_1, d_2) =& \frac{\beta^\alpha}{\Gamma(\alpha)(l-1)}\sum_{j=0}^{d_1}\int_{w_1=\tau_{d_1,j}}^{\xi(d_1,0,n(\tau_2-\tau_1),k)} A_{d_1,0,j,0}\sum_{p=0}^{2}\Bigg[C_p\ \mathbb{1}_{\mathcal{S}_6}(\boldsymbol{\zeta}, d_1) \\ & \frac{(w_1-\tau_{d_1j})^{d_1-1}}{\Gamma(d_1)}H_1(w_1, n(\tau_2-\tau_1), d_1, 0, p)\Bigg]\ dw_1, \end{aligned}$$

where $\mathcal{S}_6 = \{(\boldsymbol{\zeta}, d_1) : (n-d_1)(\tau_2-\tau_1) \le c(d_1, 0)\}$.

Case 4: If $d_1 > 0$, $d_2 > 0$, then

$$\begin{aligned}H(d_1,d_2) =& \frac{\beta^\alpha}{\Gamma(\alpha)(l-1)} \sum_{j=0}^{d_1}\sum_{k=0}^{d_2} \int_{w_2=\tau_{d_1,d_2,k}}^{\xi(d_1,d_2,k)} \int_{w_1=\tau_{d_1,j}}^{\xi(d_1,d_2,w_2,j)} A_{d_1,d_2,j,k} \\ & \left[\sum_{p=0}^{2} C_p \frac{(w_1-\tau_{d_1,j})^{d_1-1}(w_2-\tau_{d_1,d_2,k})^{d_2-1}}{\Gamma(d_1)\Gamma(d_2)} H_1(w_1,w_2,d_1,d_2,p)\right] dw_1\, dw_2,\end{aligned}$$

where $\xi(d_1,d_2,k) = \max\{\tau_{d_1,d_2,k}, c(d_1,d_2)\}$ and $\xi(d_1,d_2,w_2,k) = \max\{\tau_{d_1,k}, c(d_1,d_2,w_2)\}$.

Expression of $E[D]$

For the expression of $E[D]$, we need to consider three cases:

Case 1: If $m = 0$, then

$$E[D] = n - n\left(\frac{\beta}{\beta+\tau_2}\right)^\alpha .$$

Case 2: If $0 < m < n$, then

$$\begin{aligned}E[D] =& n + \sum_{d_1=0}^{m-1} (n-d_1)\binom{n}{d_1} \sum_{i=0}^{d_1}\binom{d_1}{i}(-1)^{d_1-i+1}\left[\frac{1}{(l-1)(\tau_2-\tau_1)(\alpha-1)}\right. \\ & \left.\left\{\frac{\beta^\alpha}{[\beta+(n-i)\tau_1+(\tau_2-\tau_1)]^{\alpha-1}} - \frac{\beta^\alpha}{[\beta+(n-i)\tau_1+l(\tau_2-\tau_1)]^{\alpha-1}}\right\}\right] \\ & + \sum_{d_1=m}^{n-1} (n-d_1)\binom{n}{d_1}\sum_{i=0}^{d_1}\binom{d_1}{i}(-1)^{d_1-i+1}\left[\left(\frac{\beta}{\beta+(n-i)\tau_1+(\tau_2-\tau_1)}\right)^\alpha\right].\end{aligned}$$

Case 3: If $m = n$, then

$$\begin{aligned}E[D] =& n - \sum_{d_1=0}^{n-1}(n-d_1)\binom{n}{d_1}\sum_{i=0}^{d_1}\binom{d_1}{i}\left[\frac{(-1)^{d_1-i+1}}{(l-1)(\tau_2-\tau_1)(\alpha-1)}\times\right. \\ & \left.\left\{\frac{\beta^\alpha}{[\beta+(n-i)\tau_1+(\tau_2-\tau_1)]^{\alpha-1}} - \frac{\beta^\alpha}{[\beta+(n-i)\tau_1+l(\tau_2-\tau_1)]^{\alpha-1}}\right\}\right].\end{aligned}$$

Expression of n_{as}

$$n_{as} = \begin{cases} 0, & \text{for } m = 0, \\ \displaystyle\sum_{d_1=0}^{m-1}\sum_{i=0}^{d_1}(n-d_1)\binom{n}{d_1}\binom{d_1}{i}(-1)^i\left(\frac{\beta}{(n-i)\tau_1+\beta}\right)^\alpha, & \text{for } m > 0\,. \end{cases}$$

Expression of $E[\tau]$

$$E[\tau] = \begin{cases} \displaystyle\sum_{i=1}^{n}\binom{n}{i}(-1)^i\left[\frac{\beta}{i(\alpha-1)} - \frac{\beta^\alpha}{i(\alpha-1)(\beta+i\tau_2)^{\alpha-1}}\right], & \text{for } m = 0, \\ \displaystyle\int_{\boldsymbol\theta}\int_{t=0}^{\tau_1}(1-F_{X_{(n)}}(t\mid\boldsymbol\theta))\, p(\boldsymbol\theta)\; dt\; d\boldsymbol\theta + \int_{\boldsymbol\theta}\int_{t=\tau_1}^{\tau_2}(1-F_{X_{(n)}}(t\mid\boldsymbol\theta))\, p(\boldsymbol\theta)\; dt\; d\boldsymbol\theta, & \text{for } m > 0, \end{cases}$$

where

$$
\begin{aligned}
&\int_{\boldsymbol{\theta}} \int_{t=0}^{\tau_1} (1 - F_{X_{(n)}}(t \mid \boldsymbol{\theta}))\ p(\boldsymbol{\theta})\ dt\ d\boldsymbol{\theta} \\
=& \int_0^\infty \int_0^{\tau_1} \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \exp(-i\lambda t) \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} \exp(-\beta\lambda)\ dt\ d\lambda \\
=& \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \int_0^\infty \lambda^{\alpha-1} \exp(-\beta\lambda) \left(\frac{1 - \exp(-i\lambda\tau_1)}{i\lambda} \right)\ d\lambda \\
=& \sum_{i=1}^{n} \binom{n}{i} (-1)^i \left[\frac{\beta}{i(\alpha-1)} - \frac{\beta^\alpha}{i(\alpha-1)(\beta+i\tau_1)^{\alpha-1}} \right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\boldsymbol{\theta}} \int_{t=\tau_1}^{\tau_2} (1 - F_{X_{(n)}}(t \mid \boldsymbol{\theta}))\ p(\boldsymbol{\theta})\ dt\ d\boldsymbol{\theta} \\
=& - \sum_{d_1=0}^{n-1} \sum_{j=0}^{d_1} \sum_{k=1}^{n-d_1} \frac{A_{d_1 n-d_1 jk}}{m-1} \int_1^l \left[\frac{\beta^\alpha}{(\beta + (n-d_1+j)\tau_1)^{\alpha-1} \phi^\delta (\alpha-1)} \right. \\
&\qquad \left. - \frac{\beta^\alpha}{(\beta + (n-d_1+j)\tau_1 + \phi^\delta k(\tau_2-\tau_1))^{\alpha-1} \phi^\delta (\alpha-1)} \right]\ d\phi.
\end{aligned}
$$

Chapter 5

RASP for competing risks data under PICS-I

This chapter considers the design of RASPs under PICS-I in the presence of competing risks. A general modeling framework is developed for PICS-I data with competing failure modes, accommodating both independent and dependent competing risks. While the assumption of independence among potential failure times is common in the literature, practical applications often involve correlation between these times (see, for example, Zhang et al. [132], Liu [77]). Under this unified framework, we derive the Fisher information matrix, which involves only first-order partial derivatives, thereby significantly reducing computational complexity. Additionally, the asymptotic properties of the MLEs are established under standard regularity conditions, which are essential for the subsequent development of RASPs.

As a special case of the general model, we consider a frailty-based dependent competing risks model. Frailty models, widely used in survival analysis, introduce unobserved heterogeneity through random effects that multiplicatively affect the hazard rates. In this setup, we assume a gamma-distributed frailty variable with mean one (to ensure identifiability; see Hougaard [44]), and conditional independence of the failure times given the frailty. An attractive feature of this approach is that the independent competing risks model is naturally recovered as a limiting case.

We adopt the producer's risk and consumer's risk approach to design RASPs under both independence and dependence structures. This method yields the required sample size and acceptance limit but assumes a fixed inspection time, which may not lead to an optimal life testing plan. To address this, we consider optimal design criteria, specifically, the c-optimality criterion—to determine an inspection time that minimizes a measure of variance. Since such designs may not always satisfy practical budgetary constraints, we further incorporate a generic cost function and develop an approach for determining c-optimal RASPs under a cost constraint.

Several existing studies have focused on optimal design under PICS-I using various optimality criteria for different lifetime distributions. For example, Lin et al. [70, 71] provided A- and D-optimal designs for log-normal and Weibull models, respectively. Roy and Pradhan [104] introduced Bayesian D-optimal designs for log-normal lifetimes, which were later extended to Bayesian c-optimal designs for both Weibull and log-normal models by Roy and Pradhan [105].

This chapter contributes to three key aspects of RASP design in the presence of competing risks. First, we present a unified modeling framework for PICS-I data that accounts for both independence and dependence among failure modes, making it well-suited for modeling ICS. Second, we derive a general expression for the Fisher information matrix and establish asymptotic properties of the MLEs, facilitating robust inference under PICS-I. Third, we develop a comprehensive design framework for RASPs under three different setups, offering flexibility and practical applicability in reliability analysis.

The remainder of this chapter is organized as follows. Section 5.1 provides the general framework for modeling PICS-I data and the asymptotic properties of the MLEs. Subsequently, Section 5.2 considers the frailty model as a special case. Section 5.3 develops the optimal RASPs under the PICS-I. Section 5.4 discusses the methodology for obtaining traditional c-optimal RASPs with and without cost constraints. Subsequently, Section 5.5 examines the effect of incorporating the dependence among the competing failure modes with a detailed numerical experiment. Section 5.6 presents an application of the developed methodology using a real-life example. Section 5.6 investigates the finite sample properties of the developed RASPs using a simulation study. Finally, Section 5.8 highlights the contributions of this work.

5.1 General framework

Here, Section 5.1.1 presents the general model and the likelihood function. Subsequently, Section 5.1.2 discusses the asymptotic properties of the MLEs.

5.1.1 Model and likelihood function

Suppose that in a life test experiment, a test unit fails as soon as one of the competing failure modes J occurs. Let X_{ij} denote the potential failure time due to the j^{th} cause of failure of the i^{th} item for $j = 1, \ldots, J$ and $i = 1, \ldots, n$. Let X_i denote the overall failure time of the i^{th} item, and the index C_i denote the cause of failure of the i^{th} item. It is easy to understand that $X_i = \min\{X_{i1}, \ldots, X_{iJ}\} = X_{iC_i}$. Let $f_j(x \mid \boldsymbol{\theta}_j)$ and $F_j(x \mid \boldsymbol{\theta}_j)$ be PDF and CDF of X_{ij}, respectively, where $\boldsymbol{\theta}_j$ is the vector of parameters corresponding to the jth cause of failure. Also, denote the joint RF of $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{iJ})$ by $\overline{R}(\boldsymbol{x} \mid \boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_l)$ is a vector of parameters, with l being the number of model parameters.

In a competing risk setup, we collect data on C_i and X_i from a life test. The joint distribution of C_i and T_i is specified in terms of the sub-distribution function $G(j, x \mid \boldsymbol{\theta}) = P(C_i = j, X_i \leq x)$ or equivalently, by the sub-survivor function $\overline{G}(j, x \mid \boldsymbol{\theta}) = P(C_i = j, X_i > x)$ (Crowder [34]). The sub-density function corresponding to the observed pair $\{C_i = j, X_i = x\}$ is then given by

$$g(j, x \mid \boldsymbol{\theta}) = \left[-\frac{\delta \overline{R}\,(\boldsymbol{x} \mid \boldsymbol{\theta})}{\delta x_j} \right]_{x\mathbf{1}_J},$$

where $\mathbf{1}_J$ is a unit vector of J components and $[\ldots]_{x\mathbf{1}_J}$ signifies that the enclosed partial derivative is to be computed at $x\mathbf{1}_J = (x, \ldots, x)$ for $j = 1, \ldots, J$.

Suppose that n identical test units having J competing failure modes are subjected to a life test under a k-point PICS-I. In this setup, we observe a number of failures for each cause after each inspection. Let D_{ij} be the number of failures at the interval $[\tau_{i-1}, \tau_i)$ due to cause j and d_{ij} be the observed value of D_{ij}, for $i = 1, \ldots, k$ and $j = 1, \ldots, J$. Therefore, $D_i = \sum_{j=1}^{J} D_{ij}$, for $i = 1, \ldots, k$. The observed competing risks data under PICS-I can be written as $\boldsymbol{x} = (d_{11}, \ldots, d_{1J}, r_1, \ldots, d_{k1}, \ldots, d_{kJ}, r_k)$.

Let q_{ij} be the probability that a test unit fails in the ith interval due to the jth failure mode. It is easy to see that

$$\begin{aligned} q_{ij} =& \frac{P\left(C=j, T \leq \tau_i\right) - P\left(C=j, T \leq \tau_{i-1}\right)}{1 - P\left(T \leq \tau_{i-1}\right)} \\ =& \frac{G\left(j, \tau_i \mid \boldsymbol{\theta}\right) - G\left(j, \tau_{i-1} \mid \boldsymbol{\theta}\right)}{R(\tau_{i-1} \mid \boldsymbol{\theta})}. \end{aligned} \tag{5.1.1}$$

Furthermore, let q_i represent the probability that a test unit fails in the ith interval. Then, we have

$$\begin{aligned} q_i =& \frac{P\left(T \leq \tau_i\right) - P\left(T \leq \tau_{i-1}\right)}{1 - P\left(T \leq \tau_{i-1}\right)} \\ =& \frac{F(\tau_i \mid \boldsymbol{\theta}) - F\left(\tau_{i-1} \mid \boldsymbol{\theta}\right)}{R(\tau_{i-1} \mid \boldsymbol{\theta})}. \end{aligned} \tag{5.1.2}$$

Note that $\sum_{j=1}^{J} q_{ij} = q_i$. From Wu and Huang [122], the expressions for $E[N_i]$, $E[D_{ij}]$ and $E[R_i]$ are obtained as follows:

$$E\left[N_i\right] = n \prod_{l=0}^{i-1} (1-q_l)(1-p_l), \quad E\left[D_{ij}\right] = n \left\{ \prod_{l=0}^{i-1} \left[(1-q_l)(1-p_l)\right] \right\} q_{ij}$$

and

$$E\left[R_i\right] = n \left\{ \prod_{l=0}^{i-1} \left[(1-q_l)(1-p_l)\right] \right\} (1-q_i) p_i,$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, J$, with $p_0 = 0$ and $q_0 = 0$ (see Roy and Pradhan [105], Wu and Huang [122] for details). Note that $D_i = \sum_{j=1}^{J} D_{ij}$ is the number of failures that occur in the time interval $(\tau_{i-1}, \tau_i]$ and $D = \sum_{i=1}^{M} D_i$ is the total number of failures in the life test. Thus, we have

$$E[D] = \sum_{i=1}^{k} E[D_i] = \sum_{i=1}^{k} \sum_{j=1}^{J} E[D_{ij}].$$

Based on the observed data $\boldsymbol{x}$, the likelihood function of $\boldsymbol{\theta}$ is given by

$$L(\boldsymbol{\theta} \mid \boldsymbol{x}) \propto \prod_{i=1}^{k} \prod_{k=1}^{n_i} \left[\prod_{j=1}^{J} q_{ij}^{\delta_{ikj}} \right] (1-q_i)^{1-\delta_{ik+}},$$

where $\delta_{ik+} = \sum_{j=1}^{J} \delta_{ikj}$ and $\sum_{k=1}^{n_i} (1-\delta_{ik+}) = n_i - d_i$. The log-likelihood function, after ignoring the proportionality constant, can be written as

$$l(\boldsymbol{\theta} \mid \boldsymbol{x}) = \log L(\boldsymbol{\theta} \mid \boldsymbol{x}) = \sum_{i=1}^{k} \sum_{k=1}^{n_i} \left[\sum_{j=1}^{J} \delta_{ikj} \log(q_{ij}) + (1-\delta_{ik+}) \log(1-q_i) \right]. \tag{5.1.3}$$

We can easily obtain the MLEs of the model parameters by numerically maximizing the log-likelihood function $l(\boldsymbol{\theta} \mid \boldsymbol{x})$.

5.1.2 Asymptotic properties of MLEs

We now present the asymptotic properties of the MLEs, which are essential in the development of RASPs. Let $\widehat{\boldsymbol{\theta}}$ denote the MLE of $\boldsymbol{\theta}$. Towards this end, we state a set of regularity conditions required to establish the asymptotic properties of $\widehat{\boldsymbol{\theta}}$.

Regularity conditions

I. The unobserved lifetimes $X_1, X_2, \ldots, X_n$ are IID with common CDF $F(\cdot \mid \boldsymbol{\theta})$ and PDF $f(\cdot \mid \boldsymbol{\theta})$.

II. The number of causes of failure of a test unit is finite.

III. $(X_1, C_1), (X_2, C_2), \ldots, (X_n, C_n)$ are IID with common joint sub-survivor function $G(\cdot, \cdot \mid \boldsymbol{\theta})$ and sub-density function $g(\cdot, \cdot \mid \boldsymbol{\theta})$, where C_i is the cause of the failure of the i^{th} item for $i = 1, \ldots, n$.

IV. The supports of $f(\cdot \mid \boldsymbol{\theta})$ and $g(\cdot, \cdot \mid \boldsymbol{\theta})$ are independent of $\boldsymbol{\theta}$.

V. The parameter space $\boldsymbol{\Theta}$ contains an open set $\boldsymbol{\Theta}_0$ of which the true parameter $\boldsymbol{\theta}^0$ is an interior point.

VI. For almost all x and for all $j = 1, 2, \ldots, J$, both $F(x \mid \boldsymbol{\theta})$ and $G(j, x \mid \boldsymbol{\theta})$ admit all third-order derivatives, $\frac{\partial^3 F(x \mid \boldsymbol{\theta})}{\partial\theta_u \partial\theta_v \partial\theta_w}$ and $\frac{\partial^3 G(j, x \mid \boldsymbol{\theta})}{\partial\theta_u \partial\theta_v \partial\theta_w}$, respectively, for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0$ and $u, v, w = 1, 2, \ldots, l$. In addition, all first, second, and third order derivatives of $F(x \mid \boldsymbol{\theta})$ and $G(j, x \mid \boldsymbol{\theta})$ with respect to the parameters are bounded for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0$.

VII. τ_i's are fixed in such a way that

  (a) $0 < q_{ij} < 1$ and $0 < q_i < 1$ for $i = 1, 2, \ldots, k$ and $j = 1, 2 \ldots, J$.

  (b) $\nabla_{\boldsymbol{\theta}} \boldsymbol{q}$ is a matrix of rank l, where $\nabla_{\boldsymbol{\theta}} \boldsymbol{q} = \left(\frac{\partial q_i}{\partial \theta_u}\right)_{l \times k}$ for $u = 1, 2, \ldots, l$ and $i = 1, 2, \ldots, k$.

The regularity conditions (I)-(VI) presented above are essentially standard requirements for establishing the asymptotic properties of the MLEs in the competing risks setup (Mukhopadhyay [87]), whereas condition (VII) ensures the positive definiteness of the Fisher information matrix. We now present the following results, which are required to obtain the Fisher information matrix.

Lemma 5.1 *The first derivative of the log-likelihood satisfies the following:*

$$E\left[\frac{\partial l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial \theta_u}\right] = 0, \ \textit{for } u = 1, \ldots, l.$$

Proof: The proof is given in Appendix 5.A.1. (Proved)

Lemma 5.2 *The second derivative of the log-likelihood function satisfies the following*

$$E\left[\frac{\partial^2 l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial \theta_u \partial \theta_v}\right] = E\left[\left(\frac{\partial l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial \theta_u}\right)\left(\frac{\partial l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial \theta_v}\right)\right], \ \textit{for } u, v = 1, \ldots, l.$$

Proof: The proof is given in Appendix 5.A.1. (Proved)

Using Lemmas 5.1 and 5.2, we readily obtain the Fisher information matrix, which is provided in the following theorem.

Theorem 5.1 *The Fisher information matrix for $\boldsymbol{\theta}$ is given by*

$$I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}) = \sum_{i=1}^{k} \left[\sum_{j=1}^{J} \frac{E[N_i]}{q_{ij}} \nabla_{\boldsymbol{\theta}}(q_{ij}) \nabla_{\boldsymbol{\theta}}(q_{ij})^T + \frac{E[N_i]}{(1-q_i)} \nabla_{\boldsymbol{\theta}}(q_i) \nabla_{\boldsymbol{\theta}}(q_i)^T \right],$$

It is interesting to note that the Fisher information matrix involves only the first-order partial derivatives, thus simplifying its computation to a great extent, as illustrated in the subsequent sections. We now present the asymptotic properties of $\widehat{\boldsymbol{\theta}}$. We first present the following results, which are necessary to prove asymptotic properties of $\widehat{\boldsymbol{\theta}}$. For convenience, we use $F(x)$, $R(x)$, $G(j,x)$, and $l(\boldsymbol{\theta})$ instead of $F(x \mid \boldsymbol{\theta})$, $R(x \mid \boldsymbol{\theta})$, $G(j,x \mid \boldsymbol{\theta})$ and $l(\boldsymbol{\theta} \mid \boldsymbol{x})$, respectively.

Lemma 5.3 *Under the regularity conditions (I)-(VII), the first-, second-, and third-order derivatives of q_{ij} and q_i are bounded.*

Proof: The proof is given in Appendix 5.A.1. (Proved)

Lemma 5.4 *Under the regularity conditions (I)-(VII), the ratio $\frac{N_i}{n}$ converges in probability to a finite number $b_i = \prod_{l=1}^{j-1}(1-p_l)(1-q_l)$, as $n \to \infty$, for $i = 1, \ldots, k$.*

Proof: A proof of the above lemma follows from Budhiraja and Pradhan [19]. (Proved)

Lemma 5.5 *If the regularity conditions (I)-(VII) hold, then*

(i) $I(\boldsymbol{\theta} \mid \boldsymbol{\zeta})$ is finite for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0$.

(ii) $I(\boldsymbol{\theta} \mid \boldsymbol{\zeta})$ is positive definite for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}_0$.

Proof: The proof is given in Appendix 5.A.1. (Proved)

Lemma 5.6 *Suppose that the regularity conditions (I)-(VII) hold. Let $\boldsymbol{\Theta}_0$ be an open subset in the parameter space $\boldsymbol{\Theta}$ containing the true parameter $\boldsymbol{\theta}^0$. Then, all third derivatives $\frac{\partial^3 l(\boldsymbol{\theta})}{\partial\theta_u \partial\theta_v \partial\theta_w}$ exist. Furthermore, there exists a bound $K_{uvw}(\boldsymbol{x})$ such that $\left|\frac{\partial^3 l(\boldsymbol{\theta})}{\partial\theta_u \partial\theta_v \partial\theta_w}\right| \leq K_{uvw}(\boldsymbol{x})$, with $E[K_{uvw}] = nk_{uvw}$, where k_{uvw} is finite for $u, v, w = 1, 2, \ldots, l$.*

Proof: The proof is given in Appendix 5.A.1. (Proved)

Theorem 5.2 *If the regularity conditions (I)-(VII) (as stated above) hold, then*

(I) $\widehat{\boldsymbol{\theta}}$ is consistent for $\boldsymbol{\theta}$.

(II) $\widehat{\boldsymbol{\theta}}$ is asymptotically normal with mean $\boldsymbol{\theta}^0$ and variance-covariance matrix $[I(\boldsymbol{\theta}^0 \mid \boldsymbol{\zeta})]^{-1}$, where $\boldsymbol{\theta}^0$ denotes the true value $\boldsymbol{\theta}$.

Proof: The proof is given in Appendix 5.A.2. (Proved)

5.2 Frailty model

In this section, we consider a frailty-based dependent competing risk model. We consider a frailty-based dependent competing risks model that allows us to incorporate the dependence among the potential failure times. The model considered here allows us to obtain an independent competing risks model as a special case. Thus, we can also study the relevance of incorporating the dependence structure among the competing failure modes while deciding RASPs.

We assume that X_{ij}, the potential failure time due to the jth failure mode follows a Wei$(\nu_j, 1/\eta_j)$ with PDF $f_j(\cdot \mid \boldsymbol{\theta}_j)$ where $\boldsymbol{\theta}_j = (\eta_j, \nu_j)$, with $\eta_j\,(>0)$ and $\nu_j\,(>0)$ being the scale and shape parameters, respectively, for $i = 1, \ldots, n$ and $j = 1, \ldots, J$. As in Roy and Mukhopadhyay [103], we further assume $\nu_1 = \cdots = \nu_J = \nu$, where ν is the common shape parameter. Towards this end, we should also note that this assumption often holds true for real data sets (see, for example, the numerical illustration provided by Liu [77]). Then, RF of X_{ij} is given by

$$R_j(x \mid \boldsymbol{\theta}_j) = \exp\left[-\left(\frac{x}{\eta_j}\right)^{\nu}\right] = \exp\left(-\Lambda_j(x)\right),$$

where $\Lambda_j(\cdot)$ is the cumulative hazard function corresponding to the jth failure mode.

A common approach to introduce the dependence between the potential failure times X_{ij}s is through the unobserved random factor Z, commonly known as the frailty which is discussed in Section 1.5. We assume that Z follows a Gamma distribution with mean 1 and variance θ (Hougaard [44]). Then, the unconditional joint RF of $\overline{\boldsymbol{X}}_i$ is given by

$$\overline{R}(\boldsymbol{x} \mid \boldsymbol{\theta}) = (1 + \theta\Delta)^{-\frac{1}{\theta}}, \tag{5.2.1}$$

where $\Delta = \sum_{j=1}^{J} \Lambda_j(x_j) = \sum_{j=1}^{J} \left(\frac{x_j}{\eta_j}\right)^{\nu}$ and $\boldsymbol{\theta} = (\eta_1, \ldots, \eta_J, \nu, \theta)$. Then, the sub-density function corresponding to the observed pair $\{C_i = j, X_i = x\}$ is given by

$$g(j, x \mid \boldsymbol{\theta}) = \left(\frac{\nu}{\eta_j}\right)\left(\frac{x}{\eta_j}\right)^{\nu-1}\left[1 + \theta\sum_{j'=1}^{J}\left(\frac{x}{\eta_{j'}}\right)^{\nu}\right]^{-\frac{1}{\theta}-1}.$$

Furthermore, the sub-survivor function corresponding to the observed pair $\{C_i = j, X_i = x\}$ is given by

$$\begin{aligned}
\overline{G}(j, x \mid \boldsymbol{\theta}) &= \int_t^{\infty} g(j, s)\mathrm{d}s \\
&= \int_t^{\infty}\left(\frac{\nu}{\eta_j}\right)\left(\frac{s}{\eta_j}\right)^{\nu-1}\left[1 + \theta\sum_{j'=1}^{J}\left(\frac{s}{\eta_{j'}}\right)^{\nu}\right]^{-\frac{1}{\theta}-1}\mathrm{d}s \\
&= \frac{1}{\sum_{j'=1}^{J}\left(\frac{\eta_j}{\eta_{j'}}\right)^{\nu}}\left[1 + \theta\sum_{j'=1}^{J}\left(\frac{x}{\eta_{j'}}\right)^{\nu}\right]^{-\frac{1}{\theta}}.
\end{aligned} \tag{5.2.2}$$

A common issue that arises with these frailty models is the problem of identifiability of the model parameters. Using Theorem 2.1 of Ghosh et al. [39], it can be easily shown that the model presented above is identifiable from the observed competing risks data on (C, T). Towards this end, we obtain RF of the lifetime X_i. RF of X_i is given by

$$R(x \mid \boldsymbol{\theta}) = P(X_i > x) = P(X_{i1} > x, \ldots, X_{iJ} > x) = \overline{R}(x\mathbf{1} \mid \boldsymbol{\theta}) = \left[1 + \theta\sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\right]^{-\frac{1}{\theta}}. \tag{5.2.3}$$

Interestingly, as $\theta \to 0$, we have

$$R(x \mid \boldsymbol{\theta}) = \exp\left[-\sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\right], \tag{5.2.4}$$

which is essentially RF of X_i under the assumption of independence among the potential failure times (Roy and Mukhopadhyay [103]). Thus, the independent competing risks model can be obtained as a special case of the shared gamma frailty model presented here. Accordingly, the sub-survivor function is given by

$$\overline{G}(j, x \mid \boldsymbol{\theta}) = \frac{1}{\sum_{j'=1}^{J}\left(\frac{\eta_j}{\eta_{j'}}\right)^{\nu}} \exp\left[-\sum_{j'=1}^{J}\left(\frac{x}{\eta_{j'}}\right)^{\nu}\right]. \tag{5.2.5}$$

The degree of dependence among potential failure times (Crowder [34, pp. 37-38]) X_js is often measured by the following ratio

$$\varrho(x) = \frac{\overline{R}\left(x\mathbf{1} \mid \boldsymbol{\theta}\right)}{\prod_{j=1}^{J} R_j\left(x \mid \boldsymbol{\theta}_j\right)} = \frac{\left[1 + \theta\sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\right]^{-\frac{1}{\theta}}}{\exp\left[-\sum_{j'=1}^{J}\left(\frac{x}{\eta_{j'}}\right)^{\nu}\right]},$$

If $\varrho(x) > 1$, there is a positive dependence among the potential failure times X_js, for all x. It is easy to show that $\varrho(x)$ is an increasing function of θ, and it attains a value of 1 only when $\theta \to 0$. Thus, any positive value of θ (i.e., $\theta > 0$) induces positive dependence among the potential failure times X_js.

5.3 Development of RASPs

Suppose n test units, randomly selected from a lot, are put on a life test under PICS-I to decide the acceptability of the lot. In practice, the decision to accept or reject such a lot is often based on the product's reliability at some pre-decided time point mutually agreed upon by the producer and consumer. Thus, a lot is accepted if the reliability of the unit $R(x)$ at some specified time point t_0 is greater than π_0 and it is rejected if it is lower than π_1 (Lam [58]). Obviously, $\pi_0 \geq \pi_1$. The appropriate values of π_0 and π_1 are decided based on an agreement between the producer and the consumer. It is easy to see that the above decision rule is equivalent to testing the following set of simple hypotheses:

$$H_0 : R(t_0) = \pi_0 \quad \text{vs} \quad H_A : R(t_0) = \pi_1. \tag{5.3.1}$$

In practice, $R(t_0)$ needs to be estimated. Let $\widehat{R}(t_0)$ be the MLE of $R(t_0)$, which is defined as $\widehat{R}(x) = R(x \mid \widehat{\theta})$. The lot is accepted if $\widehat{R}(t_0) > \pi_c$ and rejected if $\widehat{R}(t_0) \leq \pi_c$, where π_c is an appropriately chosen acceptance limit. In traditional RASPs, we need to decide n and π_c. Let α and β be the producer's risk and consumer's risk, respectively. We have

$$P\left(\widehat{R}(t_0) > \pi_c \mid H_0\right) = 1 - \alpha \quad \text{and} \quad P\left(\widehat{R}(t_0) > \pi_c \mid H_A\right) = \beta. \tag{5.3.2}$$

Now, from Theorem 5.2, we get that $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ follows asymptotically normal distribution with mean $\mathbf{0}$ and variance-covariance $I^1(\boldsymbol{\theta}|\boldsymbol{\zeta})^{-1}$, where $I^1(\boldsymbol{\theta}|\boldsymbol{\zeta}) = I(\boldsymbol{\theta}|\boldsymbol{\zeta})/n$. Furthermore, using the delta method, $\sqrt{n}[\widehat{R}(t_0) - R(t_0)]$ follows asymptotically $\mathcal{N}(0, S^2)$, where $S^2 = \boldsymbol{C}_T'[I^1(\boldsymbol{\theta}|\boldsymbol{\zeta})]^{-1}\boldsymbol{C}_T$ and $\boldsymbol{C}_T = \nabla_{\boldsymbol{\theta}} R(t_0)$ (Meeker et al. [82]). Hence

$$Z = \frac{\sqrt{n}[\widehat{R}(t_0) - R(t_0)]}{S}$$

is asymptotically $\mathcal{N}(0, 1)$. Now, as Theorem 5.1 suggests, we need to compute $\nabla_{\boldsymbol{\theta}}(q_{ij})$ and $\nabla_{\boldsymbol{\theta}}(q_i)$ to obtain $I(\boldsymbol{\theta} \mid \boldsymbol{\zeta})$. From (5.1.1) and (5.1.2), we have

$$\log(q_{ij}) = \log\left(\overline{G}\left(j, \tau_{i-1}|\boldsymbol{\theta}\right) - \overline{G}\left(j, \tau_i|\boldsymbol{\theta}\right)\right) - \log\left(R\left(\tau_{i-1}|\boldsymbol{\theta}\right)\right)$$

and

$$\log(1 - q_i) = \log\left(R\left(\tau_i|\boldsymbol{\theta}\right)\right) - \log\left(R\left(\tau_{i-1}|\boldsymbol{\theta}\right)\right), \tag{5.3.3}$$

for $j = 1, \ldots, J$ and $i = 1, \ldots, k$. Now, using (5.2.2) and (5.2.5), the term $\log(q_{ij})$ can be written as

$$\log(q_{ij}) = \log\left(R\left(\tau_{i-1}|\boldsymbol{\theta}\right) - R\left(\tau_i|\boldsymbol{\theta}\right)\right) - \psi_j\left(\boldsymbol{\theta}\right) - \log\left(R\left(\tau_{i-1}|\boldsymbol{\theta}\right)\right), \tag{5.3.4}$$

where $\psi_j(\boldsymbol{\theta}) = \log\left(\sum_{j'=1}^{J}\left(\frac{\eta_j}{\eta_{j'}}\right)^{\nu}\right)$. Now, differentiating both side w.r.t θ_u of (5.3.4) and (5.3.3), we get

$$\begin{aligned}\frac{1}{q_{ij}}\left(\frac{\partial q_{ij}}{\partial \theta_u}\right) =& \frac{1}{R\left(\tau_{i-1}|\boldsymbol{\theta}\right) - R\left(\tau_i|\boldsymbol{\theta}\right)}\left[\left(\frac{\partial R\left(\tau_{i-1}|\boldsymbol{\theta}\right)}{\partial \theta_u}\right) - \left(\frac{\partial R\left(\tau_i|\boldsymbol{\theta}\right)}{\partial \theta_u}\right)\right] - \left(\frac{\partial \psi_j\left(\boldsymbol{\theta}\right)}{\partial \theta_u}\right) \\ & - \frac{1}{R\left(\tau_{i-1}|\boldsymbol{\theta}\right)}\left(\frac{\partial R\left(\tau_{i-1}|\boldsymbol{\theta}\right)}{\partial \theta_u}\right),\end{aligned}$$

and

$$\left(\frac{1}{1 - q_i}\right)\left(\frac{\partial q_i}{\partial \theta_u}\right) = \frac{1}{R\left(\tau_{i-1}|\boldsymbol{\theta}\right)}\left(\frac{\partial R\left(\tau_{i-1}|\boldsymbol{\theta}\right)}{\partial \theta_u}\right) - \frac{1}{R(\tau_i|\boldsymbol{\theta})}\left(\frac{\partial R(\tau_i|\boldsymbol{\theta})}{\partial \theta_u}\right),$$

for $u = 1, \ldots, l$.

Thus, it is straightforward to obtain S^2, since we readily have

$$\frac{\partial R(x|\boldsymbol{\theta})}{\partial \eta_j} = \frac{\nu}{\eta_j}\left[1 + \theta \sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\right]^{-\frac{1}{\theta}-1}\left(\frac{x}{\eta_j}\right)^{\nu},$$

$$\frac{\partial R(x|\boldsymbol{\theta})}{\partial \nu} = -\left[1 + \theta \sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\right]^{-\frac{1}{\theta}-1}\sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\log\left(\frac{x}{\eta_j}\right)$$

and

$$\frac{\partial R(x|\boldsymbol{\theta})}{\partial \theta} = \left[1 + \theta \sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\right]^{-\frac{1}{\theta}}\left[\frac{1}{\theta^2}\log\left(1 + \theta \sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\right) - \frac{1}{\theta}\left(\frac{\sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}}{1 + \theta \sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}}\right)\right]$$

for the dependent competing risks model in (5.2.2). Similarly, for the independent competing risks model in (5.2.5), we have

$$\frac{\partial R(x|\boldsymbol{\theta})}{\partial \eta_j} = \frac{\nu}{\eta_j}\left(\frac{x}{\eta_j}\right)^{\nu}\exp\left(-\sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\right) \quad \text{and} \quad \frac{\partial R(x|\boldsymbol{\theta})}{\partial \nu} = -\exp\left(-\sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\right)\sum_{j=1}^{J}\left(\frac{x}{\eta_j}\right)^{\nu}\log\left(\frac{x}{\eta_j}\right).$$

Furthermore, we also note that

$$\frac{\partial \psi_j(\boldsymbol{\theta})}{\partial \eta_l} = \begin{cases} \dfrac{\left(\frac{\nu}{\eta_l}\right)\sum_{j'\neq j}\left(\frac{\eta_j}{\eta_{j'}}\right)^{\nu}}{\sum_{j'=1}^{J}\left(\frac{\eta_j}{\eta_{j'}}\right)^{\nu}} & \text{if } l = j \\ -\dfrac{\left(\frac{\nu}{\eta_l}\right)\sum_{j'\neq j}\left(\frac{\eta_j}{\eta_{j'}}\right)^{\nu}}{\sum_{j'=1}^{J}\left(\frac{\eta_j}{\eta_{j'}}\right)^{\nu}} & \text{if } l \neq j \end{cases} \quad \text{and} \quad \frac{\partial \psi_j(\boldsymbol{\theta})}{\partial \nu} = \frac{\sum_{j=1}^{j}\left(\frac{\eta_j}{\eta_{j'}}\right)^{\nu}\log\left(\frac{\eta_j}{\eta_{j'}}\right)}{\sum_{j'=1}^{J}\left(\frac{\eta_j}{\eta_{j'}}\right)^{\nu}}.$$

Towards this end, let us denote by S_0 and S_1 the values of S, under H_0 and H_A, respectively. Then, from (5.3.2), we have

$$\Phi\left(\frac{\sqrt{n}[\pi_c - \pi_0]}{S_0}\right) = \alpha \quad \text{and} \quad \Phi\left(\frac{\sqrt{n}[\pi_c - \pi_1]}{S_1}\right) = 1 - \beta.$$

Solving these two equations, we get

$$\pi_c = \frac{\pi_0 S_1 z_\beta - \pi_1 S_0 z_{1-\alpha}}{S_1 z_\beta - S_0 z_{1-\alpha}}, \tag{5.3.5}$$

where z_δ is such that $P(Z > z_\delta) = \delta$. Moreover, using the expression of π_c, we get n as follows:

$$n = \left[\frac{S_1 z_\beta - S_0 z_{1-\alpha}}{\pi_0 - \pi_1}\right]^2. \tag{5.3.6}$$

For convenience, we reproduce the entire process of finding n and π_c in Algorithm 5.1.

Algorithm 5.1: Determination of n and π_c.

Input : α, β, t_0 π_0, π_1

Output : Sample size n^*, acceptance limit π_c

Compute S_0 and S_1, the standard deviations of $\widehat{R}(t_0 \mid \boldsymbol{\theta})$ under H_0 and H_A, respectively.;

Compute the acceptance limit π_c using (5.3.5).;

Calculate the value of n using (5.3.6).;

The required sample size is taken as $n^* = \lfloor n \rfloor$.

5.4 Optimal PICS-I

In the previous section, we presented the method for obtaining the optimal n and π_c, for given α, β, π_0 and π_1. In practice, however, it is essential to suitably determine inspection times and the proportions of withdrawals to efficiently conduct life tests under PICS-I. In general, the current literature presents two primary approaches to resolve this issue. The first approach deals with achieving the estimation precision of some important reliability characteristics and presents the optimal design parameters considering a suitably constructed design criterion that aligns with this goal. This approach typically ignores the cost constraints that are ubiquitous in the context of life tests. Thus, the second approach proposes a cost-based objective function and determines the optimal design parameters by minimizing such a function. In this context, it is also important to highlight that there are a select number of studies that integrate both these approaches. However, such work is largely absent in the context of PICS-I and, more importantly, in the presence of competing causes of failure. Towards this end, we present the first approach in Section 5.4.1 and subsequently deal with the combined approach in Section 5.4.2.

5.4.1 Optimal RASP without cost constraints

Since our main focus is development of reliability-based sampling plan, we consider a design criterion that focuses on precise estimation of $\widehat{R}(x)$. This, in turn, would be helpful in accurately deciding the lot acceptance criteria presented in the previous section. Thus, we consider a design criterion that aims to minimize the variance of $\widehat{R}(x)$ at a specific time point t_0. In particular, our design criterion is given by

$$\phi(\boldsymbol{\zeta}) = S^2, \tag{5.4.1}$$

where S^2 is the standardized variance, as in the previous section (Atkinson et al. [5]). This design criterion is more commonly known as the c-optimality criterion in the literature. It is easy to see that $\phi(\boldsymbol{\zeta})$ depends on the unknown true parameter $\boldsymbol{\theta}$. Thus, in order to compute it, a suitable set of guess values for the parameters is typically used. Note that such guess values are often available in practice from similar life tests performed in the past (Meeker et al. [82]).

It is easy to see that $\phi(\zeta)$ is independent of n. To compute the optimal $\boldsymbol{\zeta}$, say $\boldsymbol{\zeta}^*$, we minimize $\phi(\boldsymbol{\zeta})$ with respect to k, $\boldsymbol{\tau}$ and $\boldsymbol{p}$. Towards this end, as is standard in the existing literature, we assume that inspections are equispaced, i.e., for $i = 1, \ldots, k$, $\tau_i - \tau_{i-1} = h$, where h is the common time interval between two successive inspections. Furthermore, we also assume that $p_1 = \ldots = p_{k-1} = p$, where p is the common proportion of withdrawals at each τ_i. Note that such PICS-I are particularly popular among reliability engineers, mainly because of the straightforwardness they offer in practical applications. Thus, we have three decision variables in this context, resulting in $\boldsymbol{\zeta} = (k, h, p)$. However, as the following theorem shows, the complexity of the optimization problem reduces further due to the following monotonicity properties of $\phi(\boldsymbol{\zeta})$.

Theorem 5.3 *The design criterion $\phi(\boldsymbol{\zeta})$, given in (5.4.1), satisfies the following monotonicity properties:*

(i) For fixed (h, p), $\phi(\boldsymbol{\zeta})$ is decreasing in k.

(ii) For fixed (k, h), $\phi(\boldsymbol{\zeta})$ is increasing in p.

Proof: The proof is given in Appendix 5.A.2. (Proved)

Theorem 5.3 suggests that $\phi(\boldsymbol{\zeta})$ attains its optimum value for the decision variables k and p at their respective limits. Thus, we must fix them *a priori* and determine the optimal value of h, say h^*, by minimizing $\phi(\boldsymbol{\zeta})$. Subsequently, we determine π_c and n using (5.3.5) and (5.3.6), respectively. For convenience, we summarize this entire process in Algorithm 5.2.

Algorithm 5.2: Determination of optimal decision variables without cost constraints

Input: α, β, t_0, π_0, π_1, k_0, p_0
Output: Optimal sample size n^* and acceptance limit π_c
Obtain h^* by minimizing S^2 ; ▷ `Expression for` S^2 `in Section 5.3`
Calculate n using formula (5.3.6) with h^*;
Set $n^* = \lfloor n \rfloor$;
Compute acceptance limit π_c using formula (5.3.5) with $\boldsymbol{\zeta}^* = (n^*, k_0, h^*, p_0)$;
return n^*, π_c

Note that h^* can be solved using any standard optimization module (e.g., *optim()* function in *R*).

5.4.2 Optimal RASP with cost constraints

Reliability experiments are often conducted under strict budgetary limitations; the approach presented in the previous subsection overlooks these budgetary constraints. To resolve this issue, we now incorporate the budgetary constraints into the optimization problem presented above and provide an enhanced solution that aligns with real-world budgetary constraints.

Let $TC(\boldsymbol{\zeta})$ denote the total cost of running a life test under PICS-I. The total cost corresponding to a life test under PICS-I is given by

$$TC(\boldsymbol{\zeta}) = nC_s + C_t E[\tau] + C_d E[D] + C_I E[I].$$

The cost coefficient C_s, C_t, C_d and C_I are defined in Chapter 2. Let C_B be the available budget for conducting the life test. Then, for given (α, β), the decision problem is to find the optimal $\boldsymbol{\zeta}^* = (n^*, k^*, h^*, p^*)$ by solving

$$\text{minimize } \phi(\boldsymbol{\zeta}) \tag{5.4.2}$$

subject to

$$TC(\boldsymbol{\zeta}) \leq C_B, \tag{5.4.3}$$

$$n = \left[\frac{S_1 z_\beta - S_0 z_{1-\alpha}}{\pi_0 - \pi_1}\right]^2, \tag{5.4.4}$$

$$k \geq s, \; n \in \mathbb{N}, \; k \in \mathbb{N}, \; h > 0 \text{ and } \; 0 \leq p < 1.$$

It is obvious that (5.4.3) represents the cost constraint, as discussed above. Note that the equality constraint in (5.4.4) is derived in Section 5.3. Furthermore, as the Regularity Condition VII(b) suggests, k must be greater than s, the number of parameters in the model. From (5.4.4), it is easy to see that n depends on k, h and p. We reduce the complexity of the decision problem by substituting (5.4.4) both in (5.4.2) and (5.4.3). Furthermore, as in Huang and Wu [45], we pre-fix the uniform withdrawal proportion p. Thus, the optimization problem involves only the decision variables k and h. Furthermore, as Theorem 5.3 suggests, $\phi(\boldsymbol{\zeta})$ is decreasing in k, while $TC(\boldsymbol{\zeta})$ is increasing in k. Now, we propose Algorithm 5.3 to compute the optimal decision variables.

Algorithm 5.3: Determination of optimal decision variables with cost constraints

Input: α, β, t_0, π_0, π_1, p_0 and k_0;

Output: Optimal n^*, k^*, h^* and acceptance limit π_c;

Initialize $k \leftarrow s$ ▷ s is the lower bound for k

while $k < k_0$ **do**

- Obtain the solution $h^*(k)$ minimizing $\phi(\boldsymbol{\zeta})$ subject to cost constraint (5.4.3);
- Compute n using (5.4.4);
- Set $n^*(k) = \lfloor n \rfloor$ ▷ Optimal n for given k;
- Compute $\phi(\boldsymbol{\zeta})$ for $\boldsymbol{\zeta} = (n^*(k), k, h^*(k), p_0)$;
- Update $k \leftarrow k + 1$

Find k^* such that

$$\phi(n^*(k^*), k^*, h^*(k^*), p_0) = \min_{s \leq k \leq k_0} \phi(n^*(k), k, h^*(k), p_0)$$

Set $n^* \leftarrow n^*(k^*)$ and $h^* \leftarrow h^*(k^*)$;

Compute acceptance limit π_c from (5.3.5) using $\boldsymbol{\zeta}^* = (n^*, k^*, h^*, p_0)$;

5.5 Numerical experiment

In this section, we conduct a detailed numerical study to assess how the degree of dependence among the potential failure times impacts the resulting RASPs. We consider the battery failure data set after recalibrating it in units of thousands of ampere-hours to determine RASPs.

The battery failure data set includes four causes of failure and provides the number of failures attributed to each cause at each inspection point, along with the corresponding number of withdrawals. The limited number of observed failures for each cause leads us to group causes "1" and "3" together as the first cause of failure. Similarly, causes "2" and "4" are considered together as the second cause of failure. Therefore, we have $J = 2$ for the subsequent analysis. We also use the independent competing risk model presented in (5.2.5) and obtain the MLEs $\widehat{\eta}_1 = 1.291$, $\widehat{\eta}_2 = 1.339$ and $\widehat{\nu} = 1.644$ by maximizing the log-likelihood function $l(\boldsymbol{\theta} \mid \boldsymbol{x})$ in (5.1.3).

As Algorithms 5.1, 5.2 and 5.3 suggest, we further need to compute S_0 and S_1, the values of S under H_0 and H_A as specified in (5.3.1). However, this would require the specification of the unknown model parameters. Towards this end, we adopt the approach of Wu et al. [121]. We note that the reliability functions $R(t_0 \mid \boldsymbol{\theta})$ in (5.2.3) is an increasing function of η_j for each $j = 1, \ldots, J$. Thus, the hypotheses in (5.3.1) can be equivalently stated as follows:

$$H_0 : \boldsymbol{\eta} = \boldsymbol{\eta}_0 \quad \text{vs} \quad H_A : \boldsymbol{\eta} = \boldsymbol{\eta}_1,$$

where $\boldsymbol{\eta}_0 = (\eta_{01}, \ldots, \eta_{0J})$ and $\boldsymbol{\eta}_1 = (\eta_{11}, \ldots, \eta_{1J})$. A lot is accepted when $\boldsymbol{\eta} > \boldsymbol{\eta}_0$ i.e, $\eta_i > \eta_{0i}$ for $i = 1, \ldots, J$ and it is rejected when $\boldsymbol{\eta} < \boldsymbol{\eta}_1$ i.e, $\eta_i < \eta_{1i}$ for $i = 1, \ldots, J$. Obviously, $\eta_{0j} \geq \eta_{1j}$, for $j = 1, \ldots, J$. For the subsequent analysis, we consider $\boldsymbol{\eta}_0 = (\eta_{01}, \eta_{02}) = (1.291, 1.339)$, which are the MLEs of η_1 and η_2. To decide $\boldsymbol{\eta}_1$, we further define the discrimination ratio d_j as $d_j = \eta_{0j}/\eta_{1j}$ and set it to a suitable set of values as detailed in the subsequent sub-sections. Throughout the analysis, we have used $\nu = 1.644$, which is the MLE of ν. Furthermore, we consider $(\alpha, \beta) = (0.05, 0.1)$. Also, the specific time point of interest t_0 is fixed at 0.5.

5.5.1 Determination of RASPs

As evident from Section 5.3, we must specify k, $\boldsymbol{\tau}$ and $\boldsymbol{p}$ first to obtain the sample size n and the acceptance limit π_c. For convenience, we consider the equispaced PICS-I with uniform withdrawal proportions introduced in Section 5.4. In particular, we consider three options for the number of inspections k (i.e., 4, 6, and 8). In each case, we further select three choices for both the inspection time interval h (i.e., 0.2, 0.3, and 0.4) and the withdrawal proportion p (i.e., 0.0, 0.2, 0.3). Moreover, we consider the same discrimination ratio for both causes and choose two possible values for the discrimination ratio d (i.e., 1.5 and 1.8). We then obtain the optimal values of π_c and n both for the independent and dependent competing risks model presented in Section 5.2. For the dependent competing risks model, we further consider two different levels of dependence considering two different choices for θ (i.e., 0.5 and 1). Table 5.1 provides the resulting optimal values of π_c and n in each case. For convenience, we represent the independent competing risks model using $\theta = 0$ in Table 5.1.

Table 5.1: Determination of RASPs

	θ	h	$d = 1.5$						$d = 1.8$					
			$k = 4$		$k = 6$		$k = 8$		$k = 4$		$k = 6$		$k = 8$	
			π_c	n^*	π_c	n^*	π_c	n^*	π_c	n^*	π_c	n^*	π_c	n^*
$p = 0$	0	0.2	0.547	32	0.546	31	0.547	31	0.482	13	0.481	13	0.481	13
		0.3	0.546	33	0.547	33	0.547	33	0.481	14	0.482	13	0.482	13
		0.4	0.548	36	0.548	36	0.548	36	0.483	15	0.484	15	0.484	15
	0.5	0.2	0.593	57	0.594	51	0.594	49	0.544	25	0.546	23	0.547	22
		0.3	0.594	53	0.594	50	0.594	49	0.547	24	0.547	22	0.547	22
		0.4	0.593	53	0.594	53	0.593	53	0.546	24	0.545	23	0.545	23
	1	0.2	0.627	76	0.628	70	0.629	67	0.587	34	0.589	32	0.590	30
		0.3	0.629	72	0.628	68	0.628	67	0.590	33	0.590	31	0.590	30
		0.4	0.628	71	0.628	70	0.628	70	0.589	32	0.588	32	0.588	32
$p = 0.2$	0	0.2	0.548	38	0.547	37	0.547	37	0.484	16	0.484	15	0.484	15
		0.3	0.547	36	0.547	36	0.547	36	0.482	15	0.482	15	0.482	15
		0.4	0.547	37	0.547	37	0.547	37	0.482	15	0.483	15	0.483	15
	0.5	0.2	0.593	71	0.594	66	0.595	63	0.544	31	0.546	29	0.547	28
		0.3	0.595	60	0.595	56	0.595	55	0.548	27	0.549	25	0.549	25
		0.4	0.594	56	0.594	55	0.594	54	0.547	25	0.546	24	0.546	24
	1	0.2	0.627	93	0.628	88	0.628	86	0.587	42	0.589	40	0.589	39
		0.3	0.629	82	0.629	77	0.629	75	0.590	38	0.591	35	0.591	35
		0.4	0.628	75	0.628	73	0.628	73	0.589	34	0.589	33	0.589	33
$p = 0.3$	0	0.2	0.548	44	0.548	42	0.548	42	0.486	18	0.486	17	0.486	17
		0.3	0.547	37	0.547	37	0.547	37	0.483	15	0.483	15	0.483	15
		0.4	0.547	38	0.547	38	0.547	38	0.482	16	0.483	15	0.483	15
	0.5	0.2	0.593	82	0.594	77	0.595	74	0.544	36	0.546	34	0.547	33
		0.3	0.595	66	0.595	61	0.595	60	0.549	29	0.549	27	0.550	27
		0.4	0.594	57	0.594	56	0.594	56	0.548	26	0.547	25	0.547	25
	1	0.2	0.627	107	0.628	102	0.628	100	0.587	48	0.588	46	0.589	45
		0.3	0.629	89	0.629	83	0.629	82	0.591	41	0.591	38	0.591	38
		0.4	0.628	77	0.628	75	0.628	75	0.590	35	0.590	34	0.589	34

It is interesting to note that as θ increases, both π_c and n increase irrespective to the choices of k, h, p, and d. Furthermore, the resulting optimal π_c and n are higher for the dependent competing risks model compared to the independent competing risks model. This is perhaps due to the fact that RF increases with θ. For given θ, h, k, the optimal π_c is rather insensitive to the choice of withdrawal proportion p. However, the optimal n increases with p both for the dependent and independent competing risks model. Furthermore, the effect of k on π_c and n seems to be insignificant, particularly for higher d. As expected, both π_c and n decrease as d increases.

5.5.2 Optimal RASP without budgetary constraint

We now consider the determination h, n and π_c using the c-optimality design criterion presented in Section 5.4. As Algorithm 5.2 suggests, we first fix k and p at the three possible values presented above and obtain the optimal h by minimizing $\phi(\boldsymbol{\zeta})$. As evident from Step 5 in Algorithm 5.2, determining the optimal h involves solving a one-dimensional optimization problem. To ensure stability, we experimented with several reasonable starting values, and the resulting RASPs were consistently robust to variations in initial parameter guesses.

Table 5.2: Optimal RASPs without budgetary constraints

p_0	θ	$k_0 = 4$				$k_0 = 6$				$k_0 = 8$			
		h^*	$10\times$ $\phi(\boldsymbol{\zeta}^*)$	n^*	π_c	h^*	$10\times$ $\phi(\boldsymbol{\zeta}^*)$	n^*	π_c	h^*	$10\times$ $\phi(\boldsymbol{\zeta}^*)$	n^*	π_c
0	0	0.197	1.649	32	0.547	0.143	1.601	31	0.547	0.112	1.579	30	0.547
	0.5	0.332	1.789	53	0.594	0.265	1.695	50	0.594	0.226	1.650	49	0.594
	1	0.348	1.767	71	0.628	0.284	1.683	68	0.629	0.246	1.643	66	0.629
0.2	0	0.272	1.857	36	0.547	0.282	1.854	36	0.547	0.284	1.850	36	0.547
	0.5	0.384	1.896	56	0.594	0.351	1.852	55	0.594	0.338	1.839	54	0.594
	1	0.40	1.857	75	0.620	0.371	1.819	73	0.628	0.36	1.806	73	0.628
0.3	0	0.317	1.931	37	0.547	0.325	1.927	37	0.547	0.325	1.926	37	0.547
	0.5	0.407	1.945	57	0.594	0.385	1.910	56	0.594	0.378	1.901	56	0.594
	1	0.421	1.892	76	0.628	0.402	1.866	75	0.628	0.396	1.865	75	0.628

It is evident that the optimal inspection interval h gets larger as θ increases, irrespective of the choices of p and k. Thus, as the level of dependence among the potential causes of failure goes up, the successive inspections will be more infrequent. Furthermore, as the withdrawal proportion p increases, the optimal interval h becomes longer irrespective of the choices of θ and k. Thus, the successive inspections in PICS-I are expected to be less frequent in comparison to a traditional ICS. Furthermore, as the number of inspections k increases, the inspection intervals are slightly wider. As observed in the first case, the sample size n increases as the level of dependence gets stronger. This is also true for π_c. Furthermore, as expected, the sample size n decreases significantly as the discrimination ratio d increases. We also note that the value of the design criterion increases as the withdrawal proportion p increases, whereas it goes down as k increases. Note that this is consistent with Theorem 5.3.

Overall, the optimal inspection interval (h) increases with both the dependence level (θ) and the withdrawal proportion (p), indicating less frequent inspections under stronger dependence or higher unit removals. The required sample size (n) and the π_c both increase as the dependence intensifies, whereas higher discrimination ratios lead to smaller sample sizes. Finally, the value of the design criterion rises with p but decreases with k, which is consistent with the theoretical findings.

5.5.3 Optimal RASP with budgetary constraint

In this subsection, we consider the budgetary constraint and obtain the optimal decision variables n, h, and π_c using Algorithm 5.3. For the purpose of illustration, we assume $C_s = 0.1$, $C_t = 5$, $C_d = 0.025$, and $C_I = 10$, where each of these cost components is as in Section 5.4.2. Furthermore, we consider four possible values of the available budget C_B, i.e., 55, 65, 85, and 95, primarily to check the sensitivity of the design parameters with respect to available budgetary allocation. Note that Step 8 of Algorithm 5.3 involves a one-dimensional constrained optimization problem, which is observed to be largely insensitive to the choice of starting values.

Table 5.3 presents the optimal values of n, k, h, π_c under various scenarios for the common discrimination ratio $d = 1.5$. Table 5.3 further reports $E[D^*]$, $E[\tau^*]$, $E[I^*]$ and $\phi(\zeta^*)$. The corresponding results for the common discrimination ratio $d = 1.8$ are given in Table 5.4.

Table 5.3: Optimal RASPs with budgetary constraints for $d = 1.5$

p_0	C_B	θ	(n^*, k^*, h^*)	π_c	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	$10 \times \phi(\zeta^*)$
		0	(32, 4, 0.196)	0.547	18.394	0.784	4.000	1.650
	55	0.5	(53, 4, 0.332)	0.594	39.973	1.328	4.000	1.789
		1	(71, 4, 0.330)	0.628	47.443	1.320	4.000	1.770
		0	(31, 5, 0.165)	0.547	18.769	0.825	5.000	1.619
	65	0.5	(51, 5, 0.293)	0.594	40.417	1.465	5.000	1.732
		1	(71, 4, 0.348)	0.628	48.797	1.392	4.000	1.766
0		0	(30, 7, 0.125)	0.547	19.230	0.875	7.000	1.588
	85	0.5	(50, 6, 0.265)	0.594	41.079	1.590	6.000	1.695
		1	(68, 6, 0.284)	0.629	51.270	1.704	6.000	1.683
		0	(30, 7, 0.125)	0.547	19.230	0.875	7.000	1.588
	95	0.5	(49, 7, 0.243)	0.594	41.332	1.701	7.000	1.669
		1	(67, 7, 0.263)	0.629	52.045	1.841	7.000	1.661
		0	(36, 4, 0.272)	0.547	20.095	1.088	4.000	1.857
	55	0.5	(56, 4, 0.384)	0.594	35.120	1.536	4.000	1.896
		1	(85, 4, 0.277)	0.629	38.166	1.108	4.000	2.087
		0	(36, 5, 0.274)	0.547	21.833	1.362	4.970	1.857
	65	0.5	(56, 4, 0.384)	0.594	35.120	1.536	4.000	1.896
		1	(75, 4, 0.400)	0.628	43.207	1.60	4.000	1.853
0.2		0	(36, 8, 0.283)	0.547	23.350	1.748	6.177	0.1850
	85	0.5	(55, 6, 0.350)	0.594	35.138	2.075	5.930	1.852
		1	(73, 6, 0.371)	0.628	43.406	2.224	5.996	1.819
		0	(36, 8, 0.283)	0.547	23.350	1.748	6.177	1.850
	95	0.5	(54, 8, 0.337)	0.594	34.526	2.465	7.314	1.839
		1	(73, 7, 0.364)	0.628	43.638	2.536	6.966	1.810
		0	(37, 4, 0.316)	0.547	19.940	1.259	3.985	1.931
	55	0.5	(57, 4, 0.406)	0.594	32.571	1.623	3.997	1.942
		1	(111, 4, 0.178)	0.627	27.896	0.712	4.000	2.847
		0	(37, 6, 0.324)	0.547	21.309	1.625	5.014	1.927
	65	0.5	(57, 4, 0.406)	0.594	32.571	1.623	3.997	1.942
		1	(76, 4, 0.420)	0.628	39.972	1.680	4.000	1.897
0.3		0	(37, 6, 0.324)	0.547	21.309	1.625	5.014	1.927
	85	0.5	(56, 9, 0.376)	0.594	32.014	2.417	6.429	1.904
		1	(75, 6, 0.401)	0.628	40.001	2.372	5.916	1.869
		0	(36, 8, 0.283)	0.547	23.350	1.748	6.177	1.849
	95	0.5	(56, 9, 0.376)	0.594	32.014	2.417	6.429	1.904
		1	(75, 7, 0.397)	0.6280	40.037	2.650	6.675	1.865

Table 5.4: Optimal RASPs with budgetary constraints for $d = 1.8$

p_0	C_B	θ	(n^*, k^*, h^*)	π	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	$\phi(\boldsymbol{\zeta}^*) \times 10$
0	55	0	(13, 4, 0.196)	0.482	7.473	0.784	4.000	1.650
		0.5	(23, 4, 0.332)	0.547	17.347	1.328	4.000	1.789
		1	(32, 4, 0.3480)	0.590	21.993	1.392	4.000	1.766
	65	0	(13, 5, 0.165)	0.482	7.871	0.825	5.000	1.619
		0.5	(23, 5, 0.293)	0.5470	18.2270	1.4650	5.0000	1.732
		1	(32, 5, 0.311)	0.590	23.200	1.555	5.000	1.716
	85	0	(12, 7, 0.125)	0.482	7.692	0.875	6.999	1.588
		0.5	(22, 7, 0.243)	0.547	18.557	1.699	6.994	1.669
		1	(31, 7, 0.263)	0.590	24.080	1.841	7.000	1.661
	95	0	(12, 8, 0.112)	0.482	7.864	0.896	7.999	1.571
		0.5	(21, 8, 0.225)	0.547	18.069	1.796	7.982	1.649
		1	(30, 8, 0.246)	0.590	23.857	1.968	8.000	1.643
0.2	55	0	(15, 4, 0.272)	0.482	8.373	1.079	3.968	1.857
		0.5	(25, 4, 0.384)	0.547	15.679	1.531	3.986	1.896
		1	(34, 4, 0.400)	0.589	19.587	1.600	3.999	1.833
	65	0	(15, 7, 0.283)	0.482	9.691	1.493	5.277	1.851
		0.5	(24, 5, 0.363)	0.548	15.263	1.769	4.874	1.867
		1	(34, 5, 0.382)	0.590	20.005	1.904	4.984	1.819
	85	0	(15, 7, 0.283)	0.482	9.691	1.493	5.277	1.851
		0.5	(24, 10, 0.332)	0.548	15.327	2.267	6.829	1.834
		1	(33, 7, 0.364)	0.590	19.727	2.439	6.701	1.811
	95	0	(15, 7, 0.283)	0.482	9.691	1.493	5.277	1.851
		0.5	(24, 10, 0.332)	0.548	15.327	2.267	6.829	1.834
		1	(33, 8, 0.359)	0.590	19.768	2.641	7.356	1.806
0.3	55	0	(15, 6, 0.324)	0.483	8.639	1.404	4.334	1.927
		0.5	(25, 4, 0.406)	0.547	14.286	1.592	3.920	1.942
		1	(35, 4, 0.420)	0.589	18.408	1.675	3.988	1.897
	65	0	(15, 6, 0.324)	0.483	8.639	1.404	4.334	1.927
		0.5	(25, 6, 0.384)	0.548	14.345	1.932	5.031	1.912
		1	(34, 5, 0.408)	0.589	18.074	1.990	4.876	1.878
	85	0	(15, 6, 0.324)	0.483	8.639	1.404	4.334	1.927
		0.5	(25, 8, 0.377)	0.548	14.298	2.034	5.394	1.905
		1	(34, 10, 0.393)	0.590	18.158	2.623	6.673	1.860
	95	0	(15, 6, 0.324)	0.483	8.639	1.404	4.334	1.927
		0.5	(25, 8, 0.377)	0.548	14.298	2.034	5.394	1.905
		1	(34, 10, 0.393)	0.590	18.158	2.623	6.673	1.860

We observe that the required sample size and acceptance limit increase as the level of dependence gets stronger for both $d = 1.5$ and 1.8. This is true irrespective of the size of the available budget and the withdrawal proportion p. Also, the inspections are expected to get infrequent as the level of dependence goes up in each case. However, the optimal number of inspections is typically insensitive to the level of dependence at a fixed budget and withdrawal proportion. However, for a given level of dependence, as the available budget increases, the number of inspections k goes up marginally. This is true for every withdrawal proportion. Interestingly, the sample size is rather insensitive to the available budget for a given level of dependence. However, the inspection interval is expected to be shorter as the available budget increases. As expected, the sample size requirement significantly drops as the discrimination ratio goes up. Furthermore, the acceptance limit also goes down with the increase in the discrimination ratio. It is also interesting to note that the expected number of failures and expected duration increase as the level of dependence becomes stronger.

As noted in Section 5.4.1, the design criterion $\phi(\boldsymbol{\zeta})$ does not depend on the sample size n. For comparative purposes, we additionally consider the traditional c-optimal design criterion $\phi(\boldsymbol{\zeta}) = S^2/n$ (see Chaloner and Verdinelli [22]), subject to constraints (5.4.3) and (5.4.4). Tables 5.5 and 5.6 present the resulting traditional optimal plans under various settings for the discrimination ratios $d = 1.5$ and $d = 1.8$, respectively.

Table 5.5: Traditional c-optimal RASPs with budgetary constraints for $d = 1.5$

p	C_B	θ	(n^*, k^*, h^*)	π_c	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	$1000 \times \phi(\zeta^*)/n^*$
0.0	55	0	(102, 3, 0.810)	0.573	101.868	2.428	2.998	3.157
		0.5	(54, 4, 0.287)	0.594	37.342	1.148	4.000	3.361
		1	(73, 4, 0.293)	0.629	45.518	1.172	4.000	2.460
	65	0	(172, 3, 0.896)	0.586	172.083	2.686	2.998	2.285
		0.5	(100, 4, 0.630)	0.599	93.477	2.520	4.000	3.046
		1	(109, 4, 0.589)	0.630	91.476	2.356	4.000	2.403
	85	0	(321, 3, 0.976)	0.600	321.054	2.927	2.999	1.540
		0.5	(236, 4, 0.777)	0.610	226.982	3.108	4.000	2.338
		1	(244, 4, 0.744)	0.636	215.839	2.976	4.000	2.019
	95	0	(396, 3, 1.000)	0.604	396.722	2.999	2.999	1.349
		0.5	(309, 4, 0.819)	0.613	298.739	3.276	4.000	2.169
		1	(320, 4, 0.791)	0.638	286.232	3.164	4.000	1.915
0.2	55	0	(107, 3, 0.801)	0.573	96.916	2.393	2.988	3.119
		0.5	(59, 4, 0.322)	0.595	32.979	1.288	4.000	3.328
		1	(85, 4, 0.277)	0.629	38.166	1.108	4.000	2.455
	65	0	(177, 3, 0.881)	0.586	163.173	2.633	2.989	2.299
		0.5	(104, 4, 0.623)	0.600	81.739	2.491	3.999	2.997
		1	(112, 4, 0.586)	0.630	77.936	2.344	4.000	2.383
	85	0	(329, 3, 0.960)	0.600	308.316	2.874	2.994	1.571
		0.5	(247, 4, 0.752)	0.611	205.997	3.008	4.000	2.308
		1	(257, 4, 0.723)	0.637	193.542	2.892	4.000	1.992
	95	0	(407, 3, 0.984)	0.604	379.188	2.945	2.995	1.378
		0.5	(322, 4, 0.789)	0.614	272.098	3.156	4.000	2.151
		1	(332, 4, 0.762)	0.638	254.411	3.048	4.000	1.900
0.3	55	0	(110, 3, 0.796)	0.573	95.002	2.365	2.971	3.106
		0.5	(63, 4, 0.325)	0.595	30.671	1.300	4.000	3.307
		1	(114, 4, 0.584)	0.630	72.216	2.336	4.000	2.372
	65	0	(182, 3, 0.874)	0.586	159.997	2.597	2.975	2.299
		0.5	(107, 4, 0.619)	0.600	76.969	2.471	3.993	2.963
		1	(262, 4, 0.709)	0.637	181.348	2.836	4.000	1.982
	85	0	(334, 3, 0.950)	0.599	301.080	2.836	2.985	1.590
		0.5	(253, 4, 0.737)	0.611	195.481	2.948	4.000	2.293
		1	(339, 4, 0.745)	0.639	239.477	2.980	4.000	1.890
	95	0	(409, 3, 0.973)	0.603	371.142	2.908	2.988	1.408
		0.5	(329, 4, 0.771)	0.614	258.440	3.084	4.000	2.140
		1	(332, 4, 0.762)	0.638	254.411	3.048	4.000	1.900

Table 5.6: Traditional c-optimal RASPs with budgetary constraints for $d = 1.8$

p	C_B	θ	(n^*, k^*, h^*)	π_c	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	$1000 \times \phi(\zeta^*)/n^*$
0.0	55	0	(100, 3, 0.829)	0.570	100.246	2.484	2.996	3.352
		0.5	(32, 4, 0.549)	0.548	29.106	2.194	3.997	7.460
		1	(33, 4, 0.343)	0.590	22.511	1.372	4.000	5.353
	65	0	(174, 3, 0.878)	0.589	174.326	2.633	2.999	2.158
		0.5	(86, 4, 0.715)	0.575	81.918	2.860	4.000	4.898
		1	(91, 4, 0.693)	0.602	79.363	2.772	4.000	4.310
	85	0	(325, 3, 0.927)	0.605	324.673	2.781	3.000	1.315
		0.5	(224, 4, 0.846)	0.594	217.186	3.384	4.000	3.401
		1	(229, 4, 0.841)	0.614	206.928	3.364	4.000	3.387
	95	0	(403, 3, 0.943)	0.610	403.090	2.829	3.000	1.110
		0.5	(298, 4, 0.886)	0.598	290.010	3.544	4.000	3.101
		1	(301, 4, 0.887)	0.617	274.204	3.548	4.000	3.201
0.2	55	0	(105, 3, 0.819)	0.570	2.440	2.979	3.311	5.432
		0.5	(35, 4, 0.564)	0.550	26.508	2.226	3.948	7.208
		1	(37, 4, 0.323)	0.591	18.603	1.292	4.000	5.274
	65	0	(179, 3, 0.866)	0.588	164.851	2.592	2.993	2.183
		0.5	(92, 4, 0.699)	0.577	75.096	2.791	3.992	4.749
		1	(96, 4, 0.678)	0.603	70.686	2.712	4.000	4.222
	85	0	(334, 3, 0.915)	0.605	310.161	2.744	2.999	1.343
		0.5	(235, 4, 0.813)	0.594	200.140	3.252	4.000	3.353
		1	(240, 4, 0.807)	0.615	187.252	3.228	4.000	3.351
	95	0	(414, 3, 0.931)	0.609	380.744	2.790	3.000	1.137
		0.5	(312, 4, 0.849)	0.599	268.606	3.396	4.000	3.072
		1	(316, 4, 0.848)	0.618	250.209	3.392	4.000	3.171
0.3	55	0	(109, 3, 0.814)	0.571	94.427	2.407	2.958	3.272
		0.5	(63, 4, 0.325)	0.595	30.671	1.300	4.000	3.307
		1	(39, 4, 0.336)	0.591	17.576	1.343	3.998	5.232
	65	0	(182, 3, 0.859)	0.588	160.647	2.562	2.983	2.204
		0.5	(107, 4, 0.619)	0.600	76.969	2.471	3.993	2.963
		1	(98, 4, 0.667)	0.604	66.068	2.667	3.999	4.176
	85	0	(336, 3, 0.907)	0.604	299.949	2.718	2.996	1.374
		0.5	(253, 4, 0.737)	0.611	195.481	2.948	4.000	2.293
		1	(246, 4, 0.787)	0.615	177.568	3.148	4.000	3.333
	95	0	(417, 3, 0.923)	0.609	373.761	2.767	2.998	1.164
		0.5	(329, 4, 0.771)	0.614	258.440	3.084	4.000	2.140
		1	(324, 4, 0.825)	0.618	238.049	3.300	4.000	3.155

For a given budget, the optimal inspection interval is typically shorter for the dependent competing risks model, irrespective of the chosen withdrawal proportion p and discrimination ratios d. As expected, for any fixed level of dependence (θ), the optimal sample size (n^*) increases with the available budget, which is consistent with standard reliability planning intuition.

5.6 Numerical illustration

In this section, we demonstrate the practical implementation of the proposed RASPs using a real-life dataset. We consider the ARC-1 VHF communication transmitter–receiver example reported by Mendenhall and Hader [83], which involves two failure modes ($J = 2$), namely confirmed and unconfirmed failures. To facilitate numerical computation, the failure times are rescaled to units of thousands of hours. Using the parameter estimates obtained from this dataset, we illustrate how the proposed RASPs can be implemented and demonstrate the associated decision-making rule for lot acceptance.

In addition to the frailty model in Section 5.2 and the independent counterpart, we also consider a Clayton copula-based formulation to account for dependence. Using the Clayton survival copula with dependence parameter $\theta > 0$, we have the joint RF

$$\overline{R}(x_1, x_2 \mid \boldsymbol{\theta}) = \left\{R_1(x_1 \mid \boldsymbol{\theta}_1)^{-\theta} + R_2(x_2 \mid \boldsymbol{\theta}_2)^{-\theta} - 1\right\}^{-1/\theta}. \tag{5.6.1}$$

Assuming a Weibull marginal RF of the form $R_j(x) = \exp\{-(x/\eta_j)^{\nu_j}\}$, $j = 1, 2$ (Zhang et al. [133]), we have

$$\overline{R}(x_1, x_2|\boldsymbol{\theta}) = \left[\exp\left\{\theta\left(\frac{x_1}{\eta_1}\right)^{\nu_1}\right\} + \exp\left\{\theta\left(\frac{x_2}{\eta_2}\right)^{\nu_2}\right\} - 1\right]^{-\frac{1}{\theta}}.$$

The corresponding sub-density function is given by

$$g(j, t|\boldsymbol{\theta}) = \left(\frac{\nu_j}{\eta_j}\right)\left(\frac{t}{\eta_j}\right)^{\nu_j - 1} \exp\left\{\theta\left(\frac{t}{\eta_j}\right)^{\nu_j}\right\}\left[\exp\left\{\theta\left(\frac{t}{\eta_1}\right)^{\nu_1}\right\} + \exp\left\{\theta\left(\frac{t}{\eta_2}\right)^{\nu_2}\right\} - 1\right]^{-\frac{1}{\theta}-1}.$$

Although the Clayton copula and Gamma frailty models capture dependence through different mechanisms, the resulting strength of dependence is comparable, since for both approaches the Kendall's tau is the same.

Now, we fit the independent, frailty-based, and copula-based models to the above-mentioned data set. As discussed in Section 5.2, we assume equal shape parameters for the frailty model. However, we briefly examine the validity of this assumption by considering frailty models with unequal shape parameters. For completeness, we also consider the corresponding independent counterparts. We obtain the MLEs and the corresponding standard errors (SEs) of the model parameters by maximizing the resulting log-likelihood function as in (5.1.3). Table 5.7 reports the MLEs together with their SEs, along with the corresponding log-likelihood value (LLV), Akaike's Information Criterion (AIC), and Bayesian Information Criterion (BIC).

Table 5.7: MLEs (SEs) and model selection criteria under independent and dependent models

	Model				
	Independent		Dependent (Frailty)		
	Equal	Unequal	Equal	Unequal	Dependent (Copula)
η_1	0.439 (0.026)	0.436 (0.025)	0.303 (0.039)	0.299 (0.037)	0.370 (0.023)
η_2	0.822 (0.076)	0.894 (0.107)	0.497 (0.086)	0.514 (0.096)	0.529 (0.086)
ν_1	1.135 (0.052)	1.194 (0.066)	1.436 (0.130)	1.529 (0.145)	1.293 (0.066)
ν_2	-	1.029 (0.083)	-	1.311 (0.142)	1.286 (0.119)
θ	-	-	0.616 (0.237)	0.646 (0.240)	1.245 (0.633)
LLV	−138.404	−137.214	−134.124	−132.571	−133.437
AIC	282.807	282.428	276.248	275.142	276.8737
BIC	294.539	298.071	291.891	294.696	296.4277

The AIC and BIC values suggest that both dependent competing risks models (frailty-based and copula-based) provide a better fit than the independent model. However, it is not possible to pick a clear winner among the dependent models. As suggested in the literature in such contexts (see, for example, Yang [126]), the BIC is typically considered for model selection. Thus, it appears that the frailty model with an equal shape parameter is sufficient for this data set. However, for the purpose of illustration and comparative assessment, we retain the independent model along with the copula model in the subsequent analysis of the optimal RASPs.

We now present the optimal RASPs by minimizing the c-optimality criterion given in (5.4.1). We adopt the same approach as in the previous section to derive S_0 and S_1. Specifically, we set the time point of interest as $t_0 = 0.15$ and use a discrimination ratio of $d = 1.5$ for both causes. In addition, RASPs are evaluated under PIC-I schemes with 4, 6, and 8 inspection points (i.e., $k = 4, 6, 8$). We consider three scenarios, i.e., no intermediate withdrawal ($p = 0$), 20% withdrawal of surviving units ($p = 0.2$), and 30% withdrawal of surviving units ($p = 0.3$).

Table 5.8: Optimal RASPs without budgetary constraints for competing models

p	Model	$k=4$				$k=6$				$k=8$			
		h^*	$10\times$ $\phi(\zeta^*)$	n^*	π_c	h^*	$10\times$ $\phi(\zeta^*)$	n^*	π_c	h^*	$10\times$ $\phi(\zeta^*)$	n^*	π_c
0	I	0.064	1.750	72	0.563	0.048	1.690	69	0.563	0.039	1.659	68	0.563
	D(F)	0.107	2.016	71	0.538	0.089	1.918	68	0.538	0.078	1.875	66	0.538
	D(C)	0.088	1.905	74	0.554	0.069	1.812	70	0.554	0.058	1.765	68	0.554
0.2	I	0.083	1.914	78	0.563	0.078	1.914	78	0.563	0.077	1.906	78	0.563
	D(F)	0.121	2.116	74	0.538	0.112	2.070	73	0.538	0.109	2.052	72	0.538
	D(C)	0.101	1.997	79	0.555	0.088	1.948	77	0.555	0.083	1.935	76	0.555
0.3	I	0.091	1.982	81	0.563	0.090	1.979	81	0.563	0.090	1.979	81	0.563
	D(F)	0.126	2.158	75	0.538	0.121	2.124	74	0.538	0.119	2.115	74	0.538
	D(C)	0.106	2.040	81	0.555	0.097	2.007	80	0.555	0.095	2.001	79	0.555

Table 5.8 presents the optimal inspection interval (h), sample size (n), and π_c for both independent (I) and dependent competing risk models. For brevity, the frailty-based model is denoted as D(F), and the copula-based model as D(C) in the table. The inspection interval (h) is generally shorter under the independent model compared to the frailty model, indicating the need for more frequent inspections when risks are assumed to be independent. In contrast, the copula model produces inspection intervals that are comparable to those of the independent model. The π_c is marginally higher under the independent model relative to the dependent specifications. Interestingly, the copula-based model results in values similar to those observed in the independent case. Finally, for all models, the required sample size (n) increases as the withdrawal proportion (p) becomes larger.

We now incorporate budgetary constraints and obtain the optimal RASPs as discussed in Section 5.4.2. For illustrative purposes, we set the cost components as $C_s = 0.1$, $C_t = 5$, $C_d = 0.025$, and $C_I = 10$. In addition, we consider the available budget as $C_b = 65$. Table 5.9 reports the n^*, k^*, h^*, and π_c. Additionally, it presents $E[D^*]$, $E[\tau^*]$, $E[I^*]$ and $\phi(\zeta^*)$.

Table 5.9: Optimal RASPs with budgetary constraints for competing risks models

p	Model	(n^*, k^*, h^*)	π_c	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	$10 \times \phi(\boldsymbol{\zeta}^*)$
0.0	I	(70, 5, 0.054)	0.563	40.337	0.270	5.000	1.713
	D(F)	(69, 5, 0.096)	0.538	55.867	0.480	5.000	1.961
	D(C)	(72, 5, 0.077)	0.554	52.221	0.385	5.000	1.850
0.2	I	(78, 5, 0.079)	0.563	41.432	0.395	5.000	1.907
	D(F)	(73, 5, 0.115)	0.538	48.605	0.575	4.998	2.084
	D(C)	(77, 5, 0.093)	0.555	45.453	0.465	5.000	1.965
0.3	I	(81, 5, 0.091)	0.563	40.640	0.455	4.998	1.979
	D(F)	(75, 5, 0.122)	0.538	45.649	0.606	4.967	2.136
	D(C)	(80, 5, 0.101)	0.555	43.142	0.504	4.994	2.017

Consistent with previous findings, the π_c is marginally higher under the independent model compared to the dependent models. The inspection interval (h) tends to be longer under the frailty-based dependent model, whereas the copula model generally yields values closer to those of the independent case. While the number of inspections (k) remains largely unaffected across all models, the required sample size (n) is slightly lower under the frailty model. This is true at each withdrawal proportion (p). Interestingly, $E[D^*]$ and $E[\tau^*]$ are higher for the frailty model, whereas the copula model produces intermediate values between the independent and frailty formulations.

To illustrate the decision-making procedure for accepting or rejecting a lot, we provide a demonstration using a simulated data set under the frailty model. For this purpose, we consider the optimal RASP $(73, 5, 0.115)$ with a withdrawal proportion of $p = 0.2$. Under this plan, a life test is conducted on 73 units, with 5 inspections performed at intervals of 0.115 time units. At each intermediate inspection, 20% of the surviving units are withdrawn from the experiment, while all remaining units are removed at the 5th inspection. Based on the observed data, a decision is made by comparing the estimated reliability with the acceptance threshold. Specifically, the lot is accepted if $\widehat{R}(0.15 \mid \widehat{\boldsymbol{\theta}}) > 0.538$.

Using the MLEs previously obtained under the frailty model, we simulate a data set following the algorithm proposed by Roy and Pradhan [106] to demonstrate the implementation of this decision rule. The simulated data set is presented in Table 5.10. For this simulated data set, the estimated model parameters and reliability are computed as:

$$\widehat{\eta}_1 = 0.292, \ \widehat{\eta}_2 = 0.374, \ \widehat{\nu} = 1.779, \ \widehat{\theta} = 0.668, \text{ and } \widehat{R}(0.15 \mid \widehat{\boldsymbol{\theta}}) = 0.648.$$

Since the estimated reliability (0.648) exceeds the acceptance threshold (0.538), the lot is accepted.

Table 5.10: Simulated PIC-I data set

i	Time interval	d_{i1}	d_{i2}	r_i
1	(0, 0.115]	11	7	11
2	(0.115, 0.230]	10	8	5
3	(0.230, 0.345]	6	4	2
4	(0.345, 0.460]	1	0	1
5	(0.460, 0.575]	3	1	3

5.7 Simulation evaluation

In this section, we evaluate the finite sample behavior of the optimal RASPs through a simulation study. We consider the optimal RASPs under budgetary constraints for subsequent analysis.

For each optimal PICS-I, we simulate 5000 data sets following the algorithm provided by Roy and Pradhan [106]. In each case, we consider the MLEs reported in the previous subsection as the true values (TVs) of the model parameters. For each data set, we obtain the MLEs of the relevant parameters by maximizing the log-likelihood function in (5.1.3). Using these MLEs, we then obtain $\widehat{R}(x)$, given by (5.2.3) and (5.2.4). The histograms of the resulting $\widehat{R}(x)$ at 0.15 for both independent and dependent competing risks models with withdrawal proportions $p = 0$, 0.2 and 0.3 are given in Figures 5.1, 5.2 and 5.3, respectively. As expected, the histograms are symmetric in nature, which is in line with our observation in Theorem 5.2.

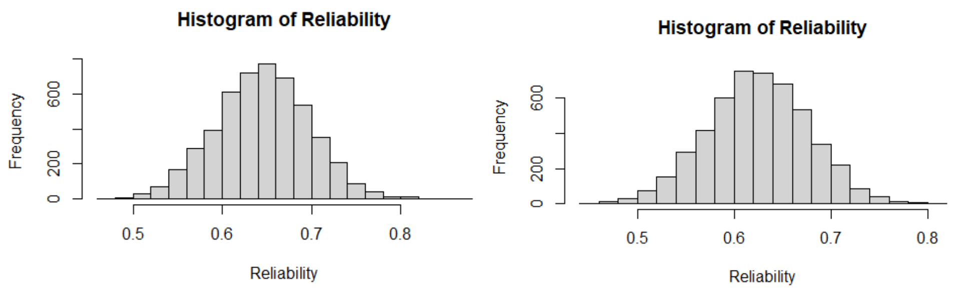


Figure 5.1: Histograms of $\widehat{R}(x)$ for independent (left) and dependent (right) models when $p = 0$

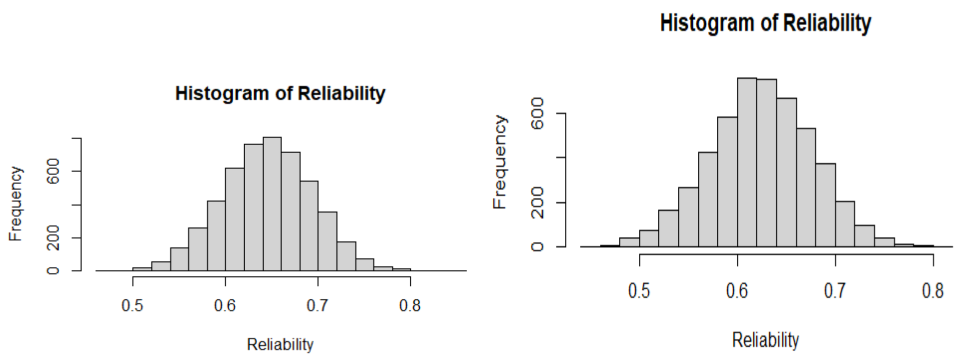


Figure 5.2: Histograms of $\widehat{R}(x)$ for independent (left) and dependent (right) models when $p = 0.2$

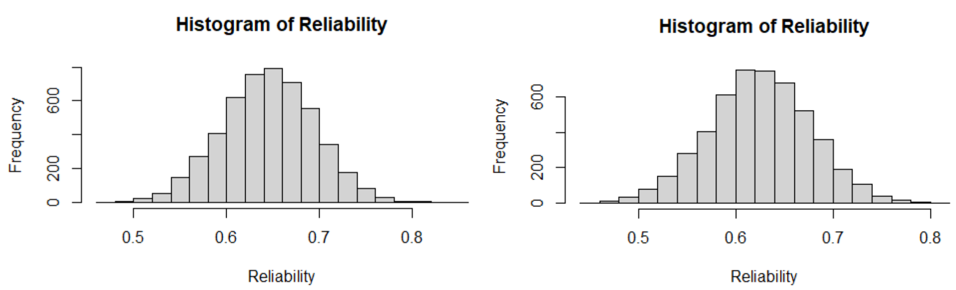


Figure 5.3: Histograms of $\widehat{R}(x)$ for independent (left) and dependent (right) models when $p = 0.3$

Table 5.11: Average estimates and RMSDs of $R(x \mid \boldsymbol{\theta})$ and S^2 along with estimates of producer's and consumer's risks for independent (I) and dependent (D) competing risks models

Model	TV of RF	p_0	ζ^*	Avg. RF	100× RMSD of RF	10× Avg. S^2	100× RMSD of S^2	$\widehat{\alpha}$	$\widehat{\beta}$
I	0.644	0.0	(70, 5, 0.054)	0.645	4.886	1.696	1.122	0.049	0.128
		0.2	(78, 5, 0.079)	0.645	4.926	1.889	1.156	0.051	0.127
		0.3	(81, 5, 0.091)	0.645	4.933	1.959	1.199	0.050	0.123
D	0.626	0.0	(69, 5, 0.096)	0.624	5.200	1.957	1.564	0.050	0.111
		0.2	(73, 5, 0.115)	0.625	5.250	2.081	1.555	0.054	0.120
		0.3	(75, 5, 0.122)	0.625	5.250	2.131	1.541	0.050	0.121

We further obtain the average estimate of RF and the corresponding root mean square deviation (RMSD) over 5000 simulation runs. Table 5.11 reports the average estimate of RF and the corresponding RMSD along with their respective TVs. We also compute the estimate of the standardized variance S^2 in each case using the MLEs of the model parameters. Table 5.11 also provides the average estimate of standardized variance S^2 and the corresponding RMSD. Furthermore, we check whether the lot is accepted or rejected in each case using the decision rule presented in Section 5.3 and obtain the estimates of the producer's risk (α) and the consumer's risk (β), as given by (5.3.2), respectively. Table 5.11 reports the estimated producer's risk ($\widehat{\alpha}$) and the consumer's risk ($\widehat{\beta}$). It is easy to note that the average estimates of RF are close to their TVs in each case. Furthermore, the precision in reliability estimates is hardly affected as the withdrawal proportion increases. A similar observation can also be drawn for S^2. We also note that the estimates of the producer's and consumer's risk are reasonably close to their respective specified values.

5.8 Discussion

In this chapter, we considered the development of RASPs under PICS-I in the presence of competing causes of failure. At the outset, a general framework for incorporating the presence of competing risks under PICS-I schemes is presented and obtained the Fisher information matrix in this context. We then discussed the asymptotic properties of the MLEs of the model parameters under a set of regularity conditions. Note that such asymptotic properties are essential for developing RASPs. The general framework presented in this work can accommodate both dependent and independent competing risks models and can prove to be really useful in modeling the interval-censored data sets, which are common in several managerial applications.

Subsequently, we discussed a frailty model as a special case of the general framework developed in this work. The frailty model incorporates the dependence among the potential failure times, which is largely neglected in designing the optimal censoring schemes. Moreover, this model allows us to obtain a popular independent competing risks model as a special case, thus enabling us to study the effect of incorporating the dependence structure among the potential failure times on RASPs. Subsequently, we developed RASPs in three important contexts for both the dependent and independent competing risks models. First, we present the traditional RASPs using the producer's and consumer's risks. These traditional RASPs require several specific information, including details such as the inspection times and the proportion of intermediate withdrawals, which the decision-makers must decide suitably while conducting the life-test experiments under PICS-I. We thus considered a suitable c-optimal design criterion which allowed us to determine these design variables. We developed RASPs both with and without budgetary constraints. Through a detailed numerical study, it is observed that the level of dependence between the potential failure times has a significant impact on the resulting RASPs. We further demonstrated the application of the developed RASPs in a practical context and studied the finite-sample properties of these RASPs through a detailed simulation study. We observed that these RASPs exhibit the desired asymptotic properties even for a modest sample size.

The numerical investigation demonstrates that model parameters have a meaningful impact on the resulting RASPs. In particular, an increase in the dependence level among competing failure modes leads to higher required sample sizes, larger acceptance limits, and longer expected test durations, regardless of the chosen scheme. The withdrawal proportion exerts a moderate influence, primarily increasing the sample size, while the discrimination ratio plays a more dominant role by reducing the sample size requirement and tightening the acceptance criteria. Under budget constraints, higher testing budgets allow shorter inspection intervals and more frequent inspections. These observations reinforce the importance of incorporating both dependence and cost considerations in planning RASPs.

5.A Appendix

5.A.1 Proof of lemmas

Proof of lemma 5.1:

Proof: By differentiating (5.1.3) with respect to θ_u, we have

$$\frac{\partial l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial \theta_u} = \sum_{i=1}^{k}\sum_{k=1}^{n_i}\left[\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\partial q_{ij}}{\partial \theta_u}\right) - \frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial q_i}{\partial \theta_u}\right)\right] = \sum_{i=1}^{k}\sum_{k=1}^{n_i} U_{ik,u}, \tag{5.A.1}$$

where $U_{ik,u} = \sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\partial q_{ij}}{\partial \theta_u}\right) - \frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial q_i}{\partial \theta_u}\right)$ and $\delta_{ik+} = \sum_{j=1}^{J}\delta_{ikj}$. Then, we get

$$E\left[\frac{\partial l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial \theta_u}\right] = \sum_{i=1}^{k} E_{N_i}\left[\sum_{k=1}^{N_i} E\left[U_{ik,u} \mid N_i\right]\right].$$

It is sufficient to prove that $E\left[U_{ik,u} \mid N_i\right] = 0$. We know $E[\delta_{ikj}] = q_{ij}$ and $E[1-\delta_{ik+}] = (1-q_i)$. Thus, we have

$$E\left[U_{ik,u} \mid N_i\right] = \sum_{j=1}^{J}\frac{\partial q_{ij}}{\partial \theta_u} - \frac{\partial q_i}{\partial \theta_u}.$$

We know $\sum_{j=1}^{J} q_{ij} = q_i$. This implies that

$$\sum_{j=1}^{J}\frac{\partial q_{ij}}{\partial \theta_u} = \frac{\partial q_i}{\partial \theta_u}.$$

Hence, $E\left[U_{ik,u} \mid N_i\right] = 0$. (Proved)

Proof of Lemma 5.2:

Proof: By differentiating (5.A.1) with respect to θ_v, we have

$$\begin{aligned}\frac{\partial^2 l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial\theta_u\partial\theta_v} =& \sum_{i=1}^{k}\sum_{k=1}^{n_i}\left[\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\partial^2 q_{ij}}{\partial\theta_u\partial\theta_v}\right) - \frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial^2 q_i}{\partial\theta_u\partial\theta_v}\right)\right] \\ &- \sum_{i=1}^{k}\sum_{k=1}^{n_i}\left[\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}^2}\right)\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)\left(\frac{\partial q_{ij}}{\partial\theta_v}\right) + \frac{(1-\delta_{ik+})}{(1-q_i)^2}\left(\frac{\partial q_i}{\partial\theta_u}\right)\left(\frac{\partial q_i}{\partial\theta_v}\right)\right] \\ =& \sum_{i=1}^{k}\sum_{k=1}^{n_i} V_{ik,uv}, \text{ say,}\end{aligned} \tag{5.A.2}$$

where

$$\begin{aligned}V_{ik,uv} =& \left[\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\partial^2 q_{ij}}{\partial\theta_u\partial\theta_v}\right) - \frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial^2 q_i}{\partial\theta_u\partial\theta_v}\right)\right] \\ &- \left[\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}^2}\right)\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)\left(\frac{\partial q_{ij}}{\partial\theta_v}\right) + \frac{(1-\delta_{ik+})}{(1-q_i)^2}\left(\frac{\partial q_i}{\partial\theta_u}\right)\left(\frac{\partial q_i}{\partial\theta_v}\right)\right].\end{aligned}$$

Now, proceeding in the same manner as in Lemma 5.1, we have

$$\sum_{i=1}^{k} E_{N_i}\left[\sum_{k=1}^{N_i} E\left[\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\partial^2 q_{ij}}{\partial\theta_u\partial\theta_v}\right) - \frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial^2 q_i}{\partial\theta_u\partial\theta_v}\right)\right]\right] = 0.$$

Therefore, taking the expectation of (5.A.2), we have

$$\begin{aligned}E\left[\frac{\partial^2 l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial\theta_u\partial\theta_v}\right] = -\sum_{i=1}^{k} E_{N_i}\Bigg[\sum_{k=1}^{N_i}\Bigg[&\sum_{j=1}^{J}\left(\frac{E[\delta_{ikj}]}{q_{ij}^2}\right)\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)\left(\frac{\partial q_{ij}}{\partial\theta_v}\right) \\ &+ \left(\frac{E[1-\delta_{ik+}]}{(1-q_i)^2}\right)\left(\frac{\partial q_i}{\partial\theta_u}\right)\left(\frac{\partial q_i}{\partial\theta_v}\right)\Bigg]\Bigg].\end{aligned}$$

Now, using $E[\delta_{ikj}] = q_{ij}$ and $E[1-\delta_{ik+}] = 1 - q_i$, we get

$$E\left[-\frac{\partial^2 l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial\theta_u\partial\theta_v}\right] = \sum_{i=1}^{k}\left[\sum_{j=1}^{J}\left(\frac{E[N_i]}{q_{ij}}\right)\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)\left(\frac{\partial q_{ij}}{\partial\theta_v}\right) + \left(\frac{E[N_i]}{(1-q_i)}\right)\left(\frac{\partial q_i}{\partial\theta_u}\right)\left(\frac{\partial q_i}{\partial\theta_v}\right)\right]. \tag{5.A.3}$$

Note that

$$\left(\frac{\partial l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial\theta_u}\right)\left(\frac{\partial l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial\theta_v}\right) = \sum_{i=1}^{k}\sum_{k=1}^{n_i}\sum_{i'=1}^{k}\sum_{k'=1}^{n_{i'}} U_{ik,u}U_{i'k',v},$$

where $U_{ik,u}$ and $U_{i'k',v}$ are as defined above. Now, suppose $i < i'$. Then, for given $n_{i'}$, n_i is fixed and hence $E[U_{ik,u}U_{i'k',v} \mid n_{i'}] = U_{ik,u}E[U_{i'k',v} \mid n_{i'}] = 0$. Similarly, when $i > i'$, $E[U_{ik,u}U_{i'k',v} \mid n_i] = U_{i'k',v}E[U_{ik,u} \mid n_i] = 0$. Also, when $i' = i$ and $k' \neq k$, we have

$$E[U_{ik,u}U_{ik',v} \mid n_i]$$

$$=E\left[\left\{\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)-\frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial q_i}{\partial\theta_u}\right)\right\}\left\{\sum_{j'=1}^{J}\left(\frac{\delta_{ik'j'}}{q_{ij'}}\right)\left(\frac{\partial q_{ij}}{\partial\theta_v}\right)-\frac{(1-\delta_{ik'+})}{(1-q_i)}\left(\frac{\partial q_i}{\partial\theta_v}\right)\right\}\right]$$
$$=E\left[\sum_{j=1}^{J}\sum_{j'=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\delta_{ik'j'}}{q_{ij'}}\right)\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)\left(\frac{\partial q_{ij}}{\partial\theta_v}\right)-\sum_{j'=1}^{J}\left(\frac{\delta_{ik'j'}}{q_{ij'}}\right)\left(\frac{\partial q_{ij}}{\partial\theta_v}\right)\frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial q_i}{\partial\theta_u}\right)\right.$$
$$\left.-\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)\frac{(1-\delta_{ik'+})}{(1-q_i)}\left(\frac{\partial q_i}{\partial\theta_v}\right)+\frac{(1-\delta_{ik+})}{(1-q_i)}\frac{(1-\delta_{ik'+})}{(1-q_i)}\left(\frac{\partial q_i}{\partial\theta_u}\right)\left(\frac{\partial q_i}{\partial\theta_v}\right)\right]=0.$$

Since $X_1,\ldots,X_n$ are IID, we have, for any fixed i,

$$E\left[\sum_{j=1}^{J}\delta_{ikj}\sum_{j'=1}^{J}\delta_{ik'j'}\right]=E\left[\sum_{j=1}^{J}\delta_{ikj}\right]E\left[\sum_{j'=1}^{J}\delta_{ik'j'}\right],$$
$$E\left[\sum_{j=1}^{J}\delta_{ikj}(1-\delta_{ik'+})\right]=E\left[\sum_{j=1}^{J}\delta_{ikj}\right]E\left[1-\delta_{ik'+})\right],$$
$$E\left[\sum_{j'=1}^{J}\delta_{ik'j'}(1-\delta_{ik+})\right]=E\left[\sum_{j'=1}^{J}\delta_{ik'j'}\right]E\left[1-\delta_{ik+})\right],$$

and

$$E\left[(1-\delta_{ik+})(1-\delta_{ik'+})\right]=E\left[(1-\delta_{ik+})\right]E\left[1-\delta_{ik'+})\right].$$

Now, it is easy to see that $E[U_{ik,u}U_{ik',v}\mid n_i]=0$. Lastly, when $i'=i$ and $k'=k$, we have

$$E[U_{ik,u}U_{ik,v}\mid n_i]$$
$$=E\left[\sum_{j=1}^{J}\sum_{j'=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\delta_{ikj'}}{q_{ij'}}\right)\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)\left(\frac{\partial q_{ij}}{\partial\theta_v}\right)-\sum_{j'=1}^{J}\left(\frac{\delta_{ikj'}}{q_{ij'}}\right)\left(\frac{\partial q_{ij'}}{\partial\theta_v}\right)\frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial q_i}{\partial\theta_u}\right)\right.$$
$$\left.-\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)\frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial q_i}{\partial\theta_v}\right)+\frac{(1-\delta_{ik+})^2}{(1-q_i)^2}\left(\frac{\partial q_i}{\partial\theta_u}\right)\left(\frac{\partial q_i}{\partial\theta_v}\right)\right].\tag{5.A.4}$$

Note that $P(\delta_{ikj}\delta_{ikj'}=1)=0$ for $j\neq j'$, and $P(\delta_{ikl}(1-\delta_{ik+})=1)=0$ for $l=1,\ldots,J$. Therefore, for $i=1,\ldots,k$, $k=1,\ldots,n_i$ and $j,j'=1,\ldots,J$, we get

$$E[\delta_{ikj}\delta_{ikj'}\mid n_i]=\begin{cases}E[\delta_{ikj}^2\mid n_i]=E[\delta_{ikj}\mid n_i]=q_{ij}, & \text{for } j=j',\\ 0, & \text{for } j\neq j',\end{cases}$$

$$E[\delta_{ikl}(1-\delta_{ik+})\mid n_i]=0,\qquad \text{for } l=j,j',$$

and

$$E[(1-\delta_{ik+})^2\mid n_i]=E[(1-\delta_{ikj})\mid n_i]=1-q_i.$$

Then, (5.A.4) can be written as

$$E[U_{ik,u}U_{ik,v}\mid n_i]=\sum_{j=1}^{J}\frac{1}{q_{ij}}\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)\left(\frac{\partial q_{ij}}{\partial\theta_v}\right)+\frac{1}{(1-q_i)}\left(\frac{\partial q_i}{\partial\theta_u}\right)\left(\frac{\partial q_i}{\partial\theta_v}\right)=\sigma_{i,uv}^2\text{ (say)}.\tag{5.A.5}$$

Thus, we have

$$\begin{aligned}
&E\left[\left(\frac{\partial l(\boldsymbol{\theta}\mid \boldsymbol{x})}{\partial\theta_u}\right)\left(\frac{\partial l(\boldsymbol{\theta}\mid \boldsymbol{x})}{\partial\theta_v}\right)\right]\\
&=\sum_{i=1}^{k} E_{N_i}\left[U_{ik,u}U_{ik,v}\mid N_i\right]\\
&=\sum_{i=1}^{k}\left[\sum_{j=1}^{J}\frac{E[N_i]}{q_{ij}}\left(\frac{\partial q_{ij}}{\partial\theta_u}\right)\left(\frac{\partial q_{ij}}{\partial\theta_v}\right)+\frac{E[N_i]}{(1-q_i)}\left(\frac{\partial q_i}{\partial\theta_u}\right)\left(\frac{\partial q_i}{\partial\theta_v}\right)\right]. \qquad (5.A.6)
\end{aligned}$$

(Proved)

Proof of lemma 5.3:

Proof: We have

$$\frac{\partial q_{ij}}{\partial\theta_u}=\frac{1}{R(\tau_{i-1})}\left[\frac{\partial\left[G(j,\tau_i)-G(j,\tau_{i-1})\right]}{\partial\theta_u}+q_{ij}\left(\frac{\partial F_T(\tau_{i-1})}{\partial\theta_u}\right)\right], \qquad (5.A.7)$$

for $u=1,2\ldots,l$. Now, let us define the following bounds:

$$\kappa=\max_{1\le i\le k}\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0}\frac{1}{R(\tau_i)}, \qquad (5.A.8)$$

$$A_u=\max_{1\le i\le k}\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0}\left|\frac{\partial F(\tau_i)}{\partial\theta_u}\right| \qquad (5.A.9)$$

and

$$B_u=\max_{1\le i\le k}\max_{1\le j\le J}\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0}\left|\frac{\partial G(j,\tau_i)}{\partial\theta_u}\right|, \qquad (5.A.10)$$

for $u=1,2,\ldots,l$. Then, by using (5.A.8), (5.A.9) and (5.A.10), we have

$$\left|\frac{\partial q_{ij}}{\partial\theta_u}\right|\le\kappa[2B_u+A_u]=C_u,\quad\text{say.}$$

Thus, the first-order derivative of q_{ij} is bounded. Now, differentiating with respect to θ_v, for $v=1,2,\ldots,l$, we get

$$\begin{aligned}
\frac{\partial^2 q_{ij}}{\partial\theta_u\partial\theta_v}&=\frac{1}{[R(\tau_{i-1})]^2}\left[\frac{\partial\left[G(j,\tau_i)-G(j,\tau_{i-1})\right]}{\partial\theta_u}+q_{ij}\left(\frac{\partial F(\tau_i)}{\partial\theta_u}\right)\right]\left(\frac{\partial F(\tau_{i-1})}{\partial\theta_v}\right)\\
&\quad+\frac{1}{R(\tau_{i-1})}\left[\frac{\partial^2\left[G(j,\tau_i)-G(j,\tau_{i-1})\right]}{\partial\theta_u\partial\theta_v}+\left(\frac{\partial q_{ij}}{\partial\theta_v}\right)\left(\frac{\partial F(\tau_{i-1})}{\partial\theta_u}\right)+q_{ij}\left(\frac{\partial^2 F(\tau_{i-1})}{\partial\theta_u\partial\theta_v}\right)\right] \qquad (5.A.11)
\end{aligned}$$

Under the regularity conditions, the second-order derivatives of $G(j,\tau_i)$ and $F(\tau_i)$ are assumed to be bounded. Therefore, we define

$$A_{uv}=\max_{1\le i\le k}\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0}\left|\frac{\partial^2 F(\tau_i)}{\partial\theta_u\partial\theta_v}\right| \qquad (5.A.12)$$

and

$$B_{uv}=\max_{1\le i\le k}\max_{1\le j\le J}\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0}\left|\frac{\partial^2 G(j,\tau_i)}{\partial\theta_u\partial\theta_v}\right|, \qquad (5.A.13)$$

for $u = 1, 2, \ldots, l$ and $v = 1, 2, \ldots, l$. Then, by using (5.A.8), (5.A.9),(5.A.10), (5.A.12) and (5.A.13) we have

$$\left|\frac{\partial^2 q_{ij}}{\partial\theta_u\partial\theta_v}\right| \leq \kappa^2[2B_u + A_u]A_v + \kappa[2B_{uv} + C_u A_u + A_{uv}] = C_{uv}, \quad \text{say.}$$

Differentiating (5.A.11) with respect to θ_w, for $w = 1, 2, \ldots, l$, we get

$$\begin{aligned}
&\frac{\partial^3 q_{ij}}{\partial\theta_u\partial\theta_v\partial\theta_w}\\
=&\frac{2}{[R(\tau_{i-1})]^3}\left[\frac{\partial\left[G(j,\tau_i) - G(j,\tau_{i-1})\right]}{\partial\theta_u} + q_{ij}\left(\frac{\partial F(\tau_i)}{\partial\theta_u}\right)\right]\left(\frac{\partial F(\tau_{i-1})}{\partial\theta_v}\right)\left(\frac{\partial F(\tau_{i-1})}{\partial\theta_w}\right)\\
&+\frac{1}{[R(\tau_{i-1})]^2}\left[\frac{\partial\left[G(j,\tau_i) - G(j,\tau_{i-1})\right]}{\partial\theta_u} + q_{ij}\left(\frac{\partial F(\tau_i)}{\partial\theta_u}\right)\right]\left(\frac{\partial^2 F(\tau_{i-1})}{\partial\theta_v\partial\theta_w}\right)\\
&+\frac{1}{[R(\tau_{i-1})]^2}\left[\frac{\partial^2\left[G(j,\tau_i) - G(j,\tau_{i-1})\right]}{\partial\theta_u\partial\theta_w} + q_{ij}\left(\frac{\partial^2 F(\tau_i)}{\partial\theta_u\partial\theta_w}\right) + \left(\frac{\partial q_{ij}}{\partial\theta_w}\right)\left(\frac{\partial F(\tau_i)}{\partial\theta_u}\right)\right]\left(\frac{\partial F(\tau_{i-1})}{\partial\theta_v}\right)\\
&+\frac{1}{[R(\tau_{i-1})]^2}\left[\frac{\partial^2\left[G(j,\tau_i) - G(j,\tau_{i-1})\right]}{\partial\theta_u\partial\theta_v} + \left(\frac{\partial q_{ij}}{\partial\theta_v}\right)\left(\frac{\partial F(\tau_{i-1})}{\partial\theta_u}\right) + q_{ij}\left(\frac{\partial^2 F(\tau_{i-1})}{\partial\theta_u\partial\theta_v}\right)\right]\left(\frac{\partial F(\tau_{i-1})}{\partial\theta_w}\right)\\
&+\frac{1}{R(\tau_{i-1})}\left[\frac{\partial^3\left[G(j,\tau_i) - G(j,\tau_{i-1})\right]}{\partial\theta_u\partial\theta_v\partial\theta_w} + \left(\frac{\partial^2 q_{ij}}{\partial\theta_v\partial\theta_w}\right)\left(\frac{\partial F(\tau_{i-1})}{\partial\theta_u}\right) + \left(\frac{\partial q_{ij}}{\partial\theta_v}\right)\left(\frac{\partial^2 F(\tau_{i-1})}{\partial\theta_u\partial_w}\right)\right.\\
&\qquad\qquad\left. + q_{ij}\left(\frac{\partial^3 F(\tau_{i-1})}{\partial\theta_u\partial\theta_v\partial\theta_w}\right) + \left(\frac{\partial q_{ij}}{\partial\theta_w}\right)\left(\frac{\partial^2 F(\tau_{i-1})}{\partial\theta_u\partial\theta_v}\right)\right].
\end{aligned}$$

Under the regularity conditions, it is assumed that the third-order derivatives of $G(j, \tau_i)$ and $F(\tau_i)$ are bounded. Therefore, we define

$$A_{uvw} = \max_{1\leq i\leq k}\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0}\left|\frac{\partial^3 F(\tau_i)}{\partial\theta_u\partial\theta_v\partial\theta_w}\right| \tag{5.A.14}$$

and

$$B_{uvw} = \max_{1\leq i\leq k}\max_{1\leq j\leq J}\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_0}\left|\frac{\partial^3 G(j,\tau_i)}{\partial\theta_u\partial\theta_v\partial\theta_w}\right|, \tag{5.A.15}$$

for $u = 1, 2, \ldots, l$, $v = 1, 2, \ldots, l$ and $w = 1, 2, \ldots, l$. Then, by using (5.A.8-5.A.15), we have

$$\begin{aligned}
\left|\frac{\partial^3 q_{ij}}{\partial\theta_u\partial\theta_v\partial\theta_w}\right| \leq& 2\kappa^3[2B_u + A_u]A_v A_w + \kappa^2[2B_u + C_u A_u]A_{vw} + \boldsymbol{\zeta}^2[2B_{uw} + A_{uw} + C_w A_u]A_v\\
&+\kappa^2[2B_{uv} + C_v A_u + A_{uv}]A_w + \kappa[3B_{uvw} + C_{vw}A_u + C_v A_{uw} + A_{uvw} + C_w A_{uv}]\\
&= C_{uvw}, \quad \text{say}
\end{aligned}$$

This shows that the third-order derivative of q_{ij} is bounded. We know that $q_i = \sum_{j=1}^{J} q_{ij}$. Therefore, we get $\left|\frac{\partial q_i}{\partial\theta_u}\right| \leq JC_u$, $\left|\frac{\partial^2 q_i}{\partial\theta_u\partial\theta_v}\right| \leq JC_{uv}$ and $\left|\frac{\partial^3 q_i}{\partial\theta_u\partial\theta_v\partial\theta_w}\right| \leq JC_{uvw}$. This shows that the first, second, and third-order derivatives of q_i are also bounded. (Proved)

Proof of lemma 5.5:

Proof: **Proof of part (i):** Using (5.A.6), we have

$$I_{uv}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}) = \sum_{i=1}^{k}\left[\sum_{j=1}^{J}\frac{E[N_i]}{q_{ij}}\frac{\partial q_{ij}}{\partial\theta_u}\frac{\partial q_{ij}}{\partial\theta_v} + \frac{E[N_i]}{(1-q_i)}\frac{\partial q_i}{\partial\theta_u}\frac{\partial q_i}{\partial\theta_v}\right],$$

for $u, v = 1, 2 \cdots, l$.

Note that for fixed n, $E[N_i] < n$, for $i = 1, \ldots, k$. Since $0 < q_{ij} < 1$, $0 < \frac{1}{q_{ij}} < \infty$, $\forall \boldsymbol{\theta} \in \boldsymbol{\Theta}_0$. Also, since $0 < q_i < 1$, $0 < \frac{1}{q_i} < \infty$, $\forall \boldsymbol{\theta} \in \boldsymbol{\Theta}_0$. From Lemma 5.3, we also have that the first-order derivatives of q_{ij} and q_i are bounded, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, J$. Thus, $I_{uv}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}) < \infty$. Hence, the proof.

Proof of part (ii): By definition of $I(\boldsymbol{\theta}|\boldsymbol{\zeta})$, it is a symmetric matrix. For any vector $\boldsymbol{b}(\boldsymbol{\theta}) \neq \mathbf{0}$, we get

$$\boldsymbol{b}(\boldsymbol{\theta})^T I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}) \boldsymbol{b}(\boldsymbol{\theta}) = \sum_{i=1}^{k} \left[\sum_{j=1}^{J} \frac{E[N_i]}{q_{ij}} \left\{ \boldsymbol{b}(\boldsymbol{\theta})^T \nabla_{\boldsymbol{\theta}}(q_{ij}) \right\}^2 + \frac{E[N_i]}{(1-q_i)} \left\{ \boldsymbol{b}(\boldsymbol{\theta})^T \nabla(q_i) \right\}^2 \right]. \tag{5.A.16}$$

Since $0 < q_i < 1$, $0 < q_{ij} < 1$ and $E[N_i] > 0$, then from (5.A.16), we can say that $I(\boldsymbol{\theta} \mid \boldsymbol{\zeta})$ is non-negative definite matrix. We know prove that it is a positive definite matrix. We use the method of contradiction for this purpose. Let us consider that there exists a vector $\boldsymbol{b}'(\boldsymbol{\theta}) \neq 0$ such that

$$\boldsymbol{b}'(\boldsymbol{\theta})^T I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}) \boldsymbol{b}'(\boldsymbol{\theta}) = \sum_{i=1}^{k} \left[\sum_{j=1}^{J} \frac{E[N_i]}{q_{ij}} \left\{ \boldsymbol{b}'(\boldsymbol{\theta})^T \nabla_{\boldsymbol{\theta}}(q_{ij}) \right\}^2 + \frac{E[N_i]}{(1-q_i)} \left\{ \boldsymbol{b}'(\boldsymbol{\theta})^T \nabla(q_i) \right\}^2 \right]. \tag{5.A.17}$$

Now (5.A.17) holds if $\boldsymbol{b}'(\boldsymbol{\theta})^T \nabla_{\boldsymbol{\theta}}(q_{ij}) = 0$ and $\boldsymbol{b}'(\boldsymbol{\theta})^T \nabla(q_i) = 0$, for all $i = 1, \ldots, k$ and $j = 1, \ldots, J$. However, this implies that $\nabla \boldsymbol{q}$ is a rank-deficient matrix, which contradicts the regularity conditions V(b). Hence $I(\boldsymbol{\theta} \mid \boldsymbol{\zeta})$ is a positive definite matrix for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. (Proved)

Proof of lemma 5.6:

Proof: Differentiating both sides of (5.A.2) with respect to θ_w, we get

$$\begin{aligned}
\frac{\partial^3 l(\boldsymbol{\theta})}{\partial\theta_u \partial\theta_v \partial\theta_w} &= \sum_{i=1}^{k} \sum_{k=1}^{n_i} \left[\sum_{j=1}^{J} \left(\frac{\delta_{ikj}}{q_{ij}} \right) \left(\frac{\partial^3 q_{ij}}{\partial\theta_u \partial\theta_v \partial\theta_w} \right) - \frac{(1-\delta_{ik+})}{(1-q_i)} \left(\frac{\partial^3 q_i}{\partial\theta_u \partial\theta_v \partial\theta_w} \right) \right] \\
&- \sum_{i=1}^{k} \sum_{k=1}^{n_i} \sum_{j=1}^{J} \left(\frac{\delta_{ikj}}{q_{ij}^2} \right) \left[\left(\frac{\partial^2 q_{ij}}{\partial\theta_u \partial\theta_v} \right) \left(\frac{\partial q_{ij}}{\partial\theta_w} \right) + \left(\frac{\partial^2 q_{ij}}{\partial\theta_u \partial\theta_w} \right) \left(\frac{\partial q_{ij}}{\partial\theta_v} \right) + \left(\frac{\partial^2 q_{ij}}{\partial\partial\theta_v \theta_w} \right) \left(\frac{\partial q_{ij}}{\partial\theta_u} \right) \right] \\
&- \sum_{i=1}^{k} \sum_{k=1}^{n_i} \frac{(1-\delta_{ik+})}{(1-q_i)^2} \left[\left(\frac{\partial^2 q_i}{\partial\theta_u \partial\theta_v} \right) \left(\frac{\partial q_i}{\partial\theta_w} \right) + \left(\frac{\partial^2 q_i}{\partial\theta_u \partial\theta_w} \right) \left(\frac{\partial q_i}{\partial\theta_v} \right) + \left(\frac{\partial^2 q_i}{\partial\theta_v \partial\theta_w} \right) \left(\frac{\partial q_i}{\partial\theta_u} \right) \right] \\
&+ 2 \sum_{i=1}^{k} \sum_{k=1}^{n_i} \left[\sum_{j=1}^{J} \left(\frac{\delta_{ikj}}{q_{ij}^3} \right) \left(\frac{\partial q_{ij}}{\partial\theta_u} \right) \left(\frac{\partial q_{ij}}{\partial\theta_v} \right) \left(\frac{\partial q_{ij}}{\partial\theta_w} \right) - \frac{(n_i-\delta_{ik+})}{(1-q_i)^3} \left(\frac{\partial q_i}{\partial\theta_u} \right) \left(\frac{\partial q_i}{\partial\theta_v} \right) \left(\frac{\partial q_i}{\partial\theta_w} \right) \right].
\end{aligned}$$

We now define

$$K_1 = \max_{1 \le i \le k} \max_{1 \le j \le J} \left(\frac{1}{q_{ij}} \right) \quad \text{and} \quad K_2 = \max_{1 \le i \le k} \left(\frac{1}{1-q_i} \right).$$

Thus, we have

$$\begin{aligned}
\left| \frac{\partial^3 l(\boldsymbol{\theta})}{\partial\theta_u \partial\theta_v \partial\theta_w} \right| \le \sum_{i=1}^{k} n_i \, &\left[(K_1 + K_2) J C_{uvw} + (K_1^2 + K_2^2) J^2 (C_{uv} C_w + C_{vw} C_u + C_{uw} C_v) \right. \\
&\left. + 2(K_1^3 J + K_2^3 J^3) C_u C_v C_w \right] = \sum_{i=1}^{k} n_i b_{uvw}, \text{ say.}
\end{aligned}$$

Let $\sum_{i=1}^{k} n_i b_{uvw} = K_{uvw}$. Now $E\left[K_{uvw}\right] = n k_{uvw}$, where $k_{uvw} = b_{uvw} \sum_{i=1}^{k} b_i$. It is easy to see that $k_{uvw} < \infty$. Hence, the proof. (Proved)

5.A.2 Proof of theorem

Proof of theorem 5.2

Proof: The second part of the theorem can be proved following the steps as in Lehmann and Casella [61, Theorem 5.1, page 463]. For the first part, it is enough to prove the following results:

(a) $\frac{1}{n}\frac{\partial l(\boldsymbol{\theta})}{\partial \theta_u} \xrightarrow{P} 0$, as $n \to \infty$.

(b) $\frac{1}{n}\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \theta_u \partial \theta_v} \xrightarrow{P} -I^1_{uv}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}) = -\sum_{i=1}^{k} b_i \sigma^2_{i,uv}$, as $n \to \infty$,
where $\sigma^2_{i,uv} = \left[\frac{1}{q_{ij}}\left(\frac{\partial q_{ij}}{\partial \theta_u}\right)\left(\frac{\partial q_{ij}}{\partial \theta_v}\right) + \left(\frac{1}{1-q_i}\right)\left(\frac{\partial q_i}{\partial \theta_u}\right)\left(\frac{\partial q_i}{\partial \theta_v}\right)\right]$

Now, we have

$$\frac{1}{n}\frac{\partial l(\boldsymbol{\theta})}{\partial \theta_u} = \sum_{i=1}^{k} \frac{n_i}{n}\left[\frac{1}{n_i}\sum_{k=1}^{n_i} U_{ik,u}\right],$$

where $U_{ik,u} = \sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\partial q_{ij}}{\partial \theta_u}\right) - \frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial q_i}{\partial \theta_u}\right)$. For given n_i, $U_{ik,u}$s are i.i.d. and $E[U_{ik,u} \mid n_i] = 0$, for $k = 1, \ldots, n_i$. Now, by the Weak Law of Large Numbers (WLLN), for given n_i, $\sum_{k=1}^{n_i} U_{ik,u}/n_i \xrightarrow{P} 0$, as $n \to \infty$. Now, by Lemma 5.3, it follows that $\frac{1}{n}\frac{\partial l(\boldsymbol{\theta})}{\partial \theta_u} \xrightarrow{P} 0$ as $n \to \infty$.

Similarly, we have

$$\frac{1}{n}\frac{\partial l^2(\boldsymbol{\theta})}{\partial \theta_u \partial \theta_v} = \sum_{i=1}^{k} \frac{n_i}{n}\left[\frac{1}{n_i}\sum_{k=1}^{n_i} V_{ik,uv}\right],$$

where $V_{ik,uv} = \left[\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}}\right)\left(\frac{\partial^2 q_{ij}}{\partial \theta_u \partial \theta_v}\right) - \frac{(1-\delta_{ik+})}{(1-q_i)}\left(\frac{\partial^2 q_i}{\partial \theta_u \partial \theta_v}\right)\right] - \left[\sum_{j=1}^{J}\left(\frac{\delta_{ikj}}{q_{ij}^2}\right)\left(\frac{\partial q_{ij}}{\partial \theta_u}\right)\left(\frac{\partial q_{ij}}{\partial \theta_v}\right) + \frac{(1-\delta_{ik+})}{(1-q_i)^2}\right.$ $\left.\left(\frac{\partial q_i}{\partial \theta_u}\right)\left(\frac{\partial q_i}{\partial \theta_v}\right)\right]$. For given n_i, $V_{ik,uv}$s are i.i.d. and $E[V_{ik,uv} \mid n_i] = -\left[\frac{1}{q_{ij}}\left(\frac{\partial q_{ij}}{\partial \theta_u}\right)\left(\frac{\partial q_{ij}}{\partial \theta_v}\right) + \left(\frac{1}{1-q_i}\right)\right.$ $\left.\left(\frac{\partial q_i}{\partial \theta_u}\right)\left(\frac{\partial q_i}{\partial \theta_v}\right)\right]$, for $k = 1, \ldots, n_i$. Now, by the WLLN, for a given n_i,

$$\sum_{k=1}^{n_i} \frac{V_{ik,uv}}{n_i} \xrightarrow{P} -\left[\frac{1}{q_{ij}}\left(\frac{\partial q_{ij}}{\partial \theta_u}\right)\left(\frac{\partial q_{ij}}{\partial \theta_v}\right) + \left(\frac{1}{1-q_i}\right)\left(\frac{\partial q_i}{\partial \theta_u}\right)\left(\frac{\partial q_i}{\partial \theta_v}\right)\right] = -\sigma^2_{i,uv},$$

as $n \to \infty$. Now, by Lemma 5.4, we get the desired result. (Proved)

Proof of theorem 5.3

Proof: **Proof of part (i):** Let $\boldsymbol{\zeta}_1 = (k, h, p)$ and $\boldsymbol{\zeta}_2 = (k+1, h, p)$. Then, it is enough to show that $\phi(\boldsymbol{\zeta}_2) \leq \phi(\boldsymbol{\zeta}_1)$.

$$\begin{aligned} & I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2) - I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1) \\ =& \sum_{j=1}^{J} \frac{E[N_{k+1}]}{q_{(k+1)j}} \nabla_{\boldsymbol{\theta}}(q_{(k+1)j}) \nabla_{\boldsymbol{\theta}}(q_{(k+1)j})^T + \frac{E[N_{k+1}]}{(1-q_{k+1})} \nabla_{\boldsymbol{\theta}}(q_{(k+1)}) \nabla_{\boldsymbol{\theta}}(q_{(k+1)})^T. \end{aligned}$$

Now, for any $\boldsymbol{b}(\boldsymbol{\theta}) \neq 0$, we have

$$\boldsymbol{b}(\boldsymbol{\theta})^T [I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2) - I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1)] \boldsymbol{b}(\boldsymbol{\theta})$$

$$= \sum_{j=1}^{J} \frac{E[N_{k+1}]}{q_{(k+1)j}} \left[\boldsymbol{b}(\boldsymbol{\theta})^T \nabla_{\boldsymbol{\theta}} q_{(k+1)j}\right]^2 + \frac{E[N_{k+1}]}{(1-q_{k+1})} \left[\boldsymbol{b}(\boldsymbol{\theta})^T \nabla_{\boldsymbol{\theta}} q_{(k+1)}\right]^2$$

This shows that $I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2) - I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1)$ is a non-negative definite matrix. This implies that $I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1) - I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2)$ is also non-negative definite matrix. Therefore, we have

$$\begin{aligned}
&\boldsymbol{b}(\boldsymbol{\theta})^T \left[I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1) - I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2)\right] \boldsymbol{b}(\boldsymbol{\theta}) \geq 0 \\
\Longrightarrow\ &\boldsymbol{b}(\boldsymbol{\theta})^T I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1) \boldsymbol{b}(\boldsymbol{\theta}) - \boldsymbol{b}(\boldsymbol{\theta})^T I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2) \boldsymbol{b}(\boldsymbol{\theta}) \geq 0 \\
\Longrightarrow\ &\phi(\boldsymbol{\zeta}_2) \leq \phi(\boldsymbol{\zeta}_1).
\end{aligned}$$

Proof of part (ii): Let $\boldsymbol{\zeta}_1 = (k, h, p_1)$ and $\boldsymbol{\zeta}_2 = (k, h, p_2)$, where $p_1 < p_2$. Then it is enough to prove that $\phi(\boldsymbol{\zeta}_1) \leq \phi(\boldsymbol{\zeta}_2)$.

$$\begin{aligned}
&I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1) - I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2) \\
=&\sum_{i=1}^{k} \left[\sum_{j=1}^{J} \frac{(E[N_i^1] - E[N_i^2])}{q_{ij}} \nabla_{\boldsymbol{\theta}}(q_{ij}) \nabla_{\boldsymbol{\theta}}(q_{ij})^T + \frac{(E[N_i^1] - E[N_i^2])}{(1-q_i)} \nabla_{\boldsymbol{\theta}}(q_i) \nabla_{\boldsymbol{\theta}}(q_i)^T\right],
\end{aligned}$$

where $E[N_i^k] = n(1-p_k)^{i-1} R(\tau_{i-1}|\boldsymbol{\theta})$, for $i = 1, \ldots, k$ and $k = 1, 2$. Now, for any $\boldsymbol{b}(\boldsymbol{\theta}) \neq 0$, we have

$$\begin{aligned}
&\boldsymbol{b}(\boldsymbol{\theta})^T [I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1) - I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2)] \boldsymbol{b}(\boldsymbol{\theta}) \\
=&\sum_{i=1}^{k} \left[\sum_{j=1}^{J} \frac{(E[N_i^1] - E[N_i^2])}{q_{ij}} \left[\boldsymbol{b}(\boldsymbol{\theta})^T \nabla_{\boldsymbol{\theta}}(q_{ij})\right]^2 + \frac{(E[N_i^1] - E[N_i^2])}{(1-q_i)} \left[\boldsymbol{b}(\boldsymbol{\theta})^T \nabla_{\boldsymbol{\theta}}(q_i)\right]^2\right]
\end{aligned}$$

Since

$$\begin{aligned}
E[N_i^1] - E[N_i^2] &= n(1-p_1)^{i-1} R(\tau_{i-1}|\boldsymbol{\theta}) - n(1-p_2)^{i-1} R(\tau_{i-1}|\boldsymbol{\theta}) \\
&= nR(\tau_{i-1}|\boldsymbol{\theta}) \left[(1-p_1)^{i-1} - (1-p_2)^{i-1}\right] \geq 0
\end{aligned}$$

This shows that $I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1) - I(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2)$ is a non-negative definite matrix. This implies that $I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2) - I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1)$ is also non-negative definite matrix. Therefore, we have

$$\begin{aligned}
&\boldsymbol{b}(\boldsymbol{\theta})^T \left[I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2) - I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1)\right] \boldsymbol{b}(\boldsymbol{\theta}) \geq 0 \\
\Longrightarrow\ &\boldsymbol{b}(\boldsymbol{\theta})^T I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_2) \boldsymbol{b}(\boldsymbol{\theta}) - \boldsymbol{b}(\boldsymbol{\theta})^T I^{-1}(\boldsymbol{\theta} \mid \boldsymbol{\zeta}_1) \boldsymbol{b}(\boldsymbol{\theta}) \geq 0 \\
\Longrightarrow\ &\phi(\boldsymbol{\zeta}_1) \leq \phi(\boldsymbol{\zeta}_2).
\end{aligned}$$

(Proved)

Chapter 6

RASP for independent competing risks data under ICS by Bayesian approach

This chapter addresses the design of BRASPs under ICS for independent competing risks data. Most of the existing BRASP literature has focused on systems with a single failure mode. However, in practical scenarios, a product may fail due to multiple causes, known as competing risks. The causes of failure may be statistically independent or dependent. Although dependence among risks is more realistic, most studies assume statistical independence to avoid identifiability issues inherent in the latent failure model. This chapter also assumes that the latent failure times are statistically independent.

Designing RASPs in the presence of competing risks is a critical problem in reliability research. Wu and Huang [122] investigated this problem under PICS-I using a classical approach. More recently, Prajapat et al. [96] proposed BRASPs under type-II and HCS-I for competing risks data. In this chapter, we extend the investigation to the design of optimal BRASPs under ICS for competing risks.

The primary contribution of this chapter is to develop an optimal BRASP by minimizing the Bayes risk for the exponential distribution under ICS. Given that reliability plays a pivotal role in the consumer's acceptance decision, the first objective is to construct BRASP using a decision function based on the MLE of the RF. The second objective is to propose an approximate BRASP by utilizing the asymptotic properties of the MLE. Finally, the third objective is to develop BRASP based on the Bayes decision function. These three approaches offer complementary strategies for designing efficient sampling plans under ICS in the presence of competing risks.

The organization of the chapter is as follows. Section 6.1 discusses the model for the exponential distribution in the presence of competing risk under ICS and the derivation of the Bayes risk for the general decision function. Section 6.2 presents the explicit form of the decision function when the decision is taken based on the MLE of the RF and also an approximate Bayes risk using the asymptotic property of the MLE of the RF. Section 6.3 discusses the Bayes decision function. Section 6.4 provides the numerical study for all cases. Section 6.5 presents the discussion.

6.1 Model and assumptions

Suppose a lot of items is considered to make a decision whether the lot to be accepted or rejected. For this, n items are put on a life test at time $\tau_0 = 0$. The lifetimes of n items $X_1, \ldots, X_n$ are IID with common CDF $F(\cdot \mid \boldsymbol{\theta})$ and PDF $f(\cdot \mid \boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_l)$ is a vector of parameters. Let us consider that every item may fail due to any one of J causes of failure. Let X_{ij} be the latent lifetime of the i^{th} item due to the j^{th} cause of failure, for $i = 1, \ldots, n$ and $j = 1, \ldots, J$. Therefore $X_i = \min\{X_{i1}, \ldots, X_{iJ}\}$. It is assumed that X_{ij} follows $\text{Exp}(\lambda_{0j})$ with PDF $f_{0j}(\cdot \mid \lambda_{0j})$. In the latent failure rate model, $X_{iJ}, \ldots, X_{iJ}$ are assumed to be independent. Therefore $X_1, \ldots, X_n$ follows $\text{Exp}(\lambda)$, where $\lambda = \lambda_{01} + \cdots + \lambda_{0J}$. Now, $\boldsymbol{\theta}$ can be taken as $\boldsymbol{\theta} = (\lambda_{01}, \ldots, \lambda_{0J})$ Let $C_i \in \{1, \cdots, J\}$ denote the cause of the failure of the i^{th} item. The joint distribution of (X_i, C_i) is given by

$$G(j, x \mid \boldsymbol{\theta}) = P(C = j, X_i \le x \mid \boldsymbol{\theta}) = \frac{\lambda_{0j}}{\lambda}\left[1 - \exp(-\lambda x)\right], \quad \lambda > 0, \ x > 0,$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, J$. Let us consider that the life test is conducted under ICS. In the ICS life testing, we only observe the number of failures due to every cause of failure at pre-specified inspection times $\tau_1, \ldots, \tau_k$, where $0 < \tau_1 < \cdots < \tau_k$. Suppose D_{mj} denotes the number of failures in the interval $(\tau_{m-1}, \tau_m]$ due to the cause j, for $m = 1, \ldots, k$ and $j = 1, \ldots, J$. Let $\boldsymbol{X} = (D_{11}, \ldots, D_{1J}, \ldots, D_{k1}, \ldots, D_{kJ})$. The observed data up to inspection time τ_k is $\boldsymbol{x} = (d_{11}, \ldots, d_{1J}, \ldots, d_{k1}, \ldots, d_{kJ})$, where d_{mj} is the observed value of D_{mj}, for $m = 1, \ldots, k$ and $j = 1, \ldots, J$. The likelihood function can be written as

$$\begin{aligned} L(\boldsymbol{x} \mid \boldsymbol{\theta}) &\propto \prod_{m=1}^{k}\prod_{j=1}^{J}\left[G(j, \tau_m \mid \boldsymbol{\theta}) - G(j, \tau_{m-1} \mid \boldsymbol{\theta})\right]^{d_{mj}}\left[1 - F(\tau_k \mid \boldsymbol{\theta})\right]^{n-d} \\ &\propto \prod_{m=1}^{k}\prod_{j=1}^{J}\left[\frac{\lambda_{0j}}{\lambda}\left\{\exp(-\lambda\tau_{m-1}) - \exp(-\lambda\tau_m)\right\}\right]^{d_{mj}} \times \left[\exp(-\lambda\tau_k)\right]^{n-d}. \end{aligned} \tag{6.1.1}$$

The vector of the decision variable of the life testing plan under interval censoring is denoted by $\boldsymbol{\zeta} = (n, \tau_1, \ldots, \tau_k, k)$. Let $a(\boldsymbol{x} \mid \boldsymbol{\zeta})$ denotes the action of acceptance sampling, which is defined in (4.1.5). The loss functions for acceptance and rejection of the lot after the life testing is defined as

$$\mathcal{L}\left(\mathcal{A} \mid \boldsymbol{x}, \boldsymbol{\theta}\right) = h(\boldsymbol{\theta}) + nC_s - (n-d)v_s + \tau C_t + IC_I \tag{6.1.2}$$

and

$$\mathcal{L}\left(\mathcal{R} \mid \boldsymbol{x}, \boldsymbol{\theta}\right) = C_r + nC_s - (n-d)v_s + \tau C_t + IC_I, \tag{6.1.3}$$

respectively. The loss function $h(\boldsymbol{\theta})$ and cost coefficients v_s, C_r, C_s, C_t and C_I are defined in Chapter 4. Here we considered quadratic loss function which is defined as $h(\boldsymbol{\theta}) = C_0 + \sum_{j=1}^{J} C_j\lambda_{0j} + \sum_{\substack{i=1 \\ i\le j}}^{J}\sum_{j=1}^{J} C_{ij}\lambda_{0i}\lambda_{0j}$, where the coefficients C_i and C_{jl}, for $i = 0, 1, \ldots, J$, $j = 1, \ldots, l$ and $l = 1, \ldots, J$ are taken in such a way that $h(\boldsymbol{\theta}) \ge 0$ for all $\boldsymbol{\theta}$. This loss function is known as a quadratic loss function. The expected loss for a given BRASP $\boldsymbol{\zeta}$ and observed data $\boldsymbol{x}$ is given by

$$\begin{aligned} \psi(\boldsymbol{\zeta}, a \mid \boldsymbol{x}, \boldsymbol{\theta}) =& \mathcal{L}(\mathcal{A} \mid \boldsymbol{x}, \boldsymbol{\theta})a(\boldsymbol{x} \mid \boldsymbol{\zeta}) + \mathcal{L}(\mathcal{R} \mid \boldsymbol{x}, \boldsymbol{\theta})\left[1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta})\right] \\ =& C_r + nC_s - (n-d)v_s + \tau C_t + IC_I + a(\boldsymbol{x} \mid \boldsymbol{\zeta})\left[h(\boldsymbol{\theta}) - C_r\right]. \end{aligned}$$

The Bayes risk of a BRASP $(\boldsymbol{\zeta}, a)$ is obtained as

$$R_B(\boldsymbol{\zeta}, a) = E_{\boldsymbol{\theta}}E_{\boldsymbol{x} \mid \boldsymbol{\theta}}\left[\psi(\boldsymbol{\zeta}, a \mid \boldsymbol{x}, \boldsymbol{\theta})\right]$$

$$= C_r + n(C_s - v_s) + v_s E[D] + C_t E[\tau] + C_I E[I] + R_A(\boldsymbol{\zeta}, a), \tag{6.1.4}$$

where

$$\begin{aligned} R_A(\boldsymbol{\zeta}, a) &= E_{\boldsymbol{\theta}} E_{\boldsymbol{x} \mid \boldsymbol{\theta}}[a(\boldsymbol{x} \mid \boldsymbol{\zeta})[h(\boldsymbol{\theta}) - C_r] \\ &= \sum_{\boldsymbol{x} \in \mathcal{X}} \int_{\boldsymbol{\theta}} [h(\boldsymbol{\theta}) - C_r] P_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta}) \, d\boldsymbol{\theta}. \end{aligned} \tag{6.1.5}$$

Here, $\mathcal{X}$ is the set of all possible $\boldsymbol{x}$ for which the consumer accepts the lot after life testing and

$$P_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\theta}) = \frac{n!}{\prod_{m=1}^{k} \prod_{j=1}^{J} d_{mj}!(n-d)!} \prod_{m=1}^{k} \prod_{j=1}^{J} [G(j, \tau_m \mid \boldsymbol{\theta}) - G(j, \tau_{m-1} \mid \boldsymbol{\theta})]^{d_{mj}} [1 - F(\tau_k \mid \boldsymbol{\theta})]^{n-d}, \tag{6.1.6}$$

Moreover, $E[D] = E_{\boldsymbol{\theta}}[E[D \mid \boldsymbol{\theta}]] = E_{\boldsymbol{\theta}}[n(1 - \exp(-\lambda \tau_k))]$, $E[\tau] = E_{\boldsymbol{\theta}}[\tau \mid \boldsymbol{\theta}] = \sum_{i=1}^{k} \tau_i P_i$, where $\mathcal{P}_i = P_i - P_{i-1}$ is the probability that the life-test terminates at τ_i with $P_i = [1 - \exp(-\lambda \tau_i)]^n$, for $i = 1, \ldots, k-1$, $P_k = 1$, and $E[I \mid \boldsymbol{\theta}] = \sum_{i=1}^{k} i \, \mathcal{P}_i$.

Now, the optimal BRASP $(\boldsymbol{\zeta}^*, a^*) = (n^*, \tau_1^*, \ldots, \tau_k^*, k^*, a^*)$ is obtained by maximizing $R_B(\boldsymbol{\zeta}, a)$. If intervals are of equal length h, the optimal BRASP is $(\boldsymbol{\zeta}^*, a^*) = (n^*, h^*, k^*, a^*)$.

6.2 BRASP based on reliability criteria decision function

Here, we obtain BRASP under the assumption that the manufacturer knows the consumer's acceptance criteria. The consumer takes his/her decision based on the reliability criteria. A lot is of acceptable quality when the reliability of the product at a specified time point t_0, $R(t_0 \mid \boldsymbol{\theta}) = 1 - F(t_0 \mid \boldsymbol{\theta})$ is greater than π_c. Let $R(t_0 \mid \widehat{\boldsymbol{\theta}})$ is the MLE of $R(t_0 \mid \boldsymbol{\theta})$. Therefore, the lot is accepted if $R(t_0 \mid \widehat{\boldsymbol{\theta}}) > \pi_c$ and rejected if $R(t_0 \mid \widehat{\boldsymbol{\theta}}) \leq \pi_c$. The likelihood function based on the competing risks data under ICS is given in (6.1.1). Then the log-likelihood function can be written as

$$l(\boldsymbol{\theta} \mid \boldsymbol{x}) = \sum_{m=1}^{k} \sum_{j=1}^{J} [-d_{mj} \{\lambda \tau_{m-1} + \log(\lambda_{0j}) - \log(\lambda) + \log(1 - \exp(-\lambda(\tau_m - \tau_{m-1})))\}] - (n-d)\lambda \tau_k.$$

Therefore, the likelihood equations are given by

$$\begin{aligned} & \frac{\partial l(\boldsymbol{\theta} \mid \boldsymbol{x})}{\partial \lambda_{0j}} = 0 \\ \implies & \sum_{m=1}^{k} \frac{d_{mj}}{\lambda_{0j}} - \sum_{m=1}^{k} d_{m+} \left[\tau_{m-1} + \frac{1}{\lambda} - \frac{(\tau_m - \tau_{m-1}) \exp(-\lambda(\tau_m - \tau_{m-1}))}{1 - \exp(-\lambda(\tau_m - \tau_{m-1}))} \right] - (n-d)\tau_k = 0, \\ \implies & \frac{d_{+j}}{\lambda_{0j}} = \frac{d}{\lambda} + g(\lambda), \end{aligned} \tag{6.2.1}$$

where $g(\lambda) = \sum_{m=1}^{k} d_{m+} \left[\tau_{m-1} - \frac{(\tau_m - \tau_{m-1}) \exp(-\lambda(\tau_m - \tau_{m-1}))}{1 - \exp(-\lambda(\tau_m - \tau_{m-1}))} \right] + (n-d)\tau_k$, $d_{m+} = \sum_{j=1}^{J} d_{mj}$, $d_{+j} = \sum_{m=1}^{k} d_{mj}$ and $d = \sum_{m=1}^{k} d_{m+} = \sum_{j=1}^{J} d_{+j}$, for $j = 1, \cdots, J$. Solving (6.2.1), for $j = 1, \ldots, J$, we get the MLEs $\widehat{\boldsymbol{\theta}} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_J)$. Now, from the likelihood equations, we get

$$\begin{aligned} & \frac{\widehat{\lambda}_{01}}{d_{+1}} = \frac{\widehat{\lambda}_{02}}{d_{+2}} = \cdots = \frac{\widehat{\lambda}_{0J}}{d_{+J}} = \frac{\widehat{\lambda}}{d} + g(\widehat{\lambda}) \\ \implies & \widehat{\lambda}_{0j} = \frac{d_{+j}}{d} \widehat{\lambda} + g(\widehat{\lambda}), \qquad j = 1, \ldots, J. \end{aligned} \tag{6.2.2}$$

We know that $\widehat{\lambda} = \widehat{\lambda}_{01} + \cdots + \widehat{\lambda}_{0J}$. Therefore, from (6.2.2), we get $g(\widehat{\lambda}) = 0$. It is easy to show that $\lim_{\lambda \to 0} g(\lambda) = -\infty$ and $\lim_{\lambda \to \infty} = \sum_{m=1}^{k} d_{m+}\tau_{m-1} + (n-d)\tau_k > 0$. Also, we can see that $g(\lambda)$ is a strictly increasing function in λ. Hence, we get a unique solution of $\widehat{\lambda}$. The decision function can be written as

$$a(\boldsymbol{x} \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{if } \exp(-\widehat{\lambda} t_0) > \pi_c, \\ 0, & \text{otherwise}, \end{cases} \tag{6.2.3}$$

Note that we only need to estimate λ for our decision function. It is seen that $\widehat{\lambda}$ exists when $d > 0$. When $d = 0$, this means that all items have survived up to time τ_k, and in that case $\widehat{\lambda}$ is taken as $1/n\tau_k$ (Bartholomew [15] and Sinha [113]). If the inspection times are of equal length, i.e, $\tau_m = mh$, $m = 1, \cdots, k$, where h is the length of each interval, we can find the explicit form of $\widehat{\lambda}$ as

$$\widehat{\lambda} = \begin{cases} -\dfrac{1}{h} \ln \left(1 - \dfrac{d}{\sum_{m=1}^{k} m d_{m+} + (n-d)k}\right), & \text{if } d > 0, \\ \dfrac{1}{nkh}, & \text{if } d = 0. \end{cases}$$

Therefore, the decision function (6.2.3) can be written as

$$a(\boldsymbol{x} \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{if } \left(\frac{d}{\sum_{m=1}^{k} m d_{m+} + (n-d)k}\right)^{t_0/h} > \pi_c, \\ 0, & \text{otherwise}, \end{cases}$$

if $d > 0$, and

$$a(\mathbf{0} \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{if } \exp\left(-\frac{t_0}{nkh}\right) > \pi_c, \\ 0, & \text{otherwise}, \end{cases}$$

if $d = 0$. In this case, the form of the decision function is known. Therefore, to find the optimal decision we need to determine the optimal value of π_c. Let π_c^* be the optimal value of π_c. In that case $a^* \equiv \pi_c^*$. Now, it is assumed that the λ follows $G(\alpha_0, \beta_0)$ with PDF $p_1(\cdot)$. Also, it is assumed that for given λ, $(\lambda_{01}/\lambda, \ldots, \lambda_{0J}/\lambda)$ follows Dirichlet distribution with joint PDF

$$p_2(\lambda_{01}/\lambda, \ldots, \lambda_{0J}/\lambda) = \frac{\Gamma(\alpha_+)}{\prod_{j=1}^{J} \Gamma(\alpha_j)} \prod_{j=1}^{J} \left(\frac{\lambda_{01}}{\lambda}\right)^{\alpha_j - 1}, \quad 0 < \frac{\lambda_{0j}}{\lambda} < 1, \alpha_j > 0, \text{ for } j = 1, \ldots, J,$$

where λ_{0j}/λ is the ratio of the failure rate due to j^{th} cause of failure to the failure rate of the item and $\alpha_+ = \sum_{j=1}^{J} \alpha_j$ and it is denoted by $Di(\alpha_1, \ldots, \alpha_J)$. Therefore, the joint prior PDF of $\boldsymbol{\theta}$ is

$$p(\boldsymbol{\theta}) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \lambda^{\alpha_0 - \alpha_+} \exp(-\beta_0 \lambda) \prod_{j=1}^{J} \frac{\Gamma(\alpha_+)}{\Gamma(\alpha_j)} \lambda_{0j}^{\alpha_j - 1}.$$

This is known as the Gamma-Dirichlet distribution and is denoted by $GD(\alpha_0, \beta_0, \alpha_1, \ldots, \alpha_J)$. We provide the following results needed for the computation of Bayes risk.

Result 6.1

$$\int_{\lambda_{01}=0}^{\infty} \cdots \int_{\lambda_{0J}=0}^{\infty} \lambda^{\alpha_0 - \alpha_+} \exp(-\beta_0 \lambda) \prod_{j=1}^{J} \lambda_{0j}^{\alpha_j - 1} \; d\lambda_{01} \cdots d\lambda_{0J} = \frac{\Gamma(\alpha_0)}{\beta_0^{\alpha_0} \Gamma(\alpha_+)} \prod_{j=1}^{J} \Gamma(\alpha_j).$$

Result 6.2 *If n items are placed on a life test and inspected at equal intervals of length h up to a maximum of k inspections, then the Bayes risk is given by*

$$R_B(\boldsymbol{\zeta}, \pi_c) = C_r + nC_s + v_s n \frac{\beta_0^{\alpha_0}}{(\beta_0 + kh)^{\alpha_0}} + (C_I + c_t h) \left[k - \sum_{i=1}^{k-1} \sum_{j=1}^{n} \binom{n}{j} (-1)^j \left(\frac{\beta_0}{\beta_0 + ijh} \right)^{\alpha_0} \right] + R_A(\boldsymbol{\zeta}, \pi_c),$$

where

$$\begin{aligned} R_A(\boldsymbol{\zeta}, \pi_c) = \sum_{\boldsymbol{x} \in \mathcal{X}} & \frac{n!}{d_{11}! d_{12}! \cdots d_{m1}! d_{k2}! (n-d)!} \sum_{i=0}^{d} (-1)^i \binom{d}{i} \left[(C_0 - C_r) \frac{\beta_0^{\alpha_0}}{(\beta_0 + ih + sh)^{\alpha_0}} \right. \\ & + \Gamma(\alpha_+) \left\{ \frac{\alpha_0 \beta_0^{\alpha_0}}{(\beta_0 + ih + sh)^{\alpha_0 + 1}} \sum_{p=1}^{J} C_p \prod_{j=1}^{J} \frac{\Gamma(\alpha_j + d_{+j} + \delta_{pj})}{\Gamma(\alpha_j) \Gamma(\alpha_+ + d + 1)} \right. \\ & \left. \left. + \frac{\alpha_0 (\alpha_0 + 1) \beta_0^{\alpha_0}}{(\beta_0 + ih + sh)^{\alpha_0 + 2}} \sum_{\substack{p=1 \\ p \le q}}^{J} \sum_{q=1}^{J} C_{pq} \prod_{j=1}^{J} \frac{\Gamma(\alpha_j + d_{+j} + \delta_{pj} + \delta_{qj})}{\Gamma(\alpha_j) \Gamma(\alpha_+ + d + 2)} \right\} \right] \end{aligned}$$

and δ_{ij} denotes the indicator function given by $\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$

Proof: The expressions of $E[D]$, $E[I]$ and $E[\tau]$ are given in Chen et al. [29]. Only we need to prove the term $R_A(\boldsymbol{\zeta}, \pi_c)$. Using (6.1.6), when the inspection intervals are of equal length, the joint PMF of $\boldsymbol{X}$ can be written

$$\begin{aligned} P_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\theta}) &= \frac{n!}{\prod_{m=1}^{k} \prod_{j=1}^{J} d_{mj}! (n-d)!} \prod_{m=1}^{k} \prod_{j=1}^{J} \left\{ \exp\left[-\lambda d_{mj} (m-1) h\right] \left[1 - \exp(-\lambda h)\right]^{d_{mj}} \left(\frac{\lambda_{0j}}{\lambda} \right)^{d_{mj}} \right\} \\ & \quad \times \exp\left[-\lambda (n-d) k h\right] \\ &= \frac{n!}{\prod_{m=1}^{k} \prod_{j=1}^{J} d_{mj}! (n-d)!} \sum_{i=0}^{d} (-1)^i \binom{d}{i} \prod_{j=1}^{J} \left(\frac{\lambda_{0j}}{\lambda} \right)^{d_{+j}} \exp\left[-(i + b(k)) \lambda_{0j} h\right], \end{aligned}$$

where $b(k) = nk - \sum_{m=1}^{k} \sum_{j=1}^{J} (k - (m-1)) d_{mj}$.

From (6.1.5), we know that

$$R_A(\boldsymbol{\zeta}, a) = \sum_{\boldsymbol{x} \in \mathcal{X}} \int_{\boldsymbol{\theta}} \left[h(\boldsymbol{\theta}) - C_r\right] \; P_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\theta}) \; p(\boldsymbol{\theta}) \; d\boldsymbol{\theta}.$$

Now using Result 6.1, we get the required expression of $R_A(\boldsymbol{\zeta}, a)$. (Proved)

For different inspection intervals, the term $R_A(\boldsymbol{\zeta}, a)$ cannot be obtained analytically. A simple Monte Carlo integration can be used to evaluate the integration. For this, a large number S_1 of observations are generated from the prior $p(\boldsymbol{\theta})$. If $p(\boldsymbol{\theta})$ follows $GD(\alpha_0, \beta_0, \alpha_1, \ldots, \alpha_J)$, then the data can be generated using Algorithm 6.1.

Algorithm 6.1: Procedure for Data Generation from Gamma-Dirichlet Distribution

Generate $\lambda^{(i)}$ from $G(\alpha_0, \beta_0)$ distribution.;

Generate $\left(\frac{\lambda_{01}^{(i)}}{\lambda^{(i)}}, \ldots, \frac{\lambda_{0J}^{(i)}}{\lambda^{(i)}} \right)$ from $Di(\alpha_1, \ldots, \alpha_J)$ distribution using R package `LaplacesDemon`.;

Compute $\boldsymbol{\theta}^{(i)} = \left(\lambda^{(i)} \times \frac{\lambda_{01}^{(i)}}{\lambda^{(i)}}, \ldots, \lambda^{(i)} \times \frac{\lambda_{0J}^{(i)}}{\lambda^{(i)}} \right)$ to obtain the generated sample.;

Let $\boldsymbol{\theta}^{(1)} \ldots, \boldsymbol{\theta}^{(S_1)}$ be the generated observations. Then

$$R_A(\boldsymbol{\zeta}, a) \approx \frac{1}{S_1} \sum_{i=1}^{S_1} \left[C_r - h\left(\boldsymbol{\theta}^{(i)}\right)\right] \sum_{\boldsymbol{x} \in \mathcal{X}} P_{\boldsymbol{X}}\left(\boldsymbol{x} \mid \boldsymbol{\theta}^{(i)}\right).$$

6.2.1 Approximate form of Bayes risk using asymptotic properties of the MLE

For large samples and different prior distributions, computing Bayes risk will be intensive and complex. We approximate the Bayes risk in a simplified form, where we can calculate Bayes risk for any choice of prior. We know that under certain regularity conditions, for large n, $\widehat{\boldsymbol{\theta}}$ follows normal distribution with mean $\boldsymbol{\theta}$ and variance $I^{-1}(\boldsymbol{\theta})$ under certain regularity conditions, where $I^{-1}(\boldsymbol{\theta})$ is the inverse of the Fisher information matrix of $\boldsymbol{\theta}$. The regularity conditions are stated in Section 5.1.2 of Chapter 5.

For interval-censored data, the Fisher information matrix for $\boldsymbol{\theta}$ can be written as

$$I(\boldsymbol{\theta}) = \sum_{m=1}^{k} \left[\sum_{j=1}^{J} \frac{n[1 - F(\tau_{m-1} \mid \boldsymbol{\theta})]}{q_{mj}} \nabla_{\boldsymbol{\theta}}(q_{mj}) \nabla_{\boldsymbol{\theta}}(q_{mj})^T + \frac{n[1 - F(\tau_{m-1} \mid \boldsymbol{\theta})]}{(1 - q_m)} \nabla_{\boldsymbol{\theta}}(q_m) \nabla_{\boldsymbol{\theta}}(q_m)^T\right], \quad (6.2.4)$$

where $\nabla_{\boldsymbol{\theta}}(q_{mj}) = \left(\frac{\partial q_{mj}}{\lambda_1}, \ldots, \frac{\partial q_{mj}}{\lambda_J}\right)$, $\nabla_{\boldsymbol{\theta}}(q_m) = \left(\frac{\partial q_m}{\lambda_1}, \ldots, \frac{\partial q_m}{\lambda_J}\right)$ and q_{mj} and q_m are defined in (5.1.1) and (5.1.2), respectively. Now we find the approximate expression of $P(\exp(-\widehat{\lambda} t_0) > \pi_c)$. Let $c(\boldsymbol{\theta}) = \exp(-\lambda t_0)$. By using the delta method, we have $c(\widehat{\boldsymbol{\theta}}) \sim \mathcal{N}(c(\boldsymbol{\theta}), S^2)$, where $S^2 = \nabla_{\boldsymbol{\theta}} c(\boldsymbol{\theta})^T I^{-1}(\boldsymbol{\theta}) \nabla_{\boldsymbol{\theta}} c(\boldsymbol{\theta})$ and $\nabla_{\boldsymbol{\theta}} c(\boldsymbol{\theta}) = \left(\frac{\partial c(\boldsymbol{\theta})}{\partial \theta_1}, \ldots, \frac{\partial c(\boldsymbol{\theta})}{\partial \theta_J}\right)$. Therefore,

$$P\left(\exp(-\widehat{\lambda} t_0) > \pi_c\right) = P\left(\frac{c(\widehat{\boldsymbol{\theta}}) - c(\boldsymbol{\theta})}{S} > \frac{\pi_c - c(\boldsymbol{\theta})}{S}\right) \approx 1 - \Phi\left(\frac{\pi_c - c(\boldsymbol{\theta})}{S}\right) = \Phi\left(\frac{c(\boldsymbol{\theta}) - \pi_c}{S}\right),$$

where Φ is the CDF of standard normal distribution.

Now, we compute $\nabla_{\boldsymbol{\theta}}(q_{mj})$ and $\nabla_{\boldsymbol{\theta}}(q_m)$ to obtain $I(\boldsymbol{\theta})$. We have

$$\frac{\partial q_{mj}}{\partial \lambda_l} = \begin{cases} \dfrac{\lambda_{0j}(\tau_m - \tau_{m-1})(1 - q_m)}{\lambda} - \dfrac{q_m \lambda_{0j}}{\lambda^2}, & \text{if } l \neq j, \\ \dfrac{\lambda_{0j}(\tau_m - \tau_{m-1})(1 - q_m)}{\lambda} + \dfrac{q_m(\lambda - \lambda_{0j})}{\lambda^2}, & l = j, \end{cases}$$

$$\frac{\partial q_m}{\partial \lambda_{0j}} = (\tau_m - \tau_{m-1}) \exp[-\lambda(\tau_m - \tau_{m-1})] \quad \text{and} \quad \frac{\partial c(\boldsymbol{\theta})}{\partial \lambda_{0j}} = t_0 \exp[-\lambda t_0],$$

for $m = 1, \ldots, k$ and $l, j = 1, \ldots, J$. Therefore,

$$R_A(\boldsymbol{\zeta}, a) \approx \frac{1}{S_1} \sum_{i=1}^{S_1} \left[C_r - h\left(\boldsymbol{\theta}^{(i)}\right)\right] \Phi\left(\frac{c(\boldsymbol{\theta}^{(i)}) - \pi_c}{S_{(i)}}\right),$$

where $S_{(i)}$ is the value of S when $\boldsymbol{\theta} = \boldsymbol{\theta}^{(i)}$

6.2.2 Finding optimal BRASP

Now, we provide upper bounds of n, k and τ_k. Since all cost-coefficients are non-negative and $C_s \geq v_s \geq 0$, this implies that $R_B(\boldsymbol{\zeta}^*, a)$ is non-negative. Therefore, the following equalities hold:

$$R_B(\boldsymbol{\zeta}^*, a) \geq n^*(C_s - v_s) + C_\tau E[\tau] + C_I E[I]$$

$$\geq n^*(C_s - v_s) + C_\tau \tau_k + C_I E[I] \tag{6.2.5}$$

$$\geq n^*(C_s - v_s) + C_I k \tag{6.2.6}$$

$$\geq n^*(C_s - v_s). \tag{6.2.7}$$

When the decision of acceptance or rejection about the lot is taken without life testing, that scenario is called no sampling case. The Bayes risks corresponding to no sampling case are given by $R_B(\mathbf{0}, 1) = E_{\boldsymbol{\theta}}[h(\boldsymbol{\theta})]$, if the lot is accepted and $R_B(\mathbf{0}, 0) = C_r$, if the lot is rejected. Therefore, $R_B(\mathbf{0}, a) = \min\{E_{\boldsymbol{\theta}}[h(\boldsymbol{\theta})], C_r\}$. Now,

$$R_B(\boldsymbol{\zeta}^*, a) \leq \min\{E_{\boldsymbol{\theta}}[h(\boldsymbol{\theta})], C_r\}. \tag{6.2.8}$$

From (6.2.7) and (6.2.8), we get

$$n_0 = \left\lfloor \frac{\min\{E_{\boldsymbol{\theta}}[h(\boldsymbol{\theta})], C_r\}}{C_s - v_s} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. From (6.2.6) and (6.2.8), we get

$$k_0(n) = \left\lfloor \frac{\min\{E_{\boldsymbol{\theta}}[h(\boldsymbol{\theta})], C_r\} - (C_s - v_s)n}{C_I} \right\rfloor.$$

From (6.2.5) and (6.2.8), we get

$$\tau_B(n, k(n)) = \frac{\min\{E_{\boldsymbol{\theta}}[h(\boldsymbol{\theta})], C_r\} - (C_s - v_s)n - C_I k}{C_\tau}.$$

Since all the decision variables are bounded, we get a finite value of $\boldsymbol{\zeta}^*$. We consider Algorithm 6.2 to determine the optimal BRASP.

Algorithm 6.2: Determination of design parameters when decision function is fixed

Input : Fix n_0, $k_0(n)$ for each n and $\tau_B(n, k(n))$ for each n, k.
Output : Optimal design parameters n^*, $k^*(n^*)$, $\boldsymbol{\tau}^*(n^*, k^*(n^*))$, $\pi_c^*(n^*, k^*(n^*))$

for n *in* 1 *to* n_0 **do**
  for k *in* 0 *to* $k_0(n)$ **do**
    ▷ `Minimize` $R_B(n, \boldsymbol{\tau}, k, \pi_c)$ `w.r.t.` $\boldsymbol{\tau}$ **and** π_c **given** n, k
    Define $\boldsymbol{h} = (h_1, \ldots, h_k)$ such that $\tau_1 = h_1, \quad \tau_i = \tau_{i-1} + h_i, \quad i = 2, \ldots, k$, with constraints:

$$h_i > 0, \quad \sum_{i=1}^{k} h_i < \tau_B(n, k).$$

    `// For equal-length intervals:` $h_i = h < \frac{\tau_B(n,k)}{k}$
    Find $(\boldsymbol{h}^*, \pi_c^*) = \arg\min_{\boldsymbol{h}, \pi_c} R_B(n, \boldsymbol{\tau}, k, \pi_c)$.
    Compute $\boldsymbol{\tau}^* = (\tau_1^*, \ldots, \tau_k^*)$ from $\boldsymbol{h}^*$.
  Find

$$k^*(n) = \arg \min_{0 \leq k \leq k_0(n)} R_B(n, \boldsymbol{\tau}^*(n, k), k, \pi_c^*(n, k)).$$

Find

$$n^* = \arg \min_{1 \leq n \leq n_0} R_B(n, \boldsymbol{\tau}^*(n, k^*(n)), k^*(n), \pi_c^*(n, k^*(n))).$$

Take $\pi_c^*(n^*, k^*(n^*)) = \pi_c^*$, $\boldsymbol{\tau}^*(n^*, k^*(n^*)) = \boldsymbol{\tau}^*$ and $k^*(n^*) = k^*$

Therefore, $(\boldsymbol{\zeta}^*, \pi_c^*)$ is the optimal BRASP when the form of the decision function is known.

6.3 BRASP based on Bayes decision function

Here, we consider that the manufacturer does not know the consumer's decision criteria. Therefore, the decision function is arbitrary. It is also assumed that a single prior assessment is used by both the manufacturer and consumer. An optimal decision function is derived that minimizes the Bayes risk among all the decision functions. This optimal function is known as a Bayes decision function. The Bayes risk $R_B(\boldsymbol{\zeta}, a)$ in (6.1.4) consists of the decision function $a(\cdot \mid \boldsymbol{\zeta})$ only in the term $R_A(\boldsymbol{\zeta}, a)$. Note that $R_A(\boldsymbol{\zeta}, a)$ in (6.1.5) can be written as

$$\begin{aligned}
R_A(\boldsymbol{\zeta}, a) =& E_{\boldsymbol{x}} E_{\boldsymbol{\theta} \mid \boldsymbol{x}}[a(\boldsymbol{x} \mid \boldsymbol{\zeta})(h(\boldsymbol{\theta}) - C_r)] \\
=& \sum_{\boldsymbol{x} \in \mathcal{X}} [a(\boldsymbol{x} \mid \boldsymbol{\zeta})] E_{\boldsymbol{\theta} \mid \boldsymbol{x}}[h(\boldsymbol{\theta}) - C_r]\ P_{\boldsymbol{X}}(\boldsymbol{x}) \\
=& \sum_{\boldsymbol{x} \in \mathcal{X}} [a(\boldsymbol{x} \mid \boldsymbol{\zeta})][\varphi(\boldsymbol{x}) - C_r]\ P_{\boldsymbol{X}}(\boldsymbol{x}),
\end{aligned}$$

where $\varphi(\boldsymbol{x}) = \int_{\boldsymbol{\theta}} h(\boldsymbol{\theta})\ p(\boldsymbol{\theta} \mid \boldsymbol{x})\ d\boldsymbol{\theta}$ and $P_{\boldsymbol{X}}(\boldsymbol{x}) = \int_{\boldsymbol{\theta}} P_{\boldsymbol{X}}(\boldsymbol{x} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta})\ d\boldsymbol{\theta}$.

To obtain the Bayes decision, we need to minimize $R_A(\boldsymbol{\zeta}, a)$ with respect to a which is equivalent to minimization of $E_{\boldsymbol{\theta}} E_{\boldsymbol{x} \mid \boldsymbol{\theta}}[a(\boldsymbol{x} \mid \boldsymbol{\zeta})(h(\boldsymbol{\theta}) - C_r)]$ with respect to a. From Section 4.2.2, the Bayes decision function $a^*(\boldsymbol{x} \mid \boldsymbol{\zeta})$ for fixed $\boldsymbol{\zeta}$ is given by

$$a^*(\boldsymbol{x} \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{if } C_r - \varphi(w_1, w_2, \boldsymbol{d}) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the prior expectation of the loss $h(\boldsymbol{\theta})$ is

$$E[h(\boldsymbol{\theta})] = \ C_0 + \sum_{p=1}^{J} C_p \frac{\alpha_0(\alpha_p + 1)}{\beta_0 \alpha_+} + \frac{\alpha_0(\alpha_0 + 1)}{\beta_0^2 \alpha_+ (\alpha_+ + 1)} \left[\sum_{\substack{p=1 \\ p<q}}^{J} \sum_{q=1}^{J} C_{pq}(\alpha_p + 1)(\alpha_q + 1) + \sum_{p=1}^{J} C_{pp}(\alpha_p + 2) \right] = \varphi(\mathbf{0})$$

Therefore, in no sampling case, the Bayes decision is

$$a^*(\mathbf{0} \mid \mathbf{0}) = \begin{cases} 1, & \text{if } C_r - \varphi(\mathbf{0}) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Result 6.3 *If n items are put on a test and inspected at equal intervals of length h, then the posterior expectation of $h(\boldsymbol{\theta})$, $\varphi(\boldsymbol{x}) = \int_{\boldsymbol{\theta}} h(\boldsymbol{\theta})\ p(\boldsymbol{\theta} \mid \boldsymbol{x})\ d\boldsymbol{\theta}$ can be written as*

$$\begin{aligned}
\varphi(\boldsymbol{x}) =& C_0 + C \sum_{p=1}^{J} C_p \left[\sum_{i=0}^{d} (-1)^i \binom{d}{i} \frac{\Gamma(\alpha_0 + 1)}{[(i + b(k))h + \beta_0]^{\alpha_0 + 1}} \prod_{j=1}^{J} \frac{\Gamma(\alpha_j + d_{+j} + \delta_{pj})}{\Gamma(\alpha_+ + d + 1)} \right] \\
&+ C \sum_{\substack{p=1 \\ p \leq q}}^{J} \sum_{q=1}^{J} C_{pq} \left[\sum_{i=0}^{d} (-1)^i \binom{d}{i} \frac{\Gamma(\alpha_0 + 2)}{[(i + b(k))h + \beta_0]^{\alpha_0 + 2}} \prod_{j=1}^{J} \frac{\Gamma(\alpha_j + d_{+j} + \delta_{pj} + \delta_{qj})}{\Gamma(\alpha_+ + d + 2)} \right],
\end{aligned}$$

where

$$C = \left[\sum_{i=0}^{d} (-1)^i \binom{d}{i} \frac{\Gamma(\alpha_0)}{[(i + b(k))h + \beta_0]^{\alpha_0}} \prod_{j=1}^{J} \frac{\Gamma(\alpha_j + d_{+j})}{\Gamma(\alpha_+ + d)} \right]^{-1}.$$

Proof: The likelihood function is given by

$$L(\boldsymbol{\theta} \mid \boldsymbol{x}) \propto \prod_{m=1}^{k}\prod_{j=1}^{J}\left\{\exp\left[-\lambda d_{mj}(m-1)h\right]\left[1-\exp(-\lambda h)\right]^{d_{mj}}\left(\frac{\lambda_{0j}}{\lambda}\right)^{d_{mj}}\right\}\exp\left[-\lambda(n-d)kh\right]$$
$$\propto \sum_{i=0}^{d}(-1)^i\binom{d}{i}\prod_{j=1}^{J}\left(\frac{\lambda_{0j}}{\lambda}\right)^{d_{+j}}\exp\left[-(i+b(k))\lambda_{0j}h\right],$$

where $b(k) = nk - \sum_{m=1}^{k}\sum_{j=1}^{J}(k-(m-1))d_{mj}$. The joint prior of $\boldsymbol{\theta}$ follows $GD(\alpha_0, \beta_0, \alpha_1, \ldots, \alpha_J)$. The posterior distribution of $\boldsymbol{\theta}$ is given by

$$p(\boldsymbol{\theta} \mid \boldsymbol{x}) = C\sum_{i=0}^{d}(-1)^i\binom{d}{i}\lambda^{\alpha_0-(\alpha_+ + d)}\prod_{j=1}^{J}\lambda_{0j}^{\alpha_j+d_{+j}-1}\exp\left[-(\beta_0+(i+b(k))h)\lambda_{0j}\right],$$

where C is the normalizing constant of the posterior distribution given by

$$C = \left[\sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{\Gamma(\alpha_0)}{[(i+b(k))h+\beta_0]^{\alpha_0}}\prod_{j=1}^{J}\frac{\Gamma(\alpha_j+d_{+j})}{\Gamma(\alpha_+ + d)}\right]^{-1}.$$

Using Result 6.1, we get the desired result. (Proved)

For the sampling case, when the lengths of inspection intervals are unequal, the posterior expectation of $h(\boldsymbol{\theta})$ cannot be obtained analytically. A simple Monte Carlo integration can be used to evaluate it. For this, a large number S_2 of observations are generated from the prior distributions $p(\boldsymbol{\theta})$. Let $\boldsymbol{\theta}^{(1)}\ldots,\boldsymbol{\theta}^{(S_2)}$ be the generated observations. Then

$$\varphi(\boldsymbol{x}) \approx \frac{\sum_{j=1}^{S_2} h\left(\boldsymbol{\theta}^{(j)}\right) L\left(\boldsymbol{\theta}^{(j)} \mid \boldsymbol{x}\right)}{\sum_{j=1}^{S_2} L\left(\boldsymbol{\theta}^{(j)} \mid \boldsymbol{x}\right)}. \tag{6.3.1}$$

Next, we provides Algorithm 6.3 for finding optimal BRASP under this scenario.

Algorithm 6.3: Determination of design parameters when decision function is arbitrary

Input : Fix n_0, $k_0(n)$ for each n and $\tau_B(n, k(n))$ for each n, k.

Output : Optimal design parameters n^*, $k^*(n^*)$, $\boldsymbol{\tau}^*(n^*, k^*(n^*))$, Bayes decision function $a^*(\cdot \mid \boldsymbol{\zeta}^*)$

for $\boldsymbol{\zeta} = (n, \boldsymbol{\tau}, k)$ **do**

  Find the Bayes decision function $a^*(\cdot \mid \boldsymbol{\zeta}) = \arg\min_{a(\cdot|\boldsymbol{\zeta})} R_B(\boldsymbol{\zeta})$ according to Section 4.2.2.

for $n = 1$ *to* n_0 **do**

  for $k = 0$ *to* $k_0(n)$ **do**

    Minimize $R_B(n, \boldsymbol{\tau}, k, a^*(\cdot \mid \boldsymbol{\zeta}))$ with respect to $\boldsymbol{\tau}$, in a manner similar to Algorithm 6.2, i.e.,

$$\boldsymbol{\tau}^*(n,k) = \arg\min_{\boldsymbol{\tau}:\tau_1<\cdots<\tau_k} R_B(n, \boldsymbol{\tau}, k, a^*(\cdot \mid (n, \boldsymbol{\tau}, k))).$$

  Determine

$$k^*(n) = \arg\min_{0\le k\le k_0(n)} R_B(n, \boldsymbol{\tau}^*(n,k), k, a^*(\cdot \mid (n, \boldsymbol{\tau}^*(n,k), k))).$$

Find the optimal sample size

$$n^* = \arg\min_{1\le n\le n_0} R_B(n, \boldsymbol{\tau}^*(n, k^*(n)), k^*(n), a^*(\cdot \mid (n, \boldsymbol{\tau}^*(n, k^*(n)), k^*(n)))).$$

Take $a^*(\cdot \mid (n^*, \boldsymbol{\tau}^*(n^*, k^*(n^*)), k^*(n^*))) = a^*$, $\boldsymbol{\tau}^*(n^*, k^*(n^*)) = \boldsymbol{\tau}^*$ and $\boldsymbol{k}^*(n^*) = k^*$

Therefore, $(\boldsymbol{\zeta}^*, a^*) = (n^*, \boldsymbol{\tau}^*, k^*, a^*)$ is the optimal BRASP when the decision function is arbitrary.

Theorem 6.1 *The BRASP $(n^*, \boldsymbol{\tau}^*, k^*, a^*)$ is the optimal BRASP among the all BRASPs.*

Proof: The proof is similar to the proof of Theorem 4.2. (Proved)

6.4 Numerical Example

Here we provide some examples of determining optimum BRASPs. For illustration, we consider two causes of failure, i.e, $J = 2$.

Example 6.1 *We consider the hyperparameter values of prior distribution as* $\alpha_0 = 2.8$, $\beta_0 = 1$, $\alpha_1 = 1.5$ *and* $\alpha_2 = 1.8$. *The coefficient of the loss function are taken as* $C_0 = 2$, $C_1 = C_2 = 4$ *and* $C_{11} = C_{12} = C_{22} = 4$. *The other cost coefficients are taken as* $C_r = 40$, $C_s = 0.5$, $v_s = 0.25$, $C_t = 0.3$ *and* $C_I = 0.1$. $t_0 = 0.1$.

Here, we denote the BRASP as type-I when the decision function is arbitrary, and type-II when the form of the decision function is known. The optimal sampling schemes corresponding to type-I and type-II RASPs are provided in Table 6.1. Also, in Table 6.1, we provide $E[D^*]$, $E[\tau^*]$, $E[I^*]$ and optimal Bayes risk R_B^* at $\boldsymbol{\zeta}^*$.

Table 6.1: Optimal BRASP

Type	$\boldsymbol{\zeta}^* = (n^*, h^*, k^*)$	π_c^*	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	R_B^*
I	(4, 0.30,3)	-	3.337	0.740	2.467	33.90826
II	(4, 0.30, 3)	0.76	3.337	0.740	2.467	33.90826

6.4.1 Effect of the parameters on the optimum solution

Next, we consider the effect of the parameters on the optimum solution. The effect of the parameters C_r, C_t, α_0 and β_0 are given in Tables 6.2, 6.3, 6.4 and 6.5, respectively.

Table 6.2: Optimal BRASPs for different values of C_r

C_r	Type	$\boldsymbol{\zeta}^* = (n^*, h^*, k^*)$	π_c^*	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	R_B^*
20	I	(0, 0, 0)	-	0	0	0	20
	II	(0, 0, 0)	1	0	0	0	20
30	I	(3, 0.41, 3)	-	2.682	0.855	2.085	28.07000
	II	(3, 0.41, 3)	0.80	2.682	0.855	2.085	28.07000
50	I	(6, 0.27, 2)	-	4.209	0.525	1.944	38.21502
	II	(6, 0.27, 2)	0.72	4.209	0.525	1.944	38.21517
60	I	(7, 0.21, 2)	-	4.378	0.416	1.982	41.44479
	II	(7, 0.22, 2)	0.68	4.478	0.435	1.978	41.44624
70	I	(3, 0.19, 2)	-	2.98	0.38	1.96	44.69447
	II	(3, 0.19, 2)	0.52	2.98	0.38	1.96	44.69447
90	I	(0, 0, 0)	-				47.66190
	II	(0,0,0)	0	0	0	0	47.66190

Note that the Bayes risks of accepting and rejecting the lot without sampling are $R_B(0,0,0,1) = \varphi(\mathbf{0})$ and $R_B(0,0,0,0) = C_r$, respectively. We know that the minimum Bayes risk is always less than $R_B(0,0,0,1)$ and $R_B(0,0,0,0)$, otherwise the lot will be accepted or rejected without sampling. Therefore, sometimes the lot can be accepted without sampling when the corresponding Bayes risk is minimal among all plans. In Table 6.2, it is seen that for $C_r = 20, 90$, the optimal solution is $(0,0,0)$, which is the no sampling case. For type-I BRASP, $R_B^* = C_r$, when $C_r = 20$. This means that the lot is rejected without life testing. When $C_r = 90$, $R_B^* = \varphi(\mathbf{0})$, which means that the lot is accepted without life testing. Also, in Table 6.2, it is observed that when C_r increases, sample size increases up to a certain value and after that it decreases. This is due to the fact that the distance between C_r and prior expectation $\varphi(\mathbf{0})$ is decreasing with C_r on $(-\infty, 47.66)$ and increasing with C_r on $(47.66, \infty)$.

Table 6.3: Optimal BRASPs for different values of C_t

C_t	Type	$\boldsymbol{\zeta}^* = (n^*, h^*, m^*)$	π_c^*	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	R_B^*
0	I	(4, 0.36, 3)	-	3.485	0.838	2.329	33.66994
	II	(4, 0.36, 3)	0.74	3.485	0.838	2.329	33.66994
0.1	I	(4, 0.36, 3)	-	3.485	0.838	2.329	33.75377
	II	(4, 0.36, 3)	0.74	3.485	0.838	2.329	33.75377
0.5	I	(5, 0.29, 2)	-	3.611	0.553	1.908	34.04424
	II	(5, 0.29, 2)	0.76	3.611	0.553	1.908	34.04424
1	I	(5, 0.25, 2)	-	3.393	0.484	1.936	34.31483
	II	(5, 0.25, 2)	0.76	3.393	0.484	1.936	34.31483

Table 6.4: Optimal BRASPs for different values of α_0

α_0	Type	$\boldsymbol{\zeta}^* = (n^*, h^*, m^*)$	π_c^*	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	R_B^*
2.2	I	(5, 0.25, 2)	-	2.951	0.490	1.961	28.28359
	II	(5, 0.40, 1)	0.670	2.615	0.400	1.000	28.28878
2.5	I	(5, 0.27, 3)	-	3.866	0.727	2.691	31.31123
	II	(5, 0.43, 1)	0.690	2.955	0.430	1.000	32.25544
3	I	(4, 0.31, 3)	-	3.444	0.741	2.390	35.38329
	II	(5, 0.30, 2)	0.770	3.779	0.566	1.887	35.45693
3.3	I	(4, 0.40, 3)	-	3.703	0.836	2.090	37.22465
	II	(5, 0.32, 2)	0.78	4.023	0.590	1.844	37.36195

Table 6.5: Optimal BRASPs for different values of β_0

β_0	Type	$\boldsymbol{\zeta}^* = (n^*, h^*, m^*)$	π_c^*	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	R_B^*
0.5	I	(0, 0, 0)	-	0	0	0	40
	II	(0, 0, 0)	1	0	0	0	40
0.75	II	(3, 0.37, 3)	-	2.764	0.716	1.934	37.74172
	II	(3, 0.37, 3)	0.790	2.764	0.716	1.934	37.74172
1.25	I	(5, 0.38, 1)	-	2.622	0.380	1.000	29.46731
	II	(5, 0.38, 1)	0.660	2.622	0.380	1.000	29.46731
1.5	I	(0, 0, 0)	-	1	0	0	24.78307
	II	(0, 0, 0)	0	1	0	0	24.78307

In Table 6.3, it is observed that when c_t increases the expected duration of the test decreases as expected. In Table 6.4, we find that for type-II BRASP the sample size n remains unchanged for the different values of α_0. However, the time interval h and number of inspections change with α_0. In Table 6.5, the similar fact is seen as in Table 6.2. Tables 6.1-6.5 show that in most of the cases optimum RASPs do not change with the types of BRASP.

6.4.2 Illustrative example using approximate Bayes risk

Example 6.2 *We consider the hyperparameter values of prior distribution as* $\alpha_0 = 2.8$*,* $\beta_0 = 1$*,* $\alpha_1 = 1.5$ *and* $\alpha_2 = 1.8$*. The acceptance cost coefficient are taken as* $C_0 = 2$*,* $C_1 = C_2 = 4$ *and* $C_{11} = C_{12} = C_{22} = 4$*. The other cost coefficients are taken as* $C_r = 40$*,* $C_s = 0.15$*,* $v_s = 0.1$*,* $C_t = 0.3$*, and* $C_I = 0$ *and* $t_0 = 0.1$*.*

The optimal BRASP is provided in Table 6.6. For different values of C_I and C_s, the optimal BRASPs using approximate Bayes Risk function are tabulated in Table 6.7.

Table 6.6: Optimal BRASP using approximate Bayes Risk function

$\zeta^* = (n^*, h^*, k^*)$	π_c^*	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	R_B^*
(13, 0.102, 5)	0.761	3.337	0.740	2.467	31.4662

Table 6.7: Optimal BRASP for different value of C_s and C_I using approximate Bayes Risk function

C_s	C_I	$\zeta^* = (n^*, h^*, k^*)$	π_c^*	$E[D^*]$	$E[\tau^*]$	$E[I^*]$	R_B^*
0.12	0.01	(17, 0.100, 4)	0.759	10.373	0.399	3.990	31.06911
0.12	0.05	(19, 0.159, 2)	0.759	10.230	0.318	2.000	31.16593
0.12	0.1	(19, 0.159, 2)	0.759	10.230	0.318	2.000	31.26591
0.15	0.01	(13, 0.103, 5)	0.761	8.938	0.507	4.924	31.51546
0.15	0.05	(14, 0.149, 3)	0.761	9.025	0.445	2.986	31.65157
0.15	0.1	(15, 0.194, 2)	0.761	9.010	0.388	1.999	31.75719
0.18	0.01	(11, 0.117, 5)	0.763	7.971	0.568	4.850	31.87559
0.18	0.05	(12, 0.166, 3)	0.763	8.130	0.493	2.969	32.02930
0.18	0.1	(12, 0.216, 2)	0.763	7.609	0.431	1.995	32.15844

It is seen that when C_I is fixed, the sample size decreases with C_s and when C_s is fixed the sample size increases with C_I. Also, the acceptance limit π_c^* increases with C_s.

6.4.3 Illustration with data

In all cases, once the value of ζ is determined, the manufacturer conducts life test to make a decision regarding the lot. In Example 6.1, the optimal plan is $(4, 0.30, 3)$ for equal length inspection intervals. Thus, the manufacturer conducts a life test with a sample size of 4, and after a time length of 0.30, the number of failures and their causes are observed. The decision to accept or reject the lot is then based on the observed number of failures. In this case, the acceptance probability and the optimal sampling parameter are identical. Therefore, the acceptance sets for both types of RASPs coincide.

To illustrate, we generated six observations using the optimal design $(4, 0.30, 3)$, which are presented in Table 5.10. For type-I BRASP, the decision for the observed data $\boldsymbol{x}$ is based on the function $\varphi(\boldsymbol{x})$, whereas for type-II BRASP, the decision is based on the estimated reliability. Table 6.8 reports the values of both functions for the observed data $\boldsymbol{x}$, along with the corresponding decisions. Specifically, under type-I BRASP, the lot is accepted if $\varphi(\boldsymbol{x}) < C_r$, while under type-II BRASP, the lot is accepted if the estimated reliability exceeds 0.76.

Table 6.8: The data sets, estimated reliability and corresponding decision about the lot

i	Data			Estimated Reliability	$\varphi(\boldsymbol{x})$	Decision
	(d_{11}, d_{12})	(d_{21}, d_{22})	(d_{31}, d_{32})			
1	(1, 1)	(1, 0)	(1, 0)	0.753	42.531	Rejects the lot
2	(2, 0)	(2, 0)	-	0.693	55.898	Rejects the lot
3	(0, 0)	(0, 0)	(0, 1)	0.971	8.439	Accepts the lot
4	(2, 1)	(0, 0)	(0, 1)	0.693	53.155	Rejects the lot
5	(0, 0)	(0, 0)	(0, 0)	0.973	6.063	Accepts the lot
6	(0, 0)	(1, 0)	(2, 1)	0.860	22.219	Accepts the lot

In Table 6.8, it is seen that for 2^{nd} observed data, all 4 items failed after the 2^{nd} inspection. Therefore, no samples are surviving for 3^{rd} inspection. Hence $E[I]$ is less than k and $E[\tau]$ is less than τ_k.

6.5 Discussions

In this work, we have addressed the determination of the optimum BRASP under ICS for competing risk data. An advantage of this censoring scheme is that continuous monitoring is not required. The methodology can be extended to the PICS-I setting. For illustration, we have considered the exponential distribution. Our findings indicate that, under ICS, the optimal BRASP based on the reliability acceptance criterion is similar to the optimal BRASP based on the Bayes decision function. Therefore, for large samples, the asymptotic properties of the MLEs can be used to approximate the Bayes risk and to obtain an optimal BRASP. The proposed methodology for approximating the Bayes risk can also be extended to other lifetime distributions with an appropriate choice of prior distributions.

Chapter 7

RASP through adaptive SSSPALT under type-II censoring for competing risk data by Bayesian approach

This chapter focuses on the determination of BRASP under type-II censoring for competing risk data, with the objective of minimizing the total expected cost with respect to a common prior distribution for the lifetime parameters and a shared utility function between the manufacturer and the consumer.

Chapter 6 addresses the determination of BRASP for competing risk data based on life testing conducted under non-accelerated conditions. However, in practical applications, many modern products are highly reliable and complex. Consequently, failures often occur due to multiple causes, and the mean time to failure under normal operating conditions tends to be very large. In such cases, conventional life testing frequently fails to provide sufficient lifetime information to support decision-making, primarily due to time and resource constraints (see Klein and Basu [52]).

To overcome this limitation, Chapter 4 examines BRASP under ASSSPALT for items subject to a single cause of failure. In the present chapter, we extend the analysis to BRASP under ASSSPALT for items subject to multiple causes of failure under type-II censoring. The adaptive test is defined in Chapter 4 as follows: a sample of size n is placed on a life test at the normal stress level s_0 at time $\tau_0 = 0$. If the number of failures up to time τ_1 is less than a pre-specified threshold m, the stress level s_0 is increased to s_1; otherwise, the stress remains unchanged after τ_1. Under type-II censoring, the life test continues until r failures are observed at the current stress level. Similar to Chapter 4, two important special cases are noted:

(i) If $m = 0$, the life test reduces to a conventional non-accelerated life test under type-II censoring.

(ii) If $m = r$, the life test reduces to a conventional SSSALT under type-II censoring.

Adaptive designs are not new in reliability testing (see, for example, Xiang et al. [124], Wang and Wu [117], You and Pham [130]); however, their integration into BRASP under step-stress testing with competing risks and censoring remains unexplored. From a production planning and quality assurance perspective, such adaptive strategies can yield more economical and informative testing procedures, facilitating better lot disposition decisions under real-world constraints.

Although classical and Bayesian inference methods under competing risks and step-stress testing have been developed by many authors (see Han and Balakrishnan [40], Balakrishnan and Han [9], Han and Kundu [41], Koley and Kundu [53]), the design of BRASPs under SSSPALT with censoring and competing risks has received little attention. This chapter studies the determination of BRASP under ASSSPALT for exponential lifetimes with type-II censoring and competing risks, referred to here as BSPAA. The proposed methodology provides a general framework for designing BRASP under type-II censoring for competing risk data. As special cases, it also accommodates BRASP under conventional SSSPALT with type-II censoring for competing risk data, referred to as CBSPA in this chapter, and BRASP under conventional non-accelerated life testing with type-II censoring for competing risk data, referred to as CBSP, which was studied by Prajapati et al. [99]. We derive the Bayes decision rule under both general and specific loss functions, evaluate its performance through numerical studies, and demonstrate their practical relevance using a real-world example.

The remainder of this chapter is organized as follows. Section 7.1 introduces the framework of the proposed BSPAA for the exponential distribution in a competing risks setting. Section 7.2 investigates the monotonicity property of the Bayes decision function under a general loss function and exploits this property to derive explicit expressions for the Bayes decision rule and the corresponding Bayes risk. Section 7.3 considers a quadratic loss function to obtain closed-form expressions for the Bayes decision function and the Bayes risk. Section 7.4 presents an algorithm for determining the optimal BSPAA. Section 7.5 evaluates the optimal BSPAA under various scenarios and analyzes the effects of different decision variables on the plan. Section 7.6 uses real data to assess the performance of the proposed BSPAA when the time point τ_1 is fixed. In both Sections 7.5 and 7.6, comparative studies are conducted between BSPAA and CBSP, as well as between BSPAA and CBSPA, under type-II censoring for competing risks data. Furthermore, Section 7.6 illustrates the conceptual foundation and practical relevance of the proposed BSPAA. Finally, Section 7.7 concludes the chapter.

7.1 Model

In this section, we develop the model for the BRASP under an ASSSPALT for type-II censored competing risk data using by Bayesian decision-theoretic approach.

7.1.1 Lifetime Distribution

Suppose n identical items are selected from the lot and put on life test at time $\tau_0 = 0$ under an initial stress level s_0, representing normal usage conditions. Let D_1 be the number of failures observed at a pre-specified inspection time τ_1. If $D_1 < m$, where m is a pre-decided threshold, the stress level is increased from s_0 to a higher level s_1. Otherwise, the stress level remains unchanged at s_0. The life test continues until r $(\leq n)$ failures are observed. Obviously, we have $m \leq r \leq n$. The time-varying stress level s experienced by a test unit at ASSSPALT is defined in 4.1.1.

Suppose that each test unit may fail due to one of J mutually exclusive and independent causes, often referred to in the literature as competing risks. Let X_{ijk} denote the potential failure time of k^{th} item due to j^{th} cause under stress level s_i, where $i = 0, 1$, $j = 1, \ldots, J$ and $k = 1, \ldots, n$. Let T_{ik} denote the lifetime of the unit at stress level s_i of k^{th} item for $i = 0, 1$ and $k = 1, \ldots, n$. Since failure occurs as soon as any one of the J competing risks triggers a failure, we have $T_{ik} = \min\{X_{i1k}, \ldots, X_{iJk}\}$. We further assume that X_{ijk} follows $\text{Exp}(\lambda_{ij})$, where $i = 0, 1$, $j = 1, \ldots, J$ and $k = 1, \ldots, n$. It is easy to see that T_{ik} follows $\text{Exp}\left(\sum_{j=1}^{J} \lambda_{ij}\right)$ Furthermore, let $C_{ik} \in \{1, \ldots, J\}$ denote the cause of failure at stress level s_i of k^{th} item. In a competing risks setup, the observed data typically consist of realizations of the random pair $(T_{i1}, C_{i1}), \ldots, (T_{in}, C_{in})$. The pairs $(T_{i1}, C_{i1}), \ldots, (T_{in}, C_{in})$ are assumed to be IID with joint PDF at stress level s_i given by

$$g_i(x, j \mid \boldsymbol{\lambda}_i) = \lambda_{ij} \exp\left[-\left(\sum_{j=1}^{J} \lambda_{ij}\right) x\right], \qquad x > 0,$$

where $\boldsymbol{\lambda}_i = (\lambda_{i1}, \ldots, \lambda_{iJ})$, for $i = 0, 1$. Suppose $X_1, \ldots, X_n$ denotes the lifetime of the items with CDF $F(\cdot \mid \boldsymbol{\theta})$ and $C_1, \ldots, C_n$ be the cause of the failure of i^{th} item under ASSSPALT. Then,

$$F(x \mid \boldsymbol{\theta}) = \begin{cases} 1 - \exp\left[-\left(\sum_{j=1}^{J} \lambda_{0j}\right) x\right], & \text{if } x \le \tau_1, \\ \left[1 - \exp\left\{-\left(\sum_{j=1}^{J} \lambda_{0j}\right) x\right\}\right]^{1-\delta} \left[1 - \exp\left\{-\left(\sum_{j=1}^{J} \lambda_{0j}\right) \tau_1 + \left(\sum_{j=1}^{J} \lambda_{1j}\right) (x - \tau_1)\right\}\right]^{\delta}, & \text{if } x > \tau_1, \end{cases} \tag{7.1.1}$$

where δ is defined in (4.1.2). The joint PDF of (X_i, C_i) is given by

$$g(x, j \mid \boldsymbol{\theta}) = \begin{cases} \lambda_{0j} \exp\left[-\left(\sum_{j=1}^{J} \lambda_{0j}\right) x\right], & \text{if } x \le \tau_1, \\ \left[\lambda_{0j} \exp\left\{-\left(\sum_{j=1}^{J} \lambda_{0j}\right) x\right\}\right]^{1-\delta} \left[\lambda_{1j} \exp\left\{-\left(\sum_{j=1}^{J} \lambda_{0j}\right) \tau_1 + \left(\sum_{j=1}^{J} \lambda_{1j}\right) (x - \tau_1)\right\}\right]^{\delta}, & \text{if } x > \tau_1. \end{cases} \tag{7.1.2}$$

Let p_{1j} and p_{2j} denote the relative risks of failure on a test unit due to cause j before and after τ_1, respectively. It is easy to see that

$$p_{1j} = \frac{\lambda_{0j}}{\left(\sum_{j=1}^{J} \lambda_{0j}\right)} \quad \text{and} \quad p_{2j} = \left[\frac{\lambda_{0j}}{\left(\sum_{j=1}^{J} \lambda_{0j}\right)}\right]^{1-\delta} \left[\frac{\lambda_{1j}}{\left(\sum_{j=1}^{J} \lambda_{1j}\right)}\right]^{\delta}.$$

7.1.2 Likelihood function

Let D_{1j} and D_{2j}, for $j = 1, \ldots, J$, be the number of failures before and after time τ_1 due to risk j, respectively. Let the random vector $\boldsymbol{D} = (D_{11}, \ldots, D_{1J}, D_{21}, \ldots, D_{2J})$ denote the number of failures. Clearly $D_1 = \sum_{j=1}^{J} D_{ij}$. Note that the lifetime of the remaining $(n - r)$ surviving units is censored at $X_{(r)}$. Then, the observed data in an ASSSPALT scheme under the type-II censoring scheme can be represented as $\boldsymbol{x} = (\boldsymbol{y}, \boldsymbol{d})$, where $\boldsymbol{y} = (x_{(1)}, \ldots, x_{(r)})$ and $\boldsymbol{d} = (d_{11}, \ldots, d_{1J}, d_{21}, \ldots, d_{2J})$ and d_{kj} is the observed value of D_{kj}, for $j = 1, \ldots, J$ and $k = 1, 2$. Let $\lambda_{1j} = \lambda_{0j} \times \phi_j$, for $j = 1, \ldots, J$. Since $\lambda_{1j} > \lambda_{0j}$, we have $\phi_j > 1$ for all j. Thus, the parameter vector of interest is given by $\boldsymbol{\theta} = (\boldsymbol{\lambda}, \boldsymbol{\phi})$, where $\boldsymbol{\lambda} = (\lambda_{01}, \ldots, \lambda_{0J})$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_J)$. Given the observed data $\boldsymbol{x}$, the likelihood function can be written as using (7.1.1) and (7.1.2)

$$L(\boldsymbol{\theta} \mid \boldsymbol{x}) \propto \begin{cases} \prod_{j=1}^{J} \lambda_{0j}^{d_{1j}} \exp\left[-\lambda_{0j}\left(\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}\right)\right], & X_{(r)} \leq \tau_1, \\ \prod_{j=1}^{J} \lambda_{0j}^{d_{+j}} \phi_j^{\delta d_{2j}} \exp\left[-\lambda_{0j}\left(\sum_{i=0}^{d_1} x_{(i)} + (n-d_{1j})\tau_1\right.\right. & \\ \left.\left. +\phi_j^{\delta}\left(\sum_{i=d_1+1}^{r} \left(x_{(i)}-\tau_1\right) + (n-r)\left(x_{(r)}-\tau_1\right)\right)\right)\right], & X_{(r)} > \tau_1, \end{cases} \tag{7.1.3}$$

where $d_{+j} = \sum_i d_{ij}$. We now consider the following sufficient statistics:

$$w_1(\boldsymbol{x}) \equiv w_1 = \begin{cases} \sum_{j=1}^{r} x_{(j)} + (n-r)x_{(r)}, & \text{if } x_{(r)} \leq \tau_1, \\ \sum_{j=0}^{d_1} x_{(j)} + (n-d_1)\tau_1, & \text{if } x_{(r)} > \tau_1, \end{cases}$$

and

$$w_2(\boldsymbol{x}) \equiv w_2 = \sum_{j=d_1+1}^{r} (x_{(j)} - \tau_1) + (n-r)(x_{(r)} - \tau_1),$$

where $d_1 = \sum_{j=1}^{J} d_{ij}$ is the observed value of D_1. Then, the likelihood function (7.1.3) reduces to

$$L(\boldsymbol{\theta} \mid \boldsymbol{x}) \propto \prod_{j=1}^{J} \lambda_{0j}^{d_{+j}} \phi_j^{\delta d_{2j}} \exp\left[-\lambda_{0j}\left(w_1 + \phi_j^{\delta} w_2\right)\right].$$

7.1.3 Bayesian decision-theoretic framework for ASSSPALT

To implement an ASSSPALT in practice, it is essential to carefully determine the key design variables. The vector of decision variables of the life testing plan under this setup is denoted by $\boldsymbol{\zeta} = (n, \tau_1, r, m)$. Based on $\boldsymbol{x}$, the RASP leads to either acceptance or rejection of the production lot. The total cost associated with the life test depends on the decision outcome. The loss functions for acceptance and rejection after the life testing are defined as

$$\mathcal{L}\left(\mathcal{A} \mid \boldsymbol{x}, \boldsymbol{\theta}\right) = h(\boldsymbol{\lambda}) + nC_s - (n-r)v_s + x_{(r)}C_t + \delta(n-d_1)C_a$$

and

$$\mathcal{L}\left(\mathcal{R} \mid \boldsymbol{x}, \boldsymbol{\theta}\right) = C_r + nC_s - (n-r)v_s + x_{(r)}C_t + \delta(n-d_1)C_a,$$

respectively. The cost components $h(\boldsymbol{\lambda})$, v_s, C_r C_s, C_a and C_t are defined in Chapter 4. The expected loss for a given RASP $(\boldsymbol{\zeta}, a)$ and observed data $\boldsymbol{x}$ is:

$$\begin{aligned} \psi(\boldsymbol{\zeta}, a \mid \boldsymbol{x}, \boldsymbol{\theta}) &= a(\boldsymbol{x} \mid \boldsymbol{\zeta})\mathcal{L}(\mathcal{A} \mid \boldsymbol{x}, \boldsymbol{\theta}) + (1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta}))\mathcal{L}(\mathcal{R} \mid \boldsymbol{x}, \boldsymbol{\theta}) \\ &= a(\boldsymbol{x} \mid \boldsymbol{\zeta})h(\boldsymbol{\lambda}) + (1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta}))C_r + nC_s + \delta(n-d_1)C_a - (n-r)v_s + x_{(r)}C_t, \end{aligned}$$

where $a(\boldsymbol{x} \mid \boldsymbol{\zeta})$ is the decision function based on the observed data $\boldsymbol{x}$ and it is defined in 4.1.5. Let the prior distribution of $\boldsymbol{\theta}$ be given by $p(\boldsymbol{\theta})$. The Bayes risk for the BRASP $(\boldsymbol{\zeta}, a)$ is defined as the expected loss over the joint distribution of the parameters and data:

$$\begin{aligned} R_B(\boldsymbol{\zeta}, a) &= E_{\boldsymbol{\theta}} E_{\boldsymbol{x}|\boldsymbol{\theta}}\left[\psi(\boldsymbol{\zeta} \mid \boldsymbol{x}, \boldsymbol{\theta})\right] \\ &= n(C_s - v_s) + rv_s + C_a E_{\boldsymbol{\theta}}[E[\delta(n-D_1) \mid \boldsymbol{\theta}]] + C_t E_{\boldsymbol{\theta}}[E[X_{(r)} \mid \boldsymbol{\theta}] + R_A(\boldsymbol{\zeta}, a), \end{aligned} \tag{7.1.4}$$

where the term $R_A(\boldsymbol{\zeta}, a)$ accounts for the decision uncertainty and is given by:

$$\begin{aligned} R_A(\boldsymbol{\zeta}, a) &= E_{\boldsymbol{\theta}} E_{\boldsymbol{x}|\boldsymbol{\theta}} \left[a(\boldsymbol{x} \mid \boldsymbol{\zeta}) h(\boldsymbol{\lambda}) + (1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta})) C_r \right] \\ &= E_{\boldsymbol{\lambda}}[h(\boldsymbol{\lambda})] + E_{\boldsymbol{\theta}} E_{\boldsymbol{x}|\boldsymbol{\theta}} \left[(1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta}))(C_r - h(\boldsymbol{\lambda})) \right]. \end{aligned} \tag{7.1.5}$$

The optimal BRASP is then defined as the BRASP that minimizes the Bayes risk:

$$(\boldsymbol{\zeta}^*, a^*) = \arg \min_{(\boldsymbol{\zeta}, a) \in \mathcal{C}} R(\boldsymbol{\zeta}, a),$$

where $\mathcal{C}$ denotes the class of all admissible BRASPs under the adaptive SSSPALT framework.

7.2 Derivation of Bayes decision function and Bayes risk

In this section, we derive the Bayes decision function that minimizes the Bayes risk $R_B(\boldsymbol{\zeta}, a)$ over all admissible decision functions. We also provide an explicit form of the Bayes risk corresponding to the BRASP defined in (7.1.4).

7.2.1 Bayes decision function

Recall that the Bayes risk $R_B(\boldsymbol{\zeta}, a)$, as defined in (7.1.4), depends on the decision function $a(\cdot \mid \boldsymbol{\zeta})$ only through the term $R_A(\boldsymbol{\zeta}, a)$. (7.1.5) can be written as:

$$R_A(\boldsymbol{\zeta}, a) = E_{\boldsymbol{\lambda}}[h(\boldsymbol{\lambda})] + E_{\boldsymbol{x}} E_{\boldsymbol{\theta} \mid \boldsymbol{x}}[(1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta}))(C_r - h(\boldsymbol{\lambda}))] \tag{7.2.1}$$

where

$$\begin{aligned} & E_{\boldsymbol{x}} E_{\boldsymbol{\theta} \mid \boldsymbol{x}}[(1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta}))(C_r - h(\boldsymbol{\lambda}))] \\ = & \sum_{\substack{\boldsymbol{d} \\ \boldsymbol{d}:\ \sum_i \sum_j d_{ij} = r}} \int_{w_1} \int_{w_2} [1 - a(w_1, w_2, \boldsymbol{d} \mid \boldsymbol{\zeta})][C_r - \varphi(w_1, w_2, \boldsymbol{d})] \ f_{(W_1, W_2, \boldsymbol{D})}(w_1, w_2, \boldsymbol{d}) \ dw_2 \ dw_1, \end{aligned}$$

with

$$f_{(W_1, W_2, \boldsymbol{D})}(w_1, w_2, \boldsymbol{d}) = \int_{\boldsymbol{\theta}} f_{(W_1, W_2, \boldsymbol{D})}(w_1, w_2, \boldsymbol{d} \mid \boldsymbol{\theta}) \ p(\boldsymbol{\theta}) d\boldsymbol{\theta} \text{ and } \varphi(w_1, w_2, \boldsymbol{d}) = \int_{\boldsymbol{\theta}} h(\boldsymbol{\lambda}) \ p(\boldsymbol{\theta} \mid w_1, w_2, \boldsymbol{d}) d\boldsymbol{\theta}.$$

To obtain Bayes decision, we need to minimize $R_A(\boldsymbol{\zeta}, a)$ with respect to $a(\cdot \mid \boldsymbol{\zeta})$, which is equivalent to minimization of $E_{\boldsymbol{\theta}} E_{\boldsymbol{x} \mid \boldsymbol{\theta}}[(1 - a(\boldsymbol{x} \mid \boldsymbol{\zeta}))(C_r - h(\boldsymbol{\lambda}))]$ with respect to $a(\cdot \mid \boldsymbol{\zeta})$. Thus, for fixed $\boldsymbol{\zeta}$, the Bayes decision function $a^*(\boldsymbol{x} \mid \boldsymbol{\zeta})$ is given by

$$a^*(w_1, w_2, \boldsymbol{d} \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{if } C_r - \varphi(w_1, w_2, \boldsymbol{d}) \geq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{7.2.2}$$

Now, we provide an alternative form for the Bayes decision function, which is useful to calculate the Bayes risk. To develop an alternative form of the Bayes decision function, we consider the following theorem.

Theorem 7.1 *If function $h(\boldsymbol{\lambda})$ is increasing (decreasing) in each component λ_i, for $i = 1, \ldots, J$ and the components of $\boldsymbol{\lambda}$ follow gamma distributions, then the posterior expectation of $h(\boldsymbol{\lambda})$, given by $\phi(w_1, w_2, \boldsymbol{d})$, satisfies the following monotonicity property:*

1. *For fixed $(w_2, \boldsymbol{d})$, $\phi(w_1, w_2, \boldsymbol{d})$ is decreasing (increasing) in w_1.*
2. *For fixed $(w_1, \boldsymbol{d})$, $\phi(w_1, w_2, \boldsymbol{d})$ is decreasing (increasing) in w_2.*

Proof: The above theorem can be proved following the steps as in Theorem 4.1. (Proved)

For fixed $(w_2 = 0, \boldsymbol{d})$, $\phi(w_1, 0, \boldsymbol{d})$ is a decreasing function in w_1. If $\varphi(0, 0, \boldsymbol{d}) > C_r$ and $\lim\limits_{w_2\to\infty} \varphi(w_1, 0, \boldsymbol{d})$ $< C_r$ there exist a unique point $c(\boldsymbol{d})$ such that

$$\begin{aligned}
\phi(w_1, 0, \boldsymbol{d}) &> C_r, &&\text{for } w_1 < c(\boldsymbol{d})\\
\phi(w_1, 0, \boldsymbol{d}) &< C_r, &&\text{for } w_1 > c(\boldsymbol{d}).
\end{aligned}$$

If $\varphi(0, 0, \boldsymbol{d}) < C_r$, then $\varphi(w_1, 0, \boldsymbol{d}) < C_r$ for all $w_2 > 0$. In that scenario, $c(\boldsymbol{d})$ can be taken as 0. Since, for fixed $(w_1 = c(\boldsymbol{d}), \boldsymbol{d})$, $\phi(c(\boldsymbol{d}), w_2, \boldsymbol{d})$ is decreasing function in w_2. Therefore

$$\phi(w_1, w_2, \boldsymbol{d}) < C_r \qquad \text{for } w_2 > 0 \text{ and } w_1 > c(\boldsymbol{d}).$$

If $\lim\limits_{w_2\to\infty} \varphi(0, w_2, d_1, d_2) > C_r$, then $c(\boldsymbol{d})$ can be taken as ∞. If $0 < w_1 < c(\boldsymbol{d})$, then $\phi(w_1, w_2, \boldsymbol{d})$ is decreasing function in w_2, for fixed $(w_1, \boldsymbol{d})$. Therefore, there exists a unique point $c_1(w_1, \boldsymbol{d})$ such that

$$\begin{aligned}
\phi(w_1, w_2, \boldsymbol{d}) &< C_r, &&\text{for } w_2 > c_1(w_1, \boldsymbol{d}) \text{ and } 0 < w_1 < c(\boldsymbol{d})\\
\phi(w_1, w_2, \boldsymbol{d}) &> C_r, &&\text{for } 0 < w_2 < c_1(w_1, \boldsymbol{d}) \text{ and } 0 < w_1 < c(\boldsymbol{d}).
\end{aligned}$$

Since the upper limit of w_1 is $n\tau_1$, we define $c'(\boldsymbol{d}) = \min\{c(\boldsymbol{d}), n\tau_1\}$. Therefore, the Bayes decision function (7.2.2) can be written as

$$a^*(w_1, w_2, \boldsymbol{d} \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{for } w_1 > c'(\boldsymbol{d}) \text{ or } 0 < w_1 < c'(\boldsymbol{d}) \text{ and } w_2 > c_1(w_1, \boldsymbol{d}),\\ 0, & \text{for } 0 < w_1 < c'(\boldsymbol{d}) \text{ and } 0 < w_2 < c_1(w_1, \boldsymbol{d}). \end{cases}$$

7.2.2 Bayes risk

We now derive the explicit Bayes risk function using the Bayes decision function developed above. Then, (7.2.1) can be re-written as

$$\begin{aligned}
R_A(\boldsymbol{\zeta}, a) =& E_{\boldsymbol{\theta}}[h(\boldsymbol{\lambda})] + \sum_{\substack{\boldsymbol{d}\\ \sum_{ij} d_{ij} = r}} \int_{\boldsymbol{\theta}} [C_r - h(\boldsymbol{\lambda})] P(W_1 < c'(\boldsymbol{d}), W_2 < c_1(W_1, \boldsymbol{d}), \boldsymbol{D} = \boldsymbol{d} \mid \boldsymbol{\theta})\ p(\boldsymbol{\theta})\ d\boldsymbol{\theta}\\
=& E_{\boldsymbol{\theta}}[h(\boldsymbol{\lambda})] + \sum_{\substack{\boldsymbol{d}\\ \boldsymbol{d}:\ \sum\limits_i \sum\limits_j d_{ij} = r}} H(\boldsymbol{d}),
\end{aligned}$$

where $H(\boldsymbol{d}) = \int_{w_2=0}^{c_1(\boldsymbol{d})} \int_{w_1=0}^{c_1(w_1,\boldsymbol{d})} \int_{\boldsymbol{\theta}} [C_r - h(\boldsymbol{\lambda}))] f_{(W_1,W_2,\boldsymbol{d})}(w_1, w_2, \boldsymbol{d} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta})\ dw_1\ dw_2\ d\boldsymbol{\theta}$. Now, we need to find the expressions of the $E_{\boldsymbol{\theta}}[E[\delta(n - D_1) \mid \boldsymbol{\theta}]]$, $E_{\boldsymbol{\theta}}[E[X_{(r)} \mid \boldsymbol{\theta}]]$ and $f_{(W_1,W_2,\boldsymbol{d})}(w_1, w_2, \boldsymbol{d} \mid \boldsymbol{\theta})$ in order to compute $R_B(\boldsymbol{\zeta}, a)$ given by (7.1.4). These expressions are derived in Appendix 7.A.1.

7.3 Bayes decision function and Bayes risk for quadratic loss function

In this section, we consider a quadratic loss function and derive the expressions for the Bayes decision function and the corresponding Bayes risk, primarily for the purpose of illustration. As is standard in the literature, we consider that λ_{0i} follows $G(\alpha_i, \beta_i)$, with PDF $p_{1i}(\cdot)$ for $i = 1 \dots J$. and ϕ_i follows $\mathcal{U}(1, l_i)$ with PDF $p_{2i}(\cdot)$ defined in (1.2.6) for $i = 1 \dots J$. The loss function is taken as a quadratic loss function $h(\boldsymbol{\lambda}) = C_0 + \sum_{j=1}^{J} C_j \lambda_{0j} + \sum_{i=1}^{J} \sum_{j=1 i \leq j}^{J} C_{ij} \lambda_{0i} \lambda_{0j}$ as considered Prajapati et al. [99]. Next, we obtain the Bayes decision function corresponding to the design parameter$\boldsymbol{\zeta} = (n, r, m, \tau_1)$ using the following result.

Result 7.1 *If $p_1, \dots, p_J$ are non-negative real number, then we get*

$$\prod_{j=1}^{J} \left(\int_{\lambda_{0j}=0}^{\infty} \int_{\phi_j=1}^{l_j} \lambda_{0j}^{d_{+j}+\alpha_j+p_j} \phi^{d_{2j}\delta} \exp\left[-\lambda_{0j}\left(w_1 + \phi_j^{\delta} w_2 + \beta_j\right)\right] d\lambda_{0j}\, d\phi_j \right)$$
$$= \begin{cases} \displaystyle\prod_{j=1}^{J} \frac{\Gamma(\alpha_j + d_{1j} + d_{2j} + p_j)(l_i - 1)}{(w_1 + w_2 + \beta_j)^{(\alpha_j + d_{1j} + d_{2j} + p_j)}}, & \text{if } \delta = 0, \\ \displaystyle\prod_{j=1}^{J} \frac{\Gamma(p_j + d_{1j} + d_{2j} + \alpha_j)}{(w_1 + \beta_j)^{p_j + d_{1j} + \alpha_j - 1} w_2^{d_{2j}+1}} & \\ \quad \times \left[I_{\rho_{1j}}(d_{2j} + 1, p_j + d_{1j} + \alpha_j - 1) - I_{\rho_{2j}}(d_{2j} + 1, p_j + d_{1j} + \alpha_j - 1)\right], & \text{if } \delta = 1, \end{cases}$$

it is denoted by $H_1(w_1, w_2, \boldsymbol{d}, \boldsymbol{p})$, where $\boldsymbol{p} = (p_1, \dots, p_J)$,$\rho_{ij} = h_{ij}/(1+h_{ij})$, for $i = 1, 2$ with $h_{1j} = w_2 l_j/(w_1 + \beta_j)$ and $h_{2j} = w_2/(w_1 + \beta_j)$, $j = 1, 2$. $I\rho(a, b)$ is the CDF of beta distribution defined in Section 1.2.5.

Proof: The proof is similar to the Result 4.1. (Proved)

Using the result, the $\varphi(\boldsymbol{x})$ can be written as

$$\varphi(\boldsymbol{x}) = C_0 + \sum_{j=0}^{J} C_j \frac{H_1(w_1, w_2, \boldsymbol{d}, \boldsymbol{p}_j)}{H_1(w_1, w_2, \boldsymbol{d}, \boldsymbol{0})} + \sum_{\substack{i=1 \\ i \leq j}}^{J} \sum_{i=1}^{J} C_{ij} \frac{H_1(w_1, w_2, \boldsymbol{d}, \boldsymbol{p}_{ij})}{H_1(w_1, w_2, \boldsymbol{d}, \boldsymbol{0})}, \tag{7.3.1}$$

where $\boldsymbol{p}_j$ is the J component vector whose j-th component is 1 and other components are 0, $\boldsymbol{p}_{ij}$ is the J component vector whose i-th and j-th components are 1 when $i \neq j$, 2 when $i = j$ and other components are 0. For example, $\boldsymbol{p}_1 = (1, 0, \dots, 0)$, $\boldsymbol{p}_{12} = (1, 1, 0, 0, \dots, 0)$ and $p_{11} = (2, 0, \dots, 0)$. Therefore, the Bayes decision is

$$a^*(w_1, w_2, \boldsymbol{d} \mid \boldsymbol{\zeta}) = \begin{cases} 1, & \text{if } \varphi(\boldsymbol{x}) \leq C_r, \\ 0, & \text{otherwise.} \end{cases} \tag{7.3.2}$$

The explicit form of R_A is given in the Appendix 7.A.2.

7.4 Finding the optimal BSPAA

We consider the following algorithm to find the optimal BSPAA. The Bayes risk is a function of three discrete variables n, r, m and one continuous variable τ_1. The solution cannot be obtained analytically. We provide graph of Bayes risk with respect to τ_1 for $n = 4, r = 3$ and $m = 0, 1, 2, 3$ in Figure 7.1 for the hyperparameter $\alpha_1 = 2.21$, $\alpha_2 = 9.59$, $\beta_1 = 100$, $\beta_2 = 100$, $l_1 = 109.202$ and $l_2 = 11.716$ and the cost components $C_t = 0.3$, $c_s = 0.6$, $v_s = 0.3$, $C_a = 0.1$, $C_r = 80$.

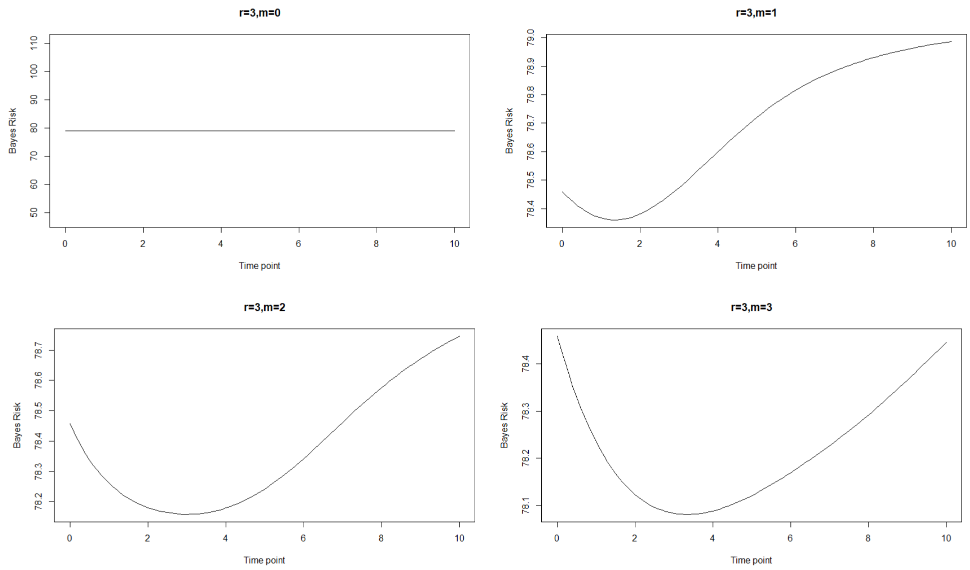


Figure 7.1: Graphical plot Bayes risk vs time point for different value of m

In Figure 7.1, it is noted that for $m = 0$, the line is parallel to the time point τ_1. This is due to the fact that for $m = 0$, BSPAA is converted to CBSP. In CBSP, the optimal design does not depend on the stress-changing time point τ_1. So when $m = 0$, we can take $\tau_1 = 0$. For $m > 0$, in Figure 7.1, it is seen that for fixed n, r, m, the Bayes risk has a unique minimum. Note that the optimal value r^* is less than or equal to n^* and the optimal value m^* is less than or equal to r^*. Therefore, we can say that the upper bound of r^* and m^* depends on n^*. From (4.4.3), the upper bound for n^* can be written as

$$n_0 = \left\lfloor \frac{\min\{E_{\boldsymbol{\lambda}}[h(\boldsymbol{\lambda})], C_r\}}{C_s - v_s} \right\rfloor . \tag{7.4.1}$$

We now provide Algorithm 7.1 to obtain optimal BSPAA.

Theorem 7.2 *The BRASPs $\boldsymbol{\zeta}^* = (n^*, r^*, m^*, \tau_1^*, a^*)$ is the optimal BSPAA among all admissible BRASPs under the ASSSPALT framework.*

Proof: The proof is similar to the proof of Theorem 4.2. (Proved)

7.5 Numerical study using real-life data and comparisons with other BRASPs

Here we illustrate the proposed method using an example and compute the optimum BSPAA. Also, we study the effect of time cost components and hyperparameters on the optimum solution of BSPAA. The optimum solution is compared with a CBSP and CBSPA. Here, we consider only two competing risks, i.e, $J = 2$.

7.5.1 An illustrative example with comparisons

Example 7.1 *The hyperparameters of the prior are taken as $\alpha_1 = 2.5$, $\beta_1 = 1.5$, $\alpha_2 = 2.2$, $\beta_2 = 2.0$ and $l_1 = l_2 = 10$. The cost coefficients of the loss function are taken as $C_0 = 2$, $C_1 = C_2 = 3$ and*

Algorithm 7.1: Procedure for finding optimal BSPAA

Input: Possible design parameters $\boldsymbol{\zeta} = (n, \tau_1, \tau_2, m)$ with $n, r, m \in \mathbb{Z}_{\geq 0}$ and $\tau_1 \geq 0$, maximum n_0

Output: Optimal design parameters $\boldsymbol{\zeta}^*$ and Bayes decision function $a^*(\cdot \mid \boldsymbol{\zeta})$ minimizing R_B

for $n = 0, 1, \ldots, n_0$, *given in (7.4.1)* **do**

  for $r = 1, \ldots, n$ **do**

    for $m = 0, 1, \ldots, r$ **do**

      for τ_1 *in domain* **do**

        Derive $a^*(\cdot \mid \boldsymbol{\zeta})$ by minimizing $R_B(\boldsymbol{\zeta}, a)$ (See Section 7.2.1);

      if $m > 0$ **then**

        Find $\tau_1^*(n, r, m)$ by minimizing (6.1.4) over $\tau > 0$:

$$\tau_1^*(n, r, m) = \arg\min_{\tau_1 \geq 0} R_B(n, r, m, \tau_1, a^*(\cdot \mid (n, r, m, \tau_1)))$$

      else

        Set $\tau_1^*(n, m) = 0$.

      Find $m^*(n, r)$ by minimizing (6.1.4) over $0 \leq m \leq r$:

$$m^*(n, r) = \arg\min_{0 \leq m \leq r} R_B(n, r, m(n, r), \tau_1^*(n, r, m(n, r)), a^*(\cdot \mid (n, r, m(n, r), \tau_1^*(n, r, m(n, r)))))$$

    Find $r^*(n)$ by minimizing (6.1.4) over $1 \leq r \leq n$:

$$r^*(n) = \arg\min_{1 \leq r \leq n} R_B(n, r(n), m^*(n, r(n)), \tau_1^*(n, r(n), m^*(n, r(n))), a^*(\cdot \mid (n, r(n), m^*(n, r(n)), \tau_1^*(n, r(n), m^*(n, r(n))))))$$

  Find n^* by minimizing (6.1.4) over $1 \leq n \leq n_0$:

$$n^* = \arg\min_{1 \leq r \leq n} R_B(n, r^*(n), m^*(n, r^*(n)), \tau_1^*(n, r^*(n), m^*(n, r^*(n))), a^*(\cdot \mid (n, r^*(n), m^*(n, r^*(n)), \tau_1^*(n, r^*(n), m^*(n, r^*(n))))))$$

Take $a^*(. \mid (n^*, r^*(n^*), m^*(r^*(n^*)), \tau_1^*(n^*, r^*(n^*), m(r^*(n^*))))) = a^*$, $m^*(r^*(n^*)) = m^*$, $r^*(n^*) = r^*$ and $\tau_1^*(n^*, r^*(n^*), m^*(r^*(n^*))) = \tau_1^*$.

$C_{11} = C_{12} = C_{22} = 4$. *The values of other cost components are* $C_a = 0.2$, $v_s = 0.25$, $C_s = 0.5$, $C_t = 5$ *and* $C_r = 40$.

The optimal design $\boldsymbol{\zeta}^* = (n^*, r^*, m^*, \tau_1^*)$ of the proposed plan and the corresponding optimal Bayes risk R_B^* are reported in Table 7.1. For comparison, Table 7.1 also presents the optimal designing parameters (n_C^*, r_C^*) and the associated optimal Bayes risk R_1^* for the CBSP, as well as the optimal designing parameters $(n_A^*, r_A^*, \tau_{1A}^*)$ and the corresponding optimal Bayes risk R_2^* for the CBSPA. In addition, the relative risk savings RRS_1 and RRS_2, defined in (4.5.1), are included in Table 7.1.

Table 7.1: Optimal design

	BSPAA		CBSPA		CBSP		$RRS(\%)$	
C_a	$(n^*, r^*, m^*, \tau_1^*)$	R_B^*	$(n_A^*, r_A^*, \tau_{1A}^*)$	R_1^*	(n_C^*, r_C^*)	R_2^*	RRS_1	RRS_2
0.2	(5, 3, 2, 0.212)	36.477	(5, 3, 0.301)	36.520	(6, 3)	36.582	0.170	0.289

7.5.2 Effect of the parameters

In this section, we investigate the impact of cost components and hyperparameters on the optimal solution of BSPAA, in comparison with the CBSP and the CBSPA. The optimal designs corresponding to various values of the test cost C_t and the hyperparameters α_1, under three different values of the acceleration cost coefficient $C_a = 0, 0.1, 0.2$, are summarized in Table 7.2 and Table 7.3, respectively. Also, for the comparison of BSPAA, CBSP and CBSPA, the RRS_1 and RRS_2 are tabulated. The other values of the cost components and hyperparameters, which are not mentioned in the tables, are kept fixed.

Table 7.2: Optimal design for different C_t

		BSPAA		CBSPA		CBSP		$RRS(\%)$	
C_t	C_a	$(n^*, r^*, m^*, \tau_1^*)$	R_B^*	(n_A^*, r_A^*, t_{1A}^*)	R_1^*	(n_C^*, r_C^*)	R_2^*	RRS_1	RRS_2
1	0	(4, 4, 4, 0.390)	34.855	(4, 4, 0.390)	34.855	(5, 4)	35.095	0.000	0.684
1	0.1	(4, 4, 4, 0.539)	34.986	(4, 4, 0.539)	34.986	(5, 4)	35.095	0.000	0.351
1	0.2	(4, 4, 4, 0.723)	35.077	(4, 4, 0.723)	35.077	(5, 4)	35.095	0.000	0.051
2	0	(5, 4, 3, 0.233)	35.250	(4, 3, 0.240)	35.293	(5, 3)	35.633	0.122	1.077
2	0.1	(5, 4, 3, 0.282)	35.441	(5, 4, 0.363)	35.457	(5, 3)	35.633	0.045	0.540
2	0.2	(4, 3, 2, 0.296)	35.581	(4, 3, 0.453)	35.593	(5, 3)	35.633	0.034	0.147
3	0	(5, 4, 4, 0.198)	35.547	(5, 4, 0.198)	35.547	(5, 3)	35.995	0.000	1.245
3	0.1	(4, 3, 3, 0.247)	35.763	(4, 3, 0.247)	35.763	(5, 3)	35.995	0.000	0.644
3	0.2	(4, 3, 2, 0.236)	35.918	(4, 3, 0.315)	35.932	(5, 3)	35.995	0.039	0.214
7	0	(5, 2, 2, 0.152)	36.599	(5, 2, 0.152)	36.599	(5, 2)	37.075	0.000	1.282
7	0.1	(4, 2, 2, 0.202)	36.795	(4, 2, 0.202)	36.795	(5, 2)	37.075	0.000	0.755
7	0.2	(4, 2, 2, 0.211)	36.949	(4, 2, 0.211)	36.949	(5, 2)	37.075	0.000	0.339
10	0	(5, 3, 3, 0.034)	36.619	(5, 3, 0.034)	36.619	(5, 2)	37.698	0.000	2.859
10	0.1	(3, 2, 2, 0.005)	36.942	(3, 2, 0.005)	36.942	(5, 2)	37.698	0.000	2.003
10	0.2	(3, 2, 2, 0.017)	37.236	(3, 2, 0.017)	37.236	(5, 2)	37.698	0.000	1.227

Table 7.3: Optimal design for different α_1

		BSPAA		CBSPA		CBSP		$RRS(\%)$	
α_1	C_a	$(n^*, r^*, m^*, \tau_1^*)$	R_B^*	(n_A^*, r_A^*, t_{1A}^*)	R_1^*	(n_C^*, r_C^*)	R_2^*	RRS_1	RRS_2
2	0	(4, 3, 3, 0.051)	32.350	(4, 3, 0.051)	32.350	(0, 0)	32.873	0.000	1.615
2	0.1	(4, 3, 3, 0.079)	32.691	(4, 3, 0.079)	32.691	(0, 0)	32.873	0.000	0.557
2	0.2	(0, 0, 0, 0)	32.873	(0, 0, 0)	32.873	(0, 0)	32.873	0.000	0.000
2.2	0	(4, 3, 3, 0.074)	33.907	(4, 3, 0.074)	33.907	(5, 2)	34.764	0.000	2.526
2.2	0.1	(4, 3, 3, 0.107)	34.224	(4, 3, 0.107)	34.224	(5, 2)	34.764	0.000	1.577
2.2	0.2	(4, 3, 3, 0.149)	34.509	(4, 3, 0.149)	34.509	(5, 2)	34.764	0.000	0.739
2.8	0	(5, 3, 3, 0.146)	37.678	(5, 3, 0.146)	37.678	(6, 3)	38.147	0.000	1.246
2.8	0.1	(5, 3, 2, 0.166)	37.916	(5, 3, 0.215)	37.928	(6, 3)	38.147	0.032	0.610
2.8	0.2	(5, 3, 3, 0.284)	38.109	(5, 3, 0.284)	38.109	(6, 3)	38.147	0.000	0.100
3	0	(5, 3, 3, 0.161)	38.638	(5, 3, 0.161)	38.638	(5, 2)	39.015	0.000	0.976
3	0.1	(5, 3, 2, 0.184)	38.844	(4, 2, 0.210)	38.853	(5, 2)	39.015	0.023	0.440
3	0.2	(4, 2, 1, 0.170)	38.955	(4, 2, 0.268)	38.967	(5, 2)	39.015	0.031	0.155

In Table 7.2, it is observed that as the test cost C_t increases, the corresponding time point τ_1^* in BSPAA decreases. This trend arises because a higher time cost necessitates reducing the duration of the life test to minimize the overall expected cost. Consequently, the test is conducted for a shorter time under the lower stress level, leading to a decrease in the optimal time point τ_1^* with increasing C_t. It is seen that RRS_2 increases with C_t and decreases with C_a, as expected. Also, in Table 7.1, it is seen that when $C_t = 5$ and $C_a = 0.2$, BSPAA is much better than the other two BRASPs under type-II censoring schemes. Also, it can be said that when C_t is low and C_a is high, CBSP is better than CABSP.

In Table 7.3, it is observed that the Bayes risk increases with α_1, as expected. When $\alpha_1 = 2$ and $C_a = 0$, the lot is accepted without any life testing under the CBSP. This is because

$$R_B^* = C_0 + C_1\frac{\alpha_1}{\beta_1} + C_2\frac{\alpha_2}{\beta_2} + C_{11}\frac{\alpha_1(\alpha_1+1)}{\beta_1^2} + C_{12}\left(\frac{\alpha_1}{\beta_1}\cdot\frac{\alpha_2}{\beta_2}\right) + C_{22}\frac{\alpha_2(\alpha_2+1)}{\beta_2^2},$$

is minimal without life testing. However, for BSPAA, life testing is still conducted. It is also noted that the relative risk saving RRS_2 generally increases with α_1, indicating improved performance of BSPAA compared to CBSP. Nevertheless, beyond a certain threshold of α_1, RRS_2 starts to decrease. This is because the difference between the rejection cost and the prior expected acceptance cost initially decreases and then increases with α_1.

7.6 Real data analysis

In the previous section, the time point is taken as a variable quantity that is determined by minimizing the Bayes risk. However, in some situations, consumers want life testing in normal operating conditions up to a certain time. If the consumer does not get sufficient failure time information, then they agree for higher stress. Therefore, in this scenario, the time point is pre-fixed. The other components of the designing parameter of $\boldsymbol{\zeta}$ is determined by minimizing the Bayes risk.

7.6.1 Data analysis

A solar lighting device sample of size 35 was put on a life test at a normal operating temperature $293K$. After a time point $\tau_1 = 5$ (in hundred hours), the temperature was changed from $293K$ to $353K$. The life test is terminated after a time point $\tau_2 = 6$ (in hundred hours). The solar device has two failure modes: one is capacitor failure, denoted by 1, and another is controller failure, denoted by 2. The data is given in Table 7.4.

Table 7.4: A competing risk data under SSSPALT

Operating Conditions	Cause	Failure time
At $293K$ (before $\tau_1 = 5$)	1	0.140, 1.582, 4.612
	2	0.783, 1.324, 1.716, 1.794, 1.883, 2.293, 2.660, 2.674, 2.725, 3.085, 3.924, 4.396, 4.892
At $353K$ (after $\tau_1 = 5$)	1	5.002, 5.112, 5.147, 5.238, 5.244, 5.247, 5.305, 5.407, 5.445, 5.483
	2	5.022, 5.082, 5.337, 5.408, 5.717

For the data given in Table 7.4, we get the MLEs using the Han and Balakrishnan [40]: $\widehat{\lambda}_{01} = 0.0222$, $\widehat{\lambda}_{02} = 0.0960$, $\widehat{\phi}_1 = 55.356$ and $\widehat{\phi}_2 = 6.359$. The standard deviations of λ_{01} and λ_{02} are taken as 0.0127 and 0.0266, respectively, using the observed Fisher information matrix. Now, using these MLEs, the RF for component i under SSSALT can be written as

$$R_i(t \mid \widehat{\boldsymbol{\theta}}_i) = \begin{cases} \exp[-\widehat{\lambda}_{0i} t], & \text{if } t \leq \tau_1, \\ \exp[-\widehat{\lambda}_{0i}\left\{\tau_1 + \widehat{\phi}_i(t-\tau_1)\right\}], & \text{if } t > \tau_1, \end{cases}$$

where $\widehat{\boldsymbol{\theta}}_i = (\widehat{\lambda}_{0i}, \widehat{\phi}_i)$ for $i = 1, 2$ and the RF for unit under SSSALT can be written as

$$R(t \mid \widehat{\boldsymbol{\theta}}) = \begin{cases} \exp[-(\widehat{\lambda}_{01} + \widehat{\lambda}_{02}) t], & \text{if } t \leq \tau_1, \\ \exp[-(\widehat{\lambda}_{01} + \widehat{\lambda}_{02})\tau_1 - (\widehat{\lambda}_{01}\widehat{\phi}_1 + \widehat{\lambda}_{02}\widehat{\phi}_2)(t-\tau_1)], & \text{if } t > \tau_1, \end{cases}$$

where $\widehat{\boldsymbol{\theta}} = (\widehat{\lambda}_{01}, \widehat{\lambda}_{02}, \widehat{\phi}_1, \widehat{\phi}_2)$. Using these functions, we draw the parametric reliability curve for component 1, component 2, and for the unit, which are represented by dotted lines in Figure 7.2. Also, using the Kaplan-Meier estimator, we draw the non-parametric reliability curve for component 1, component 2, and for the unit, which are represented by straight lines in Figure 7.2.

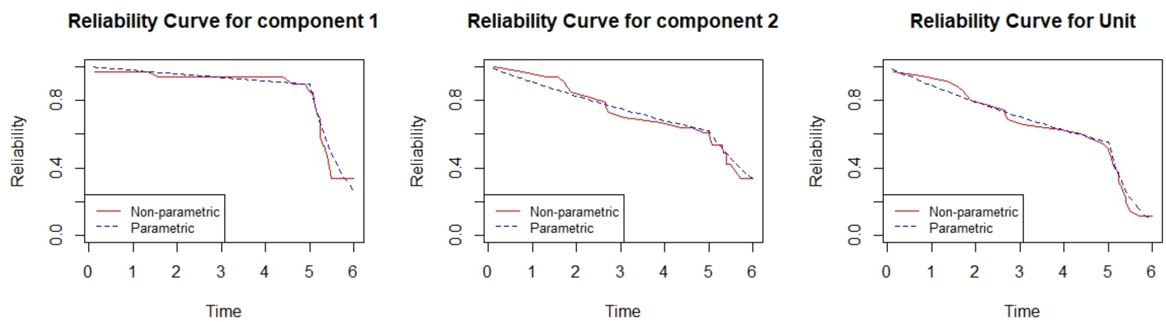


Figure 7.2: Non-parametric and parametric reliability curve

In Figure 7.2, it is seen that the parametric curve and the non-parametric curve are close to each other for all cases. Therefore, we can say that lifetimes of components 1 and 2 are well-fitted in the exponential distribution and also, the lifetime of the unit is well-fitted in our assumed distribution. Therefore, we can say that in the data, components 1 and 2 are independent in nature.

7.6.2 Illustrative Example and the effect on the solution

Example 7.2 *The hyperparameters of the priors are taken as* $\alpha_1 = 3.05$, $\alpha_2 = 13$, $\beta_1 = 137.35$, $\beta_2 = 135.44$, $l_1 = 109.072$ *and* $l_2 = 11.718$ *The values of other cost components are* $C_t = 0.1$, $c_s = 0.3$, $v_s = 0.1$, $C_a = 0.05$, $C_r = 80$. *The cost components of the loss function are taken* $C_0 = 6$, $C_1 = 200$, $C_2 = 200$, $C_{11} = 4000$, $C_{22} = 4000$, $C_{12} = 4000$. *the time point* τ_1 *is taken as* 5.

Table 7.5: Optimal design for different scenarios when $\tau_1 = 5$

BSPAA		CBSPA		CBSP		RRS	
(n^*, r^*, m^*)	R_B^*	(n_A^*, r_A^*)	R_1^*	(n_C^*, r_C^*)	R_2^*	$RRS_1(\%)$	$RRS_2(\%)$
(7, 5, 3)	77.126	(6, 5)	77.283	(7, 5)	77.236	0.204	0.142

The optimal design is given in Table 7.5. In Table 7.5, BSPAA is much better than the other two BRASPs under type-II censoring. We investigate the impact of cost components on the optimal solution of BSPAA when τ_1 is fixed. The optimal designs corresponding to various values of the test cost C_a and C_t are summarized in Tables 7.6 and 7.7, respectively.

Table 7.6: Optimal design for different scenarios for different C_a

	BSPAA		CBSPA		CBSP		$RRS(\%)$	
C_a	(n^*, r^*, m^*)	R_B^*	(n_A^*, r_A^*)	R_1^*	(n_C^*, r_C^*)	R_2^*	RRS_1	RRS_2
0	(7, 5, 3)	77.030	(6, 5)	77.119	(6, 4)	77.236	0.115	0.267
0.1	(7, 5, 2)	77.196	(6, 5)	77.443	(7, 5)	77.236	0.319	0.052
0.15	(7, 5, 1)	77.234	(5, 5)	77.583	(7, 5)	77.236	0.449	0.003
0.2	(7, 5, 0)	77.236	(4, 4)	77.708	(7, 5)	77.236	0.607	0.000

Table 7.7: Optimal design for different scenarios for different C_t

	BSPAA		CBSPA		CBSP		$RRS(\%)$	
C_t	(n^*, r^*, m^*)	R_B^*	(n_A^*, r_A^*)	R_1^*	(n_C^*, r_C^*)	R_2^*	RRS_1	RRS_2
0	(5, 5, 0)	75.850	(5, 5)	76.664	(5, 5)	75.850	1.061	0.000
0.05	(6, 5, 2)	76.661	(5, 5)	76.983	(6, 5)	76.704	0.418	0.056
0.15	(7, 5, 3)	77.481	(6, 5)	77.566	(7, 4)	77.685	0.110	0.263
0.2	(7, 4, 3)	77.806	(7, 5)	77.837	(8, 4)	78.002	0.040	0.251
0.3	(7, 3, 3)	78.309	(7, 3)	78.309	(8, 3)	78.556	0.000	0.314

In Table 7.6, it is seen that when C_a increases the value of m decreases. Also, it is seen that RRS_1 increases with C_a and RRS_2 decreases with C_a. This indicates that BSPAA converges to CBSP as expected. In Table 7.7, it is seen that for higher values of C_t, BSPAA is equivalent to CBSPA and for $C_t = 0$, BSPAA is equivalent to CBSP. The fact is already discussed in Table 7.2.

7.6.3 Decision for some simulated data

We generate type-II adaptive step-stress competing risk data based on the life testing plan $\boldsymbol{\zeta} = (7, 5, 3, 5)$ from independent exponential distributions with $\lambda_{01} = 0.0221$, $\lambda_{02} = 0.0959$ and $\phi_1 = 55.101$ and $\phi_2 = 6.358$. The data sets $\boldsymbol{x} = (x_{(1)}, x_{(2)}, x_{(3)}, x_{(4)}, x_{(5)}, d_{11}, d_{12}, d_{21}, d_{22})$ and the corresponding decisions about the lot acceptance or rejection a^* are given in Table 7.8 for illustration purposes. Also, in Table 7.8, the decision of changing the stress at τ_1, w_1 and w_2 are provided.

The BRASP can be illustrated as follows. Seven items are put on life test at $293K$. After time $\tau_1 = 5$, the number of failures is observed. If the number of failures is less than 3, the stress is increased to $353K$ and the test continues up to 5 failures observed. If the number of failures is greater than equals to 3, the stress is unchanged and the test continues up to 5 failures observed at $293K$. Then we calculate $e = \varphi(\boldsymbol{x}) - C_r$ using (7.3.1), for $i = 1, \ldots, 7$. Using (7.3.2), we can say that if $e < 0$, then $a^* = 1$ this means that the lot is accepted and if $e \geq 0$ then $a^* = 0$, this means that the lot is rejected. In data set no. 7, it is seen that the fifth failure is observed before the time $\tau_1 = 5$. Therefore, the life test is terminated at 4.69 in a hundred hours.

Table 7.8: Simulated data sets and corresponding decision about the lot

i	$x_{(1)}$	$x_{(2)}$	$x_{(3)}$	$x_{(4)}$	$x_{(5)}$	d_{11}	d_{12}	d_1	Change the stress?	d_{21}	d_{22}	e	a^*
1	0.31	2.06	5.02	5.09	5.15	1	1	2	Yes	3	0	−0.34	1
2	4.44	5.01	5.06	7.19	7.59	0	1	1	Yes	2	2	−18.13	1
3	0.21	0.55	5.30	5.41	5.69	0	2	2	Yes	1	2	11.03	0
4	0.25	0.65	1.77	9.29	18.93	0	3	3	No	0	2	10.30	0
5	2.21	2.59	2.91	13.44	20.68	1	2	3	No	2	0	−2.83	1
6	5.03	5.23	5.27	5.79	5.86	0	0	0	Yes	4	1	−16.92	1
7	0.25	0.65	3.41	4.04	4.69	3	2	3	No	0	0	7.43	0

7.7 Conclusion

This work proposes BRASP based on ASSSPALT for type-II competing censored data. Through numerical studies, it is observed that when the stress-changing time point is treated as a variable, BSPAA performs significantly better than the other two plans, CBSP and CBSPA, under certain cost configurations. In most cases, the Bayes risk of BSPAA either closely matches or coincides with that of CBSPA, highlighting its effectiveness. Notably, BSPAA consistently outperforms CBSP, especially in scenarios involving high C_t and low C_a.

Although CBSPA has not been widely explored in existing literature, the current study emphasizes its practical value alongside BSPAA. The observed RRS from CBSP reaches up to approximately 2.5%, suggesting substantial cost efficiency. Furthermore, when the stress-changing time point is fixed, BSPAA demonstrates significantly better performance compared to both CBSP and CBSPA. These findings underscore the potential of BSPAA as a robust and cost-effective methodology under censored data and competing risks in context of BRASP.

7.A Appendix

7.A.1 Expression of the terms of Bayes risk

Additional stress

n_{as} denotes the expected number of items put on higher stress levels after τ_1.

$$\begin{aligned} n_{as} &= E_{\boldsymbol{\theta}}[E[(n-D_1) \mid D_1 < m, \boldsymbol{\theta}]] \\ &= \sum_{d_1=0}^{m-1}\sum_{i=0}^{d_1}(n-d_1)\binom{n}{d_1}\binom{d_1}{i}(-1)^{d_1-i}\prod_{j=1}^{J}\left[\int_{\lambda_{0j}=0}^{\infty}\exp(-(n-i)\lambda_{0j}\tau_1)\; p_{1j}(\lambda_{0j})d\lambda_{0j}\right]. \end{aligned}$$

Expected time duration

The time duration under type-II censoring is $X_{(r)}$.

$$E[X_{(r)}|\ \boldsymbol{\theta}] = \int_{t=0}^{\tau_1} R_{X_{(r)}}(t \mid \boldsymbol{\theta})\ dt + \int_{t=\tau_1}^{\infty} R_{X_{(r)}}(t \mid \boldsymbol{\theta})\ dt.$$

For $0 < t < \tau_1$

$$R_{X_{(r)}}(t \mid \boldsymbol{\theta}) = P(\text{No. of failures in the interval } (0,t) \text{ is less than } r)$$
$$= \sum_{d_1=0}^{r-1} \binom{n}{d_1} [1 - R(t \mid \boldsymbol{\theta})]^{d_1} [R(t \mid \boldsymbol{\theta})]^{n-d_1}$$
$$= \sum_{d_1=0}^{r-1} \sum_{j=0}^{d_1} \binom{n}{d_1} \binom{d_1}{j} (-1)^{d_1-j} \exp[-\lambda(n-j)t].$$

Therefore,

$$\int_0^{\tau_1} R_{X_{(r)}}(t \mid \boldsymbol{\theta})\, dt = \sum_{d_1=0}^{r-1} \sum_{j=0}^{d_1} \binom{n}{d_1} \binom{d_1}{j} (-1)^{d_1-j} \frac{1 - \exp\left[-\left(\sum_{i=1}^{J} \lambda_{0i}\right)(n-j)\tau_1\right]}{\left(\sum_{i=1}^{J} \lambda_{0i}\right)(n-j)}.$$

For $t > \tau_1$

$$R_{X_{(r)}}(t \mid \boldsymbol{\theta})$$
$$= P(\text{No. of failures in the interval } (0,t) \text{ is less than } r \text{ and } D_1 < r \mid \boldsymbol{\theta})$$
$$= P(\text{No. of failures in the interval } (0,t) \text{ is less than } r \mid D_1 < r, \boldsymbol{\theta}) P(D_1 < r \mid \boldsymbol{\theta})$$
$$= \sum_{d_1=0}^{r-1} \sum_{d_2=0}^{r-d_1-1} \frac{n!}{d_1! d_2! (n-d)!} [1 - R(\tau_1 \mid \boldsymbol{\theta})]^{d_1} [R(\tau_1 \mid \boldsymbol{\theta}) - R(t \mid \boldsymbol{\theta})]^{d_2} [R(t \mid \boldsymbol{\theta})]^{n-d}$$
$$= \sum_{d_1=0}^{r-1} \sum_{d_2=0}^{r-d_1-1} \frac{n!}{d_1! d_2! (n-d)!} [1 - \exp(-\lambda\tau_1)]^{d_1} [\exp(-\lambda\tau_1) - \exp(-\lambda(\tau_1 + \phi(t-\tau_1)))]^{d_2} [\exp(-\lambda(\tau_1 + \phi(t-\tau_1)))]^{n-d}$$
$$= \sum_{d_1=0}^{r-1} \sum_{d_2=0}^{r-d_1-1} \sum_{j=0}^{d_1} \sum_{k=0}^{d_2} \frac{n!}{d_1! d_2! (n-d)!} (-1)^{j+k} \exp[-\lambda(n-d_1+j)\tau_1] \exp[-(\lambda_{01}\phi_1^{\delta} + \lambda_{02}\phi_2^{\delta})(n-d+k)(t-\tau_1)].$$

Therefore,

$$\int_{t=\tau_1}^{\infty} R_{X_{(r)}}(t \mid \boldsymbol{\theta})\, dt$$
$$= \sum_{d_1=0}^{r-1} \sum_{d_2=0}^{r-d_1-1} \sum_{j=0}^{d_1} \sum_{k=0}^{d_2} \frac{n!}{d_1! d_2! (n-d)!} (-1)^{j+k} \exp[-\lambda(n-d_1+j)\tau_1] \frac{1}{(\lambda_{01}\phi_1^{\delta} + \lambda_{02}\phi_2^{\delta})(n-d+k)}.$$

The expression of $f_{(W_1, W_2, \boldsymbol{D}) \mid \boldsymbol{\theta})}(w_1, w_2, \boldsymbol{d} \mid \boldsymbol{\theta})$:

The joint distribution of $\boldsymbol{D}$ is

$$P(\boldsymbol{D} = \boldsymbol{d} \mid \boldsymbol{\theta}) = A(\boldsymbol{d}) \prod_{i=1}^{2} \prod_{j=1}^{J} p_{ij}^{d_{ij}} [F(\tau_1 \mid \boldsymbol{\theta})]^{d_1} [1 - F(\tau_1 \mid \boldsymbol{\theta})]^{n-d_1},$$

where $A(\boldsymbol{d}) = \binom{n}{d_1}\binom{d_1}{d_{11}, d_{12}, \ldots, d_{1J}}\binom{d_2}{d_{21}, d_{22}, \ldots, d_{2J}}$.

Theorem 7.3 *Now using $P(\boldsymbol{D} = \boldsymbol{d} \mid \boldsymbol{\theta})$ and the joint PDF of (W_1, W_2) given $\boldsymbol{D} = \boldsymbol{d}, \boldsymbol{\theta}$ given in Balakrishnan et al. [10], we get the following expression*

$$f_{(W_1, W_2, \boldsymbol{d}) \mid \boldsymbol{\theta})}(w_1, w_2, \boldsymbol{d} \mid \boldsymbol{\theta})$$

$$
= \begin{cases} \mathbb{1}_{n\tau_1}(w_1) \binom{r}{d_{21}, d_{22}, \ldots, d_{2J}} \prod_{j=1}^{J} p_{2j}^{d_{2j}} \gamma\left(w_2, r, \phi_1\lambda_{01} + \phi_2\lambda_{02}\right), & \text{if } d_1 = 0, \\ A(\boldsymbol{d}) \sum_{j=0}^{d_1} (-1)^j \binom{d_1}{j} \prod_{i=1}^{2} \prod_{j=1}^{J} p_{ij}^{d_{ij}} \exp[-\lambda M(d_1, j)] \gamma\left(w_1 - M(d_1, j)), d_1, \lambda\right) \gamma\left(w_2, d_2, \phi_1\lambda_{01} + \phi_2\lambda_{02}\right), & \text{if } d_1 > 0, \\ \sum_{k=r}^{n} \binom{n}{k} \binom{r}{d_{11}, \ldots, d_{1J}} \sum_{j'=0}^{k} (-1)^{j'} \binom{k}{j'} \prod_{j=1}^{J} p_{1j}^{d_{1j}} \exp[-\lambda M(k, j')] \gamma(w_1 - M(k, j'), r, \lambda) \mathbb{1}_0(w_2), & \text{if } d_1 = r, \end{cases}
$$

where $M(a,b) = (n - a + b)\tau_1$ and $\mathbb{1}_x(y)$ is the indicator function defined in (2.1.11).

7.A.2 Expression of R_A for quadratic loss function

$$
E_{\boldsymbol{\lambda}}[h(\boldsymbol{\lambda})] = C_0 + \sum_{j=1}^{J} C_j \frac{\alpha_j}{\beta_j} + \sum_{\substack{i=1 \\ i<j}}^{J} \sum_{j=1}^{J} C_{ij} \frac{\alpha_i \alpha_j}{\beta_i \beta_j} + \sum_{j=1}^{J} C_{jj} \frac{\alpha_j(\alpha_j + 1)}{\beta_j^2}.
$$

For the expression of $H(\boldsymbol{d})$, we need to consider three cases:

Case 1 : When $d_1 = 0$,

$$
\begin{aligned} H(\mathbf{0}, \boldsymbol{d}_2) = \binom{r}{d_{21}, d_{22}, \ldots, d_{2J}} \int_{w_2=0}^{c_1(n*\tau_1, \boldsymbol{d})} & \mathbb{1}_{\mathcal{S}_7}(\boldsymbol{\zeta}) \frac{w_2^{d_2-1}}{\Gamma(d_2)} \Bigg[C_0 H_1(n\tau_1, w_2, \boldsymbol{d}, \mathbf{0}) \\ & + \sum_{j=0}^{J} C_j \; H_1(n\tau_1, w_2, \boldsymbol{d}, \boldsymbol{p}_j) + \sum_{\substack{i=1 \\ i\le j}}^{J} \sum_{j=1}^{J} C_{ij} \; H_1(n\tau_1, w_2, \boldsymbol{d}, \boldsymbol{p}_{ij}) \Bigg] \, dw_2, \end{aligned}
$$

where $\mathcal{S}_7 = \{\boldsymbol{\zeta} : n\tau_1 \le c(\boldsymbol{d})\}$.

Case 2 : When $\boldsymbol{d} > \mathbf{0}$,

$$
\begin{aligned} H(\boldsymbol{d}) =& A(\boldsymbol{d}) \sum_{j=0}^{d_1} (-1)^j \binom{d_1}{j} \int_{w_1=(n-d_1+j)\tau_1}^{c'(\boldsymbol{d})} \int_{w_2=0}^{c_1(w_1, \boldsymbol{d})} \frac{(w_1 - (n - d_1 + j)\tau_1)^{d_1-1} w_2^{d_2-1}}{\Gamma(d_1)\Gamma(d_2)} \Bigg[C_0 H_1(w_1, w_2, \boldsymbol{d}, \mathbf{0}) \\ & + \sum_{j=0}^{J} C_j \; H_1(w_1, w_2, \boldsymbol{d}, \boldsymbol{p}_j) + \sum_{\substack{i=1 \\ i\le j}}^{J} \sum_{j=1}^{J} C_{ij} \; H_1(w_1, w_2, \boldsymbol{d}, \boldsymbol{p}_{ij}) \Bigg] \, dw_2 \; dw_1. \end{aligned}
$$

Case 3 : When $d_2 = 0$,

$$
\begin{aligned} H(\boldsymbol{d}_1, \mathbf{0}) =& \sum_{k=r}^{n} \binom{n}{k} \binom{r}{d_{11}, \ldots, d_{1J}} \sum_{j'=0}^{k} (-1)^{j'} \binom{k}{j} \int_{w_1=(n-d_1+j)\tau_1}^{c'(\boldsymbol{d})} \frac{(w_1 - (n - d_1 + j)\tau_1)^{r-1}}{\Gamma(r)} \\ & \Bigg[C_0 H_1(w_1, 0, \boldsymbol{d}_1, \mathbf{0}, \mathbf{0}) + \sum_{j=0}^{J} C_j \; H_1(w_1, 0, \boldsymbol{d}_1, \mathbf{0}, \boldsymbol{p}_j) + \sum_{\substack{i=1 \\ i\le j}}^{J} \sum_{j=1}^{J} C_{ij} \; H_1(w_1, 0, \boldsymbol{d}_1, \mathbf{0}, \boldsymbol{p}_{ij}) \Bigg] dw_1. \end{aligned}
$$

Chapter 8

Conclusion & future works

This thesis considered the design of RASPs under various censoring schemes and testing environments. The optimal RASPs are developed under ICS and HCS-I in Chapters 2 and 3, respectively, using Bayesian decision-theoretic approaches. It is considered that the consumer and manufacturer agree on a common lifetime distribution of the product. However, they differ in their assessment of the prior distributions and utility functions because of the adversarial nature of the consumer and manufacturer.

In Chapters 2 and 3, normal operating conditions are used to conduct life tests. However, the products in recent days have been highly reliable. As a result, the mean time to failure of the product is very large. Consequently, ALT becomes necessary for designing effective RASPs. To address this issue, in Chapter 4, ASSSPALT is incorporated in the life testing and determines the optimal RASP under type-I censoring. This framework unifies both accelerated and non-accelerated testing scenarios under type-I censoring.

For complex products, failure may occur due to multiple causes. In Chapter 5, we consider the development of RASPs under PICS-I in the presence of competing causes of failure using the producer's and consumer's risk approaches. The asymptotic properties of maximum likelihood estimators are derived to develop optimal plans. A frailty-based model is employed to capture dependence among competing risks and evaluate its influence on sampling plan performance. Subsequently, RASP is extended to a Bayesian framework under ICS in Chapter 6.

Further, for highly reliabile and complex product, the design of BRASP is considered based on ASSSPALT under type-II censoring for competing risk data in Chapter 7. This framework unifies both accelerated and non-accelerated testing scenarios for competing risk data under type-II censoring. The proposed methodologies of designing RASP are illustrated using real-life data.

In Chapters 4 and 7, 2-stage step-stress test is incorporated. However, the works can be extended to more than 2-stage step-stress test. For a more than 2-stage step-stress test, the life testing scenario would be as follows. Suppose n items are put on life test at the stress level s_0 at time $t_0 = 0$. If the number of failures up to t_1 is less than a prefixed number of failures m_1, then the stress s_0 is changed to s_1, otherwise, the life test continues with stress level s_0 up to time t_2. If the cumulative number of failures up to t_2 is less than $m_2 \geq m_1$, then stress level is changed. After time t_1, if the test continued with the stress s_0, then the stress changes to s_1 and if the test continued with the stress s_1, then the stress changes to s_2. If the number of failures up to t_2 is greater than or equal to m_2, the stress level remains same and the life testing is continued up to time t_3. Continuing this procedure, the adaptive scenario can be extended to k-stage adaptive step stress life test.

Tsai et al. [116] studied an efficient sampling plan for type-I censored data where the manufacturer can take a decision and terminate the test before completing the life test. The procedure is known as the curtailment procedure. Chen et al. [30] studied the curtailment BSP for type-II censored data. Then Chen et al. [27] studied the curtailment BSP based on simple step stress ALT for type II censored data. Similarly, the works discussed in Chapters 4 and 7 can be extended to the curtailment BSP.

The work discussed in Chapter 5 can be extended in a number of important ways. We use a frailty model to incorporate the dependence among the potential failure times. However, the same can also be modeled using popular copula models (see Lo et al. [78], Nelsen [90]). It will be interesting to develop RASPs for such models. For simplicity, in Chapter 6, it is assumed that the cause of the failures is independent. However, the proposed framework can be extended to accommodate dependence using frailty or copula models.

In Chapters 4, 6 and 7, we have considered a common prior distribution for both the manufacturer and the consumer. However, in practice, they may have different prior knowledge. The methodology discussed in Chapters 4, 6 and 7 can be extended to the case where the manufacturer and consumer have different priors using the idea of Chapters 2 and 3. The exponential and Weibull distributions are used to illustrate the proposed methodologies. However, the works can be extended to other lifetime distributions and censoring schemes.

Bibliography

[1] Aggarwala, R. (2001). Progressive interval censoring: some mathematical results with application to inference. *Communications in Statistics - Theory and Methods*, 30:1921–1935.

[2] Aslam, M., Azam, M., Balamurali, S., and Jun, C.-H. (2015). An economic design of a group sampling plan for a Weibull distribution using a Bayesian approach. *Journal of Testing and Evaluation*, 43(6):1497–1503.

[3] Aslam, M. and Jun, C.-H. (2009). A group acceptance sampling plan for truncated life test having Weibull distribution. *Journal of Applied Statistics*, 36(9):1021–1027.

[4] Aslam, M., Yen, C.-H., Chang, C.-H., Al-Marshadi, A. H., and Jun, C.-H. (2019). A multiple dependent state repetitive sampling plan based on performance index for lifetime data with type II censoring. *IEEE Access*, 7:49377–49391.

[5] Atkinson, A., Donev, A., and Tobias, R. (2007). *Optimum experimental designs, with SAS*, volume 34. Oxford University Press.

[6] Azizi, F., Haghighi, F., and Gilani, N. T. (2020). Statistical inference for competing risks model under progressive interval censored Weibull data. *Communications in Statistics - Simulation and Computation*, 49:1931–1944.

[7] Bagdonavicius, V. and Nikulin, M. (2001). *Accelerated life models: modeling and statistical analysis*. Chapman and Hall/CRC.

[8] Balakrishnan, N. and Aggarwala, R. (2000). *Progressive censoring: theory, methods and applications*. Birkhauser, Boston.

[9] Balakrishnan, N. and Han, D. (2008). Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring. *Journal of Statistical Planning and Inference*, 138(12):4172–4186.

[10] Balakrishnan, N., Xie, Q., and Kundu, D. (2009). Exact inference for a simple step-stress model from the exponential distribution under time constraint. *Annals of the Institute of Statistical Mathematics*, 61:251–274.

[11] Balamurali, S., Park, H., Jun, C.-H., Kim, K.-J., and Lee, J. (2005). Designing of variables repetitive group sampling plan involving minimum average sample number. *Communications in Statistics-Simulation and Computation*, 34(3):799–809.

[12] Balasooriya, U. and Balakrishnan, N. (2000). Reliability sampling plans for lognormal distribution, based on progressively-censored samples. *IEEE Transactions on Reliability*, 49(2):199–203.

[13] Balasooriya, U. and Saw, S. L. (1998). Reliability sampling plans for the two-parameter exponential distribution under progressive censoring. *Journal of Applied Statistics*, 25(5):707–714.

[14] Balasooriya, U., Saw, S. L., and Gadag, V. (2000). Progressively censored reliability sampling plans for the Weibull distribution. *Technometrics*, 42(2):160–167.

[15] Bartholomew, D. (1963). The sampling distribution of an estimate arising in life testing. *Technometrics*, 5(3):361–374.

[16] Berger, J. O. (2013). *Statistical decision theory and Bayesian analysis*. Springer Science & Business Media.

[17] Bhattacharya, R., Pradhan, B., and Dewanji, A. (2014). Optimum life testing plans in presence of hybrid censoring: a cost function approach. *Applied Stochastic Models in Business and Industry*, 30(5):519–528.

[18] Bhattacharya, R., Pradhan, B., and Dewanji, A. (2015). Computation of optimum reliability acceptance sampling plans in presence of hybrid censoring. *Computational Statistics & Data Analysis*, 83:91–100.

[19] Budhiraja, S. and Pradhan, B. (2017). Computing optimum design parameters of a progressive type-I interval censored life test from a cost model. *Applied Stochastic Models in Business and Industry*, 33:494–506.

[20] Budhiraja, S. and Pradhan, B. (2019). Optimum reliability acceptance sampling plans under progressive type-I interval censoring with random removal using a cost model. *Journal of Applied Statistics*, 46(8):1492–1517.

[21] Chaloner, K. and Larntz, K. (1992). Bayesian design for accelerated life testing. *Journal of Statistical Planning and Inference*, 33:245–259.

[22] Chaloner, K. and Verdinelli, I. (1995). Bayesian experimental design: a review. *Statistical Science*, 10:273–304.

[23] Chen, J., Li, K.-H., and Lam, Y. (2007). Bayesian single and double variable sampling plans for the Weibull distribution with censoring. *European Journal of Operational Research*, 177(2):1062–1073.

[24] Chen, L.-S., Liang, T., and Yang, M.-C. (2020). Designing Bayesian sampling plans for censored data under products sold with warranty. *Journal of Statistical Computation and Simulation*, 90(7):1230–1250.

[25] Chen, L.-S., Liang, T., and Yang, M.-C. (2021). Optimal curtailed Bayesian sampling plans for exponential distributions with Type-I hybrid censored samples. *Communications in Statistics-Simulation and Computation*, 50(3):764–777.

[26] Chen, L.-S., Liang, T., and Yang, M.-C. (2022). Designing Bayesian sampling plans for simple step-stress of accelerated life test on censored data. *Journal of Statistical Computation and Simulation*, 92(2):395–415.

[27] Chen, L.-S., Liang, T., and Yang, M.-C. (2023). Designing efficient Bayesian sampling plans based on the simple step-stress test under random stress-change time for censored data. *Journal of Statistical Computation and Simulation*, 93(7):1104–1129.

[28] Chen, L.-S., Liang, T., and Yang, M.-C. (2026). Designing efficient Bayesian sampling plans for two-parameter exponential distribution with censored data. *Communications in Statistics-Theory and Methods*, 55(1):313–334.

[29] Chen, L.-S., Yang, M.-C., and Liang, T. (2015). Bayesian sampling plans for exponential distributions with interval censored samples. *Naval Research Logistics*, 62(7):604–616.

[30] Chen, L.-S., Yang, M.-C., and Liang, T. (2017). Curtailed Bayesian sampling plans for exponential distributions based on Type-II censored samples. *Journal of Statistical Computation and Simulation*, 87(6):1160–1178.

[31] Childs, A., Chandrasekar, B., Balakrishnan, N., and Kundu, D. (2003). Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution. *Annals of the Institute of Statistical Mathematics*, 55(2):319–330.

[32] Cohen, A. C. (1963). Progressively censored samples in life testing. *Technometrics*, 5:327–339.

[33] Condra, L. (2018). *Reliability improvement with design of experiment*. CRC Press.

[34] Crowder, M. (2001). *Classical competing risks*. Chapman and Hall/CRC.

[35] David, H. A. and Moeschberger, M. L. (1978). *The theory of competing risks*. Griffin.

[36] Devroye, L. (1984). A simple algorithm for generating random variates with a log-concave density. *Computing*, 33(3):247–257.

[37] Dodge, H. F. and Romig, H. G. (1929). A method of sampling inspection. *The Bell System Technical Journal*, 8(4):613–631.

[38] Epstein, B. (1954). Truncated life tests in the exponential case. *The Annals of Mathematical Statistics*, 25(3):555–564.

[39] Ghosh, B., Dewanji, A., and Das, S. (2024). Parametric analysis of bivariate current status data with competing risks using frailty model. *arXiv preprint arXiv:2405.05773*.

[40] Han, D. and Balakrishnan, N. (2010). Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint. *Computational Statistics & Data Analysis*, 54(9):2066–2081.

[41] Han, D. and Kundu, D. (2014). Inference for a step-stress model with competing risks for failure from the generalized exponential distribution under type-I censoring. *IEEE Transactions on reliability*, 64(1):31–43.

[42] Hanagal, D. D. (2011). *Modeling survival data using frailty models*. Springer.

[43] Happich and Michael (2001). Utility functions for life years and health status: An additional remark. Technical report, Diskussionspapier, No. 2001/6, Technische Universität Berlin, Fakultät Wirtschaft Management, Berlin.

[44] Hougaard, P. (1995). Frailty models for survival data. *Lifetime Data Analysis*, 1:255–273.

[45] Huang, S.-R. and Wu, S.-J. (2008). Reliability sampling plans under progressive type-I interval censoring using cost functions. *IEEE Transactions on Reliability*, 57:445–451.

[46] Jin, A., Yang, Z., Guo, J., Chen, C., and Liu, Z. (2026). Optimal design of multivariate accelerated degradation test based on game theory. *Quality Technology & Quantitative Management*, 23(1):1–17.

[47] Jun, C.-H., Lee, H., Lee, S.-H., and Balamurali, S. (2010). A variables repetitive group sampling plan under failure-censored reliability tests for Weibull distribution. *Journal of Applied Statistics*, 37(3):453–460.

[48] Kalbfleisch, J. D. and Prentice, R. L. (2002). *The statistical analysis of failure time data*. John Wiley & Sons, 2nd edition.

[49] Kao, J. H. (1971). MIL-STD-414 sampling procedures and tables for inspection by variables for percent defective. *Journal of Quality Technology*, 3(1):28–37.

[50] Kim, M. and Yum, B.-J. (2009). Reliability acceptance sampling plans for the Weibull distribution under accelerated Type-I censoring. *Journal of Applied Statistics*, 36(1):11–20.

[51] Kim, S.-H. and Sung, S.-I. (2023). Optimal design of cyclic-stress accelerated life tests for lognormal lifetime distribution. *Quality Technology & Quantitative Management*, 20(2):236–255.

[52] Klein, J. P. and Basu, A. P. (1981). Accelerated life testing under competing exponential failure distributions. Technical Report TR108, Department of Statistics, University of Missouri.

[53] Koley, A. and Kundu, D. (2021). Analysis of progressive Type-II censoring in presence of competing risk data under step stress modeling. *Statistica Neerlandica*, 75:115–136.

[54] Kumar, M., Bajeel, P., and Ramyamol, P. (2020). Optimal design of reliability acceptance sampling plan based on data obtained from partially accelerated life test (PALT). *International Journal of Reliability, Quality and Safety Engineering*, 27(02):2040002.

[55] Kundu, D. (2007). On hybrid censored Weibull distribution. *Journal of Statistical Planning and Inference*, 137(7):2127–2142.

[56] Kundu, D. (2014). Curtailed sampling plans. *Wiley StatsRef: Statistics Reference Online*, pages 1–11.

[57] Kundu, D. and Ganguly, A. (2017). *Analysis of step-stress models: existing results and some recent developments*. Academic Press.

[58] Lam, Y. (1990). An optimal single variable sampling plan with censoring. *Journal of the Royal Statistical Society Series D: The Statistician*, 39:53–66.

[59] Lawless, J. F. (2011). *Statistical models and methods for lifetime data*. John Wiley & Sons.

[60] Lee, A. H., Wu, C.-W., Wang, T.-C., and Kuo, M.-H. (2024). Construction of acceptance sampling schemes for exponential lifetime products with progressive type II right censoring. *Reliability Engineering & System Safety*, 243:109843.

[61] Lehmann, E. L. and Casella, G. (1998). *Theory of point estimation*. Springer.

[62] Lehmann, E. L. and Romano, J. P. (2005). *Testing statistical hypotheses*. Springer.

[63] Li, X., Gao, P., and Sun, F. (2015). Acceptance sampling plan of accelerated life testing for lognormal distribution under time-censoring. *Chinese Journal of Aeronautics*, 28(3):814–821.

[64] Liang (2014a). Designing Bayesian sampling plans with type-I progressive hybrid censored samples. *Journal of the Chinese Statistical Association*, 52(4):449–465.

[65] Liang (2014b). On designing sampling plans with type-II progressive hybrid censored samples. *Journal of the Chinese Statistical Association*, 52(2):209–221.

[66] Liang, T., Lin, C.-Y., and Chen, L.-S. (2025). Designing efficient Bayesian sampling plans for two-component exponential distributions model. *Journal of Statistical Computation and Simulation*, 95(6):1165–1189.

[67] Liang, T. and Yang, M.-C. (2013). Optimal Bayesian sampling plans for exponential distributions based on hybrid censored samples. *Journal of Statistical Computation and Simulation*, 83(5):922–940.

[68] Lieberman, G. J. and Resnikoff, G. J. (1955). Sampling plans for inspection by variables. *Journal of the American Statistical Association*, 50(270):457–516.

[69] Lim, H., Kim, Y. S., Bae, S. J., and Sung, S.-I. (2019). Partial accelerated degradation test plans for Wiener degradation processes. *Quality Technology & Quantitative Management*, 16(1):67–81.

[70] Lin, C., Balakrishnan, N., and Wu, S. J. S. (2011). Planning life tests based on progressively type-I grouped censored data from the Weibull distribution. *Communications in Statistics - Simulation and Computation*, 40:574–595.

[71] Lin, C., Wu, S. J. S., and Balakrishnan, N. (2009). Planning life tests with progressively type-I interval censored data from the lognormal distribution. *Journal of Statistical Planning and Inference*, 139:54 – 61.

[72] Lin, C.-T., Huang, Y.-L., and Balakrishnan, N. (2008). Exact Bayesian variable sampling plans for the exponential distribution based on type-I and type-II hybrid censored samples. *Communications in Statistics—Simulation and Computation®*, 37(6):1101–1116.

[73] Lin, C.-T., Huang, Y.-L., and Balakrishnan, N. (2010). Corrections on "Exact Bayesian variable sampling plans for the exponential distribution based on type-I and type-II hybrid censored samples". *Communications in Statistics-Simulation and Computation*, 39(7):1499–1505.

[74] Lin, Y.-P., Liang, T., and Huang, W.-T. (2002). Bayesian sampling plans for exponential distribution based on type I censoring data. *Annals of the Institute of Statistical Mathematics*, 54(1):100–113.

[75] Lindley, D. V. and Singpurwalla, N. D. (1991). On the evidence needed to reach agreed action between adversaries, with application to acceptance sampling. *Journal of the American Statistical Association*, 86(416):933–937.

[76] Lindley, D. V. and Singpurwalla, N. D. (1993). Adversarial life testing. *Journal of the Royal Statistical Society: Series B (Methodological)*, 55(4):837–847.

[77] Liu, X. (2012). Planning of accelerated life tests With dependent failure modes based on a gamma f model. *Technometrics*, 54:398–409.

[78] Lo, S. M., Shi, S., and Wilke, R. A. (2025). A copula duration model with dependent states and spells. *Computational Statistics & Data Analysis*, 204:108104.

[79] Lou, Y., Zheng, H., and Yang, J. (2023). Optimal design of reliability acceptance sampling plan based on constant-stress accelerated life testing considering the uncertainty of accelerated factor. *Quality and Reliability Engineering International*, 39(4):1318–1333.

[80] Lu, W. and Tsai, T.-R. (2009). Interval censored sampling plans for the gamma lifetime model. *European Journal of Operational Research*, 192(1):116–124.

[81] Lu, W. and Tsai, T.-R. (2015). Bayesian sampling plans with interval censoring. In *2015 IEEE International Conference on Industrial Engineering and Engineering Management (IEEM)*, pages 446–450. IEEE.

[82] Meeker, W. Q., Escobar, L. A., and Pascual, F. G. (2022). *Statistical methods for reliability data*. John Wiley & Sons.

[83] Mendenhall, W. and Hader, R. (1958). Estimation of parameters of mixed exponentially distributed failure time distributions from censored life test data. *Biometrika*, 45(3-4):504–520.

[84] Menke, W. W. (1969). Determination of warranty reserves. *Management Science*, 15(10):B–542.

[85] Mondal, S. and Kundu, D. (2020). Bayesian inference for Weibull distribution under the balanced joint type-II progressive censoring scheme. *American Journal of Mathematical and Management Sciences*, 39(1):56–74.

[86] Montgomery, D. C. (2020). *Introduction to statistical quality control*. John wiley & sons.

[87] Mukhopadhyay, C. (2006). Maximum likelihood analysis of masked series system lifetime data. *Journal of Statistical Planning and Inference*, 136:803–838.

[88] Mukhopadhyay, C. and Roy, S. (2016). Bayesian accelerated life testing under competing log-location-scale family of causes of failure. *Computational Statistics*, 31:89–119.

[89] Murthy, D. and Blischke, W. (1992). Product warranty management—III: a review of mathematical models. *European Journal of Operational Research*, 63(1):1–34.

[90] Nelsen, R. B. (2006). *An introduction to copulas*. Springer.

[91] Nelson, W. B. (2009). *Accelerated testing: statistical models, test plans, and data analysis*. John Wiley & Sons.

[92] Oakes, D. (1982). A model for association in bivariate survival data. *Journal of the Royal Statistical Society, Series B (Methodological)*, 44(3):414–422.

[93] Papazoglou, I. (1999). Bayesian decision analysis and reliability certification. *Reliability Engineering & System Safety*, 66(2):177–198.

[94] Pascual, F. (2010). Accelerated life test planning with independent lognormal competing risks. *Journal of Statistical Planning and Inference*, 140:1089–1100.

[95] Pham, H. (2006). *Handbook of reliability engineering*. Springer Science & Business Media.

[96] Prajapat, K., Koley, A., Mitra, S., and Kundu, D. (2023). An optimal Bayesian sampling plan for two-parameter exponential distribution under Type-I hybrid censoring. *Sankhya A*, 85(1):512–539.

[97] Prajapati, D. and Kundu, D. (2024). Bayesian acceptance sampling plan for simple step-stress model. *Communications in Statistics-Simulation and Computation*, 53(12):6411–6431.

[98] Prajapati, D., Mitra, S., and Kundu, D. (2023a). Bayesian sampling plan for the exponential distribution with generalized type-I hybrid censoring scheme. *Journal of Statistical Theory and Practice*, 17(1):5.

[99] Prajapati, D., Mitra, S., Kundu, D., and Pal, A. (2023b). Optimal Bayesian sampling plan for censored competing risks data. *Journal of Statistical Computation and Simulation*, 93(5):775–799.

[100] Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate. *Technometrics*, 5(3):375–383.

[101] Ramyamol, P. and Kumar, M. (2020). Optimal reliability acceptance sampling plan based on accelerated life test data. *International Journal of Productivity and Quality Management*, 30(3):354–370.

[102] Rasay, H., Naderkhani, F., and Golmohammadi, A. M. (2020). Designing variable sampling plans based on lifetime performance index under failure censoring reliability tests. *Quality Engineering*, 32(3):354–370.

[103] Roy, S. and Mukhopadhyay, C. (2014). Bayesian accelerated life testing under competing Weibull causes of failure. *Communications in Statistics-Theory and Methods*, 43:2429–2451.

[104] Roy, S. and Pradhan, B. (2017). Bayesian optimum life testing plans under progressive type-I interval censoring scheme. *Quality and Reliability Engineering International*, 33:2727–2737.

[105] Roy, S. and Pradhan, B. (2019). Bayesian C-optimal life testing plans under progressive type-I interval censoring scheme. *Applied Mathematical Modelling*, 70:299–314.

[106] Roy, S. and Pradhan, B. (2023). Inference for log-location-scale family of distributions under competing risks with progressive type-I interval censored data. *Statistica Neerlandica*, 77:208–232.

[107] Rufo, M. J., Martín, J., and Pérez, C. (2014). Adversarial life testing: a Bayesian negotiation model. *Reliability Engineering & System Safety*, 131:118–125.

[108] Rufo, M. J., Martín, J., and Pérez, C. J. (2016). A Bayesian negotiation model for quality and price in a multi-consumer context. *Reliability Engineering & System Safety*, 147:132–141.

[109] Sarhan, A. M., Hamilton, D. C., and Smith, B. (2010). Statistical analysis of competing risks models. *Reliability Engineering & System Safety*, 95(9):953–962.

[110] Schneider, H. (1989). Failure-censored variables-sampling plans for lognormal and Weibull distributions. *Technometrics*, 31(2):199–206.

[111] Sedyakin, N. (1966). On one physical principle in reliability theory. *Techn. Cybernetics*, 3:80–87.

[112] Singh, S. and Tripathi, Y. M. (2017). Acceptance sampling plans for inverse Weibull distribution based on truncated life test. *Life Cycle Reliability and Safety Engineering*, 6(3):169–178.

[113] Sinha, S. K. (1986). *Reliability and life testing*. John Wiley & Sons, Inc.

[114] Sklar, A. (1973). Random variables, joint distribution functions, and copulas. *Kybernetika*, 9(6):449–460.

[115] Thomas, M. U. (1983). Optimum warranty policies for nonreparable items. *IEEE transactions on reliability*, 32(3):282–288.

[116] Tsai, T.-R., Chiang, J.-Y., Liang, T., and Yang, M.-C. (2014). Efficient Bayesian sampling plans for exponential distributions with type-I-censored samples. *Journal of Statistical Computation and Simulation*, 84(5):964–981.

[117] Wang, T.-C. and Wu, C.-W. (2025). Adaptive sampling system with quick-switching mechanism indexed by lifetime performance for product reliability verification. *Annals of Operations Research*, pages 1–14.

[118] Wetherill, G. B. and Chiu, W. (1975). A review of acceptance sampling schemes with emphasis on the economic aspect. *International Statistical Review/Revue Internationale de Statistique*, pages 191–210.

[119] Wienke, A. (2010). *Frailty models in survival analysis*. Chapman and Hall/CRC.

[120] Wijsman, R. A. (1985). A useful inequality on ratios of integrals, with application to maximum likelihood estimation. *Journal of the American Statistical Association*, 80(390):472–475.

[121] Wu, S.-J., Hsu, C.-C., and Huang, S.-R. (2020). Optimal designs and reliability sampling plans for one-shot devices with cost considerations. *Reliability Engineering & System Safety*, 197:106795.

[122] Wu, S. J. and Huang, S. R. (2014). Planning progressive type-I interval censoring life tests with competing risks. *IEEE Transactions on Reliability*, 63:511–522.

[123] Wu, W., Wang, B. X., Chen, J., Miao, J., and Guan, Q. (2023). Interval estimation of the two-parameter exponential constant stress accelerated life test model under Type-II censoring. *Quality Technology & Quantitative Management*, 20(6):751–762.

[124] Xiang, S., Yang, J., and Shen, L. (2017). Designing adaptive accelerated life tests using Bayesian methods. *Quality and Reliability Engineering International*, 33(8):2269–2279.

[125] Xu, A., Fu, J., Tang, Y., and Guan, Q. (2015). Bayesian analysis of constant-stress accelerated life test for the Weibull distribution using noninformative priors. *Applied Mathematical Modelling*, 39:6183 – 6195.

[126] Yang, Y. (2005). Can the strengths of AIC and BIC be shared? A conflict between model indentification and regression estimation. *Biometrika*, 92:937–950.

[127] Yeh, L. (1990). An optimal single variable sampling plan with censoring. *Journal of the Royal Statistical Society: Series D (The Statistician)*, 39(1):53–66.

[128] Yeh, L. (1994). Bayesian variable sampling plans for the exponential distribution with type I censoring. *The Annals of Statistics*, 22(2):696–711.

[129] Yeh, L. and Choy, S. (1995). Bayesian variable sampling plans for the exponential distribution with uniformly distributed random censoring. *Journal of Statistical Planning and Inference*, 47(3):277–293.

[130] You, D. and Pham, H. (2017). Self-adaptive stress accelerated life testing scheme. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, 39(6):2095–2103.

[131] Zhang, C., Wang, L., Bai, X., and Huang, J. (2022). Bayesian reliability analysis for copula based step-stress partially accelerated dependent competing risks model. *Reliability Engineering & System Safety*, 227:108718.

[132] Zhang, L., Chen, X., Khatab, A., An, Y., and Feng, X. (2024). Joint optimization of selective maintenance and repairpersons assignment problem for mission-oriented systems operating under s-dependent competing risks. *Reliability Engineering & System Safety*, 242:109796.

[133] Zhang, X. P., Shang, J. Z., Chen, X., Zhang, C. H., and Wang, Y. S. (2014). Statistical inference of accelerated life testing with dependent competing failures based on copula theory. *IEEE Transactions on Reliability*, 63(3):764–780.

[134] Zheng, D. and Fang, X. (2017). Exact confidence limits for the acceleration factor under constant-stress partially accelerated life tests with type-I censoring. *IEEE Transactions on Reliability*, 67(1):92–104.