
Essays on Blocking Mechanisms in Economies with Club Goods

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To my parents and teachers

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ABSTRACT

The thesis intends to investigate blocking mechanisms in club economies by analyzing the notion of core and bargaining sets. We adapt the framework of Ellickson et al. [35] in this regard. Clubs in this framework are treated in a parallel fashion to private goods as articles of choice. Each club is composed of two components: (i) the profile of the club and (iii) the club project. Thus clubs are identified through their member characteristics and the local public project they undertake. The competitive equilibria notion in our setup is known as club equilibrium, where each agent maximizes their utility through their choice of *states*, which depict a combination of private goods and club memberships in our framework. It should be noted that for any state of the economy, given the club profiles, the “consistency” condition for club memberships reveals the *optimal number* of clubs that need to be formed for each type to satisfy the membership demand in that state.

To begin with, we try to answer one of the more nascent questions in the literature on general equilibrium theory by investigating the equivalence between the set of equilibrium states and the bargaining set for a club economy. We introduce the bargaining set for such an economy in line with Aumann and Maschler [6] and define a two-step veto mechanism. We establish that non-club-equilibrium states are those against which there exists a set of agents and a price vector at which they are willing to trade amongst themselves rather than consume the non-club-equilibrium state assigned to them and all other agents (weakly) prefer the non-club-equilibrium state to trade at that particular price vector. In other words, there is a Walrasian objection to non-club-equilibrium states. We further show that Walrasian objections are also justified, which helps us to establish our bargaining -equilibrium equivalence.

As a second main contribution, we follow along the lines of the first contribution to introduce the bargaining set as a solution concept to talk about the repercussions of the objection mechanism from agents. In this regard, we adapt the concept of global bargaining set introduced by Vind [81] for our club economy and provide characterizations of it in terms of the size of the (counter-) objecting coalitions thereby extending the works of Schjødtt and Sloth [70] and Hervés- Estévez and Moreno-García [48]. In the process, we obtain Schmeidler’s [71] and Vind’s [80] theorem for atomless club economies as simple corollaries. We provide further interpretations of the global bargaining set in terms of approximately robustly efficient states, a notion familiarised by Bhowmik and Kaur [22] for club economies.

The final contribution of the thesis aims to study some classical results concerning

the equivalence between the core and equilibrium states in a mixed club economy. We begin with an atomless club economy as in Ellickson et al. [35] and then allow for some agents to form cartels. The economy thus formed is characterized by the presence of both negligible and non-negligible agents to consider an oligopolistic setting as in Shitovitz [77]. Clubs in such an economy are interpreted to have no market power as the amount of inputs required to form them is insignificant compared to the aggregate endowment resources of the economy. We follow a kind of similar approach to Greenberg and Shitovitz [41] and establish the equivalence between the core (resp. club equilibria) of an atomless economy with the core (resp. club equilibria) of its associated mixed club economy, which generalizes the core-equivalence theorem of Ellickson et al. [35] to the case of a mixed club economy.

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Chapter 1

Introduction

1.1 Historical Background

The most central theme governing any economic system is the law of demand and supply. In particular, for a market system prices are governed by these forces of demand and supply. Demand is generated from the consumption side or by the consumers and supply is generated by the production side or by producers. Walras's [82] general equilibrium theory formulated a solution concept for such an economic system of demand and supply and termed it "*equilibrium*".

The equilibrium of a market system is reached when demand equals supply at any fixed point in time. Walras's theory assumes that both consumers and producers act as price takers, each consumer's objective is to maximize utility and that of a producer is to maximize profits. Any price system where demand equals supply is known as an 'equilibrium price' or the "market clearing price". If one considers a price above the market clearing one then there is a shortage in demand and if one considers a price system below the market clearing one then there is a shortage in supply. This can be visualized in Figure 1.1.

Arrow and Debreu [3] and McKenzie [65] provided a formal extension to Walras's framework by considering an economic model with finitely many consumers, producers, and commodities. Commodities in this model are defined based on three characteristics: (i) physical characteristics, (ii) the location at which a commodity is available, and (iii) the date on which a commodity is available. Under certain standard assumptions on preferences, they showed that there exists an equilibrium price for such an economy. The combination of the price along with the commodity bundles demanded by the consumers and supplied by the producers is known as the *Walrasian (competitive) equilibrium*. This is traditionally known as the existence theorem of a Walrasian

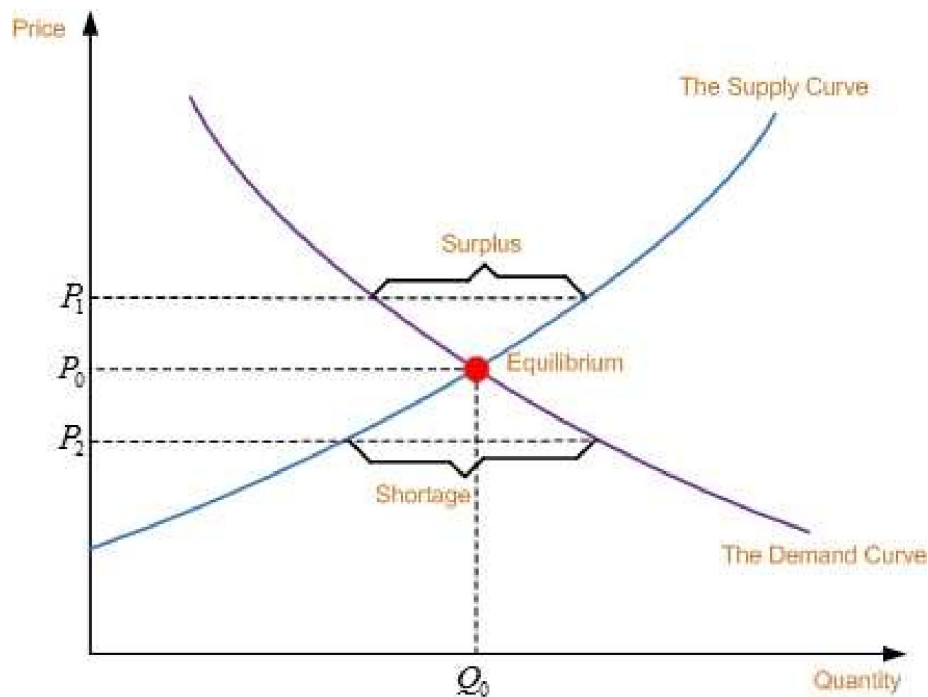


Figure 1.1: Demand and Supply

equilibrium in the literature and the economic model in [[3], [65]] is known as the *Arrow-Debreu-Mckenzie model*. Since the actions of each agent are independent of each other, the Walrasian equilibrium is a non-cooperative solution concept, and its existence relies on the usage of mathematical tools and techniques. While Arrow and Debreu [3] used an extension of the Nash equilibrium theorem in an abstract economy to guarantee the existence, McKenzie [65] provided a simpler argument by using the Brouwer fixed point theorem. However, the Arrow-Debreu-McKenzie model was devoid of any uncertainty. Thus Arrow [4] and later Debreu [29] incorporated uncertainties by allowing for finitely many states of nature and state-contingent claims in the classical model. Uncertainties in such a model are described through a family of events and a commodity on any date is also characterized by an event. They showed that if all agents have the same information about the realized state of nature, then the argument for the existence of Walrasian equilibrium in the deterministic model goes through for such a modified economy.

One of the major questions that has been asked in the literature of general equilibrium theory is how competitive the market is or in other words, if the agents in a market are left to mutual barter among themselves, can the resulting outcome be the same as that of a Walrasian equilibrium. This analysis has often been referred to as the

optimality of Walrasian equilibrium in literature. There are two widely used notions of optimality: (i) Core, and (ii) Pareto Optimality. The core of an economy is those set of commodity bundles for which there does not exist a set of agents who can form a group together to redistribute their initial endowments and arrive at a mutual trade among themselves such that everyone is better off compared to the old commodity bundle. Edgeworth [33] in his work considered a two-agent two two-commodity pure exchange economy (an economy without production) which we depict geometrically in Figure 1.1, which is known as the *Edgeworth-box* corresponding to that economy. In [33], Edgeworth considered a two-agent pure exchange economy with two commodities, which can be described geometrically. Edgeworth assumed that each agent initially possessed certain amounts of each commodity. Let a denote the total initial endowment and correspond to the point B in the xy -plane. Suppose that P is the initial endowment of the first agent concerning the xy -plane. Then, P also denotes the initial endowment of the second agent concerning the st -plane. The consumption set of the first agent corresponds to the xy -plane and that of the second agent corresponds to the st -plane. The dotted curve represents the indifference curve of the first agent, that is, any two points on this curve are equally preferable to the first agent. Likewise, the undotted curve is the indifference curve of the second agent. A *Pareto optimal allocation* is a point at which the indifference curves of the two agents are tangent to each other. The set of Pareto optimal allocations is called the *contract curve* and is the curve joining the points O and B . The intersection of this curve with the shaded region is the core of this economy and a Walrasian equilibrium is a point of the core.

Formally, Pareto optimal allocations are those allocations for which there is no mutually beneficial trade for the grand coalition by redistribution of the initial endowments such that everyone is strictly better off. The set of Pareto optimal allocations is called the *contract curve* and is the curve joining points O and B in the figure. The core is the set of allocations that lie in the intersection of the contract curve with the shaded region, and Walrasian equilibrium is a point in the core. Edgeworth [33] in his study further introduced expanded economies by introducing $2n$ many replicas of each of the n agents of the original economy. Each replicated agent is of the same type as that of the original agent, i.e., they have the same endowment and preference as that of the original agent. Edgeworth further conjectured that as n increases, the core becomes smaller, and eventually, only the Walrasian equilibrium remains. However, a significant limitation of Edgeworth's work was that it failed to create a diagrammatic depiction of the core convergence when the number of agents and commodities is arbitrarily finite, refer to [31]. Debreu and Scarf [31] formalized Edgeworth's conjecture by using advanced mathematical techniques. They showed that in the core of any

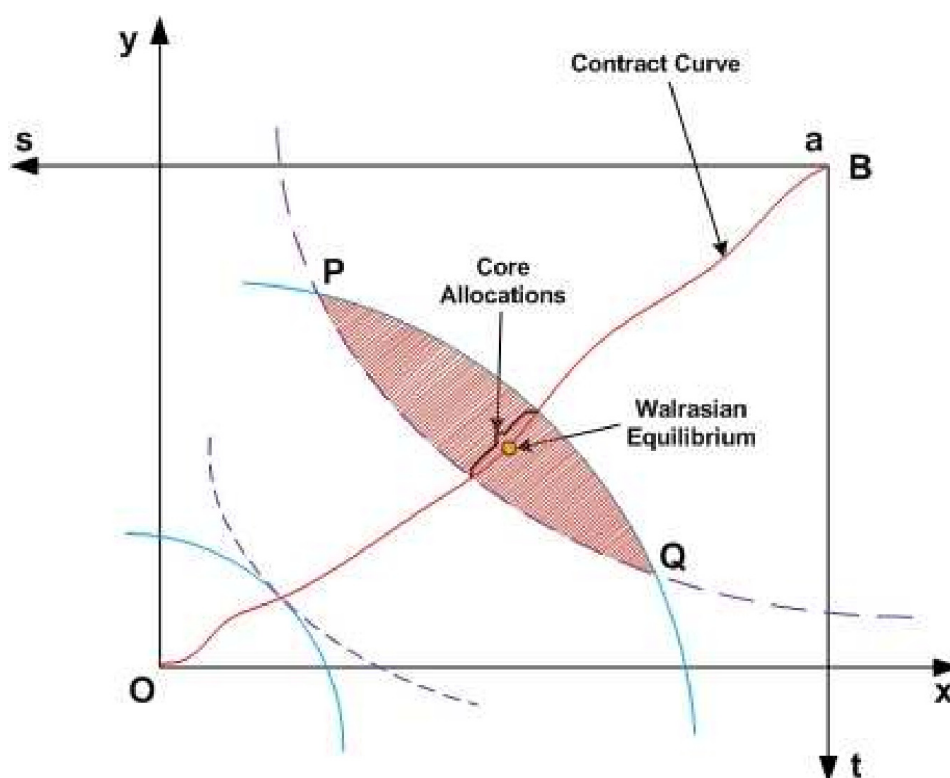


Figure 1.2: The Edgeworth Box

expanded (replicated) economies, consumption of agents of the same type is the same and referred to the intersection of the core of all the replicated economies as *Edgeworth equilibria*. They further established that under suitable price system(s), an Edgeworth equilibrium can be shown to be a Walrasian equilibrium. This, combined with the non-emptiness of the core (see [69]) yielded an alternative proof for the existence of Walrasian equilibrium.

The above analysis indicates that the key to the core allocations converging to the set of equilibrium allocations lies in the fact that as n becomes larger and larger, each agent in the economy is devoid of any market power. Following along these lines of thought, Aumann proposed a pure exchange economy with a continuum of agents modeled through the unit interval $[0, 1]$. This framework had the mathematical advantage that one can integrate over a continuum, and any change of the integrand at a single point (even a set of points with measure zero) does not affect the value of the integral. Aumann showed that Walrasian equilibrium allocations exist for such an economy and the core converges to the set of Walrasian equilibrium allocations of the economy, refer to [[7], [9]]. Aumann's core-equivalence theorem marks a seminal

result in the literature, which has been further extended to many different settings. One of the key extensions was proposed by Shitovitz [77], who, following along with Aumann's remark, claimed that no market is perfectly competitive. He represented such an economic scenario by introducing mixed models, which were characterized by the presence of both small and large agents. Shitovitz [77] showed that if there are at least two atoms in the economy with the same initial endowment and same preferences, then the core coincides with the set of Walrasian allocations of the mixed economy. See [[14], [30], [67]] for further studies of the equivalence between the core and equilibrium allocations of mixed economies.

The study on the core of an economy was further extended in three notes in the same issue of *Econometrica* in 1972. These studies provided a sharper characterization of the core by analyzing how robust the core is to the size of the blocking coalitions. Note that the core-equivalence theorem established by Aumann hinges on checking for blocking by infinitely many coalitions that can be formed from a continuum of agents, which may be a tedious process. Thus, a robustness check in terms of the size of the blocking coalition is not only interesting but also practical. The first step in this regard was marked by the work of Schmeidler [71]. Schmeidler proved that if a coalition blocks an allocation f via an allocation g , then there is an arbitrarily small-sized coalition that blocks f via g . Grodal [43] further generalized Schmeidler's theorem by restricting to coalitions that are formed by a finite union of an arbitrarily small set of agents with similar characteristics, which are presumably easier to form. Finally, Vind [80] in his work showed that if some coalition blocks a non-core allocation, then any coalition with a size smaller than the grand coalition can block the non-core allocation. Thus, for an economy with finitely many commodities, given the core equivalence, one can argue that the set of Walrasian equilibrium allocations cannot be blocked by any coalition of a given measure less than that of the grand coalition.

Hervés-Beloso and Moreno-García [46] provided another interpretation of the set of Walrasian equilibrium allocations in a pure exchange economy. Instead of exercising the veto power of arbitrarily many coalitions of different measures to obtain the core and hence the equilibrium allocations, Hervés-Beloso and Moreno-García considered the veto power of only the grand coalition exercised in a sequence of perturbed economies to characterize the set of Walrasian equilibrium allocations. These perturbed economies are obtained by a slight perturbation of the real endowments of the agents belonging to either arbitrarily small coalitions, arbitrarily large coalitions, or coalitions of a given measure less than that of the grand coalition. Precisely, Hervés-Beloso and Moreno-García proved that the set of Walrasian equilibrium allocations in a continuum economy with finitely many commodities are those that are non-dominated by the grand coalition

in any of these perturbed economies and referred to these allocations as *robustly efficient allocations*.

The notion of the core has often been criticized in the literature for highlighting the myopic nature of the agents. To deal with such short-sighted behavior Aumann and Maschler [6] introduced the concept of bargaining set to inject further credibility into the veto mechanism. They allowed for counter-objections to objections leading to ruling out frivolous objections. To further highlight the economic scenarios where the core falls short Aumann in his paper [10] provides an example of how forming atoms in continuum economies allows for non-preferable allocations for the atom in the core. Maschler [61] in his paper showed that the bargaining set is robust to such a setting and thus advantageous over the core. This formed a major motivation for studying the bargaining set in exchange economies and investigating its relation to the competitive equilibrium allocations. Mas-Colell [64] introduced the notion of bargaining set to exchange economies with a continuum of agents and established the equivalence between the set of competitive equilibrium allocations and the bargaining set. However, a parallel equivalence fails to hold in a replicated economy setting as pointed out by Anderson et al. [1]. They show that if one constructs a sequence of replicated economies as in Debreu and Scarf [31], the sequence of bargaining sets in those replicated economies fails to converge to the set of Walrasian equilibrium allocations of the original economy. Such a non-convergence mainly prevailed from the notion of counter-objection in replicated economies where agents of a certain type who were not part of the objecting coalition had to improve upon the initial allocation whereas some other agents of the same type who were part of the objecting coalition had to improve upon the objecting allocation. Later, Hervés-Estévez, and Moreno-García [49] refined the counter-objection process for replicated economies to allow agents of a certain type only to improve upon the objecting allocation in a counter-objection when agents of the same type were part of the objecting coalition. They showed that the bargaining sets in such a sequence of replicated economies converged to the set of Walrasian allocations, thereby establishing a parity to Mas-Colell [64]. It is important to observe that such equivalence is similar to that of Debreu and Scarf [31] extended to the two-step veto mechanism of the bargaining set. Liu and Zhang [60] provided a characterization of the competitive equilibrium allocations of a continuum economy with production in the spirit of Mas-Colell [64] and established that the bargaining set for such economies are equivalent to the sets of competitive allocation. One can refer to Hervés-Estévez and Moreno-García [50], Hervés-Beloso et al. [47], and Liu [59] for works on bargaining sets in finite economies with or without production and the case of finite coalitional production economy. They characterized the Walrasian equilibrium

allocations in terms of the Aubin bargaining set for such an economy.

However, the definition of bargaining set proposed by Mas-Colell [64] for atomless economies is not universally accepted and Vind [81] later proposed another notion of bargaining set and referred to it as the global bargaining set. The global bargaining set has the following appeals over its local counterpart: (i) To begin with, the global bargaining set is robust to the size of the blocking coalition. In the words of Schmeidler [71] to block an allocation by a large coalition requires communication among a large number of individuals and can be quite costly. On the other hand, one can argue as Vind [80] and restrict blocking to arbitrarily large-sized coalitions and thus support any allocation that survives such blocking as an outcome of a majority voting process. Thus, it is naturally attractive to restrict blocking to arbitrarily small or large-sized coalitions. The local bargaining set fails to be robust to the size of the blocking coalition. This further leads to the failure of the bargaining-equilibrium equivalence theorem of Mas-Colell [64] when coalition formation is restricted in the economy. (ii) Secondly, the global bargaining set is a more holistic treatment of the blocking mechanism in an economy. As pointed out by Hammond et al. [44] while a coalition S of agents blocks an allocation, it imposes an externality on the agents outside the coalition and forces them to recontract among themselves to obtain a new feasible allocation for them. Thus, in a counter-objection, one should propose an allocation that improves agents in $A \setminus S$ with respect to the new allocation and not the initial one. While the local objection and counter-objection mechanism remains silent on these issues, the global objection and counter-objection mechanism tackles them and hence is a more refined solution concept in this regard. (iii) Finally, certain properties can be studied only for a feasible allocation of an economy, and as in a global objection, the allocation proposed by the objecting coalition is not only feasible for the objecting coalition but also feasible for the entire set of agents it allows us to characterize the global bargaining set in terms of robustly efficient allocations.

Apart from the studies mentioned above on economies with purely private goods, a major part of economic analysis has been concerned with the study of social aspects of consumption in the form of public and club goods. In my thesis, I focus on club goods where clubs are voluntary groups or cliques that are formed by a group of individuals who seek to derive mutual benefit from sharing production costs, interacting with members of other characteristics, or sharing goods with certain excludable benefits. Club goods or as often referred to as locally provided public goods differ from pure public goods on the following basis: (i) Individual members voluntarily choose to be part of some clubs, whereas for pure public goods voluntarism may be absent. In the case of public goods avoidance costs can be high - for instance, one incurs harmful

damages by choosing to avoid drinking water facilities provided by the government. (ii) Secondly, club goods are mostly provided by taking into account the benefits and costs of members. Benefits arise from additional members in the form of reduced individual contributions, while costs arise from additional members in the form of crowding which therefore limits benefits and keeps club sizes finite. In comparison, the optimal sharing size for a purely public good consists of the entire population. (iii) Thirdly, as opposed to purely public goods since clubs have finite sizes there needs to be an exclusion mechanism in place that helps a club differentiate between members and non-members. Examples of such exclusion mechanisms can be membership fees, and toll booths for a certain stretch of the road among others, and (iv) Finally, in the case of club goods, one needs to determine the level of provision of the good for the members of the club whereas for public goods only the decision of provision matters and not the exact level. Early works on club goods can be traced to the works of Pigou [68] and Knight [58]. They were essentially solving for congestion on two alternative roads, where one was a narrow well-developed congested road and the other was a broad uncongested poor quality road. Their analysis was centered on the toll charges that would restrict users and hence determine “the membership size” for the developed road. The most pioneering ideas in club literature date back to the works of Buchanan [27], Tiebout [78], and Wiseman [83]. Buchanan [27] emphasized the need for formalizing clubs by highlighting the fact that in everyday life there are several public projects for which membership sizes are finite or in other words the extent of “publicness” is limited. So Buchanan’s work focused on determining the membership margin, so to speak, the size of the most desirable cost and consumption-sharing arrangement. Tiebout’s work [78] on the voting-with-the-feet hypothesis claimed that how the sizes of local jurisdictions could be determined by voluntary mobility (or membership) decisions of individual agents. Each jurisdiction in his work offered a fixed local public good different from the others. Tiebout showed that consumers with their perfect mobility partitioned themselves among the different jurisdictions according to their tastes for the local public goods. Wiseman [83] formulated the need for clubs owing to economies of scale. He suggested that for certain commodities like electricity, the price per unit drops as more and more units are produced. Thus, for such commodities coalescing individuals into large groups can lead to taking advantage of the economies of scale and thus provide it at a lower per unit cost.

In this thesis, we concentrate our attention on club economies and analyze the notion of equilibrium and optimality in these economies. Scotchmer and Wooders [75] developed a model for club economies with anonymous crowding. Anonymous crowding refers to the fact that individuals with different attributes or characteristics

can share the same facilities. They show that in an economy where all agent types differ, consumers with relevant similarity in tastes among efficient groups result in the similarity of demands for facility size and crowding and thus share the same facilities. However, with anonymous crowding, one cannot guarantee the existence of an optimal solution if the optimal club size is not an integer multiple of the size of the population. Scotchmer [74] introduced a notion of non-anonymous crowding in her work. However, with non-anonymous crowding, there is an added dimension of difficulty in forming groups containing the right composition of different types. Engl and Scotchmer [37] proposed a solution to this problem by allowing membership prices to be a linear function of attributes of the agents. The payoffs of agents in such an economy are captured through a characteristic function on the domain of attributes and hence are referred to as “hedonic” payoffs. They show that if payoffs are super-additive then the core and hedonic core are equivalent, signifying that if the attributes of two agents are close enough then so are their payoffs. Scotchmer [73] extended their work further to a setting where payoffs were not necessarily super-additive. Gilles and Scotchmer [39] made the first-ever attempt to study the decentralization of the core in a club economy framework by incorporating multiple private goods in the analysis to take care of complementarities between private and club goods in the sense that the consumer’s ranking of clubs depends on the consumption of private goods. However, the above analyses had two major limitations to them. (i) The backbone for such analysis was derived from individual optimizing decisions based on marginal benefits and marginal costs. Since the number of individuals in the economy was finite or restricted to the case of finitely many types of agents in an equal treatment economy, such an analysis was possible. Hence, entirely competitive settings as introduced by Aumann [7] would render the use of the above techniques useless, and (ii) the core “indivisible” nature of club goods was not investigated in much detail while defining equilibrium notions. Ellickson et al. [35] later proposed a unifying competitive setup with club goods which dealt with both the above-mentioned issues. They introduce an economy with a continuum of agents in line with Aumann [7] where club memberships are articles of choice.

We adopt the framework by Ellickson et al. [35]. Each individual in the economy is endowed with some external characteristics that are not only visible to other individuals in the economy but also cause externalities (positive or negative) to other members of the club he or she belongs to. Each club is identified through the composition of its members and the public project it endorses. However, unlike previous works, the trade of private goods is not restricted within the clubs. The main aspect of the framework is the parallel treatment of social interaction and consumption governed by a few

important subtleties:

- To begin with, clubs must be small relative to the economy to retain perfect competition in the model. Also, as clubs are infinitesimal, externalities arising from clubs are confined within them.
- Secondly, there are some key differences between private goods and club membership choices: (i) The space of private commodities captured by the n -dimensional Euclidean space satisfies monotonicity, like an assumption, which further aids in showing that prices for these commodities will be strictly positive. A similar inference cannot be drawn regarding the prices for club memberships. Prices for such memberships can be positive, negative, or as well as zero, and (ii) A second key and obvious difference lies in the fact that private goods are entirely divisible while membership choices are intrinsically indivisible.
- Thirdly, the feasibility of an allocation is quite different. In contrast to a typical exchange economy where the planned private goods consumption of a coalition must match aggregate endowments of the coalition, feasibility in our case requires the following: (i) the planned private goods consumption of the coalition along with the total inputs contributed to respective club projects must match aggregate endowments of the coalition; and (ii) membership choices must be consistent i.e. if a third of the population are women married to men, then a third of the population must be men married to be women.
- Lastly, the equilibrium condition in a club economy additionally requires budget balance for club types, i.e., the sum of memberships for a particular club type is just enough to pay for its inputs to club projects.¹

1.2 Research Questions

It is then evident from the historical background that the following research questions are quite evident and need to be resolved.

¹The subtleties discussed highlight that this framework closely resembles the framework of a matching market with a continuum of agents. One can observe that both models consider feasibility conditions that must hold “in the large” but are driven by individual preferences affected by what happens “in the small”. In our framework, each club (finite in size) is matched with a continuum of agents. Thus, there exists a continuum of clubs, each admitting a finite number of agents as members in equilibrium. This is closely related to the framework of matching in large economies. We elaborate on this more in our conclusion section.

Question 1.2.1. *Does the bargaining set of an atomless club economy coincide with the set of club equilibrium states?*

Question 1.2.2. *For an atomless club economy, is the local and global counter-objection mechanism of the economy robust to being blocked by arbitrarily small-sized coalitions?*

Question 1.2.3. *For an atomless club economy, is the global counter-objection mechanism of the economy robust to being blocked by arbitrarily large-sized coalitions?*

Question 1.2.4. *For an atomless club economy, can global justified objections be obtained by concentrating on blocking by only the grand coalition?*

Question 1.2.5. *For a mixed club economy $\tilde{\mathcal{E}}$, does the set of core states converge to the set of club equilibrium states?*

1.3 Thesis Contribution and Organization

The thesis contributes to the existing literature on club economies in a general equilibrium framework by attempting to answer the questions posited above. The answers constitute subsequent chapters of the thesis, which are organized as follows.

- **Chapter 2:** This chapter is dedicated to outlining the mathematical notions and methodologies that have been repeatedly employed in analyzing the research questions posed above. Ever since the formal models of Arrow and Debreu [3] and McKenzie [65] various mathematical tools have been employed to tackle various general equilibrium models. In this thesis, some notions and aspects related to Set Theory, Measure, Integration, Measurability, and Integration of correspondence have been extensively used in the analysis conducted in Chapters 4, 5 and 6.
- **Chapter 3:** This chapter is dedicated to presenting our economic model. We assume that there exists a measure space of agents containing non-negligible agents and finitely many private commodities in our setup. We then introduce the notion of clubs and its associated notions by defining club projects, the profile of a club, and the memberships associated with clubs. We introduce the set of feasible membership choices for agents in our economy and the composite consumption set consisting of private and club goods. We make certain necessary assumptions for our framework which helps us guarantee the results in the subsequent chapters. The key notion of consistency of club memberships is defined which elicits the condition for membership demand to be equal to membership supply. We

conclude the chapter by introducing various notions of club equilibrium in our framework and establish a relation in particular between the club equilibrium and club quasi-equilibrium states.

- **Chapter 4:** This chapter intends to introduce the bargaining set for our club economy. We proceed along the lines of Mas-Colell [64] to introduce the two-step veto mechanism of objection and counter-objection. Subsequently, we define the notion of bargaining set for our club economy. In this chapter, we answer Question 1.2.1 by showing that the bargaining set is equivalent to the set of club equilibrium states for an atomless club economy thereby extending the core-equivalence theorem of Ellickson et al. [35].
- **Chapter 5:** This chapter aims to investigate the robustness of a club economy's local and global counter-objection mechanism in terms of the size of the blocking coalition and whether global justified objections can be obtained by restricting blocking to only the grand coalition. For our first analysis, we answer Questions 1.2.2 and 1.2.3 and therefore establish direct extensions of the works of Schjødtt and Sloth [70] and Hervés-Estévez and Moreno-García [48]. Our results help us establish extensions of Schmeidler [71] and Vind's [80] theorem in atomless club economies. For our second check of robustness, we show that global justified objections can be characterized by approximately robustly efficient states. This in turn helps us answer Question 1.2.4 and extend the work of Hervés-Estévez and Moreno-García [48].
- **Chapter 6:** In this chapter we introduce the blocking mechanism for a mixed club economy. We aim to establish the equivalence between the core and the set of club equilibrium states in this chapter. In the process, we adopt the opposite approach of Greenberg and Shitovitz [41] and introduce an associate mixed club economy for our atomless economy. As a first step, we establish a correspondence between the core of the two economies under the assumption that a coalition exists containing negligible and non-negligible agents of the same type and for whom the average state is defined. We further establish a correspondence between the club equilibrium states of the two economies which finally aids us in establishing the central core equivalence theorem thereby answering Question 1.2.5 and extending the work of Shitovitz [77].

1.4 Bibliographic Notes

PhD research objectives aim to solve questions of interest in the relevant field of study and deliver the acquired knowledge to a larger audience. The studies conducted in the thesis were aimed at fulfilling these objectives by answering some challenging questions and presenting them at local and international conferences. The following papers are also listed as [[23], [24], [25], [26]] in the bibliography and have been written during Saha’s PhD study and this thesis is written based on the papers [[23], [24], [25]].

- (A) Bhowmik, Anuj, and Sandipan Saha. “Bargaining-equilibrium equivalence.” Accepted at the Journal of Mathematical Economics:
- (B) Bhowmik, Anuj, and Sandipan Saha. “Restricted Bargaining Sets in a Club Economy.” Accepted at the B.E. Journal of Theoretical Economics.
- (C) Bhowmik, Anuj, and Sandipan Saha. “Core Equivalence in a Mixed Club Economy.” Accepted at Social Choice and Welfare.
- (D) Bhowmik, Anuj, Sandipan Saha, and Soumi Tikader. “On Fairness and Size of Coalitions in Economies with Club Goods.” Received minor comments from the International Journal of Game Theory after second round of revision.

Saha attended the 17th Annual Conference on Economic Growth and Development which took place at the Indian Statistical Institute (Delhi), and Winter School, Delhi School of Economics in December 2022, and presented parts of the paper [C] and [D] at both the places. Saha attended the 22nd SAET (Annual Society for Advancement of Economic Theory) Conference at Panthéon Sorbonne, Paris, France, and presented parts of the paper [C] and [D]. Saha also attended the 18th Annual Conference on Economic Growth and Development which took place at the Indian Statistical Institute (Delhi), and Winter School, Delhi School of Economics in December 2023, and presented the paper [A]. Further, Saha attended the EWET (European Workshop in Economic Theory) 2024 at the University of Manchester, United Kingdom, and presented the paper [D].

Chapter 2

Mathematical Preliminaries

2.1 Set Theory

A *set* is a collection of objects, and objects constituting a set are called *elements* of the set. Typically, the letters X, Y, Z, \dots are used to denote sets and those representing elements in a set are represented by the letters x, y, z, \dots . The most common examples are the sets \mathbb{N}, \mathbb{Q} and \mathbb{R} of *positive integers*, *rational numbers* and *real numbers*, respectively. Additionally, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ is the set of non-negative real numbers and $\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$ captures the set of *extended real numbers*, where for any given $x \in \mathbb{R}$, ∞ and $-\infty$ can be thought of as $-\infty < x < \infty$. The symbol ∞ is termed as *infinity*. For any $a, b \in \mathbb{R}$, the notations

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \text{ and } (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

are referred to as *closed interval* and *open interval*, respectively. By abuse of standard terminology, \emptyset denotes the set containing no element and is referred to as the *null set* or *empty set*. A generic element x of the set X is indicated by $x \in X$ and analogously $x \notin X$ indicates that the element x is not a part of the set X . Given any two sets X and Y , an element $x \in X \setminus Y$ denotes that x belongs to the set X but not the set Y . If $x \in X$ implies that $x \in Y$, then we call X as a *subset* of Y and denote it by $X \subseteq Y$. If $Y \subseteq X$, then the *complement* of Y in X is denoted by $X \setminus Y$. We call two sets X and Y are said to be *identical* if $X \subseteq Y$ and $Y \subseteq X$ hold simultaneously. Further, $X \subset Y$ denotes the situation where “ $X \subseteq Y$ and $X \neq Y$ ”. The family of all subsets of X is called the *power set* of a set X and is denoted by $\mathcal{P}(X)$. For any collection $\{A_j : j \in J\} \subseteq \mathcal{P}(X)$, define

$$\bigcup_{j \in J} A_j = \{x \in X : x \in A_j \text{ for some } j \in J\},$$

and

$$\bigcap_{j \in J} A_j = \{x \in X : x \in A_j \text{ for all } j \in J\}.$$

are termed as the *union* and the *intersection* of the collection $\{A_j : j \in J\}$. The notations $\bigcup_{j \in J} A_j$ and $\bigcap_{j \in J} A_j$ are sometimes written as $\bigcup\{A_j : j \in J\}$ and $\bigcap\{A_j : j \in J\}$, respectively. The notation

$$\prod_{j \in J} A_j = \{(x_j : j \in J) : x_j \in A_j \text{ for all } j \in J\}.$$

is used to denote the *cartesian product* of the sets in the collection $\{A_j : j \in J\}$. In the special case of only two sets denoted by A and B , $A \cup B$, $A \cap B$, and $A \times B$ are used to denote the union, the intersection, and the Cartesian product of A and B , respectively. Given two non-empty sets if $A \cap B = \emptyset$ we call them to be *disjoint*. A collection $\{A_j : j \in J\}$ is called *pairwise disjoint* if A_i and A_j are disjoint for all $i, j \in J$ with $i \neq j$. The *symmetric difference* between any two non-empty sets A and B is defined by $A \Delta B = (A \setminus B) \cup (B \setminus A)$. A collection $\{A_j : j \in J\}$ of non-empty pairwise disjoint subsets of X is called a *partition* of the set X , if it satisfies $\bigcup_{j \in J} A_j = X$. A *relation* ' \mathcal{R} ' between two sets X and Y is a subset of $X \times Y$. If X and Y are identical then such a relation is called a *binary relation* on X and is denoted by ' \succeq '. A relation \succeq is termed as *reflexive* if $(x, x) \in \succeq$ for all $x \in X$. It is termed as *transitive* if if $(x, y) \in \succeq$ and $(y, z) \in \succeq$ imply $(x, z) \in \succeq$. It is *anti-symmetric* if $(x, y) \in \succeq$ and $(y, x) \in \succeq$ together imply $x = y$. Further, given a binary relation \succeq on a set X if for any $x, y \in X$, either $(x, y) \in \succeq$ or $(y, x) \in \succeq$ or both, we call such a binary relation is *complete*. The notation $(x, y) \in \succeq$ is also written as $x \succeq y$.

A *function* from a set X to Y is a relation that assigns to each element belonging to X exactly one element belonging to Y . Functions are usually labeled through lowercase letters such as f, g, h, \dots . A function $f : X \rightarrow \mathbb{R}$ is called a *real-valued* function. The set $\text{supp}(f) = \{x \in X : f(x) \neq 0\}$ is defined as the support of the real-valued function $f : X \rightarrow \mathbb{R}$. A real-valued function f is said to be *one-to-one* if $f(x) \neq f(y)$ for $x \neq y$, $x, y \in X$. A real-valued function f is *onto* if for each $y \in \mathbb{R}$, there exists at least one $x \in X$ such that $f(x) = y$. A function f is to be a *bijection* function if it is one-to-one and onto function. A *sequence* is a function $f : \mathbb{N} \rightarrow X$. We usually use the notation $\{x_n : n \geq 1\}$ to denote a generic sequence rather than the function $x : \mathbb{N} \rightarrow X$. A sequence $\{x_n : n \geq 1\}$ is *monotonically increasing* (resp. *monotonically decreasing*) if $x_{n+1} \geq x_n$ (resp. $x_n \geq x_{n+1}$) for all $n \geq 1$. A real-valued sequence $\{x_n : n \geq 1\}$ is *strictly increasing* (resp. *strictly decreasing*) if $x_{n+1} > x_n$ (resp. $x_n > x_{n+1}$) for all $n \geq 1$. A *subsequence* of a sequence $\{x_n : n \geq 1\}$ is a sequence $\{x_{n_k} : k \geq 1\}$, where

$\{n_k : k \geq 1\}$ is strictly increasing. For any $x, y \in X$, the notation $x > y$ means $x \geq y$ and $x \neq y$. Sometimes $x \geq y$ (resp. $x > y$) is written as $y \leq x$ (resp. $y < x$). An element y of a subset A of X is said to be *upper bound* (resp. a *lower bound*) of A if $y \geq x$ (resp. $x \geq y$) for all $x \in A$.

2.2 Metric Spaces

A metric space is a non-empty set X associated with a notion of *distance* or *metric* employed between any two points belonging to the set. Formally, a *metric* on a non-empty set X is a function $\varrho : X \times X \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- (i) $\varrho(x, x) = 0$ for all $x \in X$;
- (ii) $\varrho(x, y) = 0$ implies $x = y$;
- (iii) $\varrho(x, y) = \varrho(y, x)$ for all $x, y \in X$; and
- (iv) $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$ for all $x, y, z \in X$.

If ϱ is a metric associated with a non-empty set X , then (X, ϱ) denotes a *metric space*. A *pseudometric* on non-empty set X is a function $\varrho : X \times X \rightarrow \mathbb{R}_+$ for which conditions (i), (iii) and (iv) hold, and we denote by (X, ϱ) a *pseudometric space*. Note that the metric

$$\varrho(x, y) = \sqrt{\sum_{i=1}^{\ell} (x^i - y^i)^2}$$

along with \mathbb{R}^{ℓ} is a metric space and ϱ is referred to as the *Euclidean metric*. In particular, when $N = 1$, the notation $|x - y|$ denotes the *absolute difference* between x and y . For any $E \subseteq X$ of a metric space (X, ϱ) and $E \neq \emptyset$,

$$\text{diam}E = \sup\{\varrho(x, y) : x, y \in E\}.$$

denotes the *diameter* of the set E . A set E is said to be *bounded* if $\text{diam}E < \infty$, and is *unbounded* otherwise. The *open ball* of radius $\varepsilon > 0$ centered at a point $x \in X$ in a pseudometric space (X, ϱ) is denoted by

$$B_{\varepsilon}(x) = \{y \in X : \varrho(x, y) < \varepsilon\}.$$

Heine-Borel Theorem. A subset of a metric space (X, ϱ) is compact if and only if it is closed and bounded.

The $(\ell - 1)$ dimensional-simplex of \mathbb{R}^N is defined as

$$\Delta^\ell = \left\{ x = (x^1, \dots, x^\ell) \in \mathbb{R}^\ell : x^i \geq 0 \text{ for all } 1 \leq i \leq \ell \text{ and } \sum_{i=1}^{\ell} x^i = 1 \right\}.$$

Brouwer Fixed Point Theorem. For any continuous function $f : \Delta^\ell \rightarrow \Delta^\ell$ there is a point $x \in \Delta^\ell$ such that $f(x) = x$.

Throughout the rest of this subsection, we confine our attention to a general metric space (X, ρ) . One can recall that a sequence $\{x_n : n \geq 1\}$ converges to x in (X, ρ) if and only if $\{\rho(x_n, x) : n \geq 1\}$ converges to 0. The notation $\lim_{n \rightarrow \infty} x_n = x$ is employed to denote that the sequence $\{x_n : n \geq 1\}$ converges to x in (X, ρ) . A *Cauchy sequence* in (X, ρ) is a sequence $\{x_n : n \geq 1\}$ such that for each $\varepsilon > 0$ there is some $N \geq 1$ such that $\rho(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. We call a metric space (X, ρ) *complete* if every Cauchy sequence converges. We call a metric space compact if and only if every sequence has a convergent subsequence. If (Y, d) is a metric space, then the following conditions are equivalent:

- (i) A function $f : (X, \rho) \rightarrow (Y, d)$ is continuous.
- (ii) If $\{x_n : n \geq 1\}$ converges to x , then $\{f(x_n) : n \geq 1\}$ converges to $f(x)$.

Let (X, ρ) and (Y, d) be two metric spaces. A sequence of functions $\{f_n : n \geq 1\} : X \rightarrow Y$ converges *pointwise* to a function $f : X \rightarrow Y$ if $\{f_n(x) : n \geq 1\}$ converges to $f(x)$ for all $x \in X$. In such scenarios, we term f as the *pointwise limit* of the sequence $\{f_n : n \geq 1\}$. A sequence $\{f_n : n \geq 1\} : X \rightarrow Y$ converges *uniformly* to a function $f : X \rightarrow Y$ if for each $\varepsilon > 0$ there is some $N \geq 1$ such that $d(f_n(x), f(x)) < \varepsilon$ for all $x \in X$ and $n \geq N$. Let $\{A_n : n \geq 1\}$ be a sequence of subsets of \mathbb{R}^ℓ . A point $x \in \mathbb{R}^\ell$ is called a *limit point* of $\{A_n : n \geq 1\}$ if for $N \geq 1$ and points $x_n \in A_n$ for each $n \geq N$, the sequence $\{x_n : n \geq N\}$ converges to x . The set $\text{Li}A_n$ denotes the set of limit points of $\{A_n : n \geq 1\}$. Analogously, a point $x \in \mathbb{R}^\ell$ is called a *cluster point* of $\{A_n : n \geq 1\}$ if there exist positive integers $n_1 < n_2 < \dots$ and for each k there exists a point $x_k \in A_{n_k}$ such that the sequence $\{x_k : k \geq 1\}$ converges to x . The set of cluster points of $\{A_n : n \geq 1\}$ is denoted by $\text{Ls}A_n$. The sets $\text{Li}A_n$ and $\text{Ls}A_n$ are often referred to as the *limit infimum* and *limit supremum* of $\{A_n : n \geq 1\}$. It is clear that $\text{Li}A_n \subseteq \text{Ls}A_n$, and both $\text{Ls}A_n$ and $\text{Li}A_n$ are closed (possibly empty) sets. If $\text{Ls}A_n \subseteq \text{Li}A_n$ then $\text{Li}A_n = \text{Ls}A_n = A$ is called the *limit* of the sequence $\{A_n : n \geq 1\}$.

2.3 Measure Theory and Integration

Given a set S , a σ -algebra is a collection \mathcal{S} of subsets of S satisfying the following properties:

- (i) $S \in \mathcal{S}$;
- (ii) if $A \in \mathcal{S}$ then $S \setminus A \in \mathcal{S}$; and
- (iii) if $\{A_i : i \geq 1\} \subseteq \mathcal{S}$ then $\bigcup_{i \geq 1} A_i \in \mathcal{S}$.

The pair (S, \mathcal{S}) is called a *measurable space*, and any set belonging to \mathcal{S} is called a *measurable set*. Given a non-empty collection of σ -algebras, their intersection is also a σ -algebra. Any non-empty collection \mathcal{C} of subsets of S is contained in the σ -algebra $\mathcal{P}(S)$. The σ -algebra generated by \mathcal{C} is the smallest σ -algebra containing \mathcal{C} obtained from the intersection of all σ -algebras containing \mathcal{C} . In most cases, $\sigma(\mathcal{C})$ denotes the σ -algebra generated by \mathcal{C} . A *Borel σ -algebra* on the set X is the σ -algebra obtained from the collection of all open sets of X . Any set belonging to this σ -algebra is referred to as a *Borel measurable set* or simply a *Borel set*. We denote by $\mathcal{B}(X)$ the Borel σ -algebra of \mathbb{R}^ℓ . A function $\mu : \mathcal{S} \rightarrow \mathbb{R}^*$ is said to be a *signed charge* if it assumes a value of $-\infty$ or ∞ or neither. Note that $\mu(\emptyset) = 0$, and for any finite family $\{A_i : 1 \leq i \leq \ell\}$ of pairwise disjoint sets in \mathcal{S} , the equality

$$\mu \left(\bigcup_{i=1}^{\ell} A_i \right) = \sum_{i=1}^{\ell} \mu(A_i)$$

holds. A signed charge is a *signed measure* when for any countable collection of pairwise disjoint sets $\{A_i : i \geq 1\}$ in \mathcal{S} ,

$$\mu \left(\bigcup_{i \geq 1} A_i \right) = \sum_{i \geq 1} \mu(A_i).$$

holds. A *signed Borel charge* (resp. *measure*) is a signed charge (resp. measure) defined on the Borel σ -algebra of the set X . A signed measure is called a *measure* if it assumes only nonnegative values. In such a case, (S, \mathcal{S}, μ) is called a *measure space*. Let (X, ϱ) be a metric space. A measure μ on $\mathcal{B}(X)$ is called an *outer measure* if for all $A \in \mathcal{B}(X)$

$$\mu(A) = \inf\{\mu(U) : U \in \mathcal{T} \text{ and } A \subseteq U\}$$

holds, where \mathcal{T} denotes the collection of all open sets in X . For a given a measure space (S, \mathcal{S}, μ) , we say that the measure μ is *positive* if $\mu(A) > 0$ for at least one $A \in \mathcal{S}$,

and *complete* whenever $A \subseteq B \in \mathcal{S}$ and $\mu(B) = 0$ imply that A is measurable. In such instances, we refer to (S, \mathcal{S}, μ) as a *positive (complete) measure space*. We say μ and (S, \mathcal{S}, μ) to be finite if $\mu(S) < \infty$. We often refer to the expression μ -almost everywhere or μ -a.e. or almost all $s \in S$ to refer to an event that holds “everywhere except possibly for a set A with $\mu(A) = 0$ ”. A measurable set A is said to be an *atom* if for every measurable subset B of A either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$ holds. (S, \mathcal{S}, μ) is called an *atomless measure space* if \mathcal{S} is devoid of any atoms. Let (S, \mathcal{S}, μ) be a measure space. Define for $R \in \mathcal{S}$, $\mathcal{S}_R = \{A \in \mathcal{S} : A \subseteq R\}$ and $\mu_R : \mathcal{S}_R \rightarrow \mathbb{R}_+ \cup \{\infty\}$ by $\mu_R(A) = \mu(A)$. For a given collection $\{(S_i, \mathcal{S}_i) : 1 \leq i \leq \ell\}$ of measurable sets the *product σ -algebra* $\bigotimes_{i=1}^{\ell} \mathcal{S}_i$ is the σ -algebra defined on $\prod_{i=1}^{\ell} S_i$ and is generated by

$$\prod_{i=1}^{\ell} \mathcal{S}_i = \left\{ \prod_{i=1}^{\ell} A_i : A_i \in \mathcal{S}_i \text{ for all } 1 \leq i \leq \ell \right\}.$$

We denote it as, $\bigotimes_{i=1}^{\ell} \mathcal{S}_i = \sigma(\prod_{i=1}^{\ell} \mathcal{S}_i)$. For convenience, we denote the product σ -algebra of \mathcal{S}_1 and \mathcal{S}_2 by $\mathcal{S}_1 \otimes \mathcal{S}_2$. Further, given two measures μ_1 and μ_2 associated with \mathcal{S}_1 and \mathcal{S}_2 , the product measure μ defined by $(\mu_1 \times \mu_2)(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ for all $A_1 \in \mathcal{S}_1$ and $A_2 \in \mathcal{S}_2$.

Let (S, \mathcal{S}) and (R, \mathcal{R}) be two measurable spaces. Then a function $\varphi : S \rightarrow R$ is said to be $(\mathcal{S}, \mathcal{R})$ -measurable if $\varphi^{-1}(A) \in \mathcal{S}$ for all $A \in \mathcal{R}$. If one stands clear on \mathcal{S}, \mathcal{R} , then an $(\mathcal{S}, \mathcal{R})$ -measurable function is simply referred to as a *measurable function*. In particular, if we concentrate on a metric space Z , an $(\mathcal{S}, \mathcal{B}(Z))$ -measurable function is simply labeled as an \mathcal{S} -measurable function. If f, g are two $(\mathcal{S}, \mathcal{R})$ -measurable functions, then for $\alpha \in \mathbb{R}$ $f+g$ and αf are $(\mathcal{S}, \mathcal{R})$ -measurable. A well-established result in this domain states that the pointwise limit of a sequence of measurable functions from a measure space into a metric space is measurable. Let (S, \mathcal{S}) and (R, \mathcal{R}) be two measurable spaces. A function is said to be *jointly measurable* if it is $\mathcal{S} \otimes \mathcal{R}$ -measurable function. Define a function $\varphi(s, \cdot) : R \rightarrow X$ by $\varphi(s, \cdot)(r) = \varphi(s, r)$ for all $s \in S$. Similarly define $\varphi(\cdot, r)$ for all $r \in R$. Note that if φ is $\mathcal{S} \otimes \mathcal{R}$ -measurable then \mathcal{S} -measurability of $\varphi(\cdot, r)$ for all $r \in R$ and \mathcal{R} -measurability $\varphi(s, \cdot)$ for all $s \in S$ follows. A function $\psi : S \times X \rightarrow Z$, where Z is a metric space is said to be *Carathéodory* wherever for each $x \in X$, $\psi(\cdot, x)$ is \mathcal{S} -measurable and for each $s \in S$, $\psi(s, \cdot)$ is continuous. It is well established that a Carathéodory function $\psi : S \times X \rightarrow Z$ is $\mathcal{S} \otimes \mathcal{B}(X)$ -measurable i.e. jointly measurable. We assume for the rest of the section (X, ρ) to be a metric space and (S, \mathcal{S}, μ) be a finite measure space. A function $\varphi : S \rightarrow X$ is called a *simple function* if it assumes only a finite number of non-zero values and $A_i = \varphi^{-1}(x_i) \in \mathcal{S}$

for each $1 \leq i \leq n$. The standard expression of φ is given as $\varphi = \sum_{i=1}^n x_i \chi_{A_i}$ where,

$$\chi_{A_i}(t) = \begin{cases} 1, & \text{if } t \in A_i; \\ 0, & \text{otherwise,} \end{cases}$$

is called the *characteristic function* of A_i on S . The *integral* of φ is defined by

$$\int_S \varphi d\mu = \sum_{i=1}^n x_i \mu(A_i).$$

Further, a function $f : S \rightarrow X$ is said to be *strongly \mathcal{S} -measurable* if f is the pointwise limit of the sequence $\{\varphi_n : n \geq 1\}$ of simple functions for almost all $s \in \mathcal{S}$, i.e.

$$\lim_{n \rightarrow \infty} \|f(s) - \varphi_n(s)\| = 0.$$

The *equivalent class* of a strongly measurable function f , denoted by $[f]$, is the set of all strongly \mathcal{S} -measurable functions which are equal to f μ -almost everywhere. An important result states that if X is separable, a function $f : S \rightarrow X$ is strongly \mathcal{S} -measurable if and only if it is $(\mathcal{S}, \mathcal{B}(X))$ -measurable. A measurable function $f : S \rightarrow \mathbb{R}_+$ is referred to as *Lebesgue integrable* if

$$B = \sup \left\{ \int_S \varphi d\mu : 0 \leq \varphi \leq f \text{ and } \varphi \text{ is simple} \right\} < \infty$$

and then $\int_S f d\mu = B$ is called the *Lebesgue integral* of f over S . A function $f : S \rightarrow \mathbb{R}_+^\ell$ is said to be *Lebesgue integrable* if each $f_i : S \rightarrow \mathbb{R}_+$ is Lebesgue integrable for every $i = 1, 2, \dots, \ell$.

Remark 2.3.1. If (R, \mathcal{R}, ν) is a finite measure space and $f : (S, \mathcal{S}, \mu) \times (R, \mathcal{R}, \nu) \rightarrow X$ is an $\mathcal{S} \otimes \mathcal{R}$ -measurable function, then $\int_S f(\cdot, \cdot) d\mu$ is \mathcal{R} -measurable.

If (S, \mathcal{S}, μ) is atomless and $f : S \rightarrow \mathbb{R}_+^\ell$ is Lebesgue integrable, then $\{\int_E f d\mu : E \in \mathcal{S}\}$ is convex. This result is known in the literature as Lyapunov's convexity theorem.

Proposition 2.3.2. *If (S, \mathcal{S}, μ) is a measure space and $f : S \rightarrow \mathbb{R}_+^\ell$ is Lebesgue integrable, then for each $E \in \mathcal{S}$ with $\mu(E) > 0$ one has*

$$\frac{1}{\mu(E)} \int_E f d\mu \in \overline{\text{co}}\{f(E)\},$$

where $\overline{\text{co}}(A)$ of a set A denotes the closed convex hull of the set A , which means it is the intersection of all the closed convex sets containing the set A .

2.4 Correspondences

A *correspondence* F from X to Y is defined as associating to each $x \in X$ a subset $F(x)$ of Y and is denoted by $F : X \rightrightarrows Y$. We define Gr_F as

$$\text{Gr}_F = \{(x, y) \in X \times Y : y \in F(x), x \in X\}.$$

and refer to it as the *graph* of the correspondence. The graph of the correspondence F helps us identify a relation between the elements of X and Y . The set $F(x)$ is called the *image of F at x* . The set $\text{Dom}(F) = \{x \in X : F(x) \neq \emptyset\}$ denotes the *domain* of F . F is a *non-empty valued* correspondence if $\text{Dom}(F) = X$. For a given $U \subseteq Y$, define

$$F^-(U) = \{x \in X : F(x) \cap U \neq \emptyset\} \text{ and } F^+(U) = \{x \in X : F(x) \subseteq U\}.$$

as the *lower and upper inverses* of U by F . If Z is a set and $G : Y \rightrightarrows Z$ is a correspondence then the *composition correspondence* $G \circ F : X \rightrightarrows Z$ is defined as

$$(G \circ F)(x) = \bigcup \{G(y) : y \in F(x)\}.$$

Recall that $\mathcal{P}(Y)$ denotes the power set of Y . Then given $\mathcal{P}(Y)$, one can write a correspondence F as a function $F : X \rightarrow \mathcal{P}(Y)$. A correspondence $F : S \rightrightarrows X$ is *compact-valued* (resp. *closed-valued*, *convex-valued*) if $F(s)$ is compact (resp. closed, convex) for all $s \in S$. Given a correspondence $F : S \rightrightarrows X$, we define the closure correspondence as $(\text{cl}F) : S \rightrightarrows X$ by $(\text{cl}F)(x) = \text{cl}F(x)$.

2.4.1 Continuity of a Correspondence

A correspondence $F : X \rightrightarrows Y$ is said to be *closed at x* if given a sequence $\{x_n\}_{n \geq 1} \in X$ converging to x , there exists a sequence $\{y_n\}_{n \geq 1} \in Y$ converging to y such that if $y_n \in F(x_n)$ for every n then $y \in F(x)$. It is *closed* (has *closed graph*) if it is closed at every point of X . A correspondence $F : X \rightrightarrows Y$ is said to be *upper hemi-continuous at x* (resp. *lower hemi-continuous at x*) if for any open $V \subseteq Y$ with

$$F(x) \subseteq V \text{ (resp. } F(x) \cap V \neq \emptyset)$$

there is an open set U of X containing x such that

$$F(x') \subseteq V \text{ (resp. } F(x') \cap V \neq \emptyset)$$

for all $x' \in U$. A correspondence $F : X \rightrightarrows Y$ is called *upper hemi-continuous* (resp. *lower hemi-continuous*) if it is upper hemi-continuous at every $x \in X$ (resp. lower

hemi-continuous at every $x \in X$). A correspondence is *continuous* if it is both upper and lower hemi-continuous.

Kakutani's fixed point theorem: Let X be a nonempty compact convex subset of \mathbb{R}^ℓ and $F : X \rightrightarrows X$ be a non-empty and convex-valued correspondence. If F has a closed graph (or is closed-valued and upper hemi-continuous), then a fixed point exists of F .

2.4.2 Measurability of a Correspondence

A correspondence $F : (S, \mathcal{S}, \mu) \rightrightarrows X$ is *lower \mathcal{S} -measurable* if for every open subset U of X , $F^-(U) \in \mathcal{S}$. If one is clear on the underlying measure, then we just term f as *lower measurable*. Note that $F : (S, \mathcal{S}, \mu) \rightrightarrows X$ is lower measurable if (S, \mathcal{S}, μ) is complete, X is separable, and $\text{Gr}_F \in \mathcal{S} \otimes \mathcal{B}(X)$. A well-established fact is that F is lower measurable if and only if $\text{cl}F$ is lower measurable. We call a compact-valued correspondence $F : S \rightrightarrows \mathbb{R}^\ell$ to be lower measurable if and only if $F^-(C) \in \mathcal{S}$ for all closed subset C of X . Given $\{F_n : n \geq 1\} : S \rightrightarrows \mathbb{R}^\ell$, a sequence of correspondences, define $\bigcap_{n \geq 1} F_n : S \rightrightarrows \mathbb{R}^\ell$ by

$$\left(\bigcap_{n \geq 1} F_n \right) (s) = \bigcap_{n \geq 1} F_n(s).$$

Theorem 4.1 of Himmelberg [53] asserts that if F_n is lower measurable, closed-valued for each $n \geq 1$ and there exists at least one compact-valued F_n then $\bigcap_{n \geq 1} F_n$ is lower measurable. A *measurable selection* of a correspondence $F : S \rightrightarrows X$ is a $(\mathcal{S}, \mathcal{B}(X))$ -measurable function $f : S \rightarrow X$ such that $f(s) \in F(s)$ for almost all $s \in S$.

Kuratowski-Ryll-Nardzewski Measurable Selection Theorem. If $F : S \rightrightarrows \mathbb{R}^\ell$ is non-empty closed-valued and lower measurable, then there is a measurable function $f : S \rightarrow \mathbb{R}^\ell$ such that $f(s) \in F(s)$ for all $s \in S$.

Recall that an *integrable selection* of $F : S \rightrightarrows X$ is a measurable selection which is Lebesgue integrable as well. The integration of F in the sense of Aumann in [8] is defined by

$$\int_S F d\mu = \left\{ \int_S f d\mu : f \text{ is an integrable selection of } F \right\}.$$

If (S, \mathcal{S}, μ) is atomless, then $\int_S F d\mu$ is convex, see Yannelis [84] and

$$\text{cl} \left(\bigcup \left\{ \int_R F d\mu : R \in \mathcal{S}, \mu(R) > 0 \right\} \right)$$

is convex, refer to Evren and Husseinov [38].

Recall that $F : S \rightrightarrows \mathbb{R}_+^\ell$ is called *integrably bounded* if there is a function $\varphi : S \rightarrow \mathbb{R}_+$ such that $\|x\| \leq \varphi(s)$ for all $x \in F(s)$ and $s \in S$. If F is closed-valued and integrably bounded, then $\int_S F d\mu$ is compact, refer to Hildenband [54, p.73].

Chapter 3

The Economic Model

In this chapter, we outline the basic framework of a club economy. The model, as emphasized in the introduction, is adapted from the work of Ellickson et al. [35]. The model highlights the interaction between club goods and private goods in a pure exchange setting. We begin with two examples from Ellickson et al. [35] that help us paint the idea of a simple club economy before we go into the formal notions.

3.1 Examples

The two examples that we talk about here elicit certain key features of clubs that follow from the central works of Buchanan [27] and Arrow [5]. The first example that we talk about follows primarily from the work of Buchanan [27] and emphasizes three features : (i) although clubs are insignificant in comparison to the overall economy, studying the optimal sizes of the club is necessary, (ii) clubs are based on constant returns to scale technology, i.e. formation of two clubs of the same type require twice the number of members and twice of the input to form the clubs, and (iii) finite club sizes are vital in guaranteeing the existence of an equilibrium.

Example 3.1.1. Consider an economy with a continuum of agents uniformly distributed over $[0, 10]$. There is only one private good in the economy, and individual k is endowed with $e_k = k$ units of the private good. In addition, individuals have the choice to build and enjoy a swimming pool either by themselves or as a member of a club. One requires 6 units of private goods as input to construct a pool. The utility derived by a consumer from consuming x units of the private good is denoted by $u(x, 0) = x$ if no pool facility is enjoyed and $u(x, n) = \frac{4x}{n}$ if an individual shares a pool with $n - 1$ other members. No individual becomes a member of more than one club.

We normalize the price of private goods to 1, so consumer k has wealth k . Note that the preferences are that only the size of the club matters and not the composition. Hence, the membership cost for a pool of size n is the total inputs required for the pool split equally among the members, i.e., $q_n = \frac{6}{n}$. So if a consumer chooses no pool, then he or she achieves a utility of k ; while sharing a pool with n members yields a utility of

$$u(k - q_n; n) = \frac{4}{n} \left(k - \frac{6}{n} \right).$$

In equilibrium there is a stratification of the population based on wealth: the wealthiest consumers with income belonging in $(9, 10]$ have their own pool; consumers with income belonging in $(6, 9]$ build and use a pool with one other consumer (i.e. they are a member of a club of size 2) and consumers with the lowest income in $(0, 6]$ refrain from using a pool and consume only the private good. Hence, clubs with more than 2 members don't form in equilibrium.

The second example that we talk about highlights the need for price discrimination, i.e. agents with different characteristics must be charged different prices for their memberships. This observation is in tandem with Arrow's [5] conjecture that in the absence of price discrimination, profit maximization leads to market segmentation on one hand but also allows for integration with price discrimination.

Example 3.1.2. Consider an economy with a continuum of consumers uniformly distributed on $[0, 1]$. We identify consumers in the interval $[0, .3)$ as blue and those in the interval $[\cdot 3, 1]$ as green. Each consumer is endowed with 2 units of the private good in the economy. In addition to private goods, consumers have the opportunity to construct and use a duplex apartment. Building a duplex requires 2 units of the private good. A consumption of x units of the private good and no housing yields a utility of

$$u_B(x, 0) = u_G(x, 0) = x$$

to individuals of both characteristics (blue and green). On the other hand, a consumption of x units of private goods and a duplex apartment results in the following utilities

$$u_B(x, BB) = 4x; \quad u_B(x, BG) = 6x,$$

and

$$u_G(x, GG) = 6x; \quad u_G(x, BG) = 4x.$$

It can be noted that the utility for an individual depends not only on his or her characteristics but also on the characteristics of the other occupant of the duplex.

Here club is a duplex, and each club size has to be two. Let us denote by $q_\omega(BB)$, $q_\omega(BG)$ and $q_\omega(GG)$ the membership prices paid by a consumer of type $\omega = B, G$. At equilibrium, since clubs earn zero profit, housing prices for each of these duplexes must equal the production cost of 2; in particular, $q_B(BB) = q_G(GG) = 1$. At these prices, blue consumers earn a utility of 2 by consuming only the private good and no housing facility, a utility of 4 from choosing to stay in a segregated duplex and consuming the private good, and a utility of $6(2 - q_B(BG))$ by choosing to stay in an integrated duplex. The green individuals earn a utility of 2 by consuming only the private good and no housing facility, a utility of 6 from choosing to stay in a segregated duplex and consuming the private good, and a utility of $4(2 - q_G(BG))$ by choosing to stay in an integrated duplex. For any integrated duplex to form in equilibrium, the membership prices must satisfy $6(2 - q_B(BG)) \geq 4$ for the blue individuals, while membership prices must satisfy $4(2 - q_G(BG)) \geq 6$ for the green individuals. Equivalently, this implies that $q_B(BG) \leq \frac{4}{3}$ and $q_G(BG) \leq \frac{1}{2}$. Thus, $q_B(BG) + q_G(BG) \leq \frac{11}{6}$. This contradicts the fact that housing prices sum to the cost of production. Therefore, in equilibrium, all individuals live in segregated housing. However, segregation is not a necessary conclusion (see Ellickson et al. [35]).

3.2 Agents and Private Commodities

We assume that the space of agents for our economy is a complete, finite, positive measure space. We denote it by (A, Σ, λ) with A being the set of agents and Σ as the corresponding σ -algebra and λ denotes a nonatomic measure on Σ . The measure space is an atomless one, or in other words, comprises agents whose measure is negligible or whose influence on the market is negligible. Let N denote the set of distinct many private commodities in the market. We assume that the private commodity space is denoted by the N -dimensional Euclidean space \mathbb{R}^N . Thus, private goods in our setup are perfectly divisible.¹ The consumption set of these commodities is restricted to the non-negative orthant \mathbb{R}_+^N for each agent. Furthermore, let \mathbb{R}_{++}^N denote the strictly positive elements of \mathbb{R}^N . Under standard notations for any two bundles $x, y \in \mathbb{R}_+^N$, $x \geq y$ implies $x_i \geq y_i$ for all $i \in N$; $x > y$ implies that $x \geq y$, however $x \neq y$; and $x \gg y$ implies that $x_i > y_i$ for each $i \in N$. We denote $\|x\|_1 := \sum_{n=1}^N x_n$.

¹Without loss of generality, we assume that N also denotes the cardinality for the set of commodities.

3.3 Clubs

Each agent in the economy is identified with some external characteristics. Examples of such characteristics can be sex, religion, appearance, etc. These characteristics are important to identify individuals as members of a club, as such characteristics not only influence subjective decisions but also create externalities for other agents in the economy. We define Ω as the set of such finite external characteristics. An element $\omega \in \Omega$ denotes the characteristic of an individual agent relevant to other members. Each club can be characterized by the composition of its members, where the composition is defined as the number of individuals for each characteristic. For that we define, a map $\pi : \Omega \rightarrow \mathbb{Z}_+$, \mathbb{Z}_+ being the set of non-negative integers. We identify the composition of members of a club (concerning their characteristics) with such a map and term it as **profile** of a club. Thus, given any $\omega \in \Omega$, the number of members having characteristic ω is given by $\pi(\omega)$. Therefore, given π of a club, the total number of members is $\|\pi\|_1 := \sum_{\omega \in \Omega} \pi(\omega)$.

Club goods in the literature have often been referred to as locally provided public goods. On that note, each club in our framework engages itself in a public project (local to the club) which is referred to as an activity. These projects or activities are not traded and are ranked subjectively amongst members of the club. Thus, the absence of a commonly defined linear order makes the set of public projects in our framework an abstract one as defined by Mas-Colell [63]. The lack of a linear structure (like a Euclidean scheme) seems to prompt scenarios where there does not exist any commonly defined order. The motivation for such absence stems from the fact that individuals have different perceptions, usage, and thus a subjective ranking over the set of club goods. We denote the set of such activities by Γ . Activities are not traded and ranking amongst them may be influenced by private goods consumption. We denote a **club type** by (π, γ) , where π denotes the profile of the club and $\gamma \in \Gamma$ denotes the club good. In our economy, there exist only finitely many club types, denoted by $\mathcal{Clubs} := \{(\pi_1, \gamma_1), (\pi_2, \gamma_2), \dots, (\pi_m, \gamma_m)\}$, where m is a finite integer. Now, club projects are to be funded and operated by members of the clubs only. In the absence of the notion of money in our model, inputs to club projects are made via contributions through private goods. Thus, the requirement of inputs for a club type, denoted by **inp** (π, γ) , is a vector of \mathbb{R}_+^N .

Access to clubs is guaranteed through subscription to memberships of clubs. An agent of external characteristic $\omega \in \Omega$ can become a member of club type (π, γ) if and only if the particular club type admits individuals of characteristic ω as a non-zero fraction in the club composition, i.e., $\pi(\omega) \geq 1$ in absolute terms. We define club memberships

to be a combination consisting of an external characteristic and a particular club type. A **club membership** is thus a triplet $m = (\omega, \pi, \gamma)$, and let \mathcal{M} denote the set of all possible club memberships from which an individual may choose while optimizing. Agents are allowed to purchase memberships to multiple clubs and are even allowed to purchase multiple units of a single membership. We define a map $\ell : \mathcal{M} \rightarrow \{0, 1, 2, \dots\}$, where $\ell(\omega, \pi, \gamma)$ is the number of memberships of the type (ω, π, γ) being bought. We term the above-defined map as *list*. The following notation denotes the set of all such possible *list*:

$$Lists = \{\ell : \ell \text{ is a list}\}.$$

Letting $\mathbb{R}^{\mathcal{M}}$ denote the set of all mappings from the set \mathcal{M} to the real line, we can frequently view *Lists* as a subset of $\mathbb{R}^{\mathcal{M}}$. Throughout the rest of the paper, we also assume that the total number of memberships an individual may choose is upper bounded by M and let $Lists_M = \{\ell : \ell \text{ is a list and } \|\ell\|_1 \leq M\}$ denote such a set.

In Example 3.1.2, ‘blue’ and ‘green’ denote the characteristics of the consumers, and there exist three different types. In each type, the activity is to construct and use a duplex apartment. Possible club profiles are ‘BB’, ‘BG’, and ‘GG’. Consistent with the above notational form, this club profile ‘BB’ can be expressed as $\pi(B) = 2$ and $\pi(G) = 0$. Likewise, other profiles can be expressed in terms of π . Here the club project or activity was the “duplex”. Thus, the possible club types in the case of Example 3.1.2 are given by $\{(BB, duplex), (BG, duplex), (GG, duplex)\}$. The set of possible memberships is $\{(B, BB, duplex), (B, BG, duplex), (G, BG, duplex), (G, GG, duplex)\}$.

3.4 Club Economy

In a club economy, each agent is identified by his or her external characteristic $\omega \in \Omega$. Unlike standard private goods economies, in our framework, consumption is a combination of both private and club goods. Let $a \in A$ denote a generic agent for whom we capture the set of all such feasible consumption bundles by the set X_a . Thus $X_a \subseteq \mathbb{R}^N \times Lists$. Since club goods possess a nature of excludability in them, it is natural to assume that each agent may be barred from access to certain club types (for example, a certain swimming pool may be restricted to use only by females). We denote the set of feasible club memberships for agent a as $Lists_a \subseteq Lists_M$. Thus, for every $a \in A$ we assume $X_a = \mathbb{R}_+^N \times Lists_a$. The initial endowment distribution in the economy is given by $\{e_a\}_{a \in A}$. The map $u_a : X_a \rightarrow \mathbb{R}$ denotes the utility function of agent $a \in A$. We refer to e_a as the endowment of agent a . Endowments are said to be **desirable** if $u_a(e_a, 0) > u_a(0, l_a)$ for every agent $a \in A$ and $l_a \in Lists_a$. In simple

words, an agent will prefer to consume his or her initial endowment than consume solely club memberships.

Definition 3.4.1. A **club economy** \mathcal{E} is a mapping $a \mapsto (\omega_a, X_a, e_a, u_a)$, satisfying the following conditions:

- (A.1) The mapping $a \mapsto \omega_a$ is measurable;
- (A.2) The correspondence $a \rightrightarrows X_a$ is measurable;
- (A.3) The endowment mapping $a \mapsto e_a$ is integrable with $e_a \in \mathbb{R}_{++}^N$ for all $a \in A$; and
- (A.4) The mapping $(a, x, l) \mapsto u_a(x, l)$ is jointly measurable with $u_a(\cdot, l)$ is continuous and strongly monotonic for every $a \in A$ and $l \in Lists_a$.

3.5 States, Allocation, and Consistency

One can note that as consumption in our framework consists of both private goods and club memberships, we need to define conditions that ensure club formation against given membership demands. On that note, we introduce the notions of consistency, input contribution, and material balance in this section which further helps us to define feasible states in our economy. So, to begin with we define the notion of a state in our economy.

Definition 3.5.1. A **state** of \mathcal{E} is basically an integrable mapping $(f, l) : A \rightarrow \mathbb{R}_+^N \times \mathbb{R}_+^{\mathcal{M}}$, which specifies for any agent $a \in A$ the amount of private good consumption f_a and the club membership vector l_a .

A state is said to be **individually feasible** if $(f_a, l_a) \in X_a$ λ -a.e. on A . Club membership choices made by agents are intrinsically indivisible in nature. Equilibrium for club goods, therefore, requires the existence of the desired number of club types to match the demand of agents. To this end, we define a consistency condition in our economy. Before that, we define a **coalition** as a positive measurable subset B of A . Furthermore, a set $B' \subseteq B$ is called a **sub-coalition** of the coalition B . We proceed along the lines of (working paper) Ellickson et al. [34] to illustrate the notion of consistency. For any coalition B , a choice function $\mu : B \rightarrow Lists$ and $j \in \mathbb{N}$, let

$$E_\mu^j(\omega, \pi, \gamma) = \{a \in B : \mu_a(\omega, \pi, \gamma) = j\}.$$

Notice that $E_\mu^j(\omega, \pi, \gamma)$ denotes the set of agents in the coalition B who choose j -many memberships of the type $(\omega, \pi, \gamma) \in \mathcal{M}$. Therefore $\lambda(E_\mu^j(\omega, \pi, \gamma))$ is the proportion

of agents choosing j -many memberships of the type (ω, π, γ) . Thus, one can interpret $j\lambda(E_\mu^j(\omega, \pi, \gamma))$ to be the proportion of memberships of the type (ω, π, γ) , chosen by agents in $E_\mu^j(\omega, \pi, \gamma)$. Hence the sum

$$\sum_{j=1}^M j\lambda(E_\mu^j(\omega, \pi, \gamma))$$

denotes the memberships demanded of the type (ω, π, γ) by agents in the coalition B . We say $\mu : B \rightarrow Lists$ is *consistent* if

$$\frac{\sum_{j=1}^M j\lambda(E_\mu^j(\omega, \pi, \gamma))}{\sum_{j=1}^M j\lambda(E_\mu^j(\omega', \pi, \gamma))} = \frac{\pi(\omega)}{\pi(\omega')}$$

for each $(\pi, \gamma) \in Clubs$ and each $\omega, \omega' \in \Omega$. Equivalently, we say that $\mu : B \rightarrow Lists$ is *consistent* for the coalition B if for each $(\pi, \gamma) \in Clubs$ there exists an element $\alpha(\pi, \gamma) \in \mathbb{R}$ such that

$$\sum_{j=1}^M j\lambda(E_\mu^j(\omega, \pi, \gamma)) = \alpha(\pi, \gamma)\pi(\omega)$$

for every $\omega \in \Omega$. Thus, consistency implies that the distribution of membership choices in the population is the same as the distribution of memberships in the club type (π, γ) for every club type. We provide the formal definition of consistency below.

Definition 3.5.2. Given a (membership) vector $\bar{\mu} \in \mathbb{R}^{\mathcal{M}}$,² if for each club type $(\pi, \gamma) \in Clubs$, there exists a number $\alpha(\pi, \gamma) \in \mathbb{R}$ such that

$$\bar{\mu}(\omega, \pi, \gamma) = \alpha(\pi, \gamma)\pi(\omega)$$

for all $\omega \in \Omega$, then we call such a membership vector $\bar{\mu}$ **consistent**.

In the above definition, the number $\alpha(\pi, \gamma)$ may be interpreted as the *relative number* of clubs of type (π, γ) demanded for in $\bar{\mu}$. Thus, intuitively a consistent membership vector is one where the average membership for each club type by agents in a coalition can be satisfied by replicating the club a certain proportion ($\alpha(\pi, \gamma)$) of times in the economy. Define

²We provide the definition for a generic vector $\bar{\mu} \in \mathbb{R}^{\mathcal{M}}$. Note that an actual membership vector $\bar{\mu}$ is always a member of $\mathbb{R}_+^{\mathcal{M}}$. This follows from the definition of “list”, which specifies that each membership is purchased in (non-negative) integer quantities by every individual.

$$\mathcal{C}ons := \{ \bar{\mu} \in \mathbb{R}^{\mathcal{M}} : \bar{\mu} \text{ is consistent} \}.$$

Note that $\mathcal{C}ons$ is a vector subspace of $\mathbb{R}^{\mathcal{M}}$.³ Next, we define when a choice of club membership is consistent for a coalition of agents.

Definition 3.5.3. A choice function $\mu : B \rightarrow Lists$ is **consistent** for the coalition B if the corresponding average membership vector $\bar{\mu}_B = \int_B \mu_a d\lambda \in \mathcal{C}ons$.

Note that, as discussed above, the coordinate $\bar{\mu}_B(\omega, \pi, \gamma) = \sum_{j=1}^{\infty} j\lambda(E_{\mu}^j(\omega, \pi, \gamma))$ of $\bar{\mu}$ indicates the number of memberships (average) chosen by the members in B with external characteristics ω for club type (π, γ) . Since individuals make membership choices in nonnegative (integer) quantities, an aggregate membership vector $\bar{\mu}_B$ is always a member of the positive orthant, $\mathcal{C}ons_+ \subset \mathcal{C}ons$.

We next define feasibility for our club economy. We have already defined the feasibility condition for club memberships through the “consistency” condition. Private goods in our framework also need clearance. Juxtaposed to the traditional general equilibrium model, private goods in our setup are also used as inputs to club projects in parallel to consumption. We capture *social feasibility* for any state of the economy through the **material balance** condition. For the club type (π, γ) , we divide the inputs to the club in equal proportions among the members of the club. As in Ellickson et al. [35] for an agent $a \in A$ with membership choice l_a , let $\tau(l_a)$ denote his share for club project where

$$\tau(l_a) := \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \frac{1}{\|\pi\|_1} \text{inp}(\pi, \gamma) l_a(\omega, \pi, \gamma).$$

Definition 3.5.4. A state (f, l) is **feasible for a coalition B** if it abides by the following conditions:

- **Individual Feasibility:** $(f_a, l_a) \in X_a$ λ -a.e. on B ;
- **Material Balance:** $\int_B f_a d\lambda + \int_B \tau(l_a) d\lambda = \int_B e_a d\lambda$; and
- **Consistency:** $\int_B l_a d\lambda \in \mathcal{C}ons$.

For $B = A$ then we call it **feasible**.

³It is easy to show that the zero membership vector belongs to $\mathcal{C}ons$ as one can choose $\alpha(\pi, \gamma) = 0$ for each $(\pi, \gamma) \in Clubs$. Further, if $\bar{\mu}, \bar{\nu} \in \mathcal{C}ons$ then we have for each $(\pi, \gamma) \in Clubs$, $\bar{\mu}(\omega, \pi, \gamma) = \alpha(\pi, \gamma)\pi(\omega)$ and $\bar{\nu}(\omega, \pi, \gamma) = \beta(\pi, \gamma)\pi(\omega)$ for each $\omega \in \Omega$. Then for any $c \in \mathbb{R}$ it follows that for each $(\pi, \gamma) \in Clubs$, $c\bar{\mu} = c\alpha(\pi, \gamma)\pi(\omega)$ for each $\omega \in \Omega$ and hence $c\bar{\mu} + \bar{\nu} \in \mathcal{C}ons$ as $c\alpha(\pi, \gamma) + \beta(\pi, \gamma) \in \mathbb{R}$.

Remark 3.5.5. Given a club type $(\pi, \gamma) \in \mathcal{Clubs}$, $\pi(\omega)$ denotes the number of members a particular club of this type can accommodate, and if the membership vector $\bar{l}_B = \int_B l_a d\lambda$ is consistent, there exists a non-negative real number $\alpha(\pi, \gamma)$ that tells us how many of such clubs will be formed. In other words, $\alpha(\pi, \gamma)$ can be interpreted as how many times the club type (π, γ) needs to be replicated or formed to satisfy the demand $l_a(\omega, \pi, \gamma)$ for each $\omega \in \Omega$ and all $a \in B$. In reference to Example 3.1.1, one can observe that each wealthy consumer in $(9, 10]$ demands a swimming pool of their own, and thus one club (i.e., swimming pools of capacity one) is formed for each individual in $(9, 10]$ in equilibrium. In general, for any feasible state (f, l) , a club type (π, γ) can occur infinitely many times irrespective of whether the state (f, l) is an equilibrium or not.

3.6 Equilibrium and Optimality

In this subsection, we shall lay out some definitions. We first outline the definitions of the solution concepts for the market mechanism and then follow it up with the solution concepts from the cooperative mechanism. For the cooperative mechanism, we resort to both strong and weak notions of some concepts.

Definition 3.6.1. A **club equilibrium** of \mathcal{E} consists of a feasible state (f, l) and a price vector $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$, $p \neq 0$, such that:

- **Budget Feasibility** : For λ -a.e. on A , we have $(p, q) \cdot (f_a, l_a) = p \cdot f_a + q \cdot l_a \leq p \cdot e_a$;
- **Optimization**: For λ -a.e. on A , we have $(g_a, \mu_a) \in X_a$ and $u_a(g_a, \mu_a) > u_a(f_a, l_a)$ together imply $p \cdot g_a + q \cdot \mu_a > p \cdot e_a$.
- **Budget Balance for Club types** : For each $(\pi, \gamma) \in \mathcal{Clubs}$,

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \text{inp}(\pi, \gamma)$$

Let $\mathcal{W}(\mathcal{E})$ denote the set of club equilibrium states of the economy \mathcal{E} . A **club quasi-equilibrium** of \mathcal{E} also consists of a feasible state (f, l) and a price vector $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$, $p \neq 0$, satisfying the first and third conditions in the definition of a club equilibrium, but instead of the second condition it satisfies:

- **Quasi-optimization**: For λ -a.e. on A , we have $(g_a, \mu_a) \in X_a$ and $u_a(g_a, \mu_a) > u_a(f_a, l_a)$ together imply $p \cdot g_a + q \cdot \mu_a \geq p \cdot e_a$.

Now, for both club equilibrium and quasi-equilibrium we require that the sum of membership prices for a club must add up to what is required to pay for inputs of the club project. As stated in Ellickson et al. [35], an equivalent notion is when members instead of paying membership prices to the club, directly bear the cost of inputs among themselves. We call such membership prices as **pure transfers** and are defined as

$$\mathcal{T}rans = \{q \in \mathbb{R}^{\mathcal{M}} : q \cdot \mu = 0 \text{ for each } \mu \in \mathcal{C}ons\}.$$

Definition 3.6.2. A **pure-transfer club equilibrium** consists of a feasible state (f, l) and prices $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$, $p \neq 0$, such that:

- **Budget Feasibility for Individuals** : For λ -a.e. on A ,

$$p \cdot f_a + q \cdot l_a + p \cdot \tau(l_a) \leq p \cdot e_a;$$

- **Optimization**: For λ -a.e. on A , $(g_a, \mu_a) \in X_a$ and

$$u_a(g_a, \mu_a) > u_a(f_a, l_a) \Rightarrow p \cdot g_a + q \cdot \mu_a + p \cdot \tau(\mu_a) > p \cdot e_a; \text{ and}$$

- **Pure Transfers** : $q \in \mathcal{T}rans$.

The following lemma is adopted from Ellickson et al. [35] and it claims that a state is a club equilibrium if and only if it is a pure-transfer club equilibrium. We present the proof for the sake of completeness. The following result also holds for a quasi-club equilibrium and pure-transfer quasi-club equilibrium.

Lemma 3.6.3. Let \mathcal{E} be a club economy and $p \in \mathbb{R}_+^N \setminus \{0\}$, $q, q^* \in \mathbb{R}^{\mathcal{M}}$ be such that

$$q(\omega, \pi, \gamma) = q^*(\omega, \pi, \gamma) + \frac{1}{\|\pi\|_1} p \cdot \text{inp}(\pi, \gamma) \quad (3.6.1)$$

Then, a feasible state (f, l) is a club equilibrium state under the price vector (p, q) if and only if it is a pure-transfer club equilibrium state under the price vector (p, q^*) .

Proof. Given a price vector $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$ and any state (g, μ) , by virtue of Equation (3.6.1) it follows that

$$p \cdot g_a + q \cdot \mu_a = p \cdot g_a + q^* \cdot \mu_a + p \cdot \tau(\mu_a)$$

which thereby establishes the equivalence between the **feasibility** conditions in Definitions 3.6.1 and 3.6.2 and the **optimization** conditions in Definitions 3.6.1 and 3.6.2. Now what remains to show is that the budget balance condition implies that

$q^* \in \mathcal{T}rans$ and vice versa. We first show that the budget balance condition of Definition 3.6.1 implies that $q^* \in \mathcal{T}rans$. It follows from Equation (3.6.1) that

$$\pi(\omega)q(\omega, \pi, \gamma) = \pi(\omega)q^*(\omega, \pi, \gamma) + \frac{\pi(\omega)}{\|\pi\|_1} p \cdot inp(\pi, \gamma).$$

Summing over all $\omega \in \Omega$ we obtain

$$\sum_{\omega \in \Omega} \pi(\omega)q(\omega, \pi, \gamma) = \sum_{\omega \in \Omega} \pi(\omega)q^*(\omega, \pi, \gamma) + p \cdot inp(\pi, \gamma).$$

Since $\sum_{\omega \in \Omega} \pi(\omega)q(\omega, \pi, \gamma) = p \cdot inp(\pi, \gamma)$, we have

$$\sum_{\omega \in \Omega} \pi(\omega)q^*(\omega, \pi, \gamma) = 0. \quad (3.6.2)$$

Pick an arbitrary $\bar{\mu} \in \mathcal{C}ons$. Then there exists some $\alpha(\pi, \gamma) \in \mathbb{R}$ such that $\bar{\mu}(\omega, \pi, \gamma) = \alpha(\pi, \gamma)\pi(\omega)$ for every $\omega \in \Omega$. In view of this, Equation (3.6.2) can be re-written as

$$\sum_{\omega \in \Omega} \bar{\mu}(\omega, \pi, \gamma)q^*(\omega, \pi, \gamma) = 0.$$

Since the above equation holds for every possible club type $(\pi, \gamma) \in \mathcal{C}lubs$ it follows that $q^* \cdot \bar{\mu} = 0$. This implies that $q^* \in \mathcal{T}rans$.

To establish the converse, let $q^* \in \mathcal{T}rans$. Pick a particular club type $(\tilde{\pi}, \tilde{\gamma})$ and define for each $\omega \in \Omega$,

$$\tilde{\mu}(\omega, \pi, \gamma) := \begin{cases} 0, & \text{if } (\pi, \gamma) \neq (\tilde{\pi}, \tilde{\gamma}); \\ \tilde{\pi}(\omega), & \text{if } (\pi, \gamma) = (\tilde{\pi}, \tilde{\gamma}). \end{cases}$$

Note that $\tilde{\mu} \in \mathcal{C}ons$. As $q^* \in \mathcal{T}rans$, it follows that $q^* \cdot \tilde{\mu} = 0$. From the definition of $\tilde{\mu}$ one has

$$\sum_{\omega \in \Omega} \tilde{\pi}(\omega)q^*(\omega, \tilde{\pi}, \tilde{\gamma}) = 0.$$

Since the choice of $(\tilde{\pi}, \tilde{\gamma})$ was arbitrary, the above equation holds for every possible club type $(\pi, \gamma) \in \mathcal{C}lubs$. Then it follows from Equation (3.6.1) that

$$\sum_{\omega \in \Omega} \pi(\omega)q(\omega, \pi, \gamma) = \sum_{\omega \in \Omega} \pi(\omega)q^*(\omega, \pi, \gamma) + \sum_{\omega \in \Omega} \frac{\pi(\omega)}{\|\pi\|_1} p \cdot inp(\pi, \gamma).$$

Thus,

$$\sum_{\omega \in \Omega} \pi(\omega)q(\omega, \pi, \gamma) = p \cdot inp(\pi, \gamma)$$

for each $(\pi, \gamma) \in \mathcal{C}lubs$. This completes the proof. \square

Remark 3.6.4. It is important here to notice how the market functions compared to a classical exchange economy with no club goods. In a classical exchange economy, given a price vector $p \in \mathbb{R}_+^N$ announced by the Walrasian auctioneer, agents declare their demand, which necessarily is not unique. The Walrasian auctioneer compares aggregate demand to aggregate supply against the announcements made by the agents. If a possible combination of demands fail to match the aggregate supply of commodities then the auctioneer looks for other possible combinations of demands announced by the agents. The existence theorem in such settings guarantees that there exists a price vector for which a combination of demands exists such that aggregate demand matches aggregate supply. Now turning back to club economies we observe that given a price vector for private goods in mind, the Walrasian auctioneer calculates the cost of inputs required to form each club type. As clubs need to budget balance in equilibrium, given the cost of inputs, the Walrasian auctioneer thus calculates all possible prices for memberships of any particular club type, that is, he chooses any $q \in \mathbb{R}^{\mathcal{M}}$ such that

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \text{inp}(\pi, \gamma)$$

for all $(\pi, \gamma) \in \text{Clubs}$. Denote the set such q by $F(p)$, for a price vector p for private good. Thus, if the Walrasian auctioneer announces (p, q) then it must be the case that $q \in F(p)$. As in the case of the classical economy discussed above, an equilibrium price (p, q) is the one in which the aggregate demand satisfies the feasibility condition, that is, individual feasibility, material balance, and consistency conditions. To illustrate this, consider an economy with a set of agents $[0, 1]$ endowed with the Lebesgue σ -algebra and Lebesgue measure λ , where agents in $[0, \frac{1}{2})$ are males and that belonging to $[\frac{1}{2}, 1]$ are females. For simplicity consider that there exists only one club type (π, γ) in the economy. Given a price $p \in \mathbb{R}_+^N$ and $\text{inp}(\pi, \gamma) \in \mathbb{R}_+^N$, let $p \cdot \text{inp}(\pi, \gamma) = c$ and let the set

$$\mathcal{Q} = \{(q(m, \pi, \gamma), q(f, \pi, \gamma)) \in \mathbb{R}^{\mathcal{M}} : \pi(m) \cdot q(m, \pi, \gamma) + \pi(f) \cdot q(f, \pi, \gamma) = c\}$$

denotes the set of admissible membership prices for the male and female memberships, respectively. Further, suppose that the profile of the club requires males and females to be admitted in a one-to-one ratio. Now a particular pair $(q(m, \pi, \gamma), q(f, \pi, \gamma)) \in \mathcal{Q}$ can be such that $q(m, \pi, \gamma)$ is sufficiently low and $q(f, \pi, \gamma)$ is sufficiently high, then there may be a scenario where $\bar{\mu}(m, \pi, \gamma) > 0$ and $\bar{\mu}(f, \pi, \gamma) = 0$. As the club profile requires agents of both types to form, such a club type will not be formed, and thus such a scenario cannot be sustained as a club equilibrium. Now consider a scenario where both $q(m, \pi, \gamma)$ and $q(f, \pi, \gamma)$ are sufficiently high enough to drive both $\bar{\mu}(m, \pi, \gamma)$

and $\bar{\mu}(f, \pi, \gamma)$ to zero, then also no clubs of the type (π, γ) will be formed. However, compared to the earlier scenario, this can be sustained as a club equilibrium as here aggregate membership demands of males and females are zero (and thus, there is no demand for clubs). In contrast, in the earlier case, the aggregate male membership demand was positive, but that of the females was zero, which means that male members wanted to form clubs that could not materialize. One can then quite easily extend to the case of multiple club types and multiple characteristics for agents, and analogous reasoning holds. A key point to remember is that even though some clubs may not be formed in equilibrium, membership prices are announced a priori by the auctioneer, keeping the cost of input requirement in mind, and hence the budget balance of club types is always satisfied. Whether or not a club will be formed purely depends on the announced club membership prices and individual preferences.

In the next definition, we introduce the notion of club irreducibility, which further aids us in establishing that quasi-optimization leads to optimization. The importance of this result is captured in the proof of Proposition 4.2.5, where we show that any feasible state that is not a club equilibrium must be objected to by a set of agents willing to trade amongst themselves at the given prices.

Definition 3.6.5. Let \mathcal{E} be a club economy and let (f, l) be a feasible state. Write δ_j as the consumption bundle containing one unit of good j and no other goods. We say that (f, l) is **club linked** if whenever $I \cup J = \{1, 2, \dots, N\}$ is a partition of the set of private goods and $f_{ai} = 0$ for all $i \in I$ and almost all $a \in A$, then for almost all $a \in A$ there exist $r \in \mathbb{R}_+$, $j \in J$ such that

$$u_a(e_a + r\delta_j, 0) > u_a(f_a, l_a).$$

An economy \mathcal{E} is said to be **club irreducible** if every feasible state is club linked.

The definition implies that if the entire endowment of the private goods in the set I are used as inputs to club activities, then for almost all agents a , there exists some $j \notin I$ and a considerable amount of consumption of good j such that agent a will achieve higher utility from consuming her initial endowment and this considerable amount of good j and enjoy no club good compared to the utility from the state (f, l) . Next, we state a lemma from Ellickson et al. [35] that establishes a relation between a club equilibrium and a quasi-club equilibrium state.

Proposition 3.6.6. *Let \mathcal{E} be a club economy for which endowments are desirable. If $((f, l), (p, q))$ is a club quasi-equilibrium and (f, l) is club-linked, then $p \gg 0$ and $((f, l), (p, q))$ is a club equilibrium.*

Proof. We commence the proof by establishing that prices of all private commodities are positive. We proceed using contradiction and suppose that I is the set of indices for which $p_i > 0$, and J is the complementary set of indices consisting of commodities with prices equal to zero. Choose $i \in I$ and let $f_{ai} = 0$ for some sets of agents having a positive measure. Then any of these agents can choose to sell a small amount of commodity i to buy unlimited amounts of commodity j , which in turn yields them higher utility. This would contradict the quasi-equilibrium conditions. Hence, one can infer that for each $i \in I$, $f_{ai} = 0$ for almost all $a \in A$. Club linkedness ensures that each agent would like to consume their initial endowments along with some large enough quantities of commodity j rather than the quasi-equilibrium state (f_a, l_a) . By **(A.3)** we can infer that the aggregate endowment for a nonnull set of agents is strictly positive. Further by continuity of preferences, one can argue that each of these consumers will consume a very large fraction of their endowments along with large quantities of commodity j instead of (f_a, l_a) , contradicting quasi-equilibrium conditions. Thus, the prices of all private commodities are strictly positive. Suppose that the state (f, l) is not a club equilibrium state. Then there exists a nonnull set of agents who are quasi-optimizing and not optimizing. Then for each such agent a , there exists a state $(f'_a, l'_a) \in X_a$ such that $u_a(f'_a, l'_a) > u_a(f_a, l_a)$ and $p \cdot f'_a + q \cdot l'_a \geq p \cdot e_a$. The desirability of endowments guarantees that $p \cdot f'_a > 0$. Continuity of preferences over private goods ensures that there exists a state $(f''_a, l'_a) \in X_a$ such that $p \cdot f''_a < p \cdot f'_a$ and $u_a(f''_a, l'_a) > u_a(f_a, l_a)$. This implies $p \cdot f''_a + q \cdot l'_a < p \cdot e_a$. This is a contradiction. \square

Chapter 4

Bargaining Set

In this chapter, we introduce a two-step veto mechanism in an atomless club economy. The first step of the mechanism concerns the objection mechanism whereas the second step deals with counter-objection to the objection mechanism. While the objection mechanism is standard in the literature and helps us obtain the core, the primary motivation for counter-objection follows from the myopic nature of the agents, who can again regroup to form a different coalition and decide upon another mutually beneficial trade in comparison to the one proposed in the objection stage. The seminal work of Aumann and Maschler [6] talks about such counter-objections and introduces the notion of a bargaining set as a solution to a cooperative game. Later, Mas-Colell [64] extended the notion of bargaining set to the case of exchange economies. We extend the notion of bargaining set to the case of club economies and show that it is equivalent to the set of club equilibrium states and generalize the core-equivalence of Ellickson et al. [35]. This chapter is based on the paper [23].

4.1 Justified and Walrasian Objection

We now introduce the central concept of this chapter in the form of the bargaining set. Our notion of bargaining set is a generalization of the one proposed by Mas-Colell [64] to the case of economies with club goods.

Definition 4.1.1. A feasible state (f, l) of the economy \mathcal{E} is said to be **weakly objected** if there exists some coalition B and a feasible state (g, μ) such that (g, μ) is feasible for B , $u_a(g_a, \mu_a) \geq u_a(f_a, l_a)$ for almost all $a \in B$ and $u_a(g_a, \mu_a) > u_a(f_a, l_a)$ for all agents $a \in B'$ for some sub-coalition B' of B . We say that a pair $(S, (g, \mu))$ constitutes an **objection** to a state (f, l) if the state (f, l) is weakly objected by the

coalition S via the state (g, μ) . A feasible state (f, l) is said to be in the **core** of the economy \mathcal{E} if there exists no objection against it.

Remark 4.1.2. In our economy \mathcal{E} , there exist finitely many possible club types, which are listed as $\{(\pi_1, \gamma_1), (\pi_2, \gamma_2), \dots, (\pi_m, \gamma_m)\}$. Now, given any agent ‘ a ’ of characteristic ‘ ω ’ and $l_a \in Lists_a$ one can say that he or she purchases $l_a(\omega, \pi_i, \gamma_i)$ many memberships of the club type (π_i, γ_i) where $\pi_i(\omega) \geq 1$. Now, each club contains finitely many agents who act as members of the club and impose an externality effect on other members of the club, as discussed earlier. Hence, one can note that since an individual ‘ a ’ can purchase only finitely many memberships, individual ‘ a ’ becomes effective while recontracting for the club good, as they are recontracting with only finitely many other members. This is closely related to the framework proposed by Hammond et al. [44]. In their framework, blocking is defined only through finite coalitions. Thus, each agent although insignificant in the whole economy, becomes significant while recontracting with other agents in a coalition to make mutually beneficial arrangements. In the context of club economies, Allouch et al. [2] propose a framework of Tiebout economies where, as opposed to our framework, consumers are allowed to form jurisdiction (clubs) that have positive but finite measure and thus are significant in the economy. Further, the set of agents in their framework is partitioned into different jurisdictions, and each agent is part of only one jurisdiction (club). Allouch et al. [2] adapt the notion of *f-core* from Hammond et al. [44] and showed that given the jurisdictions, members of the jurisdiction can re-contract among themselves in either finite coalitions or in Aumann coalitions to obtain the *f-core*.¹ They show that the set of allocations in the *f-core* satisfying the equal treatment property is equivalent to the set of equilibrium allocations in their Tiebout economy. However, there is a difference between blocking in our framework and blocking in theirs. (i) In our framework the trade of private goods is not restricted within a club and hence not restricted among finitely many agents. Thus, whenever a coalition S blocks a state (f, l) through a state (g, μ) , the trade of private goods takes place within the overall coalition S . Since S is a positive measurable set we have that $(f|_{A \setminus S}, g|_S, l|_{A \setminus S}, \mu|_S)$ differs from (f, l) over a positive measurable coalition of agents. This implies that a redistribution of the initial endowment distribution within the Aumann coalition (or what we refer to as a coalition in our framework) S affects the average state of the economy as a whole. This stands in sharp contrast to the blocking by coalitions containing finitely many agents where

¹It is well-known that in models with a continuum of consumers and finite, but unbounded coalition sizes, the set of feasible allocations may not be closed. Thus, the feasible set is taken as the closure of the set of allocations that are achievable by trade only within finite coalitions. When this closure is taken, the set of allocations that are “f-feasible” is equivalent to the set of Aumann-feasible allocations.

the average allocation or consumption of the economy is unchanged, as in Allouch et al. [2] or Hammond et al. [44].(ii) Secondly, the dependence of the agent's decision on a club profile reflects the externalities imposed by the characteristics of other members (finite) in the club on his or her preference. This is a case of local externality as opposed to the notion of widespread externality where agents are concerned with the aggregate consumption as a whole.

However, as already spoken about in the beginning of the chapter, objections can be myopic and thus we introduce repercussions to objections and we define the notion of counter-objections below.

Definition 4.1.3. We say that an objection $(S, (g, \mu))$ to a feasible state (f, l) has a **counter-objection** if there exist some coalition T and a state (h, ν) such that (h, ν) is feasible for T and the following hold for almost all $a \in T$:

- (i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$ for all $a \in T \cap S$; and
- (ii) $u_a(h_a, \nu_a) > u_a(f_a, l_a)$ for all $a \in T \setminus S$.

We call an objection to be **justified** if there exists no counter-objection to it. The **bargaining set** of the economy \mathcal{E} is the set of all feasible states for which there does not exist any justified objection. We denote the bargaining set of the economy \mathcal{E} by $\mathcal{B}(\mathcal{E})$.

Now we introduce a veto mechanism through the market system, known as the Walrasian objection. Again, our notion is a generalization of that in Mas-Colell [64] to the case of an economy with club goods. Roughly speaking, it simply means that given a price pair $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$ there exists a set of agents who are willing to trade private goods amongst themselves rather than accept the proposed state and the others are willing to stay put.

Definition 4.1.4. We say that an objection $(S, (g, \mu))$ to a feasible state (f, l) is **Walrasian** if there exists a non-zero price vector $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$ such that for almost all $a \in A$:

- (i) $p \cdot h_a + q \cdot \nu_a \geq p \cdot e_a$ whenever $u_a(h_a, \nu_a) \geq u_a(g_a, \mu_a)$ and $a \in S$;
- (ii) $p \cdot h_a + q \cdot \nu_a \geq p \cdot e_a$ whenever $u_a(h_a, \nu_a) \geq u_a(f_a, l_a)$ and $a \in A \setminus S$; and
- (iii) **Budget Balance for Club types** : For each $(\pi, \gamma) \in \mathcal{C}lubs$,

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot inp(\pi, \gamma).$$

However, for the convenience of the proof, we resort to the pure-transfers case. We define the notion of pure-transfer Walrasian objection below.

Definition 4.1.5. We say that an objection $(S, (g, \mu))$ to a feasible state (f, l) is **pure-transfer Walrasian** if there exists a non-zero price vector $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$ such that for almost all $a \in A$:

- (i) $p \cdot h_a + q \cdot \nu_a + p \cdot \tau(\nu_a) \geq p \cdot e_a$ whenever $u_a(h_a, \nu_a) \geq u_a(g_a, \mu_a)$ and $a \in S$;
- (ii) $p \cdot h_a + q \cdot \nu_a + p \cdot \tau(\nu_a) \geq p \cdot e_a$ whenever $u_a(h_a, \nu_a) \geq u_a(f_a, l_a)$ and $a \in A \setminus S$;
and
- (iii) $q \in \mathcal{T}rans$.

Note that the formation of Walrasian objection requires a Walrasian auctioneer to first announce prices and then a coalition of agents join forces to mutually trade among each other to achieve a better state for themselves. Now, although in the process some club types may be formed and some may not, the budget balance is satisfied for all the club types. This primarily follows from the Remark 3.6.4 where we observe that an auctioneer announces membership prices keeping in mind the cost of input requirements of a club type. Thus, as is in the case of club equilibrium the budget balance condition is satisfied in the case of a Walrasian objection irrespective of whether a specific club is formed or not.

Lemma 4.1.6. Let \mathcal{E} be a club economy and $q, q^* \in \mathbb{R}^{\mathcal{M}}$ be such that

$$q^*(\omega, \pi, \gamma) = q(\omega, \pi, \gamma) + \frac{1}{\|\pi\|_1} p \cdot \text{inp}(\pi, \gamma)$$

Then $(S, (g, \mu))$ is a Walrasian objection under price (p, q) if and only if $(S, (g, \mu))$ is a pure-transfer Walrasian objection under price (p, q^*) .

Proof. The proof is immediate from Proposition 3.6.3. □

We now introduce another form of club irreducibility. This follows from the fact that club irreducibility alone is not enough to guarantee the equivalence between the bargaining set and the set of club equilibrium states in our case. Hence, we introduce a stronger notion and term it the “strong club irreducibility” of an economy. Below, we formally define the notion of strong club irreducibility.

Definition 4.1.7. A club economy \mathcal{E} is said to be **strong club irreducible** if for any coalition S , the economy restricted to S , $\mathcal{E}|_S$ ² is club irreducible.

²The economy restricted to the coalition S , $\mathcal{E}|_S$ is same as the economy \mathcal{E} defined in Definition 3.4.1 with only the set of agents being restricted to those belonging to coalition S .

We now further generalize Proposition 3.6.6 defined in the previous chapter.

Lemma 4.1.8. *Let \mathcal{E} be a club economy which is strong club irreducible. Further, let S be a coalition and (f, l) be a feasible state for the coalition S . Then if (f_a, l_a) is quasi-optimizing for almost all $a \in S$ under a price vector (p, q) , (f_a, l_a) is optimizing for almost all $a \in S$ under the same price vector (p, q) .*

Proof. Since the economy \mathcal{E} is strong club irreducible, it follows that the economy $\mathcal{E}|_S$ is club irreducible. Then as (f, l) is feasible in $\mathcal{E}|_S$ and (f_a, l_a) is quasi-optimizing under the price vector (p, q) for almost all $a \in S$, it follows from Proposition 3.6.6 above that $p \gg 0$ and (f_a, l_a) is optimizing for almost all $a \in S$ under the price vector (p, q) . The case for pure-transfer optimizing follows analogously. \square

Proposition 4.1.9. *Let \mathcal{E} be a club economy satisfying Definition 3.4.1 and for which endowments are desirable too. Further, assume \mathcal{E} to be strong club irreducible. Then for any pure-transfer Walrasian objection $(S, (g, \mu))$ to a non-equilibrium state (f, l) , we have the following for almost all $a \in A$:*

- (i) $p \cdot h_a + q \cdot \nu_a + p \cdot \tau(\nu_a) > p \cdot e_a$ whenever $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$ and $a \in S$; and
- (ii) $p \cdot h_a + q \cdot \nu_a + p \cdot \tau(\nu_a) > p \cdot e_a$ whenever $u_a(h_a, \nu_a) > u_a(f_a, l_a)$ and $a \in A \setminus S$.

Proof. As $(S, (g, \mu))$ is a pure-transfer Walrasian objection at some price pair (p, q) to the non-equilibrium state (f, l) , it follows that (g, μ) is a feasible state in the economy $\mathcal{E}|_S$. Further, it follows from condition (i) of Definition 4.1.5 that (g, μ) is a quasi-club equilibrium state in the economy $\mathcal{E}|_S$. Then from strong club irreducibility, one can infer that (g, μ) is club-linked in $\mathcal{E}|_S$ and thus from Proposition 3.6.6 we have that $p \gg 0$ and (g, μ) is a club-equilibrium state in $\mathcal{E}|_S$. This establishes condition (i) of our proposition.

Next, we establish Condition (ii). Ignoring a measure zero set of agents, for every agent $a \in A \setminus S$ and for every $(h_a, \nu_a) \in X_a$ such that $u_a(h_a, \nu_a) > u_a(f_a, l_a)$, we have from Condition (ii) of Definition 4.1.5 that

$$p \cdot h_a + q \cdot \nu_a + p \cdot \tau(\nu_a) \geq p \cdot e_a. \quad (4.1.1)$$

We first establish that $h_a \neq 0$ for every agent $a \in A \setminus S$ and for every $(h_a, \nu_a) \in X_a$ satisfying $u_a(h_a, \nu_a) > u_a(f_a, l_a)$. If not, pick any arbitrary $a \in A \setminus S$, then by desirability of endowments one has $u_a(e_a, 0) > u_a(h_a, \nu_a) > u_a(f_a, l_a)$. Then, from the continuity of preferences over private goods, there exists an $\alpha \in (0, 1)$ close enough to 1 such that

$u_a(\alpha e_a, 0) > u_a(f_a, l_a)$ and $p \cdot (\alpha e_a) < p \cdot e_a$ as $p \cdot e_a > 0$.³ This contradicts Equation (4.1.1) and thus $h_a \neq 0$. Define

$$R = \{a \in A \setminus S : u_a(h_a, \nu_a) > u_a(f_a, l_a) \implies p \cdot h_a + q \cdot \nu_a + p \cdot \tau(\nu_a) = p \cdot e_a\}.$$

We want to show that $\lambda(R) = 0$. Suppose not. Pick any $a \in R$. By continuity of preferences over private goods, there exists an $\beta \in (0, 1)$ close enough to 1 such that $u_a(\beta h_a, \nu_a) > u_a(f_a, l_a)$ and $p \cdot \beta h_a + q \cdot \nu_a + p \cdot \tau(\nu_a) < p \cdot h_a + q \cdot \nu_a + p \cdot \tau(\nu_a) = p \cdot e_a$.⁴ This again contradicts Equation (4.1.1). Thus, we have established Condition (ii) of our proposition and this completes the proof. \square

4.2 Main result

In this section, we establish our main result, namely the equivalence between the set of club equilibrium states and the bargaining set of a club economy. One can further note from Ellickson et al. [35] that club equilibrium states are a subset of the core states and as the core is a subset of the bargaining set, our equivalence will guarantee that $\mathcal{W}(\mathcal{E}) = \mathcal{C}(\mathcal{E}) = \mathcal{B}(\mathcal{E})$ holds in a club economy with a continuum of agents. Observe that one can obtain $\mathcal{W}(\mathcal{E}) = \mathcal{C}(\mathcal{E})$ as an immediate corollary of our result which implies that our equivalence result extends the core equivalence result of Ellickson et al. [35]. Further, as $\mathcal{C}(\mathcal{E}) = \mathcal{B}(\mathcal{E})$, one can also claim that objections that may have repercussions have not been formed. We now proceed to establish our equivalence result. To do so, we first show that any objection to a state that isn't subject to any repercussion from the cooperative mechanism is devoid of any repercussion from the market mechanism. The next proposition formalizes it. To that end we state a lemma from Ellickson et al. [35] that shall aid us in the proof of our proposition. We state and prove the lemma for the sake of completeness.

Lemma 4.2.1. *Let $C \subseteq A$ be such that $\lambda(C) > 0$ and let $\mu : C \rightarrow Lists_M$ be a measurable function. Then there is a measurable subset $C' \subseteq C$ such that*

$$\int_{C'} \mu_a d\lambda \in \mathcal{C}ons \text{ and } \lambda(C') \geq \lambda(C) - \frac{1}{D} \text{dist} \left(\int_C \mu_a d\lambda, \mathcal{C}ons \right). \quad (4.2.1)$$

where the notations have the following interpretations

$$\mathcal{D} = \{\mathcal{L} \subset Lists_M : \text{conv}(\mathcal{L}) \cap \mathcal{C}ons = \emptyset\},$$

$$D = \inf \{\text{dist}(\text{conv}(\mathcal{L}), \mathcal{C}ons) : \mathcal{L} \in \mathcal{D}\}.$$

³This follows from the fact that $p \gg 0$ and $e_a \in \mathbb{R}_{++}^N$ for all $a \in A$.

⁴This follows from the fact that $p \gg 0$ and $h_a \neq 0$ for every agent $a \in A \setminus S$.

Proof. If $Lists_M \subset \mathcal{C}ons$, then $dist(\int_C \mu_a d\lambda, \mathcal{C}ons) = 0$. Then one can choose $\mathcal{D} = \emptyset$ and $D = \infty$ and $C = C'$. we therefore assume that $Lists_M \not\subset \mathcal{C}ons$. For $l \in List_M$, define $C_l = \{a \in C : \mu_a = l\}$. Let $\mathcal{L} = \{l \in Lists_M : \lambda(C_l) > 0\}$. One can note that $\sum_{l \in \mathcal{L}} \lambda(C_l) = \lambda(C)$ and

$$\int_C \mu_a d\lambda = \sum_{l \in \mathcal{L}} \lambda(C_l) \cdot l = \lambda(C) \sum_{l \in \mathcal{L}} \frac{\lambda(C_l)}{\lambda(C)} l.$$

It then follows that

$$\int_C \mu_a d\lambda \in \lambda(C) conv(\mathcal{L})$$

If $conv(\mathcal{L}) \cap \mathcal{C}ons = \emptyset$, then right-hand side of the second part of Equation (4.2.1) is nonpositive, which then leads to choose $C'_l = \emptyset$ for each l and $C' = \cup_{l \in \mathcal{L}} C'_l = \emptyset$. Hence, we assume that $conv(\mathcal{L}) \cap \mathcal{C}ons \neq \emptyset$. We consider the following linear programming problem:

$$\text{maximize } \sum_{l \in \mathcal{L}} \beta_l$$

$$\text{subject to } 0 \leq \beta_l \leq \lambda(C_l) \text{ and } \sum_{l \in \mathcal{L}} \beta_l \cdot l \in \mathcal{C}ons.$$

Observe that the origin belongs to the feasible set which guarantees that the above problem has a solution. Let $\{\beta_l : l \in \mathcal{L}\}$ be any such solution.

For each l , write $\alpha_l = \lambda(C_l) - \beta_l \geq 0$. Define $\mathcal{L}' = \{l : \alpha_l > 0\}$. If $\mathcal{L}' = \emptyset$, we have our solution, so we assume otherwise. Choose $\alpha = \min \{\alpha_l : l \in \mathcal{L}'\}$. If $conv(\mathcal{L}') \cap \mathcal{C}ons \neq \emptyset$, there exists nonnegative real numbers ε_l summing to 1 with $\sum_{l \in \mathcal{L}'} \varepsilon_l \cdot l \in \mathcal{C}ons$. Set $\beta_l^* = \beta_l + \varepsilon_l \alpha$ for $l \in \mathcal{L}'$ and $\beta_l^* = \beta_l$ for $l \notin \mathcal{L}'$. Then $\{\beta_l^* : l \in \mathcal{L}\}$ satisfies the constraints and yields a larger value of the objective function contradicting that $\{\beta_l\}$ is the solution to the problem. Thus $conv(\mathcal{L}') \cap \mathcal{C}ons = \emptyset$. For each $l \in Lists_M$, nonatomicity of λ guarantees that we can choose $C'_l \subset C_l$ such that $\lambda(C'_l) = \beta_l$. Set $C' = \bigcup_{l \in \mathcal{L}} C'_l$. Then it follows that $\int_{C'} \mu_a d\lambda = \sum_{l \in \mathcal{L}} \lambda(C'_l) \cdot l \in \mathcal{C}ons$. Since $\mathcal{C}ons$ is a linear subspace, it follows from its properties that for every $x \in \mathbb{R}^n$, $y \in \mathcal{C}ons$, $r \in \mathbb{R}_+$, we have $dist(x - y, \mathcal{C}ons) = dist(x, \mathcal{C}ons)$ and $dist(rx, \mathcal{C}ons) = r \cdot dist(x, \mathcal{C}ons)$. Hence,

it follows that

$$\begin{aligned}
\text{dist} \left(\sum_{l \in \mathcal{L}} \lambda(C_l)l, \mathcal{C} \text{ ons} \right) &= \text{dist} \left(\left(\sum_{l \in \mathcal{L}} \lambda(C_l)l - \sum_{l \in \mathcal{L}} \beta_l l \right), \mathcal{C} \text{ ons} \right) \\
&= \text{dist} \left(\sum_{l \in \mathcal{L}'} (\lambda(C_l) - \beta_l)l, \mathcal{C} \text{ ons} \right) \\
&= \text{dist} \left(\sum_{l \in \mathcal{L}'} \alpha_l l, \mathcal{C} \text{ ons} \right) \\
&= \text{dist} \left(\sum_{l \in \mathcal{L}} \alpha_l \sum_{l \in \mathcal{L}'} \left[\frac{\alpha_l}{\sum_{l \in \mathcal{L}'} \alpha_l} \right] l, \mathcal{C} \text{ ons} \right) \\
&= \left(\sum_{l \in \mathcal{L}'} \alpha_l \right) \text{dist} \left(\sum_{l \in \mathcal{L}'} \left[\frac{\alpha_l}{\sum_{l \in \mathcal{L}'} \alpha_l} \right] l, \mathcal{C} \text{ ons} \right) \\
&\geq \left(\sum_{l \in \mathcal{L}'} \alpha_l \right) \text{dist}(\text{conv}(\mathcal{L}'), \mathcal{C} \text{ ons}) \\
&= \left(\sum_{l \in \mathcal{L}'} [\lambda(C_l)l - \beta_l l] \right) \text{dist}(\text{conv}(\mathcal{L}'), \mathcal{C} \text{ ons}) \\
&\geq D \sum_{l \in \mathcal{L}'} [\lambda(C_l)l - \beta_l l] \\
&= D \sum_{l \in \mathcal{L}} [\lambda(C_l)l - \beta_l l] \\
&= D[\lambda(C) - \lambda(C')].
\end{aligned}$$

□

Proposition 4.2.2. *Let \mathcal{E} be a club economy for which endowments are desirable and uniformly bounded above. Further, let \mathcal{E} be strong club irreducible and let $(S, (g, \mu))$ be an objection to a state (f, l) . Then the following statements are equivalent:*

- (1) $(S, (g, \mu))$ is a Walrasian objection.
- (2) $(S, (g, \mu))$ is a justified objection.

Proof. The proof is decomposed into two parts. For the first part, we show that if an objection is a Walrasian objection, it is also justified. The second part deals with the converse of the first part, namely, any justified objection is Walrasian.

(1) \Rightarrow (2)

Let $(S, (g, \mu))$ be a Walrasian objection to (f, l) under the price pair $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^M$. Thus, there exists a price pair $(p, q^*) \in \mathbb{R}^N \times \mathbb{R}^M$ such that $(S, (g, \mu))$ is a pure-transfer Walrasian objection to (f, l) . By means of contradiction, suppose that $(S, (g, \mu))$ is not a justified objection. Then there exists a counter-objection $(T, (h, \nu))$ to $(S, (g, \mu))$ such that

- (i) $\int_T h_a d\lambda + \int_T \tau(\nu_a) d\lambda = \int_T e_a d\lambda$;
- (ii) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$ for almost all $a \in T \cap S$;
- (iii) $u_a(h_a, \nu_a) > u_a(f_a, l_a)$ for almost all $a \in T \setminus S$; and
- (iv) $\int_T \nu_a d\lambda \in \mathcal{C}ons$.

From (i), one can immediately obtain that

$$\int_T p \cdot h_a d\lambda + \int_T p \cdot \tau(\nu_a) d\lambda = \int_T p \cdot e_a d\lambda. \quad (4.2.2)$$

Since, $\int_T \nu_a d\lambda \in \mathcal{C}ons$ and $q^* \in \mathcal{T}rans$ it follows that $q^* \cdot \int_T \nu_a d\lambda = 0$. Thus we can rewrite Equation (4.2.2) as

$$\int_T p \cdot h_a d\lambda + \int_T q^* \cdot \nu_a d\lambda + \int_T p \cdot \tau(\nu_a) d\lambda = \int_T p \cdot e_a d\lambda. \quad (4.2.3)$$

Since $(S, (g, \mu))$ is a pure-transfer Walrasian objection to the state (f, l) by virtue of Proposition 4.1.9 we have that if $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$ for almost all $a \in T \cap S$, then it follows that

$$p \cdot h_a + q^* \cdot \nu_a + p \cdot \tau(\nu_a) > p \cdot e_a \text{ for almost all } a \in T \cap S.$$

Analogous reasoning yields that $u_a(h_a, \nu_a) > u_a(f_a, l_a)$ for almost all $a \in T \setminus S$ implies

$$p \cdot h_a + q^* \cdot \nu_a + p \cdot \tau(\nu_a) > p \cdot e_a \text{ for almost all } a \in T \setminus S.$$

Combining the above two inequalities, we obtain

$$\int_T p \cdot h_a d\lambda + \int_T q^* \cdot \nu_a d\lambda + \int_T p \cdot \tau(\nu_a) d\lambda > \int_T p \cdot e_a d\lambda.$$

This is a contradiction to Equation (4.2.3). Hence, it follows that $(S, (g, \mu))$ is a justified objection to (f, l) .

(2) \Rightarrow (1)

This part of the proof follows along similar lines of the proof of Theorem 5.1 of Ellickson et al. [35]. We present it here for the sake of completeness. Let $(S, (g, \mu))$ be a justified objection to the state (f, l) . For each agent $a \in A$, we define the set

$$\Xi_a := \{(h_a, \nu_a) : u_a(h_a, \nu_a) \geq u_a(\tilde{g}_a, \tilde{\mu}_a)\},$$

where $(\tilde{g}_a, \tilde{\mu}_a)$ is defined as follows

$$(\tilde{g}_a, \tilde{\mu}_a) := \begin{cases} (g_a, \mu_a), & \text{if } a \in S; \\ (f_a, l_a), & \text{if } a \in A \setminus S. \end{cases}$$

Define the net preferred set associated with each agent a as :

$$\psi(a) := \{(h_a, \nu_a) \in X_a : (h_a + e_a - \tau(\nu_a), \nu_a) \in \Xi_a\}.$$

Letting $\varphi(a) := \psi(a) \cup \{0\}$ and we define the aggregate net preferred set as

$$\Psi = \int_A \varphi(a) d\lambda.$$

Define

$$C^* := \left\{ (\bar{h}, \bar{\nu}) \in X_a : \bar{h} < -\frac{W}{D} \text{dist}(\bar{\nu}, \mathcal{C}ons) \mathbf{1} \right\},$$

where W is a positive real number such that $e_a \leq W\mathbf{1}$ for all $a \in A$.

Step I: $\Psi \cap C^* = \emptyset$

We work with the contrapositive argument. Thus, let $\Psi \cap C^* \neq \emptyset$ and pick an element $(h^*, \nu^*) \in \Psi \cap C^*$. This implies the existence of a measurable selection (h, ν) of the correspondence φ such that $(h^*, \nu^*) = \int_A (h_a, \nu_a) d\lambda$. Define $B = \{a \in A : (h_a, \nu_a) \in \psi(a)\}$. It follows from the definition of C^* and the fact $\Psi \cap C^* \neq \emptyset$ that $\lambda(B) > 0$. Following from Lemma 4.2.1 we can find a coalition $B^* \subseteq B$ such that

$$\int_{B^*} \nu_a d\lambda \in \mathcal{C}ons, \tag{4.2.4}$$

and

$$\lambda(B^*) \geq \lambda(B) - \frac{1}{D} \text{dist} \left(\int_B \nu_a d\lambda, \mathcal{C}ons \right). \tag{4.2.5}$$

Since $(h^*, \nu^*) \in C^*$ and endowments are upper bounded by $W\mathbf{1}$, we get the following set of inequalities

$$\int_B h_a d\lambda < -\frac{W}{D} \text{dist} \left(\int_B \nu_a d\lambda, \mathcal{C}ons \right) \mathbf{1}, \quad (4.2.6)$$

and

$$\int_B h_a d\lambda \geq -\lambda(B)W\mathbf{1}. \quad (4.2.7)$$

Using Equations (4.2.6) - (4.2.7), we can further write

$$\lambda(B) > \frac{1}{D} \text{dist} \left(\int_B \nu_a d\lambda, \mathcal{C}ons \right).$$

It then follows from Equation (4.2.5) that $\lambda(B^*) > 0$. Also following again from the fact that endowments are upper bounded by $W\mathbf{1}$, we can infer that

$$\int_{B \setminus B^*} h_a d\lambda \geq -\lambda(B \setminus B^*)W\mathbf{1}. \quad (4.2.8)$$

Since

$$\int_B h_a d\lambda = \int_{B^*} h_a d\lambda + \int_{B \setminus B^*} h_a d\lambda,$$

we obtain, in view of Equations (4.2.8), that

$$\int_{B^*} h_a d\lambda \leq \int_B h_a d\lambda + \lambda(B \setminus B^*)W\mathbf{1}.$$

Further, by applying Equation (4.2.5) and Equation (4.2.6), we can find some $z \in \mathbb{R}_+^N \setminus \{0\}$ such that $\int_{B^*} h_a d\lambda = -z$. We thus define a state $(\tilde{h}, \tilde{\nu}) : B^* \rightarrow \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$ by letting

$$(\tilde{h}_a, \tilde{\nu}_a) := \left(h_a + \frac{z}{\lambda(B^*)} - \tau(\nu_a) + e_a, \nu_a \right).$$

It is then readily verified that

$$\int_{B^*} \tilde{h}_a d\lambda + \int_{B^*} \tau(\tilde{\nu}_a) d\lambda = \int_{B^*} e_a d\lambda.$$

This ensures the material balance condition for the state $(\tilde{h}, \tilde{\nu})$ for B^* and from Equation (4.2.4), we can conclude that $\int_{B^*} \tilde{\nu}_a d\lambda \in \mathcal{C}ons$. By the strong monotonicity of preferences with respect to the private goods and the definition of ψ , it follows that

(i) $u_a(\tilde{h}_a, \tilde{\nu}_a) > u_a(g_a, \mu_a)$ for λ -a.e. on $S \cap B^*$; and

(ii) $u_a(\tilde{h}_a, \tilde{\nu}_a) > u_a(f_a, l_a)$ for λ -a.e. on $(A \setminus S) \cap B^*$.

Thus, $(B^*, (\tilde{h}, \tilde{\nu}))$ constitutes a counter-objection to $(S, (g, \mu))$. This leads to a contradiction to the fact that $(S, (g, \mu))$ is a justified objection to the state (f, l) . Thus, we conclude that $\Psi \cap C^* = \emptyset$.

Step II: Constructing a price pair (p, q) with $p \in \mathbb{R}_+^N \setminus \{0\}$ and $q \in \mathcal{T}rans$

By the separation theorem, we can find a vector $(p, q) \in \mathbb{R}^N \times \mathbb{R}^{\mathcal{M}}$, $(p, q) \neq (0, 0)$ such that

$$(p, q) \cdot (\bar{h}, \bar{\nu}) \leq 0 \quad \text{for each } (\bar{h}, \bar{\nu}) \in C^*, \text{ and}$$

$$(p, q) \cdot (\bar{h}, \bar{\nu}) \geq 0 \quad \text{for each } (\bar{h}, \bar{\nu}) \in \Psi.$$

Because C^* contains the cone $-(\mathbb{R}_+^N \setminus \{0\}) \times \{0\}$, it follows that $p \geq 0$. Because C^* contains the subspace C , where

$$C := \{(\bar{h}, \bar{\nu}) \in X_a : \bar{h} < 0 \text{ and } \bar{\nu} \in \mathcal{C}ons\},$$

it follows that $p \cdot \bar{h} + q \cdot \bar{\nu} \leq 0$ for all $\bar{h} < 0$ and all $\bar{\nu} \in \mathcal{C}ons$. Thus, $q \cdot \bar{\nu} \leq 0$ for all $\bar{\nu} \in \mathcal{C}ons$, which in view of fact the $\mathcal{C}ons$ is a subspace, further implies that q vanishes on $\mathcal{C}ons$. Hence, $q \in \mathcal{T}rans$. We claim that $p \neq 0$. If not, let $p = 0$. It follows from the fact $(p, q) \neq (0, 0)$ that $q \neq 0$. Hence there is a $\beta \in \mathbb{R}^{\mathcal{M}}$ such that $q \cdot \beta > 0$. For $\varepsilon > 0$ sufficiently small, we have $(-1, \varepsilon\beta) \in C^*$. So $(p, q) \cdot (-1, \varepsilon\beta) \leq 0$. However, $(p, q) \cdot (-1, \varepsilon\beta) = (0, q) \cdot (-1, \varepsilon\beta) = \varepsilon q \cdot \beta$ which, by our choice of β , is positive. This is a contradiction, so we conclude that $p \neq 0$, as desired.

Step III: $(S, (g, \mu))$ is a Walrasian objection.

To prove this, we define

$$\tilde{X}_a := \{(y_a, l'_a) : u_a(y_a, l'_a) > u_a(\tilde{g}_a, \tilde{\mu}_a) \text{ and } p \cdot y_a + q \cdot l'_a + p \cdot \tau(l'_a) < p \cdot e_a\}.$$

Let

$$\mathbb{E} := \left\{ a \in A : \tilde{X}_a \neq \emptyset \right\}.$$

Choose a countable set $D \subset \mathbb{R}_+^N$ dense in \mathbb{R}_+^N . This infers that $a \in \mathbb{E}$ if and only if there exists $(h_a, \nu_a) \in D \times \mathcal{L}ists_a$ such that

$$u_a(h_a, \nu_a) > u_a(\tilde{g}_a, \tilde{\mu}_a) \text{ and } p \cdot h_a + p \cdot \tau(\nu_a) + q \cdot \nu_a < p \cdot e_a. \quad (4.2.9)$$

Given that $a \rightrightarrows X_a$ is a measurable correspondence, we can conclude that $a \rightrightarrows Lists_a$ is a measurable correspondence. For any $(x, \beta) \in D \times \mathbb{Z}_+^{\mathcal{M}}$, we define

$$H(x, \beta) := \{a \in A : \beta \in Lists_a, u_a(x, \beta) > u_a(\tilde{g}_a, \tilde{\mu}_a) \text{ and } p \cdot x + p \cdot \tau(\beta) + q \cdot \beta < p \cdot e_a\}.$$

It can be readily verified that $H(x, \beta)$ is Σ -measurable and

$$\bigcup \{H(x, \beta) : (x, \beta) \in D \times \mathbb{Z}_+^{\mathcal{M}}\} \subseteq E.$$

The other inclusion

$$\mathbb{E} \subseteq \bigcup \{H(x, \beta) : (x, \beta) \in D \times \mathbb{Z}_+^{\mathcal{M}}\}$$

follows from the fact that D is dense in \mathbb{R}_+^N and utility functions are continuous. Hence,

$$\mathbb{E} = \bigcup \{H(x, \beta) : (x, \beta) \in D \times \mathbb{Z}_+^{\mathcal{M}}\}$$

and therefore \mathbb{E} is a measurable set. Thus, the choices (h_a, ν_a) in Equation (4.2.9) can be chosen to depend measurably on $a \in \mathbb{E}$. Suppose $\lambda(\mathbb{E}) > 0$. Define $(\tilde{h}, \tilde{\nu}) : A \rightarrow \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$ by

$$(\tilde{h}_a, \tilde{\nu}_a) = \begin{cases} (h_a + \tau(\nu_a) - e_a, \nu_a) & \text{if } a \in \mathbb{E}; \\ (0, 0), & \text{otherwise.} \end{cases}$$

This implies that

$$(p, q) \cdot \int_A (\tilde{h}_a, \tilde{\nu}_a) d\lambda = \int_A (p, q) \cdot (\tilde{h}_a, \tilde{\nu}_a) d\lambda < 0.$$

On the other hand, $\int_A (\tilde{h}_a, \tilde{\nu}_a) d\lambda \in \Psi$ implies

$$(p, q) \cdot \int_A (\tilde{h}_a, \tilde{\nu}_a) d\lambda \geq 0.$$

This leads to a contradiction. Hence, $\lambda(\mathbb{E}) = 0$, which means that

- (i) for λ -a.e. on S if $u_a(y_a, \beta_a) > u_a(g_a, \mu_a)$ then $p \cdot y_a + q \cdot \beta_a + p \cdot \tau(\beta_a) \geq p \cdot e_a$; and
- (ii) for λ -a.e. on $A \setminus S$ if $u_a(y_a, \beta_a) > u_a(f_a, l_a)$ then $p \cdot y_a + q \cdot \beta_a + p \cdot \tau(\beta_a) \geq p \cdot e_a$.

By the continuity and strong monotonicity of preferences with respect to private goods only, we can infer that $(S, (g, \mu))$ constitutes a pure-transfer Walrasian objection to the state (f, l) . Then, given Lemma 4.1.6, we can claim that there exists a price pair $(p, q^*) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$ such that $(S, (g, \mu))$ is a Walrasian objection to the state (f, l) . \square

Remark 4.2.3. It is important to emphasize here that justified objections must be Walrasian helps us identify that the notion of objection cannot be defined in terms of strong blocking. This fact was highlighted in Remark 5 of Mas-Colell [64] who showed that the phenomenon holds in a classical exchange economy devoid of any club goods. As a classical exchange economy is a special case of our economy a similar argument can be made for our economy as well.

We now present a key proposition, which claims that any feasible state that fails to qualify as an equilibrium of the club economy \mathcal{E} , must be objected to by a coalition of agents who are willing to trade among themselves at the given prices to achieve a mutually beneficial outcome and all other agents (weakly) prefer the non-club-equilibrium state to trading at the given prices. The proof of this result relies on some arguments for the existence of a club equilibrium. We proceed along the same line and hence parts of our proof are reiterations of the proof of Theorem 6.1 of Ellickson et al. [35]. But before that, we state a lemma from Ellickson et al. [35] that aids in the proof of our next proposition. To this end, we define a subset $L \subset Lists_M$ to be **strictly balanced** if there are strictly positive real numbers $\{\beta_L(l) : l \in L\}$ (called balancing weights) such that $\sum_{l \in L} \beta_L(l)l \in \mathcal{Cons}$.

Lemma 4.2.4. *Let $R^* > 0$ be a constant such that if $L \subset Lists_M$ is a strictly balanced collection and $q \in \mathcal{Trans}$ is a pure transfer then*

$$\max_{l \in L} q \cdot l \geq -R^* \min_{l \in L} q \cdot l.$$

Proof. For each strictly balanced collection of lists L , we fix balancing weights $\{\varepsilon'_L(\cdot)\}$. It follows from the definition of balancing weights that they are strictly positive, the sum $\beta_L = \sum_{l \in L} \varepsilon'_L(l) > 0$. Define $\varepsilon(\cdot) = \frac{1}{\beta} \varepsilon'_L(\cdot)$. Since \mathcal{Cons} is a linear subspace, $\{\varepsilon_L(\cdot)\}$ also acts as a family of balancing weights for L , and $\sum_{l \in L} \varepsilon_L(l) = 1$. Define

$$R^* := \{\varepsilon_L(l) : L \text{ is a strictly balanced collection, } l \in L\}.$$

R^* is well-defined and $R^* > 0$ as the balancing weights are strictly positive and the collection forms a finite set. We now show that R^* has the desired property. Consider the collection $\{\varepsilon_L(\cdot)\}$ of balancing weights with a strictly balance collection L . Note that

$$\sum_{l \in L} \varepsilon_L(l)q \cdot l = q \cdot \sum_{l \in L} \varepsilon_L(l)l = 0.$$

Set $L_+ = \{l \in L : q \cdot l \geq 0\}$ and $L_- = L \setminus L_+$. Collecting all the positive terms on the left-hand side and all the negative ones on the right-hand side we get

$$\sum_{l \in L_+} \varepsilon_L(l)q \cdot l = - \sum_{l \in L_-} \varepsilon_L(l)q \cdot l. \quad (4.2.10)$$

Since the coefficients $\varepsilon_L(l) > 0$ and sum to 1, we have

$$\max_{l \in L} q \cdot l = \left[\sum_{l \in L} \varepsilon_L(l) \right] \max_{l \in L} q \cdot l \geq \left[\sum_{l \in L} \varepsilon_L(l) \right] \max_{l \in L_+} q \cdot l \geq \sum_{l \in L_+} \varepsilon_L(l) q \cdot l. \quad (4.2.11)$$

Since $q \cdot l < 0$ for each $l \in L_-$, we have

$$- \sum_{l \in L_-} \varepsilon_L(l) q \cdot l \geq - \min_{l \in L_-} \varepsilon_L(l) \min_{l \in L_-} q \cdot l \geq - \min_{l \in L} \varepsilon_L(l) \min_{l \in L} q \cdot l. \quad (4.2.12)$$

Combining Equations (4.2.10), (4.2.11), and (4.2.12) with the definition of R^* yields the desired result. \square

Proposition 4.2.5. *Let \mathcal{E} be a club economy for which endowments are desirable and uniformly bounded above. Further, let \mathcal{E} be strong club irreducible. Then any feasible state that is not a club equilibrium, has a Walrasian objection against it.*

Proof. A significant proportion of the proof follows along similar lines of the proof of Theorem 6.1 of Ellickson et al. [35], however, there are significant departures from it as well. Let (f, l) be a feasible state which is not a club equilibrium. It follows from Lemma 3.6.3 that (f, l) is not a pure-transfer club equilibrium state.

Step 1: Constructing modified economies

Without loss of generality, we assume that $\lambda(A) = 1$. We assume that $W_0 \mathbf{1}$ is a **uniform bound** on individual endowments, where W_0 is a positive real number and $\mathbf{1} = (1, 1, \dots, 1)$. Let $W = \max\{W_0, 1\}$. Apart from the original set of agents, we introduce an auxiliary set of agents in our economy. For each integer $k > 0$, choose a family $\{A_\omega^k : \omega \in \Omega\}$ of pairwise disjoint Lebesgue measurable subsets of \mathbb{R} , each of measure $\frac{1}{k}$ with $A \cap A_\omega^k = \emptyset$, for all $\omega \in \Omega$. Define

$$B^k := \bigcup \{A_\omega^k : \omega \in \Omega\}$$

and consider a measure space of agents (B^k, \mathcal{B}^k, m) , where \mathcal{B}^k is the σ -algebra of Lebesgue measurable subsets of B^k and m is the Lebesgue measure on \mathcal{B}^k . For each $k \geq 1$, we thus introduce an economy \mathcal{E}^k whose space of agents is defined by the measure space $(A^k, \Sigma^k, \lambda^k)$, where

- (1) $A^k := A \cup B^k$;
- (2) $\Sigma^k := \Sigma \otimes \mathcal{B}^k = \{C \cup D^k : C \in \Sigma, D^k \in \mathcal{B}^k\}$; and
- (3) $\lambda^k : \Sigma^k \rightarrow \mathbb{R}_+$, where $\lambda^k(C) := \lambda(C \cap A) + m(C \cap B^k)$.

We assume that for each agent $a \in A$, his or her external characteristics, choice set, initial endowment, and utility function are the same as that in the original economy \mathcal{E} . For each agent $a \in A_\omega^k$, we define

$$\begin{aligned} \omega_a &:= \omega; \\ X_a &:= \mathbb{R}_+^N \times \{l(\omega', \pi, \gamma) : l(\omega', \pi, \gamma) = 0 \text{ if } \omega' \neq \omega \text{ and } \|l\|_1 \leq M\}; \\ e_a &:= W\mathbf{1}; \text{ and} \\ u_a(x_a, \mu_a) &:= \|x_a\|_1 \text{ for all } (x_a, \mu_a) \in X_a. \end{aligned}$$

Before proceeding, we state two important conditions that aid our proof. Choose a small enough $\varepsilon \in (0, \frac{1}{N})$ such that⁵

$$[1 - (N - 1)\varepsilon] \left[\frac{W}{kN\varepsilon} - W \left(1 + \frac{|\Omega|}{k} \right) \right] - \varepsilon(N - 1)W \left(1 + \frac{|\Omega|}{k} \right) > 0 \quad (4.2.13)$$

After choosing ε , we choose a large enough $R \in \mathbb{R}_+$ such that $R > 2\|\tau(\ell)\|_1 M^*$ for all $\ell \in Lists_M$ and

$$[1 - (N - 1)\varepsilon] \left[\frac{R}{2kNM^*} - W \left(1 + \frac{|\Omega|}{k} \right) \right] - \varepsilon(N - 1)W \left(1 + \frac{|\Omega|}{k} \right) > 0 \quad (4.2.14)$$

where $M^* = \max \{\|\pi\|_1 : (\pi, \gamma) \in Clubs\}$. It is worthwhile to point out that both ε, R depend on k .

Step 2: Constructing aggregate excess demand

Since the utilities for added agents are strongly monotone in private goods, equilibrium prices of the economy \mathcal{E}^k cannot be arbitrarily low.⁶ Thus, for each $0 < \varepsilon \leq \frac{1}{N}$, we define the price space for private goods to be

$$\Delta_\varepsilon := \left\{ p \in \mathbb{R}_+^N : \sum_{n \in N} p_n = 1 \text{ and } p_n \geq \varepsilon \text{ for each } n \in N \right\}.$$

⁵ $|\Omega|$ denotes the number of elements in Ω .

⁶Since endowments of the added agents are $W\mathbf{1}$, the income of an added agent is at least W (since $\|p\|_1 = 1, p \cdot e_a = W$), as they can receive subsidies from memberships. Now, since they have perfectly substitutable preferences over private goods they choose the good with the least price. Without loss of generality let it be good 1. Now if the price of good 1 is arbitrarily low, then the demand $d_1 \geq \frac{W}{p_1}$ will be arbitrarily high. However, the aggregate demand for good 1 is bounded above by $W\lambda^k(A^k)$, a finite quantity and hence it may happen that $\int_{B^k} d_1 d\lambda^k > W\lambda^k(A^k)$ for an arbitrarily low value of p_1 . This implies that the aggregate demand of good 1 is more than that of supply in the economy for an arbitrarily low price of good 1 and thus the price for private goods must be bounded. One can analogously argue that for the same reason, membership prices have to be bounded so that memberships received are not arbitrarily high. More precisely, a negative membership price with high absolute value generates high income in the form of subsidy, which then leads to the aggregate demands of some private goods arbitrarily high.

For any $q \in \mathcal{T}rans$, if a particular membership price is positive and large enough, there must be some other membership price that is large enough and negative. Moreover, from the utilities of the added agents, it follows that such agents derive no utilities from consuming club goods. Consequently, such agents shall purchase memberships that provide them with large subsidies. This, however, leads to excess demand for private goods in the economy. Thus, membership prices must belong to a compact domain, which is defined as

$$\mathcal{Q}_R := \{q \in \mathcal{T}rans : |q_m| \leq R \text{ for all } m \in \mathcal{M}\}.$$

Fix an $\varepsilon > 0$, for a given $(p, q) \in \Delta_\varepsilon \times \mathcal{Q}_R$, the **budget set** for agent $a \in A$ is defined as

$$\mathbb{B}(a, p, q) := \{(x_a, \mu_a) \in X_a : p \cdot x_a + q \cdot \mu_a + p \cdot \tau(\mu_a) \leq p \cdot e_a\}.$$

The corresponding **demand** and **excess demand** sets are defined as

$$\mathbb{D}(a, p, q) := \arg \max \{u_a(x_a, \mu_a) : (x_a, \mu_a) \in \mathbb{B}(a, p, q)\},$$

and

$$\zeta(a, p, q) := \{(x_a + \tau(\mu_a), \mu_a) - (e_a, 0) : (x_a, \mu_a) \in \mathbb{D}(a, p, q)\}.$$

Since memberships are bounded above by M and $q \in \mathcal{Q}_R$, the subsidies from memberships are bounded above by RM . It follows that $\mathbb{B}(a, p, q)$ is compact. The continuity of u_a further implies that $\mathbb{D}(a, p, q) \neq \emptyset$, and hence $\zeta(a, p, q) \neq \emptyset$. Since individual endowments are bounded above by W , for all $a \in A^k$, the excess demand for each such agent satisfies the following inequalities:

$$-W \leq \zeta(a, p, q) \leq \frac{1}{\varepsilon} (W + RM).$$

Thus, aggregate excess demands for private goods for economy \mathcal{E}^k belong to the following compact subset of \mathbb{R}^N .

$$\mathcal{X} := \left\{ x \in \mathbb{R}^N : -\lambda^k(A^k) W \leq x_n \leq \lambda^k(A^k) \frac{1}{\varepsilon} (W + RM), \text{ for each } n \in N \right\}.$$

Aggregate demand for club memberships belongs to the set

$$\mathcal{C} := \left\{ \bar{\mu} \in \mathbb{R}_+^{\mathcal{M}} : \sum_{m \in \mathcal{M}} \bar{\mu}(m) \leq \lambda^k(A^k) M \right\}.$$

We define the aggregate excess demand correspondence $Z : \Delta_\varepsilon \times \mathcal{Q}_R \rightrightarrows \mathbb{R}^N \times \mathbb{R}_+^{\mathcal{M}}$ as

$$Z(p, q) := \int_{A^k} \zeta(a, p, q) d\lambda^k.$$

Step 3: Constructing modified aggregate excess demand

We define the modified aggregate excess demand correspondence $Z^* : \Delta_\varepsilon \times \mathcal{Q}_R \rightrightarrows \mathbb{R}^N \times \mathbb{R}_+^{\mathcal{M}}$ as

$$Z^*(p, q) := \int_{A^k} \zeta^*(a, p, q) d\lambda^k,$$

where for $a \in B^k$, $\zeta^*(a, p, q) := \zeta(a, p, q)$ and for $a \in A$,⁷

$$\zeta^*(a, p, q) := \begin{cases} \zeta(a, p, q), & \text{if } u_a(\mathbb{D}(a, p, q)) > u_a(f_a, l_a); \\ \zeta(a, p, q) \cup \{(0, 0)\}, & \text{if } u_a(\mathbb{D}(a, p, q)) = u_a(f_a, l_a); \text{ and} \\ \{(0, 0)\}, & \text{if } u_a(\mathbb{D}(a, p, q)) < u_a(f_a, l_a). \end{cases}$$

For a given $(p, q) \in \Delta_\varepsilon \times \mathcal{Q}_R$, define

$$S_{p,q} := \{a \in A : (f_a, l_a) \notin \mathbb{D}(a, p, q)\}.$$

Since (f, l) is not a pure transfer club equilibrium state, we must have $\lambda(S_{p,q}) > 0$.

Step 4: Constructing a pure-transfer club equilibrium of \mathcal{E}^k

We define a correspondence $\Phi^* : \Delta_\varepsilon \times \mathcal{Q}_R \times \mathcal{X} \times \mathcal{C} \rightrightarrows \Delta_\varepsilon \times \mathcal{Q}_R \times \mathcal{X} \times \mathcal{C}$ by

$$\Phi^*(p, q, x, \bar{\mu}) = [\arg \max \{(p^*, q^*) \cdot (x, \bar{\mu}) : (p^*, q^*) \in \Delta_\varepsilon \times \mathcal{Q}_R\}] \times Z^*(p, q).$$

By a standard argument,⁸ the correspondence Φ^* satisfies upper hemi-continuity and is non-empty, compact, and convex valued. Hence from Kakutani's fixed point theorem, we can guarantee the existence of a fixed point $((p^k, q^k), (\bar{z}^k, \bar{\mu}^k)) \in \Delta_\varepsilon \times \mathcal{Q}_R \times \mathcal{X} \times \mathcal{C}$ of Φ^* that maximizes the value of $(p^*, q^*) \cdot (x, \bar{\mu})$, where

$$(\bar{z}^k, \bar{\mu}^k) = \int_{A^k} (z_a^k, \mu_a^k) d\lambda^k$$

for some element $(z_a^k, \mu_a^k) \in \zeta^*(a, p^k, q^k)$ λ^k -a.e. on A^k . Walras law implies that $(p^k, q^k) \cdot (\bar{z}^k, \bar{\mu}^k) = 0$.

Step 5: To show that $\bar{z}^k = 0$ and $\bar{\mu}^k \in \mathcal{C}ons$

We first aim to show that $q^k \cdot \bar{\mu}^k = 0$. We show this by means of contradiction. So,

⁷Here, $u_a(\mathbb{D}(a, p, q)) = u_a(x_a, \mu_a)$ for $(x_a, \mu_a) \in \mathbb{D}(a, p, q)$.

⁸The proof is similar to that of establishing the properties of the excess demand correspondence in a classical continuum economy. One can refer to Proposition 3, page 149 of Hildenbrand [54] for a detailed proof. See Urbinati [?] for a recent work.

suppose not. Since, $0 \cdot \bar{\mu}^k = 0$, maximality and the definition of Φ^* imply that $q^k \cdot \bar{\mu}^k > 0$. Further, from maximality we can claim that $q^k \in \text{bdy } \mathcal{Q}_R$ and thus $|q_m^k| = R$ for some $m \in \mathcal{M}$. Now budget balance for club types implies that if some prices are large in magnitude and positive for some memberships then there must exist memberships for which prices are large in magnitude and negative. Let $m^* \in \mathcal{M}$ be a membership such that $q_{m^*}^k \leq -\frac{R}{M^*}$. An agent $b \in A_{\omega^*}^k$ could thus obtain a subsidy of $\frac{R}{M^*}$ by choosing the membership m^* . For such an agent, private goods are perfectly substitutable, and deriving no utility from club memberships will consume only the least expensive private good(s) and club memberships with nonpositive prices. As $R > 2\|\tau(l)\|_1 M^*$ for each private good the wealth used on inputs to clubs is less than $\frac{R}{2M^*}$. Thus b 's demand for the least expensive private good (without loss of generality, let it be good 1) is at least

$$\mathbb{D}_1(b, p^k, q^k) \geq \frac{R}{2NM^*}.$$

As $\lambda(A_{\omega^*}^k) = \frac{1}{k}$ and individual excess demands are bounded below by $-W\mathbf{1}$, aggregate excess commodity demand \bar{z}^k satisfies

$$\bar{z}_1^k \geq \frac{1}{k} \frac{R}{2NM^*} - W \left(1 + \frac{|\Omega|}{k} \right),$$

and

$$\bar{z}_n^k \geq -W \left(1 + \frac{|\Omega|}{k} \right), \text{ if } n > 1.$$

Define $p \in \Delta_\varepsilon$ by

$$p_1 = 1 - (N - 1)\varepsilon,$$

and

$$p_n = \varepsilon \text{ if } n > 1.$$

Some basic algebra yields that

$$p \cdot \bar{z}^k \geq [1 - (N - 1)\varepsilon] \left[\frac{R}{2kNM^*} - W \left(1 + \frac{|\Omega|}{k} \right) \right] - \varepsilon(N - 1)W \left(1 + \frac{|\Omega|}{k} \right).$$

Now it follows from Equation (4.2.14) that the right hand side of the above inequality is strictly positive. It can be further observed that

$$(p, 0) \cdot (\bar{z}^k, \bar{\mu}^k) > 0 = (p^k, q^k) \cdot (\bar{z}^k, \bar{\mu}^k).$$

This contradicts maximality. Hence, $q^k \cdot \bar{\mu}^k = 0$.

Next we aim at establishing $\bar{\mu}^k \in \mathcal{C}ons$. If not, we could find a $q^* \in \mathcal{T}rans$ such that $q^* \cdot \bar{\mu}^k > 0$ and hence one can find a $q^{**} \in \mathcal{Q}_R$ such that $q^{**} \cdot \bar{\mu}^k > 0$, contradicting maximality.

We claim that $p_n^k > \varepsilon$ for each n . So we suppose not. Once again we establish our claim by considering the excess demand of the auxiliary agents $A^k \setminus A$. As before, each such agent will consume only the least expensive private good(s) and club membership with nonpositive prices. Thus agent b 's demand for the least expensive private good which again we suppose without loss of generality to be good 1 is at least

$$\mathbb{D}_1(b, p^k, q^k) \geq \frac{W}{N\varepsilon}.$$

Thus, the aggregate excess demand \bar{z}^k satisfies

$$\bar{z}_1^k \geq \frac{1}{k} \frac{W}{N\varepsilon} - W \left(1 + \frac{|\Omega|}{k} \right),$$

and

$$\bar{z}_n^k \geq -W \left(1 + \frac{|\Omega|}{k} \right), \text{ if } n > 1.$$

Defining p as before, a little algebra yields

$$p \cdot \bar{z}^k \geq [1 - (N - 1)\varepsilon] \left[\frac{W}{kN\varepsilon} - W \left(1 + \frac{|\Omega|}{k} \right) \right] - \varepsilon(N - 1)W \left(1 + \frac{|\Omega|}{k} \right).$$

Equation (4.2.13) guarantees that the right-hand side of the above equation is strictly positive, and thus proceeding as before, we obtain

$$(p, 0) \cdot (\bar{z}^k, \bar{\mu}^k) > 0 = (p^k, q^k) \cdot (\bar{z}^k, \bar{\mu}^k).$$

This leads to a contradiction. Thus, $p_n^k \geq \varepsilon$ for each n .

We show that $\bar{z}^k = 0$. Since $(p^k, q^k) \cdot (\bar{z}^k, \bar{\mu}^k) = 0$ and $q^k \cdot \bar{\mu}^k = 0$, it follows that $p^k \cdot \bar{z}^k = 0$. Hence, if $\bar{z}^k \neq 0$ there are indices i, j such that $z_i^k < 0$ and $z_j^k > 0$. Define \hat{p} by

$$\hat{p}_i = p_i^k - \frac{1}{2}(p_i^k - \varepsilon),$$

$$\hat{p}_j = p_j^k + \frac{1}{2}(p_i^k - \varepsilon),$$

and

$$\hat{p}_n = p_n^k, n \neq i, j.$$

As $p_i^k > \varepsilon$, it follows that $\hat{p}_i = \frac{1}{2}(p_i^k + \varepsilon) > \varepsilon$. Thus, $\hat{p} \in \Delta_\varepsilon$. Because $p^k \cdot z^k = 0$, it follows that $\hat{p} \cdot z^k > 0$, a contradiction to maximality. Thus, we can conclude that $\bar{z}^k = 0$.

By definition, there exists a selection (z_a^k, μ_a^k) from $\zeta^*(a, p^k, q^k)$ such that

$$(\bar{z}^k, \bar{\mu}^k) = \int_{A^k} (z_a^k, \mu_a^k) d\lambda^k.$$

Set $x_a^k := z_a^k + e_a - \tau(\mu_a)$ for $a \in A^k$. Then $(x_a^k, \mu_a^k) \in \mathbb{D}(a, p^k, q^k)$ for all $a \in C_k$, where

$$C_k := \{a \in A : (z_a^k, \mu_a^k) \in \zeta(a, p^k, q^k)\}$$

has a positive measure due to the fact that $S_{p^k, q^k} \subseteq C_k$. Note that C_k can also be written as $C_k = \{a \in A : (x_a^k, \mu_a^k) \in \mathbb{D}(a, p^k, q^k)\}$.

Step 6: Constructing a bounded sequence $\{\bar{q}^k : k \geq 1\}$ of membership prices

Since membership prices q^k are bounded above by R , which in turn varies with k , we need to guarantee that a bounded sequence of membership prices exists. So passing to a subsequence if necessary we may assume without loss of generality that for each $\ell \in Lists_M$ ⁹ the sequence $\{q^k \cdot \ell : k \geq 1\}$ converges to a limit G_ℓ which may be finite or infinite. We define the following sets:

$$\mathbb{L} = \{\ell \in Lists_M : q^k \cdot \ell \rightarrow G_\ell \in \mathbb{R}\};$$

$$\mathbb{L}_+ = \{\ell \in Lists_M : q^k \cdot \ell \rightarrow +\infty\};$$

and

$$\mathbb{L}_- = \{\ell \in Lists_M : q^k \cdot \ell \rightarrow -\infty\}.$$

Consequently, for each $\ell \in \mathbb{L}$, the sequence $\{q^k \cdot \ell : k \geq 1\}$ is bounded. This implies the existence of a large enough $\bar{G} \in \mathbb{R}$ such that $|q^k \cdot \ell| \leq \frac{\bar{G}}{|\mathbb{L}|}$ ¹⁰ for each $k \geq 1$ and all $\ell \in \mathbb{L}$. Define $T : \mathcal{T}rans \rightarrow \mathbb{R}^{\mathbb{L}}$ as $T(q)_\ell = q \cdot \ell$. Recognized that T is a linear transformation, and we denote by $\text{ran}T := T(\mathcal{T}rans) \subseteq \mathbb{R}^{\mathbb{L}}$ the range of T . Now, let $\ker T := T^{-1}(0) \subseteq \mathcal{T}rans$, denote the null space of T or the kernel of T . It follows from the fundamental theorem of linear algebra that there exists a subspace H of $\mathcal{T}rans$ such that $H \cap \ker T = \{0\}$ and $H + \ker T = \mathcal{T}rans$. Denote by $T|_H$, the linear transformation T when the domain is restricted to H . It follows that the map $T|_H : H \rightarrow \text{ran}T$ is one-to-one and onto, and thus admits an inverse. Let $S : \text{ran}T \rightarrow H$ denote the inverse.

⁹ $Lists_M = \{\ell \in Lists : \|\ell\|_1 \leq M\}$.

¹⁰Note that $|\mathbb{L}|$ denotes the number of elements belonging to the set \mathbb{L} .

Since S is a linear functional, it must be continuous. Thus, there exists a constant K such that $\|S(x)\|_1 \leq K\|x\|_1$ for each $x \in \text{ran}T$. Let R^* be a constant constructed as in Lemma 4.2.4. Choose a large enough k_0 such that for all $k \geq k_0$, we have the following:

$$q^k \cdot \ell > 2K\bar{G}M + W \text{ if } \ell \in \mathbb{L}_+;$$

and

$$q^k \cdot \ell < -2K\bar{G}M - \frac{W}{R^*} \text{ if } \ell \in \mathbb{L}_-.$$

We define ST as a composition of S with T . For each $k \geq k_0$, define

$$\bar{q}^k := ST(q^k) - ST(q^{k_0}) + q^{k_0} \in \mathcal{T}rans. \quad (4.2.15)$$

Since $|q^k \cdot \ell| \leq \frac{\bar{G}}{|\mathbb{L}|}$ for all $\ell \in \mathbb{L}$, we have

$$\|ST(q^k)\|_1 \leq K\|T(q^k)\|_1 \leq K\bar{G}.$$

Similarly, $\|ST(q^{k_0})\|_1 \leq K\bar{G}$. Thus, from Equation (4.2.15), we have

$$|\bar{q}^k \cdot \ell| \leq 2K\bar{G}M + |q^{k_0} \cdot \ell|$$

for all $k > k_0$ and all $\ell \in \text{Lists}_M$. Thus, the prices of lists are bounded. In view of the fact that singleton memberships are themselves lists, we conclude that $\{\bar{q}^k : k \geq 1\} \subseteq \mathcal{T}rans$ is also a bounded sequence.

Step 7: $(x_a^k, \mu_a^k) \in \mathbb{D}(a, p^k, \bar{q}^k)$ for all $a \in C_k$ for all sufficiently large k

We know that for any k and $a \in C_k$, $(x_a^k, \mu_a^k) \in \mathbb{D}(a, p^k, q^k)$. First, we assert that for any $k > k_0$, $\mu_a^k \in \mathbb{L}$ for any $a \in C_k$. If $a \in C_k$, since endowments are bounded above by $W\mathbf{1}$ it follows quite trivially that $q^k \cdot \mu_a^k \leq W$. Thus, $\mu_a^k \notin \mathbb{L}_+$. Again, since the collection $\{\mu_a^k : a \in A^k\}$ is strictly balanced¹¹ and $q^k \in \mathcal{T}rans$, it follows from Lemma 4.2.4 that

$$\min \{q^k \cdot \mu_a^k : a \in A^k\} \geq -\frac{1}{R^*} \max \{q^k \cdot \mu_a^k : a \in A^k\} \geq -\frac{W}{R^*}.$$

Consequently, $\mu_a^k \notin \mathbb{L}_-$ for each $a \in C_k$. Hence, $\mu_a^k \in \mathbb{L}$ for each $a \in C_k$. On the other hand, it follows from Equation (4.2.15) that

$$T(\bar{q}^k) = TST(q^k) - TST(q^{k_0}) + T(q^{k_0}) = T(q^k). \quad (4.2.16)$$

¹¹As mentioned earlier the total number of memberships purchased by any agent is bounded above by M . Thus for any feasible membership vector $\alpha \in \mathbb{Z}^{\mathcal{M}}$ it must be that $0 \cdot \mathbf{1}_{\mathcal{M}} \leq \alpha \leq M \cdot \mathbf{1}_{\mathcal{M}}$, where $\mathbf{1}_{\mathcal{M}}$ is the \mathcal{M} -dimensional row vector of ones. For any given α , define $S_\alpha := \{a \in A^k : \mu_a = \alpha\}$ and $\mathbb{J} = \{\alpha : \lambda(S_\alpha) > 0\}$. Then $\int_{A^k} \mu_a^k d\lambda = \sum_{\alpha \in \mathbb{J}} \alpha \cdot \lambda(S_\alpha)$. Since $\int_{A^k} \mu_a^k d\lambda \in \mathcal{C}ons$ we can claim that the collection $\{\mu_a^k : a \in A^k\}$ is strictly balanced.

The last equality in the above equation follows from the fact that since $S, T|_H$ are inverses, the composition TS is an identity. Since $\mu_a^k \in \mathbb{L}$ for each $a \in C_k$, in view of Equation (4.2.16), we have $\bar{q}^k \cdot \mu_a^k = q^k \cdot \mu_a^k$ for each $a \in C_k$. Thus, $(x_a^k, \mu_a^k) \in \mathbb{B}(a, p^k, \bar{q}^k)$ for all $a \in C_k$. To show that (x_a^k, μ_a^k) is optimal at (p^k, \bar{q}^k) for each $a \in C_k$, we first choose $k_1 \geq k_0$ such that

$$q^k \cdot \ell < q^{k_0} \cdot \ell - 2K\bar{G}M \quad (4.2.17)$$

for all $\ell \in \mathbb{L}_-$ and $k \geq k_1$.¹² Choose and fix $a \in C_k$. Suppose that there exists some $(y, \nu) \in X_a$ such that $(y, \nu) \in \mathbb{B}(a, p^k, \bar{q}^k)$ and $u_a(y, \nu) > u_a(x_a^k, \mu_a^k)$. Budget feasibility of (y, ν) at prices (p^k, \bar{q}^k) implies that $\bar{q}^k \cdot \nu \leq W$ and hence, by Equation (4.2.15), we have $q^{k_0} \cdot \nu \leq W + 2K\bar{G}M$ as $\|ST(q^k)\|_1 \leq K\bar{G}$ and $\|ST(q^{k_0})\|_1 \leq K\bar{G}$. Since $q^k \cdot \ell > 2K\bar{G}M + W$ for all $\ell \in \mathbb{L}_+$, we have $\nu \notin \mathbb{L}_+$. For $\ell \in \mathbb{L}_-$ and $k > k_1$, by Equation (4.2.15) and Equation (4.2.17), we simply obtain

$$\bar{q}^k \cdot \ell > q^{k_0} \cdot \ell - 2K\bar{G}M > q^k \cdot \ell.$$

Thus, $\bar{q}^k \cdot \ell > q^k \cdot \ell$ for $\ell \in \mathbb{L}_-$. Thus, we have $\bar{q}^k \cdot \ell \geq q^k \cdot \ell$ for all $\ell \in \mathbb{L} \cup \mathbb{L}_-$. This along with the fact that $(y, \nu) \in \mathbb{B}(a, p^k, \bar{q}^k)$ further imply $(y, \nu) \in \mathbb{B}(a, p^k, q^k)$. Coupled with our assumption that $u_a(y, \nu) > u_a(x_a^k, \mu_a^k)$, it implies that $(x_a^k, \mu_a^k) \notin \mathbb{D}(a, p^k, q^k)$, contradicting the definition of C_k . Therefore, we conclude that $(x_a^k, \mu_a^k) \in \mathbb{D}(a, p^k, \bar{q}^k)$ for all $a \in C_k$ and $k \geq k_1$.

Step 8: Constructing Walrasian objection

Since $\{(p^k, \bar{q}^k, \bar{\mu}^k) : k \geq 1\}$ is a bounded sequence, passing to a subsequence, if necessary we may assume that $p^k \rightarrow p^* \in \Delta$, $\bar{q}^k \rightarrow q^* \in \mathcal{T}rans$, and $\bar{\mu}^k \rightarrow \bar{\mu}^* \in \mathcal{C}ons$. Since the sequence $\{\mu^k : k \geq 1\}$ is uniformly bounded, $\{\mu^k : k \geq 1\}$ uniformly integrable. By Schmeidler's version of Fatou's Lemma, there exists some $(z^*, \mu^*) : A \rightarrow \mathbb{R}^N \times \mathbb{R}_+^{\mathcal{A}}$ such that

- (i) $(z_a^*, \mu_a^*) \in Ls \{(z_a^k, \mu_a^k) : k \geq 1\}$ for λ -a.e. on A ;
- (ii) $\int_A (z_a^*, \mu_a^*) d\lambda \leq \liminf_{k \rightarrow \infty} \int_A (z_a^k, \mu_a^k) d\lambda$; and
- (iii) $\bar{\mu}^* = \int_A \mu_a^* d\lambda$.

Define S_0 as the set of agents $a \in A$ for whom $(x_a^*, \mu_a^*) := (z_a^* + e_a - \tau(\mu_a^*), \mu_a^*)$ is quasi-optimizing under the prices (p^*, q^*) .

Claim 1. $\lambda(S_0) > 0$. To see this, define

$$\widetilde{X}_a := \{(y_a, \nu_a) \in X_a : u_a(y_a, \nu_a) > u_a(f_a, l_a)\};$$

¹²Since $Lists_M$ has finitely many elements, so does \mathbb{L}_-

and

$$X_a^k := \left\{ (y_a, \nu_a) \in \widetilde{X}_a : p^k \cdot y_a + \bar{q}^k \cdot \nu_a + p^k \cdot \tau(\nu_a) < p^k \cdot e_a \right\}.$$

Let

$$R := \left\{ a \in A : p^* \cdot y_a + q^* \cdot \nu_a + p^* \cdot \tau(\nu_a) < p^* \cdot e_a \text{ for some } (y_a, \nu_a) \in \widetilde{X}_a \right\};$$

and

$$R_m := \left\{ a \in A : \bigcap_{k=m}^{\infty} X_a^k \neq \emptyset \right\}$$

for all $m \geq 1$. From the definition of R_m , it follows that $\{R_m : m \geq 1\}$ is an ascending sequence of coalitions such that

$$R \subseteq \bigcup \{R_m : m \geq 1\}.$$

We know that (f, l) is not a pure-transfer equilibrium state under the price vector (p^*, q^*) . Therefore since (f, l) is club-linked we conclude that (f, l) is not a pure-transfer quasi-equilibrium state under the price vector (p^*, q^*) . Therefore, we must have $\lambda(R) > 0$. We show that $\lambda(R \setminus S_0) = 0$. To this end, pick an element $a \in R$ such that¹³

$$(z_a^*, \mu_a^*) \in Ls \left\{ (z_a^k, \mu_a^k) : k \geq 1 \right\}. \quad (4.2.18)$$

Then there is an $m_0 \geq 1$ such that $a \in R_{m_0}$, which further implies $u_a(\mathbb{D}(a, p^k, \bar{q}^k)) > u_a(f_a, l_a)$ for all $k \geq m_0$. It follows from the definition of $\zeta^*(a, p^k, \bar{q}^k)$ that $\zeta^*(a, p^k, \bar{q}^k) = \zeta(a, p^k, \bar{q}^k)$, which further implies $a \in C_k$ and thus, $(x_a^k, \mu_a^k) \in \mathbb{D}(a, p^k, \bar{q}^k)$ for all $k \geq m_0$, where $x_a^k := z_a^k + e_a - \tau(\mu_a^k)$. Thus, defining $(x_a^*, \mu_a^*) := (z_a^* + e_a - \tau(\mu_a^*), \mu_a^*)$, from Equation (4.2.18) we note that

$$(x_a^*, \mu_a^*) \in Ls \left\{ (x_a^k, \mu_a^k) : k \geq 1 \right\}.$$

Then there is a subsequence $\{k_r : r \geq 1\}$ of positive integers such that

$$(x_a^*, \mu_a^*) = \lim_{r \rightarrow \infty} (x_a^{k_r}, \mu_a^{k_r}).$$

It can be readily verified that (x_a^*, μ_a^*) is quasi-optimizing under the price (p^*, q^*) and thus $a \in S_0$. Since $(z_a^*, \mu_a^*) \in Ls \left\{ (z_a^k, \mu_a^k) : k \geq 1 \right\}$ λ -a.e. on A , we conclude that $a \in S_0$ for almost all $a \in R$ i.e. $\lambda(R \setminus S_0) = 0$. Since, $\lambda(R) > 0$ we have $\lambda(S_0) > 0$.

Claim 2: $(z_a^*, \mu_a^*) = (0, 0)$ for almost all $a \in A \setminus S_0$. To show this, by contradiction, suppose that $(z_a^*, \mu_a^*) \neq (0, 0)$ for all $a \in S_{00}$ for some sub-coalition S_{00} of $A \setminus S_0$.

¹³This is due to Condition (i) and the fact that $\lambda(R) > 0$.

As $(z_a^*, \mu_a^*) \in Ls \{(z_a^k, \mu_a^k) : k \geq 1\}$ for λ -a.e. on A , there exists some $a \in S_{00}$ and a subsequence $\{k_r : r \geq 1\}$ of positive integers such that $(z_a^{k_r}, \mu_a^{k_r}) \neq (0, 0)$ for all $r \geq 1$ and

$$(z_a^*, \mu_a^*) = \lim_{r \rightarrow \infty} (z_a^{k_r}, \mu_a^{k_r}).$$

As $(z_a^{k_r}, \mu_a^{k_r}) \in \zeta(a, p^{k_r}, \bar{q}^{k_r})$, we have $(x_a^{k_r}, \mu_a^{k_r}) := (z_a^{k_r} + e_a - \tau(\mu_a^{k_r}, \mu_a^{k_r})) \in \mathbb{D}(a, p^{k_r}, \bar{q}^{k_r})$. If we define $(x_a^*, \mu_a^*) = (z_a^* + e_a - \tau(\mu_a^*, \mu_a^*))$, then we have

$$(x_a^*, \mu_a^*) = \lim_{r \rightarrow \infty} (x_a^{k_r}, \mu_a^{k_r}).$$

Hence (x_a^*, μ_a^*) is quasi-optimizing for individual ‘ a ’ under the price vector (p^*, q^*) . This implies $a \in S_0$, which is a contradiction.

Claim 3: $p^* \gg 0$ and (x^*, μ^*) is feasible for S_0 . To see this, by **Claim 2** we note that

$$\int_{S_0} \mu_a^* d\lambda = \int_A \mu_a^* d\lambda = \bar{\mu}^* \in \mathcal{C}ons,$$

and

$$\int_{S_0} z_a^* d\lambda = \int_A z_a^* d\lambda \leq \lim_{k \rightarrow \infty} \int_A z_a^k d\lambda = \lim_{k \rightarrow \infty} \bar{z}^k = 0.$$

Since $q^* \in \mathcal{T}rans$, we have

$$q^* \cdot \int_{S_0} \mu_a^* d\lambda = 0. \quad (4.2.19)$$

Thus, from above, we obtain

$$\int_{S_0} (x_a^* + \tau(\mu_a^*)) d\lambda \leq \int_{S_0} e_a d\lambda. \quad (4.2.20)$$

If the above equation holds without equality, then one can claim that there exists a state (x^{**}, μ^*) for the coalition S_0 such that

- (a) $x_a^{**} \geq x_a^*$ for all $a \in S_0$;
- (b) (x_a^{**}, μ_a^*) is quasi-optimizing for every agent $a \in S_0$ under the price (p^*, q^*) ;¹⁴ and
- (c) $\int_{S_0} (x_a^{**} + \tau(\mu_a^*)) d\lambda = \int_{S_0} e_a d\lambda$.

¹⁴This follows from Condition (a) and the fact that (x_a^*, μ_a^*) is quasi-optimizing for every agent $a \in S_0$ under the price (p^*, q^*) .

It follows from Condition (b), Condition (c), and Remark 4.1.8 that $p^* \gg 0$. This along with Equation (4.2.19) and the fact that the Equation (4.2.20) holds without equality imply that

$$p^* \cdot \int_{S_0} x_a^* d\lambda + p^* \cdot \int_{S_0} \tau(\mu_a^*) d\lambda + q^* \cdot \int_{S_0} \mu_a^* d\lambda < p^* \cdot \int_{S_0} e_a d\lambda. \quad (4.2.21)$$

Note that as (x_a^*, μ_a^*) is quasi-optimizing for every agent $a \in S_0$ under the price (p^*, q^*) we can show by strict monotonicity of preferences that

$$p^* \cdot x_a^* + q^* \cdot \mu_a^* + p^* \cdot \tau(\mu_a^*) \geq p^* \cdot e_a.$$

for all $a \in S_0$, which further yields that

$$p^* \cdot \int_{S_0} x_a^* d\lambda + p^* \cdot \int_{S_0} \tau(\mu_a^*) d\lambda + q^* \cdot \int_{S_0} \mu_a^* d\lambda \geq p^* \cdot \int_{S_0} e_a d\lambda. \quad (4.2.22)$$

However, Equation (4.2.21) and Equation (4.2.22) cannot hold simultaneously. Hence, our supposition is wrong and Equation (4.2.20) holds with equality. It then follows that (x^*, μ^*) is a feasible state for S_0 and (x^*, μ^*) is quasi-optimizing for every agent $a \in S_0$ under the price (p^*, q^*) . Therefore, by Lemma 4.1.8, we have $p^* \gg 0$.

Claim 4: $\zeta^*(\mathbf{a}, p^*, q^*) \neq \zeta(\mathbf{a}, p^*, q^*)$ for almost all $\mathbf{a} \in \mathbf{A} \setminus \mathbf{S}_0$. Define

$$\tilde{R} := \{a \in A \setminus S_0 : \zeta^*(a, p^*, q^*) = \zeta(a, p^*, q^*)\}.$$

We want to show that $\lambda(\tilde{R}) = 0$. Suppose not. Choose an arbitrary $a \in \tilde{R}$ and an element $(\tilde{z}_a, \tilde{\nu}_a) \in \zeta(a, p^*, q^*)$. Then by the definition of $\zeta^*(a, p^*, q^*)$, we have $u_a(\tilde{y}_a, \tilde{\nu}_a) > u_a(f_a, l_a)$, where $(\tilde{y}_a, \tilde{\nu}_a) := (\tilde{z}_a + e_a - \tau(\tilde{\nu}_a), \tilde{\nu}_a)$. As $(\tilde{y}_a, \tilde{\nu}_a) \in \mathbb{D}(a, p^*, q^*)$, desirability of endowments guarantees that $\tilde{y}_a \neq 0$ and as $p^* \gg 0$ we have $p^* \cdot \tilde{y}_a > 0$. By the continuity of preferences, there exists some $\delta \in (0, 1)$ such that

$$u_a(\delta \tilde{y}_a, \tilde{\nu}_a) > u_a(f_a, l_a) \text{ and } p^* \cdot \delta \tilde{y}_a + q^* \cdot \tilde{\nu}_a + p^* \cdot \tau(\tilde{\nu}_a) < p^* \cdot e_a.$$

Hence

$$p^k \cdot \delta \tilde{y}_a + \bar{q}^k \cdot \tilde{\nu}_a + p^k \cdot \tau(\tilde{\nu}_a) < p^k \cdot e_a$$

holds for all large enough k . So, $u_a(\mathbb{D}(a, p^k, \bar{q}^k)) > u_a(\delta \tilde{y}_a, \tilde{\nu}_a)$ for all large enough k and thus it follows that $u_a(\mathbb{D}(a, p^k, \bar{q}^k)) > u_a(f_a, l_a)$ for all large enough k . From the definition of $\zeta^*(a, p^k, \bar{q}^k)$ we have $\zeta^*(a, p^k, \bar{q}^k) = \zeta(a, p^k, \bar{q}^k)$, and hence one can claim that $(z_a^k, \mu_a^k) \in \zeta(a, p^k, \bar{q}^k)$ for all large enough k . Applying similar arguments as in the last part of the proof of **Claim 1** one can guarantee that

$$(x_a^*, \mu_a^*) = \lim_{r \rightarrow \infty} (x_a^{k_r}, \mu_a^{k_r}),$$

where $x_a^* = z_a^* + e_a - \tau(\mu_a^*)$ and $x_a^{k_r} := z_a^{k_r} + e_a - \tau(\mu_a^{k_r})$ for some subsequence $\{k_r : r \geq 1\}$ of natural numbers. It is easy to verify that (x_a^*, μ_a^*) is quasi-optimizing for individual 'a' under the price vector (p^*, q^*) and thus $a \in S_0$. This is a contradiction and hence $\lambda(\tilde{R}) = 0$.

Claim 5: $(S_0, (x^*, \mu^*))$ is a Walrasian objection to (f, l) . Choose an $a \in S_0$ and let $(y_a, \nu_a) \in X_a$ be such that $u_a(y_a, \nu_a) \geq u_a(x_a^*, \mu_a^*)$. Since (x_a^*, μ_a^*) is quasi-optimizing under the price vector (p^*, q^*) we have

$$p^* \cdot y_a + q^* \cdot \nu_a + p^* \cdot \tau(\nu_a) \geq p^* \cdot e_a.$$

By **Claim 4** it follows that $\zeta^*(a, p^*, q^*) \neq \zeta(a, p^*, q^*)$ holds λ -a.e. on $A \setminus S_0$. Choose such an $a \in A \setminus S_0$. Let $(y_a, \nu_a) \in X_a$ be such that $u_a(y_a, \nu_a) \geq u_a(f_a, l_a)$. As $\zeta^*(a, p^*, q^*) \neq \zeta(a, p^*, q^*)$, it follows from the definition of $\zeta^*(a, p^*, q^*)$, that $u_a(f_a, l_a) \geq u_a(\mathbb{D}(a, p^*, q^*))$. Consequently, $u_a(y_a, \nu_a) \geq u_a(\mathbb{D}(a, p^*, q^*))$. Thus

$$p^* \cdot y_a + q^* \cdot \nu_a + p^* \cdot \tau(\nu_a) \geq p^* \cdot e_a.$$

As $\zeta^*(a, p^*, q^*) \neq \zeta(a, p^*, q^*)$ holds λ -a.e. on $A \setminus S_0$, it follows that for almost all $a \in A \setminus S_0$, if $u_a(y_a, \nu_a) \geq u_a(f_a, l_a)$ then

$$p^* \cdot y_a + q^* \cdot \nu_a + p^* \cdot \tau(\nu_a) \geq p^* \cdot e_a.$$

Therefore, we have established that $(S_0, (x^*, \mu^*))$ is a pure-transfer Walrasian objection to (f, l) . Further, by Lemma 4.1.6 we can claim that there exists a membership price vector \tilde{q}^* such that $(S_0, (x^*, \mu^*))$ forms a Walrasian objection to (f, l) under the price vector (p^*, \tilde{q}^*) . \square

In the next theorem, we present our main equivalence result with the help of the results obtained in the above two propositions.

Theorem 4.2.6. *Let \mathcal{E} be a club economy for which endowments are desirable and uniformly bounded above. Further, let \mathcal{E} be strong club irreducible. Then the bargaining set of the economy \mathcal{E} coincides with the set of club equilibrium states.*

Proof. We first intend to show that $\mathcal{W}(\mathcal{E}) \subseteq \mathcal{B}(\mathcal{E})$. This is quite obvious. From the definition of bargaining set, it follows that the core of the economy $\mathcal{C}(\mathcal{E}) \subseteq \mathcal{B}(\mathcal{E})$. It can be noted from Theorem 4.1 of Ellickson et al. [35] that $\mathcal{W}(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E})$. Hence, it automatically follows that $\mathcal{W}(\mathcal{E}) \subseteq \mathcal{B}(\mathcal{E})$. For the reverse direction, we use the contrapositive argument. So let (f, l) be a feasible state such that $(f, l) \notin \mathcal{W}(\mathcal{E})$. Then from Proposition 4.2.5 we conclude that there exists a Walrasian objection $(S, (g, \mu))$ against it. Further from Proposition 4.2.2 we can claim that $(S, (g, \mu))$ is a justified objection to (f, l) . This further implies that $(f, l) \notin \mathcal{B}(\mathcal{E})$. This completes the proof. \square

Chapter 5

Restricted Bargaining Sets

The last chapter examined the decentralization of the bargaining set introduced by Aumann and Maschler [6] and adapted by Mas-Colell [64] to the case of exchange economies. In this chapter, we introduce a new notion of the bargaining set according to Vind [81] further in an atomless club economy and refer to the bargaining set by Mas-Colell [64], the **local bargaining set**, and the one defined by Vind [81] as the **global bargaining set**. Seminal studies by Schmeidler (1972) and Vind (1972) show that the set of core allocations is robust to blocking by arbitrarily small or large-sized coalitions. These characterizations of the core encourage a similar analysis for the bargaining set, as the underlying veto mechanism is fortified by forward-looking behavior. In that spirit Schjødtt and Sloth [70] and Hervés-Estévez and Moreno-García [48] provided a characterization of the global bargaining set in terms of the size of the blocking coalition. Schjødtt and Sloth (1994) showed that if one restricts (counter-) objecting coalition to be of arbitrarily small measure, then while the local bargaining set may get strictly enlarged, the global bargaining set is unaltered. Hervés-Estévez and Moreno-García [48] took somewhat of a similar approach to Vind [80] and showed that it is sufficient to consider counter-blocking by arbitrarily large-sized coalitions only to obtain the global bargaining set in economies with only private commodities. We follow in the footsteps of the literature and attempt to provide characterizations of the global bargaining set in the spirit of Schjødtt and Sloth [70] and Hervés-Estévez and Moreno-García [48]. Furthermore, Hervés-Estévez and Moreno-García [48] showed in an atomless economy with only private commodities, that the set of globally justified allocations is equivalent to the set of robustly efficient allocations. In this chapter, although we can show that any robustly efficient allocation qualifies as a globally justified objection the converse fails to hold. One can attribute such a failure to analogous reasons for which a club equilibrium state fails to qualify as a robustly efficient one

(see Remark 5.3.3). To tackle this failure, we posit a stronger notion of “sequential robust efficiency” and show that one can establish that the set of global justified allocations is a subset of sequentially robustly efficient allocations. However, we are uncertain about the converse at this moment. Thus, to provide a complete characterization we adapt the notion of “approximate robust efficiency” from Bhowmik and Kaur [22] and characterize the global justified objections in an atomless club economy in terms of approximately robustly efficient allocations thereby extending the work by Hervés-Estévez and Moreno-García [48]. This chapter is based on the paper [24].

5.1 Bargaining Sets

In this section, we formally define both the local and global bargaining sets proposed by Mas-Colell [64] and Vind [81]. We redefine the notion of bargaining set introduced by Mas-Colell [64] for the sake of completeness. We further define our notions of restricted bargaining sets by formally introducing our notions of restricted objections and counter-objections.

Definition 5.1.1. A **local objection** to the state (f, l) in the economy \mathcal{E} is a pair $(S, (g, \mu))$, where (g, μ) is a state which is feasible for the coalition S , such that $u_a(g_a, \mu_a) \geq u_a(f_a, l_a)$ for all $a \in S$ and

$$\lambda(\{a \in S : u_a(g_a, \mu_a) > u_a(f_a, l_a)\}) > 0.$$

A state (f, l) is said to be in the **core** of the economy \mathcal{E} if there does not exist a local objection to it. Equivalently, the core of the economy \mathcal{E} could be defined as the set of states (f, l) , such that there do not exist any global objection to (f, l) , where a **global objection** to the state (f, l) in the economy \mathcal{E} is a pair $(S, (g, \mu))$ if (g, μ) is a feasible state and $(S, (g|_S, \mu|_S))$ is also a local objection to (f, l) .

Definition 5.1.2. A **local counter-objection** to a local objection $(S, (g, \mu))$ of a state (f, l) is a pair $(T, (h, \nu))$, where (h, ν) is a feasible state for the coalition T , and $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$ λ -a.e. on $S \cap T$; and $u_a(h_a, \nu_a) > u_a(f_a, l_a)$ λ -a.e. on $T \setminus S$. A local objection is said to be a **local justified objection** if there is no local counter-objection against it. The **local bargaining set** of the economy \mathcal{E} , denoted by $\mathcal{B}^\ell(\mathcal{E})$, is the set of feasible states of the economy \mathcal{E} against which there does not exist any local justified objection.

Definition 5.1.3. A **global counter-objection** to a global objection $(S, (g, \mu))$ of a state (f, l) is a pair $(T, (h, \nu))$, where (h, ν) is a feasible state for the coalition T , and

$u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$ λ -a.e. on T . A global objection is said to be a **global justified objection** if there is no global counter-objection against it. The **global bargaining set** of the economy \mathcal{E} , denoted by $\mathcal{B}^g(\mathcal{E})$, is the set of feasible states of the economy \mathcal{E} against which there does not exist any global justified objection.

Remark 5.1.4. There are three key differences between the local and global bargaining sets:

- The global objection mechanism juxtaposed to the local notion not only proposes a feasible state for the objecting coalition but proposes a feasible state for the overall economy as a whole. Thus, objecting to any feasible state is harder in this scenario.
- The local counter-objection mechanism proposes a state that improves upon the original state for agents who belong only to the counter-objecting coalition and not the objecting one. The global counter-objection mechanism on the other hand proposes that all agents in the counter-objecting state must improve upon the objecting state necessarily. However, this follows quite trivially from the definition of global objection.
- The local and global bargaining set both contain the core of an atomless club economy \mathcal{E} . Now, Mas-Colell [64] showed that the local bargaining set coincides with the set of equilibrium allocation, which generalizes the core-Walras equivalence theorem for exchange economies. However, a similar result fails to hold for the global bargaining set, and in fact, it is larger than the set of equilibrium allocations as remarked by Vind [81].

5.1.1 The $\varepsilon\delta$ -Bargaining Sets

Schjødtt and Sloth [70] introduced a notion of $\varepsilon\delta$ -bargaining set by restricting the sizes of both objecting and counter-objecting coalitions in tandem with the core characterization proposed by Schmeidler [71]. In what follows, we adapt this notion of restricted objection and counter-objection in the spirit of Schjødtt and Sloth [70] and introduce the $\varepsilon\delta$ -bargaining set for our club economy.

Definition 5.1.5. Given a $\delta > 0$, we say that $(S, (g, \mu))$ is a **δ -feasible local objection** to (f, l) if $(S, (g, \mu))$ is a local objection to (f, l) and $\lambda(S) \leq \delta$. Given an $\varepsilon > 0$ we say that $(T, (h, \nu))$ is an **ε -feasible local counter-objection** to $(S, (g, \mu))$ if $(T, (h, \nu))$ is a local counter-objection to $(S, (g, \mu))$ and $\lambda(T) \leq \varepsilon$. Given $\varepsilon, \delta > 0$, a δ -feasible

local objection $(S, (g, \mu))$ to (f, l) is said to be ε -**justified** if there does not a ε -feasible local counter-objection $(T, (h, \nu))$ to it. The **local $\varepsilon\delta$ -bargaining set**, denoted by $\mathcal{B}_{\varepsilon\delta}^l(\mathcal{E})$, is the set of states against which there exists no ε -justified δ -feasible local objection. The **global $\varepsilon\delta$ -bargaining set**, denoted by $\mathcal{B}_{\varepsilon\delta}^g(\mathcal{E})$, is defined analogously.

5.2 The Sizes of (Counter-) Objecting Coalitions

Seminal studies by Schmeidler [71] and Vind [80] characterize core allocations as robust to blocking by arbitrarily small or large-sized coalitions. One can repeat a similar exercise for the global bargaining set. To begin with, we show that it is sufficient to consider counter-objection by arbitrarily small-sized coalitions. Note that such a characterization holds for the local bargaining set as well. However, when one restricts both the objecting and counter-objecting coalitions to arbitrary small measures, the local bargaining set is altered. The global bargaining set on the other hand is robust to such restrictions. This section studies all these issues.

Lemma 5.2.1. *Let \mathcal{E} be a club economy and (f, l) be a feasible state in \mathcal{E} . Further, let $(S, (g, \mu))$ be a (local) global objection to (f, l) and $(T, (h, \nu))$ be a (local) global counter-objection to $(S, (g, \mu))$. Then for every $\varepsilon \in (0, \lambda(T))$, there exists a coalition R such that $\lambda(R) = \varepsilon$ and $(R, (h, \nu))$ forms a (local) global counter-objection to $(S, (g, \mu))$.*

Proof. Let $\varepsilon \in (0, \lambda(T))$. Since $(T, (h, \nu))$ is a counter-objection to $(S, (g, \mu))$, we have

$$\int_T h_a d\lambda + \int_T \tau(\nu_a) d\lambda = \int_T e_a d\lambda \text{ and } \int_T \nu_a d\lambda \in \mathcal{C}ons.$$

We now define a vector-valued measure κ on $\Sigma|_T$ as

$$\kappa(B) := \left\{ \left(\lambda(B), \int_B (e_a - h_a + \tau(\nu_a)) d\lambda, \int_B \nu_a d\lambda \right) : B \in \Sigma|_T \right\}.$$

Therefore, we have

$$\kappa(\emptyset) = (0, 0, 0) \text{ and } \kappa(T) = \left(\lambda(T), 0, \int_T \nu_a d\lambda \right).$$

It follows from Lyapunov's convexity theorem that $\{\kappa(B) : B \in \Sigma|_T\}$ is a convex set. For $\alpha = \frac{\varepsilon}{\lambda(T)}$, the convexity of $\{\kappa(B) : B \in \Sigma|_T\}$ guarantees existence of a coalition $R \subseteq T$ such that

$$\kappa(R) = (1 - \alpha) \cdot \kappa(\emptyset) + \alpha \cdot \kappa(T) = \left(\varepsilon, 0, \alpha \cdot \int_T \nu_a d\lambda \right).$$

Therefore, we have that $\lambda(R) = \varepsilon$ and the following conditions are satisfied

- (i) $\int_R h_a d\lambda + \int_R \tau(\nu_a) d\lambda = \int_R e_a d\lambda$; and
- (ii) $\int_R \nu_a d\lambda \in \mathcal{C}ons$, as $\mathcal{C}ons$ is a subspace.

Thus, $(R, (h, \nu))$ forms a counter-objection to $(S, (g, \mu))$. This completes the proof. \square

Lemma 5.2.2. *Let \mathcal{E} be a club economy and let (f, l) be a feasible state in \mathcal{E} . Any (local) global objection $(S, (g, \mu))$ to the state (f, l) is justified if and only if it is ε -justified.*

Proof. The 'If' part follows from the previous lemma and the 'Only if' part follows from the fact that ε -feasible counter-objection is also a counter-objection. \square

Theorem 5.2.3. *Let \mathcal{E} be a club economy. For any $\varepsilon, \delta > 0$ we have $\mathcal{B}^l(\mathcal{E}) \subseteq \mathcal{B}_{\varepsilon\delta}^l(\mathcal{E})$ and $\mathcal{B}^g(\mathcal{E}) \subseteq \mathcal{B}_{\varepsilon\delta}^g(\mathcal{E})$.*

Proof. By abuse of notation, let us denote by $\mathcal{B}(\mathcal{E})$ both the local and global bargaining sets of the economy \mathcal{E} . Likewise, the notation $\mathcal{B}_{\varepsilon\delta}(\mathcal{E})$ is employed to denote both local and global $\varepsilon\delta$ -bargaining sets of the economy \mathcal{E} . Suppose $(f, l) \in \mathcal{B}(\mathcal{E}) \setminus \mathcal{B}_{\varepsilon\delta}(\mathcal{E})$. This means that there exists an ε -justified δ -objection to the feasible state (f, l) . Let $(S, (g, \mu))$ be an ε -justified δ -objection to (f, l) , where $\lambda(S) \leq \delta$. By Lemma 5.2.2, it follows that $(S, (g, \mu))$ itself constitutes a justified objection to (f, l) . This is a contradiction as $(f, l) \in \mathcal{B}(\mathcal{E})$. So $\mathcal{B}(\mathcal{E}) \subseteq \mathcal{B}_{\varepsilon\delta}(\mathcal{E})$. \square

Remark 5.2.4. Consider $\delta \geq \lambda(A)$. Lemma 5.2.2 guarantees that any justified objection is ε -justified. Thus, for the local bargaining set $\mathcal{B}^l(\mathcal{E})$ of the economy it can be readily verified that $\mathcal{B}_{\varepsilon\delta}^l(\mathcal{E}) \subseteq \mathcal{B}^l(\mathcal{E})$. Combined with Theorem 5.2.3 it can be inferred that $\mathcal{B}_{\varepsilon\delta}^l(\mathcal{E}) = \mathcal{B}^l(\mathcal{E})$. However, as established in Theorem 3 of Schjødtt and Sloth (1994) the equivalence fails to hold if $\delta < \lambda(A)$ for a typical exchange economy with only private commodities. It can be observed as an atomless economy with only private commodities corresponds to a special case for our club economy, their argument holds in our framework and we can claim that if $\delta < \lambda(A)$, $\mathcal{B}_{\varepsilon\delta}^l(\mathcal{E}) \neq \mathcal{B}^l(\mathcal{E})$ for our atomless club economy as well.

Note that, although the equivalence fails to hold for the local bargaining set, one can show that the global bargaining set $\mathcal{B}^g(\mathcal{E})$ is robust to the choice of any $0 \leq \delta \leq \lambda(A)$. This was a key observation made by Schjødtt and Sloth [70] for an atomless economy with only private commodities. We establish a similar result for a club economy.

Theorem 5.2.5. *Let \mathcal{E} be a club economy. For any $\varepsilon, \delta > 0$, we have $\mathcal{B}^g(\mathcal{E}) = \mathcal{B}_{\varepsilon\delta}^g(\mathcal{E})$*

Proof. By Theorem 5.2.3, we have $\mathcal{B}^g(\mathcal{E}) \subseteq \mathcal{B}_{\varepsilon\delta}^g(\mathcal{E})$. We show that $\mathcal{B}_{\varepsilon\delta}^g(\mathcal{E}) \subseteq \mathcal{B}^g(\mathcal{E})$. Let $(f, l) \in \mathcal{B}_{\varepsilon\delta}^g(\mathcal{E})$. Then, either (f, l) has no objection, and if there exists a δ -objection to it, there exists an ε -counter-objection to it. If (f, l) has no objection, then $(f, l) \in \mathcal{C}(\mathcal{E}) \subseteq \mathcal{B}^g(\mathcal{E})$. So let, $(S, (g, \mu))$ be an objection to (f, l) . Lyapunov's convexity theorem guarantees the existence of a sub-coalition \tilde{S} of S such that $\lambda(\tilde{S}) \leq \delta$ and $(\tilde{S}, (g, \mu))$ constitutes a δ -feasible objection to (f, l) . Since $(f, l) \in \mathcal{B}_{\varepsilon\delta}^g(\mathcal{E})$, there exists an ε -feasible counter-objection $(T, (h, \nu))$ to $(\tilde{S}, (g, \mu))$. It can be easily observed that $(T, (h, \nu))$ constitutes an objection to $(S, (g, \mu))$, which means that $(f, l) \in \mathcal{B}^g(\mathcal{E})$. This completes the proof of the theorem. \square

One can further look into the blocking mechanism under the specs of a normative notion and check whether the set of states in the bargaining set can be obtained as outcomes of a ‘‘majority voting’’ mechanism. Hervés-Estévez and Moreno-García [48] in their paper show that given any $0 < \alpha < \lambda(A)$ it is sufficient to consider only counter-objections by coalitions of a fixed size ‘ α ’ to obtain the global bargaining set. Their idea was inspired by the work of Vind [80] who showed that in an atomless economy with only private commodities, one can always construct a blocking coalition to a non-core allocation whose measure is equal to α for all $0 < \alpha < \lambda(A)$. Our next result is an extension of Lemma 3.2 of Hervés-Estévez and Moreno-García [48] to our framework.

Proposition 5.2.6. *Let \mathcal{E} be a club economy. Further, let (f, l) be a state in \mathcal{E} and $(S, (g, \mu))$ a global objection to (f, l) . Let $(T, (h, \nu))$ be a global counter-objection to $(S, (g, \mu))$. Then for all $\alpha \in (0, \lambda(A))$ there exists a global counter-objection $(E, (y, \kappa))$ to $(S, (g, \mu))$ such that $\lambda(E) = \alpha$.*

Proof. Since $(T, (h, \nu))$ is a counter-objection to $(S, (g, \mu))$ then it must follow that

- (i) $(h_a, \nu_a) \in X_a$ λ -a.e. on T ;
- (ii) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$ λ -a.e. in T ;
- (iii) $\int_T h_a d\lambda + \int_T \tau(\nu_a) d\lambda = \int_T e_a d\lambda$; and
- (iv) $\int_T \nu_a d\lambda \in \mathcal{C}ons$.

By Lemma 5.2.1, without loss of generality, we assume that $\lambda(T) < \alpha < \lambda(A)$. Define δ such that

$$\delta := 1 - \frac{\alpha - \lambda(T)}{\lambda(A \setminus T)}.$$

By the continuity of monotonicity of preferences, we can find a function $\zeta : T \rightarrow \mathbb{R}_+^N$ such that $u_a(\zeta_a, \nu_a) > u_a(g_a, \mu_a)$ λ -a.e. on T and

$$\int_T \zeta_a d\lambda = \int_T h_a d\lambda - z.$$

By Lemma 3.1 in Bhowmik and Kaur [22], there exists an allocation (ξ, ν') such that

- (i) $u_a(\xi_a, \nu'_a) > u_a(g_a, \mu_a)$, λ -a.e. on T ;
- (ii) $\int_T \xi_a d\lambda = \int_T (\delta \cdot \zeta_a + (1 - \delta) \cdot g_a) d\lambda$; and
- (iii) $\int_T \nu'_a d\lambda = \int_T (\delta \cdot \nu_a + (1 - \delta) \cdot \mu_a) d\lambda$.

Furthermore, by the Lyapunov convexity theorem, there exists a coalition $R \subseteq A \setminus T$ such that

- (1) $\lambda(R) = (1 - \delta)\lambda(A \setminus T)$;
- (2) $\int_R \mu_a d\lambda = (1 - \delta) \int_{A \setminus T} \mu_a d\lambda$; and
- (3) $\int_R (g_a + \tau(\mu_a) - e_a) d\lambda = (1 - \delta) \int_{A \setminus T} (g_a + \tau(\mu_a) - e_a) d\lambda$.

Lastly, let us define $E := T \cup R$ and an allocation $(y, \kappa) : A \rightarrow \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$ such that

$$(y_a, \kappa_a) = \begin{cases} (\xi_a, \nu'_a), & \text{for } a \in T; \\ \left(g_a + \frac{z\delta}{\lambda(R)}, \mu_a\right), & \text{for } a \notin T. \end{cases}$$

It can be readily verified that

$$\int_E \kappa_a d\lambda = \int_T \nu'_a d\lambda + \int_R \mu_a d\lambda.$$

Since $\int_T \nu'_a \in \mathcal{C}ons$ and $\int_A \mu_a d\lambda \in \mathcal{C}ons$, we have $\int_E \kappa_a d\lambda \in \mathcal{C}ons$. Therefore, we have

$$\int_E \tau(\kappa_a) d\lambda = \int_T \tau(\nu'_a) d\lambda + \int_R \tau(\mu_a) d\lambda.$$

Using the above equality, we derive that

$$\begin{aligned} \int_E [y_a + \tau(\kappa_a) - e_a] d\lambda &= \alpha \int_S [h_a + \tau(\nu'_a) - e_a] d\lambda + (1 - \alpha) \int_A [g_a + \tau(\mu'_a) - e_a] d\lambda \\ &= 0 \end{aligned}$$

Thus, $(E, (y, \kappa))$ forms a global counter-objection to $(S, (g, \mu))$ and $\lambda(E) = \alpha$. \square

Let $\alpha\text{-}\mathcal{B}^g(\mathcal{E})$ denote the set of feasible states in $\mathcal{B}^g(\mathcal{E})$ such that if they are globally objected then they are counter-objected by coalitions in $\mathcal{C}_\alpha = \{T \in \Sigma : \lambda(T) = \alpha\}$. In view of the above proposition, one can show that the global bargaining set is unaltered when one restricts counter-objecting coalitions to \mathcal{C}_α . The next theorem formalizes our finding.

Theorem 5.2.7. *Let \mathcal{E} be a club economy. Then for every $\alpha \in (0, \lambda(A))$ we have $\alpha\text{-}\mathcal{B}^g(\mathcal{E}) = \mathcal{B}^g(\mathcal{E})$.*

Proof. We just need to verify that $\mathcal{B}^g(\mathcal{E}) \subseteq \alpha\text{-}\mathcal{B}^g(\mathcal{E})$. To this end, let $(f, l) \in \mathcal{B}^g(\mathcal{E})$. If (f, l) has no objection, then $(f, l) \in \mathcal{C}(\mathcal{E}) \subseteq \mathcal{B}^g(\mathcal{E})$. Thus, we assume that (f, l) has a global objection $(S, (g, \mu))$, which is globally counter-objected by $(T, (h, \nu))$. Then, by Proposition 5.2.6, we can guarantee the existence of a global counter-objection $(E, (y, \kappa))$ such that it forms a global counter-objection to $(S, (g, \mu))$ and $\lambda(E) = \alpha$. Hence, $(f, l) \in \alpha\text{-}\mathcal{B}^g(\mathcal{E})$. \square

Remark 5.2.8. Consider a non-core state (g, μ) in the economy \mathcal{E} and take any $\alpha \in (0, \lambda(A))$. Then applying Lyapunov's convexity theorem as in the proof of Lemma 5.2.1 one can obtain for a given $0 < \varepsilon < \lambda(A)$ a coalition S such that $\lambda(S) = \varepsilon$ and (g, μ) is a feasible state for the coalition S . Now define a state $(f, l) : A \rightarrow \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{A}}$ as

$$(f_a, l_a) := \begin{cases} (0, 0), & \text{if } a \in S; \\ (\frac{1}{\lambda(A \setminus S)} \int_A e_a d\lambda, 0), & \text{if } a \in A \setminus S. \end{cases}$$

Note that $(S, (g, \mu))$ forms a global objection to the state (f, l) . Now, as (g, μ) is a non-core state there exists an objection $(T, (h, \nu))$ to it which then eventually forms a global counter-objection to the objection $(S, (g, \mu))$ of the state (f, l) . It then follows that $(f, l) \in \mathcal{B}^g(\mathcal{E})$ and by Theorem 5.2.7 we have $(f, l) \in \alpha\text{-}\mathcal{B}^g(\mathcal{E})$. Hence, there exists a counter-objection $(\tilde{T}, (\tilde{h}, \tilde{\nu}))$ to the objection $(S, (g, \mu))$ of the state (f, l) where $\lambda(\tilde{T}) = \alpha$. Observe that $(\tilde{T}, (\tilde{h}, \tilde{\nu}))$ forms an objection to the non-core state (g, μ) . Thus, any non-core state can be blocked by a coalition of size α for every $\alpha \in (0, \lambda(A))$. This gives us exact versions of Schmeidler's and Vind's theorem for an atomless club economy.

Given the above remark, we can give the following theorem which helps us to obtain a sharper interpretation of the core-equivalence theorem (Theorem 5.1) of Ellickson et al. [35] as a simple corollary.

Theorem 5.2.9. *Let \mathcal{E} be a club economy in which endowments are desirable and uniformly bounded above. In particular, if \mathcal{E} is club irreducible, then the following statements are equivalent:*

- (i) *A state (f, l) is a club equilibrium.*
- (ii) *(f, l) cannot be blocked by an arbitrary small-sized coalition.*
- (iii) *(f, l) cannot be blocked by an arbitrary large-sized coalition.*
- (iv) *(f, l) cannot be blocked by a coalition of size α for each $0 < \alpha < \lambda(A)$.*

5.3 The Veto Power of the Grand Coalition

In this section, we attempt to characterize the bargaining set in light of the veto power of the grand coalition. In that regard, we propose an alternative veto mechanism as in Hervés-Beloso and Moreno-García [46] where one only allows for blocking by the grand coalition in a collection of infinitely many perturbed economies formed by perturbing the initial endowments of a certain coalition of agents in the economy. Hervés-Estévez and Moreno-García [48] in their paper provided a characterization of global justified objections through robustly efficient allocations in a typical exchange economy with only private commodities. We extend their characterization result to the case of an atomless club economy. We begin by showing that if a state is robustly efficient in a club economy then it is a global justified objection. However, the converse fails to hold, pushing us to introduce a stronger notion of “sequential robust efficiency”. We show that any global justified objection is sequentially robustly efficient but whether the converse holds is unclear. Thus, in an attempt to give a complete characterization result, we adapt the notion of “approximate robust efficiency” from Bhowmik and Kaur (2023), by relaxing slightly the consistency condition and showing that the global bargaining set consists of states that cannot be objected by approximately robustly efficient states. To this end, for any feasible state (g, μ) , coalitions S, B with $B \in \Sigma_S$ and real number $\alpha \in (0, 1]$, we define an economy $\mathcal{E}(S, B, g, \mu, \alpha)$ whose initial endowment of private goods and club memberships are given below:

$$e_a(S, g, \alpha) := \begin{cases} e_a, & \text{if } a \in A \setminus S; \\ (1 - \alpha)e_a + \alpha g_a, & \text{if } a \in S, \end{cases}$$

and

$$\beta_a(B, \mu) := \begin{cases} \mu_a, & \text{if } a \in B; \\ 0, & \text{if } a \in A \setminus B. \end{cases}$$

Definition 5.3.1. A state (g, μ) is said to be **dominated** in an economy $\mathcal{E}(S, B, g, \mu, \alpha)$ if there exists a state (h, ν) such that

- (i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$ λ -a.e. on A ;
- (ii) $\int_A h_a d\lambda + \int_A \tau(\nu_a) d\lambda = \int_A e(S, g, \alpha) d\lambda + \int_A \tau(\beta_a(B, \mu)) d\lambda$; and
- (iii) $\int_A \nu_a d\lambda, \int_B \mu_a d\lambda \in \mathcal{C}ons$.

A state (g, μ) is said to be **robustly efficient** if it is not dominated in any perturbed economy $\mathcal{E}(S, B, g, \mu, \alpha)$ for every $0 < \alpha \leq 1$ and coalitions S, B .

Proposition 5.3.2. *Let \mathcal{E} be an atomless club economy and let $(S, (g, \mu))$ be a global objection to the state (f, l) in the economy \mathcal{E} . Then $(S, (g, \mu))$ is a global justified objection if (g, μ) is a robustly efficient state of \mathcal{E} .*

Proof. Let (g, μ) be a robustly efficient state of \mathcal{E} . Suppose, by way of contradiction, we assume that $(S, (g, \mu))$ is not a justified objection. By Proposition 5.2.6, there exists a coalition T with $\lambda(T) < \lambda(A)$ and a state (h, ν) such that

- (i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$, λ -a.e. on T ;
- (ii) $\int_T g_a d\lambda + \int_T \tau(\nu_a) d\lambda = \int_T e_a d\lambda$; and
- (iii) $\int_T \nu_a d\lambda \in \mathcal{C}ons$.

By (A.4), there exists a function $\zeta : T \rightarrow \mathbb{R}_+^N$ and some $z \in \mathbb{R}_+^N \setminus \{0\}$ such that $u_a(\zeta_a, \nu_a) > u_a(g_a, \mu_a)$, λ -a.e. on T , and

$$\int_T \zeta_a d\lambda = \int_T h_a d\lambda - z.$$

Choose an element $\alpha \in (0, 1]$. In view of Lemma 3.1 of Bhowmik and Kaur [22], we can find some state (φ, κ) such that

- (A) $u_a(\varphi_a, \kappa_a) > u_a(g_a, \mu_a)$, λ -a.e. on T ;
- (B) $\int_T \varphi_a d\lambda = \int_T (\alpha \zeta_a + (1 - \alpha)g_a) d\lambda$; and
- (C) $\int_T \kappa_a d\lambda = \int_T (\alpha \nu_a + (1 - \alpha)\mu_a) d\lambda$.

By the Lyapunov convexity theorem, we can choose some coalition $C \in \Sigma_{A \setminus T}$ such that $\lambda(C) = \alpha \lambda(A \setminus T)$ and

$$\int_C \mu_a d\lambda = \alpha \int_{A \setminus T} \mu_a d\lambda.$$

Consider a coalition B such that¹ $C \subseteq B$ and $\int_B \mu_a d\lambda \in \mathcal{C}ons$. Let us define two functions $\xi : A \rightarrow \mathbb{R}_+^N$ and $\eta : A \rightarrow \mathbb{R}^{\mathcal{M}}$ by letting

$$\xi_a := \begin{cases} \varphi_a, & \text{if } a \in T; \\ g_a + \frac{\alpha z}{\lambda(A \setminus T)}, & \text{if } a \in A \setminus T; \end{cases}$$

and

$$\eta_a := \begin{cases} \kappa_a + \mu_a, & \text{if } a \in T \cap (B \setminus C); \\ \kappa_a, & \text{if } a \in T \setminus (B \setminus C); \\ 2\mu_a, & \text{if } a \in (A \setminus T) \cap (B \setminus C); \\ \mu_a, & \text{if } a \in (A \setminus T) \setminus (B \setminus C). \end{cases}$$

In view of (A.4), it follows that $u_a(\xi_a, \eta_a) > u_a(g_a, \mu_a)$, λ -a.e. on A . Next, we now show that (g, μ) is dominated in the economy $\mathcal{E}(A \setminus T, B, g, \mu, \alpha)$. To see this, we note that

$$\begin{aligned} \int_A (\eta_a - \beta_a(B, \mu)) d\lambda &= \int_T \kappa_a d\lambda + \int_{B \setminus C} \mu_a d\lambda + \int_{A \setminus T} \mu_a d\lambda - \int_B \mu_a d\lambda \\ &= \alpha \int_T \nu_a d\lambda + (1 - \alpha) \int_T \mu_a d\lambda + \int_{A \setminus T} \mu_a d\lambda - \int_C \mu_a d\lambda \\ &= \alpha \int_T \nu_a d\lambda + (1 - \alpha) \int_T \mu_a d\lambda + \int_{A \setminus T} \mu_a d\lambda - \alpha \int_{A \setminus T} \mu_a d\lambda \\ &= \alpha \int_T \nu_a d\lambda + (1 - \alpha) \int_T \mu_a d\lambda + (1 - \alpha) \int_{A \setminus T} \mu_a d\lambda \\ &= \alpha \int_T \nu_a d\lambda + (1 - \alpha) \int_A \mu_a d\lambda. \end{aligned}$$

Since $\int_T \nu_a d\lambda$, and $\int_A \mu_a d\lambda$ both are in $\mathcal{C}ons$, we have

$$\int_A (\eta_a - \beta_a(B, \mu)) d\lambda \in \mathcal{C}ons.$$

It follows that

$$\int_A [\tau(\eta_a) - \tau(\beta_a(B, \mu))] d\lambda = \alpha \int_T \tau(\nu_a) d\lambda + (1 - \alpha) \int_A \tau(\mu_a) d\lambda.$$

¹One such B is A . Note also that if $\lambda(C)$ is sufficiently small, then $\lambda(B)$ can be chosen to be small.

Consequently,

$$\begin{aligned}
& \int_A \xi_a d\lambda + \int_A [\tau(\eta_a) - \tau(\beta_a(B, l))] d\lambda - \int_A e_a(A \setminus T, g, \alpha) d\lambda \\
&= \int_T (\alpha \zeta_a + (1 - \alpha) g_a) d\lambda + \int_{A \setminus T} g_a d\lambda + \alpha z + \alpha \int_T \tau(\nu_a) d\lambda + (1 - \alpha) \int_A \tau(\mu_a) d\lambda \\
&\quad - \int_T e_a d\lambda - \int_{A \setminus T} ((1 - \alpha) e_a + \alpha g_a) d\lambda \\
&= \alpha \int_T [h_a + \tau(\nu_a) - e_a] d\lambda + (1 - \alpha) \int_A [g_a + \tau(\mu_a) - e_a] d\lambda \\
&= 0
\end{aligned}$$

This completes the proof. \square

Remark 5.3.3. We consider Example 3.1.1 to show that the converse of the above result is not true. Define the state $(g, \mu) : A \rightarrow \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{A}}$ as

$$g_a := \begin{cases} k & \text{if } a \in [0, 6]; \\ k - 3 & \text{if } a \in (6, 9]; \\ k - 6 & \text{if } a \in (9, 10]. \end{cases}$$

and

$$\mu_a := \begin{cases} (0, 0) & \text{if } a \in [0, 6]; \\ (0, 1) & \text{if } a \in (6, 9]; \\ (1, 0) & \text{if } a \in (9, 10]. \end{cases}$$

where the first coordinate in μ denotes the demand for membership of the club type “own pool” and the second coordinate denotes the demand for membership of the club type “shared pool with one other consumer”. Recall that (g, μ) constitutes an equilibrium of the economy. Note that by Theorem 5.1 of Ellickson et al. [35] (g, μ) constitutes a core state of the economy and thus has no objection against it. Now consider the coalition $S = (9, 10]$. It can be observed that $(S, (g, \mu))$ constitutes a global objection to the initial endowment state, and since it has no further objection to it one can claim that $(S, (g, \mu))$ constitutes a globally justified objection to the initial endowment state. In what follows, we can proceed in similar lines of Example 3.7 of Bhowmik and Kaur [22] to show that (g, μ) fails to qualify as a robustly efficient state of the economy. \square

The above non-equivalence highlights the fact that the class of global justified objections of an economy \mathcal{E} is a larger class than the class of robustly efficient states. This is primarily attributed to the weaker nature of the objection mechanism embodied in the

definition of robustly efficient states. Thus, we propose a stronger notion of objection in the form of sequential robust efficiency where any state that fails to qualify as a sequentially robustly efficient state needs to be dominated in a sequence of economies and not just in a single economy as in robust efficiency. It then follows that such a class of states is a larger class than robustly efficient states and one can expect that the converse of Proposition 5.3.2 to hold for such a particular class. To begin with, we first introduce the notion of a sequentially robustly efficient state.

We know that the cardinality of the set $\mathcal{Clubs} = \{(\pi, \gamma)\}$ is finite in our setup. Moreover, the maximum number of memberships that one particular agent can buy is bounded above by some exogenously given finite number M . So, denote by $\mathbb{Y} = \{y_1, y_2, \dots, y_k\}$ the set of feasible membership choice vectors lying between the vectors $0\mathbf{1}$ and $M\mathbf{1}$ in \mathbb{Z}_+^M , where $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{Z}_+^M$. Fix any feasible state (g, μ) , we note that $\mu_a \in \mathbb{Y}$ for all $a \in A$. Thus, given a coalition S and $y_i \in \mathbb{Y}$, we define $S^i := \{a \in S : \mu_a = y_i\}$ as the set of agents whose chose membership vector y_i . Further, define $\mathbb{I}_S^\mu := \{i : \lambda(S^i) > 0\}$.²

Definition 5.3.4. A state (g, μ) is said to be **sequentially ε -dominated** if there exist a sequence $\{\mathcal{E}(S_n, B_n, g, \mu, \alpha_n) : n \geq 1\}$ of economies and a sequence $\{(h^n, \nu^n) : n \geq 1\}$ of states such that (g, μ) is dominated by (h^n, ν^n) in $\mathcal{E}(S_n, B_n, g, \mu, \alpha_n)$ and the following conditions are satisfied:

- (i) there is a coalition R such that $u_a(\psi_a^n, \nu_a^n) > u_a(g_a, \mu_a)$ for all $\psi_a^n \in h_a^n + \mathbb{B}(0, \varepsilon)$ with $a \in R$ and $n \geq 1$;
- (ii) $\mathbb{I}_{B_n}^\mu = \mathbb{I}_{S_n}^\mu$ and $\lambda(B_n^i) \geq \alpha_n \cdot \lambda(S_n^i)$ for all $n \geq 1$ and $i \in \mathbb{I}_{S_n}^\mu$; and
- (iii) $\{(\alpha_n, \lambda(B_n)) : n \geq 1\}$ converges to $(0, 0)$.

A state (g, μ) is called **ε -sequentially robustly efficient** if it is not sequentially ε -dominated. Furthermore, a state (g, μ) is said to be **sequentially robustly efficient** if it is ε -sequentially robustly efficient for all $\varepsilon > 0$.

If we denote by $\text{RE}_\varepsilon(\mathcal{E})$ the set of ε -sequentially robustly efficient states and $\widetilde{\text{RE}}(\mathcal{E})$ the set of sequentially robustly efficient states then $\{\text{RE}_\varepsilon(\mathcal{E}) : \varepsilon > 0\}$ is an ascending sequence satisfying

$$\widetilde{\text{RE}}(\mathcal{E}) = \bigcap \{\text{RE}_\varepsilon(\mathcal{E}) : \varepsilon > 0\}.$$

In our next theorem, we show that any global justified objection is sequentially robustly efficient. However, whether the converse holds or not is unclear to us at this moment.

²Note that the definition of the set S^i is contingent upon the membership vector μ , but we refrain from defining it as S_μ^i to avoid notational complexities.

Theorem 5.3.5. *Let \mathcal{E} be an atomless club economy in which endowments are desirable and uniformly bounded above. Further, let \mathcal{E} be club irreducible. Suppose (f, l) is a feasible state of \mathcal{E} and $(S, (g, \mu))$ is a global justified objection to (f, l) . Then (g, μ) is a sequentially robustly efficient state of \mathcal{E} .*

Proof. Assume that $(S, (g, \mu))$ is a justified objection to the state (f, l) . Thus, $(g, \mu) \in \mathcal{C}(\mathcal{E})$ and by Theorem 5.1 of Ellickson et al. [35], we can claim that (g, μ) is a club equilibrium state of the economy \mathcal{E} . Let (p, q) be an equilibrium price. Suppose, by way of contradiction, that (g, μ) is not an ε - sequentially robustly efficient state for some $\varepsilon > 0$. This implies that there exists a sequence $\{\mathcal{E}(S_n, B_n, g, \mu, \alpha_n) : n \geq 1\}$ of economies and a sequence $\{(h^n, \nu^n) : n \geq 1\}$ of states such that (g, μ) is dominated by (h^n, ν^n) in $\mathcal{E}(S_n, B_n, g, \mu, \alpha_n)$, which means

- (i) $u_a(h_a^n, \nu_a^n) > u_a(g_a, \mu_a)$, λ -a.e. on A ;
- (ii) $\int_A h_a^n d\lambda + \int_A \tau(\nu_a^n) d\lambda = \int_A e(S_n, g, \alpha_n) d\lambda + \int_A \tau(\beta_a(B_n, \mu)) d\lambda$; and
- (iii) $\int_A \nu_a^n d\lambda, \int_A \beta_a(B_n, \mu) d\lambda \in \mathcal{C}ons$.

In addition, the following conditions are satisfied:

- (iv) there is a coalition R such that $u_a(\psi_a^n, \nu_a^n) > u_a(g_a, \mu_a)$ for all $\psi_a^n \in h_a^n + \mathbb{B}(0, \varepsilon)$ with $a \in R$ and $n \geq 1$; and
- (v) $\mathbb{I}_{B_n}^\mu = \mathbb{I}_{S_n}^\mu$ and $\lambda(B_n^i) \geq \alpha_n \lambda(S_n^i)$ for all $n \geq 1$ and $i \in \mathbb{I}_{S_n}$; and
- (vi) $\{(\alpha_n, \lambda(B_n)) : n \geq 1\}$ converges to $(0, 0)$.

For each $n \geq 1$, there is a sub-coalition C_n of B_n such that $\lambda(C_n^i) = \alpha_n \lambda(S_n^i)$ for all $i \in \mathbb{I}_S$. Thus, we have

$$\int_{B_n} \mu_a d\lambda - \alpha_n \int_{S_n} \mu_a d\lambda = \int_{B_n \setminus C_n} \mu_a d\lambda.$$

Since $\{\lambda(B_n) : n \geq 1\}$ converges to 0, we have $\{q \cdot \int_{B_n \setminus C_n} \mu_a d\lambda : n \geq 1\}$ converges to 0. Let $n_0 \geq 1$ be an integer such that

$$q \cdot \int_{B_{n_0} \setminus C_{n_0}} \mu_a d\lambda < \frac{\varepsilon \lambda(R)}{2N}.$$

Letting

$$\delta := \frac{q}{\lambda(R)} \int_{B_{n_0} \setminus C_{n_0}} \mu_a d\lambda,$$

we note that $\delta < \frac{\varepsilon}{2N}$. It follows that $z_0 := (\delta, \dots, \delta) \in \mathbb{B}(0, \varepsilon)$. Thus we consider $\tilde{h} : A \rightarrow \mathbb{R}_+^N$ such that

$$\tilde{h}_a = \begin{cases} h_a^{n_0} - z_0, & \text{if } a \in R; \\ h_a^{n_0}, & \text{otherwise.} \end{cases}$$

As a consequence, we have

$$\int_A \tilde{h}_a d\lambda = \int_A h_a^{n_0} d\lambda - \lambda(R)z_0.$$

It follows from (iv) that

$$p \cdot \tilde{h}_a + q \cdot \nu_a^{n_0} > p \cdot e_a \geq p \cdot g_a + q \cdot \mu_a,$$

λ -a.e. on A . Thus,

$$(1 - \alpha_{n_0})(p \cdot \tilde{h}_a + q \cdot \tilde{\nu}_a) > (1 - \alpha_{n_0})p \cdot e_a \text{ and } \alpha_{n_0}(p \cdot \tilde{h}_a + q \cdot \tilde{\mu}_a) > \alpha_{n_0}(p \cdot g_a + q \cdot \mu_a).$$

Consequently,

$$p \cdot \tilde{h}_a + q \cdot \nu_a^{n_0} > p \cdot e(S_{n_0}, g, \alpha_{n_0}) + \alpha_{n_0}q \cdot \mu_a,$$

λ -a.e. on S_{n_0} , and hence

$$\int_{S_{n_0}} (p \cdot \tilde{h}_a + q \cdot \nu_a^{n_0}) d\lambda > \int_{S_{n_0}} p \cdot e(S_{n_0}, g, \alpha_{n_0}) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot \mu_a d\lambda.$$

This further implies that

$$\int_A (p \cdot \tilde{h}_a + q \cdot \nu_a^{n_0}) d\lambda > \int_A p \cdot e(S_{n_0}, g, \alpha_{n_0}) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot \mu_a d\lambda,$$

which immediately yields that³

$$\int_A (p \cdot h_a^{n_0} + q \cdot \nu_a^{n_0}) d\lambda - \lambda(R)\delta > \int_A p \cdot e(S_{n_0}, g, \alpha_{n_0}) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot \mu_a d\lambda.$$

This is equivalent to

$$\int_A (p \cdot h_a^{n_0} + q \cdot \nu_a^{n_0}) d\lambda > \int_A p \cdot e(S_{n_0}, g, \alpha_{n_0}) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot \mu_a d\lambda + \lambda(R)\delta.$$

³Since $\|p\|_1 = 1$, we have $p \cdot z_0 = \delta$

Thus, we have that

$$\int_A (p \cdot h_a^{n_0} + q \cdot \nu_a^{n_0}) d\lambda > \int_A p \cdot e(S_{n_0}, g, \alpha_{n_0}) d\lambda + \int_{B_{n_0}} q \cdot \mu_a d\lambda. \quad (5.3.1)$$

Now it can be observed that

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] d\lambda = p \cdot \int_A \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \frac{1}{\|\pi\|_1} \text{inp}(\pi, \gamma) [\nu_a^{n_0}(\omega, \pi, \gamma) - \beta_a(B_{n_0}, \mu)(\omega, \pi, \gamma)] d\lambda.$$

Now from (iii) we have that $\int_A (\nu_a^{n_0} - \beta_a(B_{n_0}, \mu)) d\lambda \in \mathcal{C}ons$ and thus there exists a real number $\delta(\pi, \gamma)$ such that

$$\int_A (\nu_a^{n_0}(\omega, \pi, \gamma) - \beta_a(B_{n_0}, \mu)(\omega, \pi, \gamma)) d\lambda = \delta(\pi, \gamma) \pi(\omega).$$

The above equation along with the fact that $\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p \cdot \text{inp}(\pi, \gamma)$ ⁴ implies

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] d\lambda = \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \frac{1}{\|\pi\|_1} \left[\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) \right] \delta(\pi, \gamma) \pi(\omega).$$

A simple algebraic manipulation yields

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] d\lambda = \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \frac{\pi(\omega)}{\|\pi\|_1} \sum_{\omega \in \Omega} \delta(\pi, \gamma) \pi(\omega) q(\omega, \pi, \gamma).$$

We further observe that

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] d\lambda = \sum_{\omega \in \Omega} \frac{\pi(\omega)}{\|\pi\|_1} \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \delta(\pi, \gamma) \pi(\omega) q(\omega, \pi, \gamma),$$

which is equivalent to

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] d\lambda = \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \delta(\pi, \gamma) \pi(\omega) q(\omega, \pi, \gamma).$$

This further yields that

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] d\lambda = \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} \int_A [\nu_a^{n_0}(\omega, \pi, \gamma) - \beta_a(B_{n_0}, \mu)(\omega, \pi, \gamma)] q(\omega, \pi, \gamma) d\lambda.$$

⁴Refer to the definition of a club equilibrium.

which means

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] d\lambda = \int_A \sum_{(\omega, \pi, \gamma) \in \mathcal{M}} q(\omega, \pi, \gamma) [\nu_a^{n_0}(\omega, \pi, \gamma) - \beta_a(B_{n_0}, \mu)(\omega, \pi, \gamma)] d\lambda,$$

equivalently,

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] d\lambda = \int_A q \cdot [\nu_a^{n_0} - \beta_a(B_{n_0}, \mu)] d\lambda.$$

It can be observed from the definition of the perturbed economy that

$$\int_A \beta_a(B_{n_0}, \mu) d\lambda = \int_{B_{n_0}} \mu_a d\lambda.$$

Thus, it follows from (ii) that

$$\int_A (p \cdot h_a^{n_0} + q \cdot \nu_a^{n_0}) d\lambda = \int_A p \cdot e(S_{n_0}, g, \alpha_{n_0}) d\lambda + \int_{B_{n_0}} q \cdot \mu_a d\lambda.$$

This contradicts (5.3.1). □

Remark 5.3.6. Consider a club equilibrium state (g, μ) of the economy. Then one can conclude from Theorem 4.1 of Ellickson et al. [35] that (g, μ) belongs to the core of the economy. Thus, (g, μ) has no objection. Now, one can proceed as in Remark 5.2.8 to construct a coalition S with $\lambda(S) < \lambda(A)$ and show that $(S, (g, \mu))$ forms an objection to the feasible state (f, l) , where

$$(f_a, l_a) := \begin{cases} (0, 0), & \text{if } a \in S; \\ (\frac{1}{\lambda(A \setminus S)} \int_A e_a d\lambda, 0), & \text{if } a \in A \setminus S. \end{cases}$$

As (g, μ) belongs to the core of the economy, it follows that $(S, (g, \mu))$ has no counter-objection to it and is a global justified objection. Further, by Theorem 5.3.5 one can claim that (g, μ) and hence any club equilibrium state is sequentially robustly efficient.

In order to provide a complete characterization result of global justified objections with respect to some notion of robust efficiency as in Hervés-Estévez and Moreno-García [48], we resort to the notion of “approximate robust efficiency” from Bhowmik and Kaur [22]. The notion is “approximate” in the sense that instead of taking both the initial and final membership vector to be consistent as in the case of robust efficiency, we assume the net membership vector, defined as the difference between the final and initial membership vector to be consistent. We now introduce the notion of

“approximate domination” and “approximately robustly efficient” states formally.⁵ To this end, consider an arbitrary economy $\tilde{\mathcal{E}}$ which is the same as \mathcal{E} except for initial endowment being an arbitrary state $(\tilde{e}, \tilde{l}) : A \rightarrow \mathbb{R}_+^L \times \mathbb{R}^M$. We say that a state (g, μ) is **approximately dominated** by (h, ν) in $\tilde{\mathcal{E}}$ if the following conditions are satisfied:

- (i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$, λ -a.e. on A ;
- (ii) $\int_A h_a d\lambda + \int_A \tau(\nu_a) d\lambda = \int_A \tilde{e}_a d\lambda + \int_A \tau(\tilde{l}_a) d\lambda$; and
- (iii) $\int_A (\nu_a - \tilde{l}_a) d\lambda \in \mathcal{C}ons$.

To introduce the notion of an approximately robustly efficient state, for any $\mu : A \rightarrow \mathbb{R}^M$, coalition S and real number $\alpha \in (0, 1]$, we define

$$\mathcal{A}(\mu, S, \alpha) := \left\{ B \in \Sigma_S : \int_B \mu_a d\lambda = \alpha \int_S \mu_a d\lambda \right\}.$$

Observe that by the Lyapunov convexity theorem, it follows that $\mathcal{A}(\mu, S, \alpha) \neq \emptyset$.

Definition 5.3.7. A state (g, μ) in \mathcal{E} is said to be **approximately robustly efficient** if it is not approximately dominated in $\mathcal{E}(S, B, g, \mu, \alpha)$ for every $0 < \alpha \leq 1$, and coalitions B, S with $B \in \mathcal{A}(\mu, S, \alpha)$.

Theorem 5.3.8. *Let \mathcal{E} be a club economy in which endowments are desirable and uniformly bounded above. Further, let \mathcal{E} be club irreducible. Suppose (f, l) is a feasible state in the economy \mathcal{E} and $(S, (g, \mu))$ is a global objection to the state (f, l) . Then $(S, (g, \mu))$ is a global justified objection if and only if (g, μ) is approximately robustly efficient.*

Proof. Let $(S, (g, \mu))$ be a justified global objection to the state (f, l) in the economy \mathcal{E} . Thus, (g, μ) belongs to the core of the economy \mathcal{E} . Hence, from Theorem 5.1 of Ellickson et al. [35], we can claim that (g, μ) is a club equilibrium state of the economy \mathcal{E} . Suppose, by way of contradiction, let us assume that (g, μ) is not approximately robustly efficient. This means that there exists some $\alpha \in (0, 1]$, coalition S and sub-coalition $B \in \mathcal{A}(l, S, \alpha)$ such that (g, μ) is dominated in $\mathcal{E}(S, B, g, \mu, \alpha)$. Thus, there exist a state (h, ν) such that

⁵It is important to highlight the need for an approximation of the notion of robust efficiency. Ellickson et al. [35] commented that due to the non-convex nature of the club goods, the second welfare theorem fails to hold for a club goods economy. This in conjunction with the contrapositive of Remark 3.2 in Hervés-Beloso and Moreno-García [46] establishes that one cannot expect a characterization of equilibrium states through the notion of robust efficiency for such economies. Thus, Bhowmik and Kaur [22] proposed such an approximate notion.

- (i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$, λ -a.e. on A ;
- (ii) $\int_A h_a d\lambda + \int_A \tau(\nu_a) d\lambda = \int_A e(S, g, \alpha) d\lambda + \int_A \tau(\beta_a(B, \mu)) d\lambda$; and
- (iii) $\int_A (\nu_a - \beta_a(B, \mu)) d\lambda \in \mathcal{C}ons$.

Let $(p, q) \in \mathbb{R}_+^L \times \mathbb{R}^M$ be an equilibrium price corresponding to the state (g, μ) . From (i), we have

$$p \cdot h_a + q \cdot \nu_a > p \cdot e_a \geq p \cdot g_a + q \cdot \mu_a, \lambda\text{-a.e. on } A.$$

Thus,

$$(1 - \alpha)(p \cdot h_a + q \cdot \nu_a) > (1 - \alpha)p \cdot e_a \text{ and } \alpha(p \cdot h_a + q \cdot \nu_a) > \alpha(p \cdot g_a + q \cdot \mu_a).$$

It follows that

$$(p \cdot h_a + q \cdot \nu_a) > (1 - \alpha)p \cdot e_a + \alpha p \cdot g_a + \alpha q \cdot \mu_a, \lambda\text{-a.e. on } S.$$

Hence,

$$\int_A (p \cdot h_a + q \cdot \nu_a) d\lambda > \int_A p \cdot e_a(S, g, \alpha) d\lambda + \int_A q \cdot \beta_a(B, \mu) d\lambda. \quad (5.3.2)$$

From (iii), we have

$$\int_A [\nu_a - \beta_a(B, \mu)] d\lambda \in \mathcal{C}ons.$$

Then, invoking analogous arguments of the proof of Theorem 5.3.5, we obtain $\int_A p \cdot [\tau(\nu_a) - \tau(\beta_a(B, \mu))] d\lambda = \int_A q \cdot (\nu_a - \beta_a(B, \mu))$. Thus, from (ii) we have

$$\int_A (p \cdot h_a + q \cdot \nu_a) d\lambda = \int_A (p \cdot e_a(S, g, \alpha) + q \cdot \beta_a(B, \mu)) d\lambda,$$

which contradicts Equation (5.3.2). Hence, (g, μ) is an approximately robustly efficient state.

Conversely, let $(S, (g, \mu))$ be a global objection to the state (f, l) such that (g, μ) is an approximately robustly efficient state. Suppose, by way of contradiction, that $(S, (g, \mu))$ is not justified. Hence, by Proposition 5.2.6, there exists a coalition T with $\lambda(T) < \lambda(A)$ and a state (h, ν) such that

- (i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$, λ -a.e. on T ;
- (ii) $\int_T h_a d\lambda + \int_T \tau(\nu_a) d\lambda = \int_T e_a d\lambda$; and

(iii) $\int_T \nu_a d\lambda \in \mathcal{C}ons$.

Now, by Assumption (A.4), there exists a function $\zeta : T \rightarrow \mathbb{R}_+^N$ and some $z \in \mathbb{R}_+^N \setminus \{0\}$ such that $u_a(\zeta_a, \nu_a) > u_a(g_a, \mu_a)$, λ -a.e. on T , and

$$\int_T \zeta_a d\lambda = \int_T h_a d\lambda - z.$$

Let $\alpha \in (0, 1]$. Applying Lemma 3.1 of Bhowmik and Kaur [22], we can find some state (φ, κ) such that

(I) $u_a(\varphi_a, \kappa_a) > u_a(g_a, \mu_a)$, λ -a.e. on T ;

(II) $\int_T \varphi_a d\lambda = \int_T (\alpha \zeta_a + (1 - \alpha)g_a) d\lambda$; and

(III) $\int_T \kappa_a d\lambda = \int_T (\alpha \nu_a + (1 - \alpha)\mu_a) d\lambda$.

Choose a coalition $B \in \mathcal{A}(\mu, S, \alpha)$. We define two functions $\xi : A \rightarrow \mathbb{R}_+^N$ and $\eta : A \rightarrow \mathbb{R}^{\mathcal{M}}$ by letting

$$\xi_a := \begin{cases} \varphi_a, & \text{if } a \in T; \\ g_a + \frac{\alpha z}{\lambda(A \setminus T)}, & \text{if } a \in A \setminus T; \end{cases}$$

and

$$\eta_a := \begin{cases} \kappa_a, & \text{if } a \in T; \\ \mu_a, & \text{if } a \in A \setminus T. \end{cases}$$

We note that

$$\begin{aligned} \int_A (\eta_a - \beta_a(B, \mu)) d\lambda &= \int_T \kappa_a d\lambda + \int_{A \setminus T} \mu_a d\lambda - \int_B \mu_a d\lambda \\ &= \alpha \int_T \nu_a d\lambda + (1 - \alpha) \int_T \mu_a d\lambda + \int_{A \setminus T} \mu_a d\lambda - \alpha \int_{A \setminus T} \mu_a d\lambda \\ &= \alpha \int_T \nu_a d\lambda + (1 - \alpha) \int_A \mu_a d\lambda \end{aligned}$$

Since $\int_T \nu_a d\lambda, \int_A \mu_a d\lambda \in \mathcal{C}ons$ and $\mathcal{C}ons$ is a linear space, we have

$$\int_A (\eta_a - \beta_a(B, \mu)) d\lambda \in \mathcal{C}ons.$$

Furthermore,

$$\int_A [\tau(\eta_a) - \tau(\beta_a(B, \mu))] d\lambda = \alpha \int_T \tau(\nu_a) d\lambda + (1 - \alpha) \int_A \tau(\mu_a) d\lambda.$$

Finally, note that the following equations guarantees material balance for the state (ξ, η)

$$\begin{aligned}
& \int_A \xi_a d\lambda + \int_A [\tau(\xi_a) - \tau(\eta_a(B, \mu))] d\lambda - \int_A e_a(A \setminus T, g, \alpha) d\lambda \\
&= \int_T (\alpha \zeta_a + (1 - \alpha)g_a) d\lambda + \int_{A \setminus T} g_a d\lambda + \alpha z + \alpha \int_T \tau(\nu_a) d\lambda + (1 - \alpha) \int_A \tau(\mu_a) d\lambda \\
&- \int_T e_a d\lambda - \int_{A \setminus T} ((1 - \alpha)e_a + \alpha g_a) d\lambda \\
&= \alpha \int_T [h_a + \tau(\nu_a) - e_a] d\lambda + (1 - \alpha) \int_A [g_a + \tau(\mu_a) - e_a] d\lambda \\
&= 0
\end{aligned}$$

Hence, (g, μ) is not approximately robustly efficient, which is a contradiction. This completes the proof. \square

The following theorem is then immediate from above and Theorem 3.12 of Bhowmik and Kaur [22] and shows that only club equilibrium states can constitute global justified objections.

Theorem 5.3.9. *Let \mathcal{E} be a club economy in which endowments are desirable and bounded from above. Further, let \mathcal{E} be club irreducible. Suppose (f, l) is a feasible state in the economy \mathcal{E} and $(S, (g, \mu))$ is a global objection to the state (f, l) . Then the following statements are equivalent:*

- (i) (g, μ) is a club equilibrium state.
- (ii) $(S, (g, \mu))$ is a global justified objection to (f, l) .
- (iii) (g, μ) is an approximately robustly efficient state.

Remark 5.3.10. From Theorem 5.3.8, it follows that any feasible state belonging to the global bargaining set cannot be objected by a state that is non-dominated in any of the perturbed economies. One can observe from the proof of Theorem 5.3.8 that it is possible to choose the parameter α arbitrarily close to zero and still guarantee that the conclusion of Theorem 5.3.8 holds. Further, by Proposition 5.2.6 one can choose the size of the counter-objecting coalition to be arbitrarily small or large. Then in Theorem 5.3.8 the perturbed economies $\mathcal{E}(S, B, g, \mu, \alpha)$ can be restricted to those where the endowments of private goods are modified on coalitions which have arbitrarily small or large measure and thus the endowments of the perturbed economy is very close to that

of the original one. Further, it also follows that if endowments are modified over a small coalition, then the aggregate initial membership vector $\int_A \beta_a(B, \mu) d\lambda$ is very close to $0 \in \mathcal{C}ons$ and if endowments are modified over a large coalition then $\int_A \beta_A(B, \mu) d\lambda$ is very close to $\int_A \mu_a d\lambda$ which again is consistent. Therefore, the global bargaining set consists of states that cannot be objected to by a state that is non-dominated in economies obtained by perturbing slightly the initial endowments of the agents belonging to either arbitrarily small coalitions, arbitrarily large coalitions, or coalitions of a fixed measure. Note that one can also state Theorem 5.3.8 in the following way: Let $\bar{\alpha} \in (0, \lambda(A))$ and (f, l) be a feasible state in the economy \mathcal{E} . Then (f, l) is in the global bargaining set if and only if (f, l) cannot be globally objected by some $(S, (g, \mu))$ where (g, μ) is a non-dominated state for every economy $\mathcal{E}(S, B, g, \mu, \bar{\alpha})$.

Chapter 6

Core of a Mixed Club Economy

In this chapter, the economic model introduced in the last two chapters will be further generalized to contain a mixed measure space of agents. In particular, we introduce a mixed club economy from an atomless one by allowing negligible agents to form countably many cartels. We follow such a particular construction to ensure that in any feasible state (in particular, core and equilibrium state), clubs formed have no market power. We further assume that negligible agents who come together to form cartels must have the same type i.e. same endowment, same consumption set, same utility, and additionally same external characteristic.¹ In what follows, we identify a class of states ‘ \mathcal{A} ’ such that a state $(f, l) \in \mathcal{A}$ is a core state if and only if it is an equilibrium state of our mixed club economy. To define our class more formally, we denote by A_1 the set of negligible agents who come together to form countably many cartels, and $A_0 = A \setminus A_1$ ² is the set of remaining negligible agents, and λ is an atomless measure

¹Note that our approach is the complete opposite of the one taken in Shitovitz [77] and Greenberg and Shitovitz [41]. In their framework, they first model a mixed economy and then introduce an associated atomless economy by splitting the atoms into an ocean of negligible agents that have the same type as that of the atom. Since in our economy, we begin with an atomless economy we have to assume that agents who come together to form the cartels are of the same type. Now, alongside we need to also assume that all agents in a cartel must have the same external characteristics to guarantee the following: (i) As consumption sets in our economy are defined on both private goods and club membership choices, having agents of different characteristics within the cartel will not allow us to define a common consumption set for the cartel, and (ii) if agents inside the cartel are of different characteristics then it is not possible to define an external characteristic for our atomic agent or the cartel. Further, we emphasize that as the main goal of this chapter is to establish the core-equivalence theorem in a mixed club economy setting, we limit ourselves to such a simple cartel formation process and keep studying cartels with heterogeneous agents and interaction among them outside the purview of this chapter.

² $A = A_0 \cup A_1$ is the set of agents in our atomless club economy and $\tilde{A} = A_0 \cup \tilde{A}_1$ is the set of agents in the mixed club economy, where \tilde{A}_1 denotes the collection of disjoint cartels formed by negligible

on an σ -algebra defined on A . Further, let $R \supseteq A_1$ be a coalition containing agents who are of the same type and denote the set of all such R by \mathcal{R} . Then, given such a coalition R , one can say that a state belongs to our particular class if the average club membership vector for this coalition belongs to the consumption set of the agents in the coalition. Assuming $\lambda(A_1) > 0$, we define more formally:

$$\mathcal{A} = \left\{ (f, l) : (\widehat{f}_R, \widehat{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda, \frac{1}{\lambda(R)} \int_R l_a d\lambda \right) \in X_R, \text{ for some } R \in \mathcal{R} \right\}.^3$$

Note that for a state (f, l) outside the above class, it may happen that (f, l) is a core state but not an equilibrium one, and thus our equivalence theorem fails to hold in such a scenario as is evident from Remark 6.3.6 in this chapter.

In order to establish our main result, we adopt an assumption from Basile et al. [13], but for our atomless economy, which states that there exists an $R \in \mathcal{R}$ such that $\lambda(R \setminus A_1) > 0$.⁴ This assumption plays a key role in establishing that if a state belongs to the core of a mixed economy, then its corresponding state belongs to the core of the atomless economy. The argument for the same follows from the contrapositive argument. We argue that if a state (f, l) outside the core of the atomless economy is blocked by a coalition S , then the corresponding state of (f, l) ⁵ can be blocked in the mixed economy in three possible scenarios:

- (i) Firstly, if $S \subseteq A_0$, then one can block the corresponding state of (f, l) in the mixed economy with coalition S only and our assumption is not needed.⁶
- (ii) Secondly, if $S \cap A_1 = \bigcup_{i \geq 1} B_i$, where each B_i forms a cartel \widetilde{B}_i in the mixed economy, then it can be observed that the coalition $\widetilde{S} = (S \cap A_0) \cup (\bigcup_{i \geq 1} \widetilde{B}_i)$ blocks the corresponding state of (f, l) in the mixed economy, and our assumption is not needed again.

agents in A_1 and $\widetilde{\lambda}$ denotes a measure on an σ -algebra defined on \widetilde{A} .

³Note that X_R denotes the common consumption set of the agents in the coalition R . Further, if $\lambda(A_1) = 0$, then we have a standard atomless club economy and one can take \mathcal{A} to be the set of all feasible allocations of the economy. Then the core-equivalence theorem follows from the work of Ellickson et al. [35] in such a setting. Thus, we assume that $\lambda(A_1) > 0$ and hence \mathcal{A} is well defined.

⁴In the case of a mixed club economy, our assumption translates into $\widetilde{\lambda}(\widetilde{R} \setminus \widetilde{A}_1) > 0$, which is exactly the same as Basile et al. [13] who adopted this assumption in a mixed economy with public goods, where one needs to deal directly with the size of large agents due to the failure of Vind's theorem. Using this assumption, they established the equivalence between the core and equilibrium in such an economy.

⁵The corresponding state of (f, l) assigns (f, l) only to agents in A_0 and the cartels are assigned the average consumption of the coalition who groups to form them (see Greenberg and Shitovitz [77]).

⁶As the state for agents in A_0 is exactly the same as (f, l) in the corresponding state of (f, l) in the mixed economy.

- (iii) Finally, if $S \cap A_1 \neq \bigcup_{i \geq 1} B_i$, where each B_i forms a cartel \tilde{B}_i in the mixed economy, then with the help of Lyapunov's convexity theorem and our assumption one can replace this set of negligible agents with a suitable sub-coalition of agents from $R \setminus A_1$ to construct a coalition $\hat{S} \subseteq A_0$ that blocks (f, l) . Since $\hat{S} \subseteq A_0$, the coalition \hat{S} also blocks the corresponding state of (f, l) in the mixed economy.

Thus, for an allocation (g, μ) in the core of the mixed economy, its corresponding state belongs to the core of the atomless economy, and one can infer from the core-equivalence theorem of Ellickson et al. [35] that such a state is a club equilibrium state in the atomless economy. It is then routine to check that the allocation (g, μ) is a club equilibrium state of the mixed economy, and our main result follows quite immediately. It is important to emphasize that it is not possible to replace the above-mentioned assumption with the one used in Shitovitz [77] and Greenberg and Shitovitz [41] as is evident in Example 6.3.5 of this chapter where we show that the existence of two atoms of the same type does not guarantee the equivalence between the core and equilibria of a mixed club economy. This is primarily based on the core indivisible nature of club memberships. Note that in a mixed economy, the consumption of a non-negligible agent is determined by the size of the agent. Further, as club membership choices can be manipulated only in integer amounts, the size of a non-negligible agent plays a crucial role while recontracting happens among agents within a coalition. Thus, we need to directly address the size of the non-negligible agents, and we do so as highlighted in condition (iii) above by allowing some coalition of agents in A_0 to mimic the type of the agents in A_1 . This further allows recontracting to happen among agents in a smaller coalition where mutual trade among themselves helps them to become better off. Our result thus generalizes the core-equivalence theorem of Ellickson et al. [35] to a mixed economy setup with club goods where the blocking is defined by a coalition via an arbitrary allocation, which may not necessarily belong to \mathcal{A} . This chapter is based on the paper [25].

6.1 Objection Mechanism and the Core

In this section, we revisit the objection mechanism of our economy and reintroduce the notion of the core of a club economy. As opposed to the earlier two chapters we concentrate on the notion of strong objection and hence the weak core for this chapter. We begin with the formal definition of strong objection in our economy below.

Definition 6.1.1. : A feasible state (f, l) of the economy \mathcal{E} is said to be **strongly objected** if there exists some coalition B and a state (g, μ) such that (g, μ) is feasible

for B and $u_a(g_a, \mu_a) > u_a(f_a, l_a)$ for almost all $a \in B$. A feasible state (f, l) is said to be in the **weak core** or what we simply called core of the economy \mathcal{E} if it is not strongly objected.

6.2 Constructing a Mixed Economy

We now associate a mixed economy $\tilde{\mathcal{E}}$ to our atomless club economy \mathcal{E} . Before that, we talk about certain properties of our atomless economy \mathcal{E} . Given an atomless club economy \mathcal{E} , we call two agents $a, b \in A$ are of the **same type** if $(\omega_a, X_a, e_a, u_a) = (\omega_b, X_b, e_b, u_b)$. We assume that there exists a coalition R in \mathcal{E} containing agents who are of the same type. In the coalition R , there exists a countable collection of pairwise disjoint measurable sets $\{T_1, T_2, \dots\}$ of positive measure where each T_n forms a cartel in the sense that the small agents in T_n have joined forces to act as a single agent. We denote such a large agent or cartel by \tilde{T}_n . In contrast to the atomless economy, there exists no positive measurable subset of \tilde{T}_n with measure less than that of \tilde{T}_n that can form and engage in trade with other coalitions in the market. We define $\tilde{A}_1 := \bigcup \{\tilde{T}_n : n \geq 1\}$ as the union of large or non-negligible agents and $A_0 := A \setminus \tilde{A}_1$ as the set of small or negligible agents in our mixed economy with club goods. We then define the measure space $(\tilde{A}, \tilde{\Sigma}, \tilde{\lambda})$, where $\tilde{A} = A_0 \cup \tilde{A}_1$, the associated σ -algebra $\tilde{\Sigma}$ is the direct sum of the two σ -algebras i.e.

$$\tilde{\Sigma} = \left\{ C \cup D : C \cap D = \emptyset, C \in \Sigma_{A_0}, D \in \tilde{\Sigma}_{\tilde{A}_1} \right\}^7$$

and the associated measure $\tilde{\lambda} : \tilde{\Sigma} \rightarrow \mathbb{R}_+$ is defined as

$$\tilde{\lambda}(C) = \lambda(C \cap A_0) + \tilde{\lambda}(C \cap \tilde{A}_1)$$

for any $C \in \tilde{\Sigma}$. The consumption space of $\tilde{\mathcal{E}}$ is taken to be the same as in \mathcal{E} . Therefore, the associated mixed economy $\tilde{\mathcal{E}}$ is defined as $a \mapsto (\tilde{\omega}_a, \tilde{X}_a, \tilde{e}_a, \tilde{u}_a)$, where

$$\tilde{\omega}_a := \begin{cases} \omega_a, & \text{if } a \in A_0; \\ \omega_{T_n}, & \text{if } a = \tilde{T}_n \text{ and } n \geq 1, \end{cases}$$

$$\tilde{X}_a := \begin{cases} X_a, & \text{if } a \in A_0; \\ X_{T_n}, & \text{if } a = \tilde{T}_n \text{ and } n \geq 1, \end{cases}$$

⁷ $\tilde{\Sigma}_{\tilde{A}_1}$ is the σ -algebra defined over the set of atoms.

$$\tilde{e}_a := \begin{cases} e_a, & \text{if } a \in A_0; \\ e_{T_n}, & \text{if } a = \widetilde{T}_n \text{ and } n \geq 1, \end{cases}$$

and

$$\tilde{u}_a := \begin{cases} u_a, & \text{if } a \in A_0; \\ u_{T_n}, & \text{if } a = \widetilde{T}_n \text{ and } n \geq 1, \end{cases}$$

where $(\omega_{T_n}, X_{T_n}, e_{T_n}, u_{T_n})$ denotes the common value of $(\omega_a, X_a, e_a, u_a)$ for all $a \in T_n$.⁸ In our mixed economy, we need to assume that the preferences of the large agents are convex and so in addition to the four conditions listed in Definition 3.4.1, our mixed economy satisfies the following additional assumption:

(C) The utility function $\tilde{u}_a : \tilde{X}_a \rightarrow \mathbb{R}$ is quasi-concave for all $a \in \tilde{A}_1$ ⁹ in the following sense:¹⁰ for all $\lambda \in \mathbb{R}$, we have

$$\tilde{X}_a \cap \overline{\text{co}}\{(y, \mu) \in \tilde{X}_a : \tilde{u}_a(y, \mu) \geq \beta\} \subseteq \{(y, \mu) \in \tilde{X}_a : \tilde{u}_a(y, \mu) \geq \beta\}$$

Note that such an assumption is not uncommon and has been widely used in the literature (see Shitovitz [77], Greenberg and Shitovitz [41], Basile et al. [13] among others). Further observe that as our large agents are formed in the form of cartels, one can replace the above assumption by the following assumption in the atomless economy \mathcal{E} : each small agent in $A_1 = \cup \{T_n : n \geq 1\}$ must have a preference that satisfies the assumption (C) above. Note that this assumption is an addition to the set of assumptions in Ellickson et al. [35] for an atomless club economy. As a next step, we want to establish a connection between the equilibrium states and core of the atomless club economy and its associated mixed economy. To do so we first need to establish a connection between the states of these two economies. Given a feasible state (\tilde{f}, \tilde{l}) in $\tilde{\mathcal{E}}$, we define a feasible state $(f, l) := \Xi(\tilde{f}, \tilde{l})$ in \mathcal{E} as

$$(f_a, l_a) := \begin{cases} (\tilde{f}_a, \tilde{l}_a), & \text{if } a \in A_0; \\ (\tilde{f}_{T_n}, \tilde{l}_{T_n}), & \text{if } a \in T_n \text{ and } n \geq 1. \end{cases}$$

Now, given a feasible state (f, l) in \mathcal{E} , the aggregate private good consumption for any $T_n \subseteq R$ is $\int_{T_n} f_a d\lambda$ and the aggregate club membership consumption is $\int_{T_n} l_a d\lambda$. For the corresponding state (\tilde{f}, \tilde{l}) , of the mixed economy one can argue along the lines of Shitovitz [77] to claim that $(\tilde{f}_{\widetilde{T}_n}, \tilde{l}_{\widetilde{T}_n})$ is not the bundle assigned to agent \widetilde{T}_n but the

⁸Note that as agents in R of same type, we have $(\omega_{T_n}, X_{T_n}, e_{T_n}, u_{T_n})$ equal to $(\omega_R, X_R, e_R, u_R)$ for each $n \geq 1$.

⁹Note that $a \in \tilde{A}_1$ is some atom \widetilde{T}_n .

¹⁰ $\overline{\text{co}}(Z)$ stands for the closed convex hull of the set Z .

“bundle density”. The actual bundle that is assigned to \widetilde{T}_n is $(\widetilde{f}_{\widetilde{T}_n}, \widetilde{l}_{\widetilde{T}_n})\widetilde{\lambda}(\widetilde{T}_n)$.¹¹ Thus the aggregate private good consumption for \widetilde{T}_n is given by $\widetilde{f}_{\widetilde{T}_n}\widetilde{\lambda}(\widetilde{T}_n)$. As agents in T_n join forces to form a cartel it must be that $\widetilde{f}_{\widetilde{T}_n}\widetilde{\lambda}(\widetilde{T}_n) = \int_{T_n} f_a d\lambda$. As $\widetilde{\lambda}(\widetilde{T}_n) = \lambda(T_n)$ we obtain that

$$\widetilde{f}_{\widetilde{T}_n} = \frac{1}{\lambda(T_n)} \int_{T_n} f_a d\lambda.$$

One can analogously argue for the club membership choice of the atom \widetilde{T}_n to be

$$\widetilde{l}_{\widetilde{T}_n} = \frac{1}{\lambda(T_n)} \int_{T_n} l_a d\lambda$$

if such a choice is defined i.e. if $\frac{1}{\lambda(T_n)} \int_{T_n} l_a d\lambda \in \mathbb{Z}_+^{\mathcal{M}}$. Thus, given a feasible state (f, l) in \mathcal{E} , with $\frac{1}{\lambda(T_n)} \int_{T_n} l_a d\lambda \in \mathbb{Z}_+^{\mathcal{M}}$ for all $n \geq 1$ we define the corresponding feasible state $(\widetilde{f}, \widetilde{l}) := \varphi(f, l)$ in $\widetilde{\mathcal{E}}$ as

$$(\widetilde{f}_a, \widetilde{l}_a) := \begin{cases} (f_a, l_a), & \text{if } a \in A_0; \\ \left(\frac{1}{\lambda(T_n)} \int_{T_n} f_a d\lambda, \frac{1}{\lambda(T_n)} \int_{T_n} l_a d\lambda \right), & \text{if } a = \widetilde{T}_n \text{ and } n \geq 1. \end{cases}$$

One can similarly argue that for any non-negligible agent $a \in \widetilde{A}_1$ and a membership choice \widetilde{l}_a , $\tau(\widetilde{l}_a)$ denotes not the actual input contribution but rather the input contribution density for the agent ‘ a ’. The actual input contribution of agent ‘ a ’ is denoted as $\tau(\widetilde{l}_a) \cdot \widetilde{\lambda}(\{a\})$. Now although one can establish correspondences between the two economies’ agents and states, they are not the same. Understanding the difference between the two economies is crucial. The following remark summarizes it.

Remark 6.2.1. Consider an atomless economy \mathcal{E} with the set of agents denoted by $[0, 5]$ and endowed with the Lebesgue measure and Lebesgue σ -algebra. We assume that agents in $[0, 3)$ are males (M) and those in $[3, 5]$ are females (F). There exists only one club type (π, γ) in the economy whose profile is represented as $\pi(M) = \pi(F) = 1$. Note that the set of memberships in the economy is given by $\mathcal{M} := \{(m, \pi, \gamma), (f, \pi, \gamma)\}$. Define the membership vector $l : A \rightarrow \mathbb{Z}_+$ as

$$l_a = \begin{cases} (1, 0), & \text{if } a \in [0, 2); \\ (0, 0), & \text{if } a \in [2, 3); \text{ and} \\ (0, 1), & \text{if } a \in [3, 5]. \end{cases}$$

¹¹Note that utilities are formally defined over bundle densities and not on actual bundles. Since large agents of the same type may have different measures, it follows that though the utilities of large agents of the same type are the same on bundle densities, they will differ on bundles.

Then one can observe that the aggregate membership demand is $\int_{[0,5]} l_a d\lambda = (2, 2)$, where the first coordinate denotes the aggregate male membership demand and the second coordinate denotes the aggregate female membership demand. Since $\pi(M) = \pi(F) = 1$ it follows from the consistency condition that $\alpha(\pi, \gamma) = 2$. Now consider a mixed economy $\tilde{\mathcal{E}}$ which is formed from the atomless economy. The set of negligible agents is now $[0, 3)$ endowed with the Lebesgue measure and Lebesgue σ -algebra. There exists only one atom in the economy denoted by $\{6\}$ which is a cartel formed by negligible agents in $[3, 5]$ and $\tilde{\lambda}(\{6\}) = 2$. As before agents in $[0, 3)$ are males and $\{6\}$ is a female. The club type is (π, γ) and its profile is $\pi(M) = \pi(F) = 1$ as before. Define the membership vector $\tilde{l} : \tilde{A} \rightarrow \mathbb{Z}_+$ as

$$\tilde{l}_a = \begin{cases} (1, 0), & \text{if } a \in [0, 2); \\ (0, 0), & \text{if } a \in [2, 3); \text{ and} \\ (0, 1), & \text{if } a = \{6\}. \end{cases}$$

Note that the membership vector $\int_{[0,3] \cup \{6\}} \tilde{l}_a d\tilde{\lambda} \in \mathcal{Cons}$ with $\alpha(\pi, \gamma) = 2$. It follows from the definition of $\tau(\cdot)$ that $\frac{1}{2}inp(\pi, \gamma)$ is the input contribution density in our economy. Each male agent in $[0, 2]$ in the economy contributes $\frac{1}{2}inp(\pi, \gamma)$ from his initial endowment for the formation of the clubs. Note that such a contribution is infinitesimal as the initial endowment of each male agent is infinitesimal compared to the aggregate endowment of private goods in the economy. On the other hand, $\frac{1}{2}inp(\pi, \gamma) \cdot \tilde{\lambda}(\{6\})$ denotes the actual input contribution by agent $\{6\}$ which juxtaposed to each male agent contribution is not infinitesimal. However, the input contribution made by the agent $\{6\}$ dictates the formation of not just one club of type (π, γ) but measure 2 of such clubs. Now as $\{6\}$ is a cartel formed, then one can think that the individual small agents in $\{6\}$ will distribute themselves over the number of clubs formed in such a way that each club will contain only one male and one female.

Consider the atomless economy described above. Define the membership vector $\mu : A \rightarrow \mathbb{Z}_+$ as

$$\mu_a = \begin{cases} (1, 0), & \text{if } a \in [0, 3); \\ (0, 2), & \text{if } a \in [3, 4); \text{ and} \\ (0, 1), & \text{if } a \in [4, 5]. \end{cases}$$

Then one can observe that the aggregate membership demand is $\int_{[0,5]} l_a d\lambda = (3, 3)$. Since $\pi(M) = \pi(F) = 1$ it follows from the consistency condition that $\alpha(\pi, \gamma) = 3$. Now consider again a mixed economy where agents in $[3, 5]$ form the cartel $\{6\}$ and $\tilde{\lambda}(\{6\}) = 2$. It follows from the consistency condition that for the club to form in the mixed economy, the aggregate female membership demand must equal 3 which in turn

will lead to $\alpha(\pi, \gamma)$ to be equal to 3. Given that the measure of the large agent $\{6\}$ is 2, membership demand for her has to be equal to $\frac{3}{2}$. As $\frac{3}{2} \notin \mathbb{Z}_+$, such a membership demand for the large agent is not defined. In fact, there exists no membership demand profile in the mixed economy for which $\alpha(\pi, \gamma) = 3$. On the other hand, consider a sub-coalition E of $[3, 5]$ with $\lambda(E) = \frac{3}{2}$ in the atomless club economy \mathcal{E} . Note that club formation can materialize with $\alpha(\pi, \gamma) = 3$ in this economy if each agent in E demands 2 unit of club membership each. In the mixed economy $\tilde{\mathcal{E}}$ the formation of cartels by a coalition of negligible agents restricts the formation of sub-coalitions with appropriate measure and feasible membership demand that can eventually lead to $\alpha(\pi, \gamma) = 3$.

6.3 Main Result

In this section, we establish the equivalence theorem between the core and the club equilibrium for a mixed economy with club goods thereby extending the equivalence result of Ellickson et al. [35]. To do so, we establish a correspondence between the two economies' core and the set of club equilibrium states. We begin by developing the connection between club equilibrium states of the continuum economy and its associated mixed economy. The following theorem summarizes that.

Theorem 6.3.1. *Let \mathcal{E} be an atomless club economy and let $\tilde{\mathcal{E}}$ be the mixed economy derived from it. Further, let $\tilde{\mathcal{E}}$ satisfy condition (C). Then the following statements hold:*

- (i) $(\tilde{f}, \tilde{l}) \in \mathcal{W}(\tilde{\mathcal{E}}) \Rightarrow (f, l) := \Xi(\tilde{f}, \tilde{l}) \in \mathcal{W}(\mathcal{E})$.
- (ii) $(f, l) \in \mathcal{W}(\mathcal{E}) \implies (\tilde{f}, \tilde{l}) := \varphi(f, l) \in \mathcal{W}(\tilde{\mathcal{E}})$.

Proof. The proof of (i) is trivial. For the proof of (ii), let $(f, l) \in \mathcal{W}(\mathcal{E})$. Suppose that the corresponding equilibrium price is (p, q) , which implies that (f_a, l_a) maximizes the individual a 's budget set for λ -a.e. on A . For each $n \geq 1$ and for all $a, b \in T_n$, the budget sets of agents a and b are the same, which further implies

$$p \cdot \tilde{f}_{T_n} + q \cdot \tilde{l}_{T_n} \leq p \cdot \tilde{e}_{T_n}.$$

where $\tilde{f}_{T_n} = \frac{1}{\lambda(T_n)} \int_{T_n} f_a d\lambda$ and $\tilde{l}_{T_n} = \frac{1}{\lambda(T_n)} \int_{T_n} l_a d\lambda$. By Proposition 2.3.2, one has

$$\left(\frac{1}{\lambda(T_n)} \int_{T_n} f_a d\lambda, \frac{1}{\lambda(T_n)} \int_{T_n} l_a d\lambda \right) \in \overline{\text{co}}\{(f_a, l_a) : a \in T_n\}.$$

As all agents in T_n are of the same type and $(f, l) \in \mathcal{W}(\mathcal{E})$ it follows that $u_a(f_a, l_a) = \beta$ for some $\beta \in \mathbb{R}$ and for all $a \in T_n$. This further implies

$$\left(\frac{1}{\lambda(T_n)} \int_{T_n} f_a d\lambda, \frac{1}{\lambda(T_n)} \int_{T_n} l_a d\lambda \right) \in \tilde{X}_{\tilde{T}_n} \cap \text{co}\{(h, \nu) \in \tilde{X}_a : \tilde{u}_{\tilde{T}_n}(h, \nu) \geq \beta\}.$$

By condition **(C)**, we have

$$\tilde{u}_{\tilde{T}_n} \left(\frac{1}{\lambda(T_n)} \int_{T_n} f_a d\lambda, \frac{1}{\lambda(T_n)} \int_{T_n} l_a d\lambda \right) \geq \beta.$$

Thus, we have $\tilde{u}_{\tilde{T}_n}(\tilde{f}_{\tilde{T}_n}, \tilde{l}_{\tilde{T}_n}) \geq \beta$. As $(f, l) \in \mathcal{W}(\mathcal{E})$ it must be that $\tilde{u}_{\tilde{T}_n}(\tilde{f}_{\tilde{T}_n}, \tilde{l}_{\tilde{T}_n}) = u_a(f_a, l_a) = \beta$ for all $a \in T_n$ and consequently, $(\tilde{f}_{\tilde{T}_n}, \tilde{l}_{\tilde{T}_n}) \in \mathcal{W}(\mathcal{E})$. \square

Note that as large agents take the form of cartels in our mixed economy there exists no proper sub-coalitions of these cartels that can form and engage in trade with other coalitions in the economy. This restricts the number of coalitions that can form in the mixed economy, making blocking more difficult. Thus any state (f, l) that cannot be blocked in the atomless club economy can be expected to have no objection against its corresponding state (\tilde{f}, \tilde{l}) in the mixed economy. In other words, if a state belongs to the core of \mathcal{E} , then its corresponding state belongs to the core of the economy \mathcal{E} . In order to establish the same we need to first show that in any core state of the atomless club economy, the state assigns indifferent bundles to agents who are of the same type. This result will play a key role in establishing some key results, in particular in Theorem 6.3.4 where we choose $S = T_i$ for some $i = 1, 2, \dots$ and in Proposition 6.3.8 where we choose $S = R$. Note that agents belonging to both of these sets are of the same type and their preferences satisfy the condition **(C)**.

Lemma 6.3.2. *Let \mathcal{E} be an atomless club economy satisfying Definition 3.4.1 and let agents in the coalition R have preferences satisfying condition **(C)**. Suppose that S is a coalition such that members of S are of the same type as the agents in T_1 . Assume further that (f, l) is a state in the core of the economy \mathcal{E} . If¹²*

$$(\hat{f}_S, \hat{l}_S) := \left(\frac{1}{\lambda(S)} \int_S f_a d\lambda, \frac{1}{\lambda(S)} \int_S l_a d\lambda \right) \in X_S,$$

then $u_a(f_a, l_a) = u_S(\hat{f}_S, \hat{l}_S)$ λ -a.e. on S .

¹²As \hat{f}_S always belongs to \mathbb{R}_+^N , in order to guarantee that $(\hat{f}_S, \hat{l}_S) \in X_S$ it is necessary that \hat{l}_S belongs to \mathbb{Z}_+^M .

Proof. Define

$$B := \left\{ a \in S : u_S(\widehat{f}_S, \widehat{l}_S) > u_S(f_a, l_a) \right\}$$

and

$$C := \left\{ a \in S : u_S(\widehat{f}_S, \widehat{l}_S) < u_S(f_a, l_a) \right\}.$$

Recognized that B and C are Σ -measurable sets. We shall complete the proof by showing that none of these sets has a positive measure. To this end, we first assume that $\lambda(B) > 0$. By the continuity of u_S , we can choose a sub-coalition D of B and some $z \in \mathbb{R}_+^N \setminus \{0\}$ such that $u_S(\widehat{f}_S - z, \widehat{l}_S) > u_S(f_a, l_a)$ for all $a \in D$.¹³ Define $r_0 \in (0, 1]$ by letting $r_0 := \frac{\lambda(D)}{\lambda(S)}$. By the Lyapunov convexity theorem, there exists a sub-coalition E of $A \setminus S$ such that

$$(iii) \int_E (f_a - e_a) d\lambda = r_0 \int_{A \setminus S} (f_a - e_a) d\lambda; \text{ and}$$

$$(iv) \int_E l_a d\lambda = r_0 \int_{A \setminus S} l_a d\lambda.$$

Let $G := D \cup E$. Define $g : A \rightarrow \mathbb{R}_+^N$ and $\mu : A \rightarrow \mathbb{R}^{\mathcal{M}}$ by

$$g_a := \begin{cases} \widehat{f}_S - z, & \text{if } a \in D; \\ f_a + \frac{z\lambda(D)}{\lambda(E)}, & \text{otherwise,} \end{cases}$$

and

$$\mu_a := \begin{cases} \widehat{l}_S, & \text{if } a \in D; \\ l_a, & \text{otherwise.} \end{cases}$$

It is claimed that (f, l) is blocked by G via (g, μ) . To see this, first, note that

$$\int_G \mu_a d\lambda = r_0 \int_A l_a d\lambda \in \mathcal{C}ons.$$

As a consequence of this, we have

$$\int_G \tau(\mu_a) d\lambda = r_0 \int_A \tau(l_a) d\lambda.$$

As

$$\int_D (g_a - e_a) d\lambda = r_0 \int_S (f_a - e_a) d\lambda - z\lambda(D),$$

it is just routine to verify that

$$\int_G (g_a + \tau(\mu_a) - e_a) d\lambda = r_0 \int_A (f_a + \tau(l_a) - e_a) d\lambda = 0.$$

¹³Since (f, l) is a core allocation and the initial endowment state is desirable, it follows that $f_a \neq 0$ λ -a.e. on R . Thus, $\widetilde{f}_R \neq 0$.

Therefore, (f, l) is blocked by the coalition G , which violates the hypothesis of our lemma. This leads to a contradiction. Thus, we conclude that $\lambda(B) = 0$. We now assume that $\lambda(C) > 0$. By the continuity of u_R , we can find some $\alpha \in (0, 1)$ and some sub-coalition \widehat{G} of C such that $u_S(\alpha f_a, l_a) > u_S(\widehat{f}_S, \widehat{l}_S)$ λ -a.e. on \widehat{G} . Define $h : A \rightarrow \mathbb{R}_+^N$ by setting $h_a := \alpha f_a$ if $a \in \widehat{G}$; and $h_a := f_a$, otherwise. By Proposition 2.3.2, one has

$$\left(\frac{1}{\lambda(S)} \int_S h_a d\lambda, \frac{1}{\lambda(S)} \int_S l_a d\lambda \right) \in \overline{\text{co}}\{(h_a, l_a) : a \in S\},$$

which further implies

$$\left(\frac{1}{\lambda(S)} \int_S h_a d\lambda, \frac{1}{\lambda(S)} \int_S l_a d\lambda \right) \in X_S \cap \overline{\text{co}}\{(y, \mu) \in X_S : u_S(y, \mu) \geq u_S(\widehat{f}_S, \widehat{l}_S)\}.$$

By condition **(C)**, we have¹⁴

$$u_S \left(\frac{1}{\lambda(S)} \int_S h_a d\lambda, \frac{1}{\lambda(S)} \int_S l_a d\lambda \right) \geq u_S(\widehat{f}_S, \widehat{l}_S).$$

In the light of the strong monotonicity of u_S , we have

$$\begin{aligned} u_S(\widehat{f}_S, \widehat{l}_S) &> u_S \left(\frac{1}{\lambda(S)} \int_S h_a d\lambda, \frac{1}{\lambda(S)} \int_S l_a d\lambda \right) \\ &\geq u_S(\widehat{f}_S, \widehat{l}_S), \end{aligned}$$

which is a contradiction. Therefore, we have $\lambda(C) = 0$. Hence, we conclude that $u_a(f_a, l_a) = u_S(\widehat{f}_S, \widehat{l}_S)$ λ -a.e. on S . \square

Remark 6.3.3. In the case of a mixed economy $\widetilde{\mathcal{E}}$ with club goods, if the coalition S is such that $\widetilde{A}_1 \subseteq S$, then one can show that the conclusion of the above lemma holds true.

Theorem 6.3.4. *Let \mathcal{E} be an atomless club economy satisfying Definition 3.4.1 and let agents in the coalition R have preferences satisfying condition **(C)**. Further, let (f, l) be a state of the economy such that $\varphi(f, l)$ exists in the associated mixed economy $\widetilde{\mathcal{E}}$. If $(f, l) \in \mathcal{C}(\mathcal{E})$ then $(\widetilde{f}, \widetilde{l}) := \varphi(f, l) \in \mathcal{C}(\widetilde{\mathcal{E}})$.*

Proof. Let $(f, l) \in \mathcal{C}(\mathcal{E})$. On the contrary, let us assume that $(\widetilde{f}, \widetilde{l}) := \varphi(f, l) \notin \mathcal{C}(\widetilde{\mathcal{E}})$. Then there exists a coalition \widetilde{S} and a state $(\widetilde{y}, \widetilde{\mu})$ such that:

$$(i) \int_{\widetilde{S}} \widetilde{y}_a d\widetilde{\lambda} + \int_{\widetilde{S}} \tau(\widetilde{\mu}_a) d\widetilde{\lambda} = \int_{\widetilde{S}} \widetilde{e}_a d\widetilde{\lambda};$$

¹⁴By our hypothesis, types of members of S are the same as those in R .

(ii) $\tilde{u}_a(\tilde{y}_a, \tilde{\mu}_a) > \tilde{u}_a(\tilde{f}_a, \tilde{l}_a)$ $\tilde{\lambda}$ -a.e. on \tilde{S} ; and

(iii) $\int_{\tilde{S}} \tilde{\mu}_a d\tilde{\lambda} \in \mathcal{C}ons$.

We define $(y, \mu) := \Xi(\tilde{y}, \tilde{\mu})$ and $S := (\tilde{S} \cap A_0) \cup \bigcup \{T_i : \tilde{T}_i \subseteq \tilde{S}\}$. It is claimed that (f, l) is blocked by S via (y, μ) . To this end, pick an arbitrary i with $\tilde{T}_i \subseteq \tilde{S}$. As

$$\left(\tilde{f}_{T_i}, \tilde{l}_{T_i}\right) = \left(\frac{1}{\lambda(T_i)} \int_{T_i} f_a d\lambda, \frac{1}{\lambda(T_i)} \int_{T_i} l_a d\lambda\right)$$

it follows from Lemma 6.3.2 that $u_a(f_a, l_a) = u_{T_i}(\tilde{f}_{\tilde{T}_i}, \tilde{l}_{\tilde{T}_i})$ λ -a.e. on T_i . This implies that $u_a(y_a, \mu_a) = u_{T_i}(\tilde{y}_{\tilde{T}_i}, \tilde{\mu}_{\tilde{T}_i}) > u_{T_i}(\tilde{f}_{\tilde{T}_i}, \tilde{l}_{\tilde{T}_i}) = u_a(f_a, l_a)$ λ -a.e. on T_i . Therefore, (f, l) is blocked by S via (y, μ) , which leads to a contradiction. Hence, our supposition was wrong and (\tilde{f}, \tilde{l}) belongs to $\mathcal{C}(\tilde{\mathcal{E}})$. \square

The converse of the above theorem may not necessarily hold. Shitovitz [77] in his seminal work showed that in a mixed economy with a single large trader, there exists an allocation in the core that is not a core allocation in the continuum economy (see Example 1).¹⁵ Shitovitz [77] got around this difficulty by assuming that there exist at least two large agents and they are of the same type, and established the core-equivalence theorem for a mixed economy with only private goods (see Theorem B). Greenberg and Shitovitz [41] used a similar assumption to give a simpler proof of Shitovitz's result. However, such an assumption does not guarantee the result in our case due to the presence of club goods, as observed in the example below.

Example 6.3.5. Consider an economy \mathcal{E} with negligible agents distributed uniformly on $[0, 7]$ endowed with the Lebesgue measure and the Lebesgue σ -algebra. Further, assume that agents in $(\frac{1}{2}, 7]$ are of the same type. Consumers in the interval $[0, \frac{1}{2}]$ are males (M) while those in $(\frac{1}{2}, 7]$ are females (F). There are two private goods in the economy. The initial endowment distribution of the economy is

$$e_a = \begin{cases} (3, 0), & \text{if } a \in [0, \frac{1}{2}]; \\ (0, 3), & \text{if } a \in (\frac{1}{2}, 7]. \end{cases}$$

There exists only a single club type (π, γ) in the economy. The profile π for the club is represented as $\pi(M) = \pi(F) = 1$, and the input requirement for forming one such

¹⁵Note that the example shows that there exists an allocation in the core of the mixed economy that is not an equilibrium allocation. As a private good economy is a special case of our framework, by Theorem 6.3.1 one can claim that such an allocation is not an equilibrium allocation in the continuum economy and hence not in the core of the continuum economy.

club is given as $inp(\pi, \gamma) = (1, 1)$. The utility functions of the agents are given as

$$u_a(x_a, l_a) = \begin{cases} (\sqrt{x_1} + \sqrt{x_2}) \times 5l_1, & \text{if } a \in [0, \frac{1}{2}]; \\ (\sqrt{x_1} + \sqrt{x_2}) \times 5l_2, & \text{if } a \in (\frac{1}{2}, 7], \end{cases}$$

where x_1, x_2 denote the consumption of good 1 and good 2, respectively and l_1, l_2 denote the number of memberships purchased by the male agents and female agents, respectively. Now assume that agents in $(1, 4]$ join forces to act as a cartel and agents in $(4, 7]$ join forces to form another cartel. We denote by $\tilde{\mathcal{E}}$ the new economy thus formed. Note that $\tilde{\mathcal{E}}$ consists of both negligible and non-negligible agents, where the atomic agents of the economy are denoted by $\{8\}$ and $\{9\}$ with $\tilde{\lambda}(\{8\}) = \tilde{\lambda}(\{9\}) = 3$. We define a state $(\tilde{f}, \tilde{l}) : \tilde{A} \rightarrow \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{A}}$ for our economy $\tilde{\mathcal{E}}$ as follows.

$$\tilde{f}_a := \begin{cases} (\frac{1}{2}, \frac{1}{2}), & \text{if } a \in [0, 1]; \\ (0, 3), & \text{otherwise,} \end{cases}$$

and

$$\tilde{l}_a := \begin{cases} (2, 0), & \text{if } a \in [0, \frac{1}{2}]; \\ (0, 2), & \text{if } a \in (\frac{1}{2}, 1]; \\ (0, 0), & \text{otherwise.} \end{cases}$$

Note that (\tilde{f}, \tilde{l}) is a feasible state in the economy $\tilde{\mathcal{E}}$. Further, (\tilde{f}, \tilde{l}) belongs to the core of the mixed club economy $\tilde{\mathcal{E}}$. This follows from the following argument. Given that an input vector of one unit of each private good is required for a club to form no sub-coalition of $[0, \frac{1}{2}]$ or $(\frac{1}{2}, 1] \cup \{8\} \cup \{9\}$ can form to improve upon (\tilde{f}, \tilde{l}) . Further, observe that both the large agents obtain zero utility at the state (\tilde{f}, \tilde{l}) . Suppose that agent $\{8\}$ now forms a coalition with the set of agents in $[0, \frac{1}{2}]$. Agent, $\{8\}$ at the minimum will demand one unit of club membership which would result in an aggregate female membership demand of 3. For a club to form there must exist 3 male membership demand in aggregate. This implies that each agent in $[0, \frac{1}{2}]$ will have to demand 6 units of club membership each and $\alpha(\pi, \gamma) = 3$. As $inp(\pi, \gamma) = (1, 1)$ for one club, the total input requirement will be $(3, 3)$. However, the aggregate endowment for the coalition $[0, \frac{1}{2}] \cup \{8\}$ is $(\frac{3}{2}, 9)$. Thus no clubs will be formed and (\tilde{f}, \tilde{l}) cannot be blocked. Further, no subset of agents of $[0, \frac{1}{2}]$ can form a coalition with $\{8\}$ to block (\tilde{f}, \tilde{l}) as the aggregate endowment for good 1 would be even less than $\frac{3}{2}$ and agents have to demand more than 6 units of club membership to ensure consistency condition as the aggregate female membership demand is 3. Analogous reasoning yields that any sub-coalition of $[0, \frac{1}{2}]$ can not form a coalition with $\{9\}$ to improve upon the state

(\tilde{f}, \tilde{l}) . Note that as endowments are not sufficient for clubs to form with a coalition of the form $[0, \frac{1}{2}] \cup \{8\}$ or $[0, \frac{1}{2}] \cup \{9\}$, they will also not be sufficient for a coalition of the form $[0, \frac{1}{2}] \cup \{8\} \cup \{9\}$, as atoms are not endowed with good 1. We now check whether any coalition $\tilde{S} = \tilde{S}_1 \cup \tilde{S}_2$, where $\tilde{S}_1 \subseteq [0, \frac{1}{2}]$ and $\tilde{S}_2 \subseteq (\frac{1}{2}, 1]$ can block the state (\tilde{f}, \tilde{l}) . To begin with, observe that for any such coalition \tilde{S} it is not possible to decrease club membership consumption and improve utility by increasing private goods consumption. This is because club memberships have a higher weightage in the utility functions and endowment constraints do not allow for a large increase in the consumption of private goods. Now suppose that the coalition \tilde{S} is such that each member in \tilde{S}_1 demands three units of club membership and each member in \tilde{S}_2 demands two units of club membership each. For clubs to form it must be that $3\tilde{\lambda}(\tilde{S}_1) = 2\tilde{\lambda}(\tilde{S}_2)$. It then follows that the aggregate endowment for the coalition \tilde{S} is $(3, \frac{9}{2})\tilde{\lambda}(\tilde{S}_1)$ and the aggregate input requirement for club formation is $(3, 3)\tilde{\lambda}(\tilde{S}_1)$. Then the remaining endowment for private good consumption is $(0, \frac{3}{2})\tilde{\lambda}(\tilde{S}_1)$ is not sufficient for the coalition to block the state (\tilde{f}, \tilde{l}) . Thus, if any $\tilde{S}_1 \subseteq [0, \frac{1}{2}]$ wants to increase their utility by means of higher club membership consumption while any $\tilde{S}_2 \subseteq (\frac{1}{2}, 1]$ maintain the same club membership consumption compared to (\tilde{f}, \tilde{l}) , then it follows from the aggregate endowment of the coalition \tilde{S} that there isn't enough private commodities for the agents to improve upon (\tilde{f}, \tilde{l}) and vice-versa.

Now we return to the original continuum economy \mathcal{E} . For the state (\tilde{f}, \tilde{l}) in the economy $\tilde{\mathcal{E}}$ we define the associated continuum economy state $(f, l) : A \rightarrow \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$ as

$$f_a := \begin{cases} (\frac{1}{2}, \frac{1}{2}), & \text{if } a \in [0, 1]; \\ (0, 3), & \text{otherwise,} \end{cases}$$

and

$$l_a := \begin{cases} (2, 0), & \text{if } a \in [0, \frac{1}{2}]; \\ (0, 2), & \text{if } a \in (\frac{1}{2}, 1]; \\ (0, 0), & \text{otherwise.} \end{cases}$$

Now choose a coalition $S = [0, \frac{1}{2}] \cup (1, \frac{3}{2}]$. Note that $\lambda(S) = 1$ and $\int_S e_a d\lambda = (\frac{3}{2}, \frac{3}{2})$. We define the state $(g, \mu) : S \rightarrow \mathbb{R}_+^N \times \mathbb{R}^{\mathcal{M}}$ as

$$g_a := \begin{cases} (\frac{3}{4}, \frac{1}{2}), & \text{if } a \in [0, \frac{1}{2}]; \\ (\frac{1}{4}, \frac{1}{2}), & \text{if } a \in (1, \frac{3}{2}], \end{cases}$$

and

$$\mu_a := \begin{cases} (2, 0), & \text{if } a \in [0, \frac{1}{2}]; \\ (0, 0), & \text{if } a \in (\frac{1}{2}, 1]; \\ (0, 2), & \text{otherwise.} \end{cases}$$

It can be easily verified that $u_a(g_a, \mu_a) > u_a(f_a, l_a)$ for all $a \in S$. Thus, $(f, l) \notin \mathcal{C}(\mathcal{E})$.

Remark 6.3.6. It is evident from the above example that merely assuming that there exist two atoms who are of the same type as in Shitovitz [77] and Greenberg and Shitovitz [41] is not sufficient to guarantee the converse of Theorem 6.3.4 in our framework. This is due to the core indivisible nature of club membership choices. In the above example, when non-negligible agent $\{8\}$ or $\{9\}$ tries to improve upon the state (\tilde{f}, \tilde{l}) , he or she has to purchase one unit of membership for the club at the minimum. Given that $\tilde{\lambda}(\{8\}) = 3$, the actual membership demand by agent $\{8\}$ will be at least 3. For club formation to materialize, an aggregate male membership demand of 3 must exist, which in turn leads to $\alpha(\pi, \gamma) = 3$ and requires an input of $(3, 3)$. This leaves the coalition $[0, \frac{1}{2}] \cup \{8\}$ with insufficient private goods to improve upon (\tilde{f}, \tilde{l}) . In the case of purely private goods exchange economies with perfectly divisible commodities, conditions like strong monotonicity ensure that one can increase the consumption of any one of the private commodities very slightly for agent $\{8\}$ to improve upon (\tilde{f}, \tilde{l}) and thus leave other members in the coalition with sufficient endowment to improve upon (\tilde{f}, \tilde{l}) . However, in our case one needs to increase club membership consumption to improve the utility of agent $\{8\}$ and that can be done only by increasing it by at least one unit which then leaves the others in the coalition with insufficient endowment to improve upon (\tilde{f}, \tilde{l}) . This fact leads to the failure of Shitovitz's [77] assumption and hence the core-equivalence theorem in our framework. A failure of the core-equivalence theorem can also happen in exchange economies with public goods as observed by Basile et al. [13]. They adopt two different conditions to resolve this problem: (i) the presence of infinitely many atoms in the economy and, (ii) there exists a coalition \tilde{E} that contains all non-negligible agents and some positive measure of negligible agents where all agents are of the same type and show that in any core state, (f, l) the utility enjoyed by the agents in the coalition \tilde{E} is the same and is equal to the utility derived from the average consumption in the state (f, l) . We adopt the second approach of Basile et al. [13] to establish our equivalence theorem. Recall that there exists a coalition R in our atomless economy containing agents of the same type from which the non-negligible agents are formed through the formation of cartels by negligible agents in the coalition. Hence, in order to obtain a positive result we have to: (i) assume that $\lambda(R \setminus A_1) > 0$, and (ii) establish that the utility enjoyed by the agents in the coalition R in any core

state (f, l) is the same and is equal to the utility derived from the average consumption in the state (f, l) . Thus, we need to assume that the average membership choice in the state (f, l) is defined. Observe that in our Example 6.3.5, $R = (\frac{1}{2}, 7]$ and $\lambda(R \setminus A_1) = \frac{1}{2}$. However, $\int_R l_a d\lambda = (0, 1)$ and $\lambda(R) = \frac{13}{2}$. Thus,

$$\frac{1}{\lambda(R)} \int_R l_a d\lambda = \left(0, \frac{2}{13}\right) \notin \mathbb{Z}_+^2,$$

which implies $(\widehat{f}_R, \widehat{l}_R) \notin X_R$ and the equivalence breaks down. Thus, in addition to the existence of a coalition like R , we need to restrict ourselves to states (f, l) of \mathcal{E} such that $(\widehat{f}_R, \widehat{l}_R) \in X_R$ in order to establish our core-equivalence theorem.

We now proceed to establish our formal result of the core-equivalence theorem. However, before that, we establish a key proposition that helps us to establish our central result. The following result, which appeared in Bhowmik and Kaur [22], plays a crucial role in the proof of the proposition. We present the proof below for the sake of completeness.

Lemma 6.3.7. *Suppose that \mathcal{E} is an atomless economy. Let (f, l) and (g, μ) be two states of \mathcal{E} such that $u_a(g_a, \mu_a) > u_a(f_a, l_a)$ λ -a.e. on S for some coalition S . Then given any $0 < \alpha < 1$ there is a state (h, ν) such that*

- (i) $u_a(h_a, \nu_a) > u_a(f_a, l_a)$ λ -a.e. on S ;
- (ii) $\int_S h_a d\lambda = \int_S (\alpha g_a + (1 - \alpha)f_a) d\lambda$; and
- (iii) $\int_S \nu_a d\lambda = \int_S (\alpha \mu_a + (1 - \alpha)l_a) d\lambda$.

Proof. Consider a vector measure $v : \Sigma_S \rightarrow \mathbb{R}^{N+1} \times \mathbb{R}^{\mathcal{M}}$ such that

$$v(B) := \left\{ \left(\lambda(B), \int_B (g_a - f_a) d\lambda, \int_B (\mu_a - l_a) d\lambda \right) : B \in \Sigma_S \right\}.$$

Pick any $\alpha \in (0, 1)$. In view of Lyapunov's Convexity theorem, there exists a sub-coalition E of S such that $v(E) = \alpha v(S)$, which implies

- (a) $\lambda(E) = \alpha \lambda(S)$;
- (b) $\int_E (g_a - f_a) d\lambda = \alpha \int_S (g_a - f_a) d\lambda$; and
- (c) $\int_E (\mu_a - l_a) d\lambda = \alpha \int_S (\mu_a - l_a) d\lambda$.

It follows from (iv) in Definition 3.4.1 that there exists a function $g^* : E \rightarrow \mathbb{R}_+^N$ and some $z \in \mathbb{R}_+^N \setminus \{0\}$ such that $u_a(g_a^*, \mu_a) > u_a(f_a, l_a)$ λ -a.e. on E and

$$\int_E g_a^* d\lambda = \int_E g_a d\lambda - z.$$

Finally, define $h : S \rightarrow \mathbb{R}_+^N$ and $\nu : S \rightarrow \mathbb{R}^{\mathcal{M}}$ by letting

$$h_a := \begin{cases} g_a^*, & \text{if } a \in E; \\ f_a + \frac{z}{\lambda(S \setminus E)}, & \text{if } a \in S \setminus E, \end{cases}$$

and

$$\nu_a := \begin{cases} \mu_a, & \text{if } a \in E; \\ l_a, & \text{if } a \in S \setminus E. \end{cases}$$

From the strong monotonicity of utility functions, it follows directly that $u_a(h_a, \nu_a) > u_a(f_a, l_a)$ λ -a.e. on S . Further, in the presence of (b) and (c), we can readily verify that

$$\int_S h_a d\lambda = \int_S (\alpha g_a + (1 - \alpha) f_a) d\lambda$$

and

$$\int_S \nu_a d\lambda = \int_S (\alpha \mu_a + (1 - \alpha) l_a) d\lambda.$$

This completes the proof. \square

The next proposition is an extension of Proposition 6 of Basile et al. [13] to the case of club goods which helps us to establish the converse of Theorem 6.3.4. We denote by A_1 the set of agents in the atomless club economy \mathcal{E} who join forces to form the different cartels in \tilde{A}_1 .

Proposition 6.3.8. *Let \mathcal{E} be an atomless club economy and there exists a coalition R such that agents in R are of the same type, $A_1 \subseteq R$, and $\lambda(R \setminus A_1) > 0$. Assume that (f, l) is a feasible state of \mathcal{E} such that $(f, l) \notin \mathcal{C}(\mathcal{E})$ and*

$$(\hat{f}_R, \hat{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda, \frac{1}{\lambda(R)} \int_R l_a d\lambda \right) \in X_R.$$

Then (f, l) can be blocked by a coalition contained in A_0 .

Proof. Since $(f, l) \notin \mathcal{C}(\mathcal{E})$, there exists a coalition $S \in \Sigma$ and a state (g, ν) such that $u_a(g_a, \nu_a) > u_a(f_a, l_a)$ λ -a.e. on S ,

$$\int_{S \cap A_0} (g_a - e_a + \tau(\nu_a)) d\lambda + \int_{S \cap A_1} (g_a - e_a + \tau(\nu_a)) d\lambda = 0 \quad (6.3.1)$$

and

$$\int_{S \cap A_0} \nu_a d\lambda + \int_{S \cap A_1} \nu_a d\lambda \in \mathcal{C}ons. \quad (6.3.2)$$

If $\lambda(S \cap A_1) = 0$ then we are done. So, we assume that $\lambda(S \cap A_1) > 0$, and notice that ν only takes finitely many values.¹⁶ Let the range of ν be $\{\nu^1, \dots, \nu^l\}$. For each $1 \leq j \leq l$, define $K_j := \{a \in A : \nu_a = \nu^j\}$. Notice that K_j is Σ -measurable for all $1 \leq j \leq l$. Define

$$\mathbb{J} := \{j : 1 \leq j \leq l \text{ and } \lambda(S \cap A_1 \cap K_j) > 0\}.$$

For each $j \in \mathbb{J}$, denote $F_j := S \cap A_1 \cap K_j$. Consequently, we have¹⁷

$$S \cap A_1 \sim \bigcup \{F_j : j \in \mathbb{J}\}.$$

Thus, from Equation (6.3.1) and Equation (6.3.2), we have

$$\int_{S \cap A_0} (g_a - e_a + \tau(\nu_a)) d\lambda + \sum_{j \in \mathbb{J}} \lambda(F_j) (g^j - e^j + \tau(\nu^j)) = 0 \quad (6.3.3)$$

and

$$\int_{S \cap A_0} \nu_a d\lambda + \sum_{j \in \mathbb{J}} \lambda(F_j) \nu^j \in \mathcal{C}ons, \quad (6.3.4)$$

where $e^j := e_a$ for $a \in F_j$ and $j \in \mathbb{J}$, and

$$g^j := \frac{1}{\lambda(F_j)} \int_{F_j} g_a d\lambda$$

for all $j \in \mathbb{J}$. We denote by $G \subseteq A_0$ the coalition $R \setminus A_1$.

Case 1. $\lambda(S \cap A_1) \leq \lambda(G)$. In this case, we can choose a sub-coalition G_1 of G such that $\lambda(G_1) = \lambda(F_1)$. Since $\lambda(S \cap A_1 \setminus F_1) \leq \lambda(G \setminus G_1)$, we can analogously choose another sub-coalition $G_2 \subseteq G \setminus G_1$ such that $\lambda(G_2) = \lambda(F_2)$. Continuing this way, for each $j \geq 2$, we can choose a sub-coalition $G_j \subseteq G \setminus (G_1 \cup G_2, \dots, \cup G_{j-1})$ such that $\lambda(G_j) = \lambda(F_j)$. Thus, Equation (6.3.3) and Equation (6.3.4) can be expressed as

$$\int_{S \cap A_0} (g_a - e_a + \tau(\nu_a)) d\lambda + \sum_{j \in \mathbb{J}} \lambda(G_j) (g^j - e^j + \tau(\nu^j)) = 0 \quad (6.3.5)$$

¹⁶It follows from the fact that there is a uniform upper bound M on the number of club memberships that each agent can have and there is a finite set of possible club types.

¹⁷We use the notation $B \sim C$ whenever $\lambda(B \Delta C) = 0$, i.e. the measure of the symmetric difference between the two sets B and C is zero.

and

$$\int_{S \cap A_0} \nu_a d\lambda + \sum_{j \in \mathbb{J}} \lambda(G_j) \nu^j \in \mathcal{C}ons. \quad (6.3.6)$$

Case 2. $\lambda(S \cap A_1) > \lambda(G)$. In this case, we first choose $\alpha \in (0, 1]$ such that $\lambda(G) = \alpha \lambda(S \cap A_1)$. As in **Case 1**, there is a partition $\{\widehat{G}_j : j \in \mathbb{J}\}$ of G such that $\lambda(\widehat{G}_j) = \alpha \lambda(F_j)$ for all $j \in \mathbb{J}$. Applying the Lyapunov convexity theorem, we can find a sub-coalition E of $S \cap A_0$ such that

$$\int_E (g_a - e_a + \tau(\nu_a)) d\lambda = \alpha \int_{S \cap A_0} (g_a - e_a + \tau(\nu_a)) d\lambda$$

and

$$\int_E \nu_a d\lambda = \alpha \int_{S \cap A_0} \nu_a d\lambda.$$

Thus, it follows from Equation (6.3.3) and Equation (6.3.4) that

$$\int_E (g_a - e_a + \tau(\nu_a)) d\lambda + \sum_{j \in \mathbb{J}} \lambda(\widehat{G}_j) (g^j - e^j + \tau(\nu^j)) = 0 \quad (6.3.7)$$

and

$$\int_E \nu_a d\lambda + \sum_{j \in \mathbb{J}} \lambda(\widehat{G}_j) \nu^j \in \mathcal{C}ons. \quad (6.3.8)$$

Therefore, by Equations (6.3.5)-(6.3.8), we conclude that there exist a coalition $B_0 \subseteq S \cap A_0$ and a collection $\{\overline{G}_j : j \in \mathbb{J}\} \subseteq \Sigma_G$ of pairwise disjoint coalitions such that

$$\int_{B_0} (g_a - e_a + \tau(\nu_a)) d\lambda + \sum_{j \in \mathbb{J}} \lambda(\overline{G}_j) (g^j - e^j + \tau(\nu^j)) d\lambda = 0 \quad (6.3.9)$$

and

$$\int_{B_0} \nu_a d\lambda + \sum_{j \in \mathbb{J}} \lambda(\overline{G}_j) \nu^j \in \mathcal{C}ons. \quad (6.3.10)$$

Define $G_0 := \bigcup \{\overline{G}_j : j \in \mathbb{J}\}$ and $(g^*, \nu^*) : G_0 \rightarrow \mathbb{R}_+^N \times \mathbb{R}_+^M$ by letting $(g_a^*, \nu_a^*) := (g^j, \nu^j)$, if $a \in \overline{G}_j$. By Remark 6.3.3, it follows that $u_a(f_a, l_a)$ is constant λ -a.e. on R . As a consequence, we have $u_a(g_a^*, \nu_a^*) > u_a(f_a, l_a)$ λ -a.e. on G_0 . If $\lambda(B_0 \cap G_0) = 0$, then it is evident from Equation (6.3.9) and Equation (6.3.10) that $B_0 \cup G_0$ blocks (f, l) via (h, \bar{l}) , which is defined as

$$(h_a, \bar{l}_a) := \begin{cases} (g_a^*, \nu_a^*), & \text{if } a \in G_0; \\ (g_a, \nu_a), & \text{otherwise.} \end{cases}$$

This is a contradiction. Thus, we assume that $\lambda(B_0 \cap G_0) \neq 0$. We define a measurable set $C := (B_0 \setminus G_0) \cup (G_0 \setminus B_0)$, and a state (h^*, l^*) such that

$$(h_a^*, l_a^*) := \begin{cases} (g_a^*, \nu_a^*), & \text{if } a \in G_0; \\ (g_a, \nu_a), & \text{otherwise.} \end{cases}$$

By the Lyapunov convexity theorem, there is some $C_0 \in \Sigma_C$ such that

$$\int_{C_0} (h_a^* - e_a + \tau(l_a^*)) d\lambda = \frac{1}{2} \int_C (h_a^* - e_a + \tau(l_a^*)) d\lambda$$

and

$$\int_{C_0} l_a^* d\lambda = \frac{1}{2} \int_C l_a^* d\lambda.$$

By Lemma 6.3.7, there is a state (φ, l'') such that

$$\int_{B_0 \cap G_0} \varphi_a d\lambda = \frac{1}{2} \int_{B_0 \cap G_0} (g_a + g_a^*) d\lambda$$

and

$$\int_{B_0 \cap G_0} l_a'' d\lambda = \frac{1}{2} \int_{B_0 \cap G_0} (\nu_a + \nu_a^*) d\lambda.$$

It follows that

$$\int_{B_0 \cap G_0} \tau(l_a'') d\lambda = \frac{1}{2} \int_{B_0 \cap G_0} (\tau(\nu_a) + \tau(\nu_a^*)) d\lambda.$$

Therefore, we conclude that

$$\int_{B_0 \cap G_0} (\varphi_a - e_a + \tau(l_a'')) d\lambda = \frac{1}{2} \int_{B_0 \cap G_0} (g_a + g_a^* - 2e_a + \tau(\nu_a) + \tau(\nu_a^*)) d\lambda.$$

In view of above, we can re-write Equation (6.3.9) and Equation (6.3.10) as follows

$$\int_{C_0} (h_a^* - e_a + \tau(l_a^*)) d\lambda + \int_{B_0 \cap G_0} (\varphi_a - e_a + \tau(l_a'')) d\lambda = 0$$

and

$$\int_{C_0} l_a^* d\lambda + \int_{B_0 \cap G_0} l_a'' d\lambda \in \mathcal{C}ons.$$

Therefore, (f, l) is blocked by $C_0 \cup (B_0 \cap G_0) \in \Sigma_{A_0}$ via the state (y, ξ) , where (y, ξ) is defined by

$$(y_a, \xi_a) := \begin{cases} (h_a^*, l_a^*), & \text{if } a \in C_0; \\ (\varphi_a, l_a''), & \text{otherwise.} \end{cases}$$

This completes the proof. □

In light of the above proposition, we can conclude that if a state belongs to the core of a mixed economy, it also belongs to the core of the continuum economy from which it has been derived. The intuition behind the result is that as agents in the coalition, R are all of the same type, if a coalition S containing some atom(s) blocks a state (f, l) we can replace the set of agents in $S \cap A_1$ with a coalition in $R \setminus A_1$ and thus block (f, l) with a coalition of agent contained in A_0 . This is possible because of the assumption $\lambda(R \setminus A_1) > 0$ and we know from Proposition 6.3.2 that in any core state agents of the same type derive the same utility. We now formally state this result.

Theorem 6.3.9. *Let \mathcal{E} be an atomless club economy and there exists a coalition R such that agents in R are of the same type, $A_1 \subseteq R$, and $\lambda(R \setminus A_1) > 0$. Further, let $\tilde{\mathcal{E}}$ be the corresponding mixed economy. Assume that (\tilde{f}, \tilde{l}) is a feasible state of $\tilde{\mathcal{E}}$ such that*

$$(\hat{f}_R, \hat{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda, \frac{1}{\lambda(R)} \int_R l_a d\lambda \right) \in X_R,$$

where $(f, l) := \Xi(\tilde{f}, \tilde{l})$. Then $(\tilde{f}, \tilde{l}) \in \mathcal{C}(\tilde{\mathcal{E}}) \Rightarrow (f, l) \in \mathcal{C}(\mathcal{E})$.

Proof. The theorem directly follows from the proof of Proposition 6.3.8. \square

We now establish the equivalence theorem for a mixed economy with club goods by virtue of Theorem 6.3.9 and Theorem 6.3.1.

Theorem 6.3.10. *Let \mathcal{E} be an atomless club economy in which endowments are desirable and uniformly bounded above. Suppose there exists a coalition R such that agents in R are of the same type, $A_1 \subseteq R$, and $\lambda(R \setminus A_1) > 0$. Further, let \mathcal{E} be club irreducible and $\tilde{\mathcal{E}}$ be the corresponding mixed economy satisfying condition (C). Assume that (\tilde{f}, \tilde{l}) is a state of $\tilde{\mathcal{E}}$ such that*

$$(\hat{f}_R, \hat{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda, \frac{1}{\lambda(R)} \int_R l_a d\lambda \right) \in X_R,$$

where $(f, l) := \Xi(\tilde{f}, \tilde{l})$. Then $(\tilde{f}, \tilde{l}) \in \mathcal{W}(\tilde{\mathcal{E}})$ if and only if $(\tilde{f}, \tilde{l}) \in \mathcal{C}(\tilde{\mathcal{E}})$.

Proof. The implication $(\tilde{f}, \tilde{l}) \in \mathcal{W}(\tilde{\mathcal{E}}) \Rightarrow (\tilde{f}, \tilde{l}) \in \mathcal{C}(\tilde{\mathcal{E}})$ is immediate. For the other direction, let $(\tilde{f}, \tilde{l}) \in \mathcal{C}(\tilde{\mathcal{E}})$. Then from Theorem 6.3.9, it follows that $(f, l) := \Xi(\tilde{f}, \tilde{l}) \in \mathcal{C}(\mathcal{E})$. Then by Theorem 5.1 of Ellickson et al. [35], we can infer that $(f, l) \in \mathcal{W}(\mathcal{E})$. Hence, in view of Theorem 6.3.1, we have $(\tilde{f}, \tilde{l}) := \varphi(f, l)$ belongs to $\mathcal{W}(\tilde{\mathcal{E}})$. This completes the proof. \square

Remark 6.3.11. One can observe that for establishing the core-equivalence theorem for a mixed economy $\tilde{\mathcal{E}}$ with club goods, we base the assumptions mainly on an atomless club economy \mathcal{E} . This follows from the fact that the mixed economy $\tilde{\mathcal{E}}$ is formed from the atomless economy \mathcal{E} and thus the assumptions of club irreducibility, endowments being desirable and uniformly bounded above are automatically satisfied for $\tilde{\mathcal{E}}$. However, one can establish the core-equivalence result by basing the assumptions the other way around. So let $\tilde{\mathcal{E}}$ be a mixed economy with club goods satisfying condition **(C)** in which endowments are desirable and uniformly bounded above. Let $\tilde{R} \in \tilde{\Sigma}$ be a coalition of agents of the same type, and $\tilde{\lambda}(\tilde{R} \setminus \tilde{A}_1) > 0$. Further, let $\tilde{\mathcal{E}}$ be club irreducible and $(\tilde{f}, \tilde{l}) \in \mathcal{C}(\tilde{\mathcal{E}})$. Then as endowments are desirable and uniformly bounded above in $\tilde{\mathcal{E}}$ they are desirable and uniformly bounded above in \mathcal{E} as well.¹⁸ It then follows from Theorem 6.3.9 we have $(f, l) \in \mathcal{C}(\mathcal{E})$. As (\tilde{f}, \tilde{l}) is club-linked so is (f, l) .¹⁹ It follows from Theorem 5.1 of Ellickson et al. [35] that $(f, l) \in \mathcal{W}(\mathcal{E})$ and thus from Theorem 6.3.1, we have $(\tilde{f}, \tilde{l}) \in \mathcal{W}(\tilde{\mathcal{E}})$.

¹⁸As e_a is a step function in the associated atomless economy for each $a \in A$

¹⁹As (f, l) is nothing but a step function in the associated atomless economy \mathcal{E} .

Chapter 7

Conclusion

In this thesis, we have considered and analyzed blocking mechanisms in an economy with club goods and multiple private goods. We begin by refining the veto mechanism to a two-step objection counter-objection one and define the bargaining set of our economy. We extend the seminal result of Mas-Colell [64] to show that the bargaining set of an atomless club economy is equivalent to the set of club equilibrium states. In the next chapter, we attempt to provide characterizations of the bargaining set in terms of the size of the (counter-) objecting coalitions and the veto power of the grand coalition. In the process, we encounter certain shortcomings of the bargaining set introduced by Mas-Colell [64] and thus resort to the notion of the bargaining set introduced by Vind [81]. We show that the global bargaining set is unaltered when one considers objection and counter-objection by arbitrarily small-sized coalitions. Further, we show that it is sufficient to consider counter-objections by arbitrarily large-sized coalitions to obtain the global bargaining set. We conclude this chapter by showing that states belonging to the global bargaining set cannot be objected to by the grand coalition in a sequence of perturbed economies. In the final chapter, we introduce the core of a mixed club economy and established that it is equivalent to the set of club equilibrium states of the economy. Our core-equivalence theorem thus extends the seminal result of Shitovitz [77] who assumed the presence of at least two large agents of the same type to guarantee the result. However, we adopt a different approach and directly address the issue of the size of the large agents. We proceed along the lines of Basile et al. [13] in this regard and establish our core-equivalence theorem under the assumption that there exists a coalition containing both small and large agents of the same type for whom the average state belongs to the common consumption set.

7.1 Future Research

Below we propose some future possible research that will further extend the findings of this thesis.

- The framework of a market with clubs proposed by Ellickson et al. [35] bears a close resemblance to that of a matching market, a feature that follows from the nature of club goods. One can observe that juxtaposed to a traditional general equilibrium model where the market system clears through prices, for a club economy market clearance through prices solely does not determine equilibrium. A particular agent cannot demand any club membership, he or she has to be allowed by the club as well (like a male agent cannot purchase membership in a female-only swimming club). This nature is typical in matching markets. In our framework, there exists a continuum of agents on one side who are matched to clubs each of which is finite in size. Thus there exists a continuum of clubs each admitting a finite number of agents as members in equilibrium. This is closely related to the framework of matching in large economies. In particular, our framework corresponds to that of Azevedo and Hatfield [11] where there exists a continuum of agents on both sides of the market (agents and clubs). In such a setting they show that a stable matching exists as long as one side of the market has substitutable preferences.¹ Azevedo and Leshno [12] show that stable matching in such large economies usually corresponds to the limit of stable matchings in large finite economies, where individuals exhibit substitutability in preferences. However, one cannot guarantee the same when agents exhibit complementary preferences.² However, there lies a major difference between our framework and the framework considered in large matching scenarios. In a classical matching scenario (finite or continuum) the set of agents is usually partitioned into certain groups, where each group (agent) is matched to one (many) other agents from

¹Substitutability of preferences means that a club never admits an agent when more agents are available. In other words, if a club does not allow an agent to be a member, it never admits the agent as a member from a larger set of agents.

²Che et al. [28] demonstrated through an example that for finite economies the existence of exact stable matchings is falsified in the case of complementary preferences. They further motivate that a continuum of agents helps get rid of this issue and show that not only do stable matchings exist for complementary preferences but also they correspond to the limit of approximate stable matchings in large finite economies. However, as pointed out by Greinecker and Kah [42] such an individualistic model is hard to relate to a large finite economy as the number of agents changes drastically and there is no common space for comparisons. They overcome this difficulty by considering a distributional approach to large economies introduced by Hart et al. [45] and Hildenbrand [54] in general equilibrium theory.

the other side. For instance, in Che et al. [28] each college is matched with a group of students and no student can be matched with more than one college. In contrast, our framework allows for agents to be part of multiple club types and agents also derive utility from being part of different clubs at the same time. For example, consider an economy with a continuum of agents $[0, 1]$, where agents in $[0, \frac{1}{4})$ are males, $[\frac{1}{4}, \frac{1}{2})$ are females and those in $[\frac{1}{2}, 1]$ are children. Let there be two kinds of swimming pools (clubs) in the economy, where the first type admits only males and females and the second type only admits females and children. Now, consider females belonging to the set $[\frac{1}{4}, \frac{1}{3}]$. They can be matched with a certain set of male agents when they become members of the swimming pool of the first type and simultaneously they can be matched with a certain set of children when they choose to become members of the swimming pool of the second type. A recent work by Scotchmer and Shanon [76] provides a unified framework by proposing a club economy in line with Ellickson et al. [35] where club formation is modeled through a random matching process. They consider a continuum of agents being matched to finitely many group types where a random matching process models group formation in their framework. Thus, their work resembles that of the large matching literature discussed in the introduction section. In their paper, given individual membership choices, a random matching function yields a state of the economy where certain groups are formed by certain agents getting matched (unmatched). It is worthwhile to point out that unlike our framework an agent's characteristics are not all verifiable and they announce it strategically while purchasing membership of a particular group. Further, in contrast to our framework membership choices are binary, i.e. whether an agent subscribes to a membership or not is important, the number of such a particular membership being purchased is not consequential. One can introduce the notion of core and bargaining set in such a setup and check whether the findings of this thesis go through for such an asymmetric information club economy.

- A major strand of general equilibrium literature deals with the study of infinite dimensional commodity spaces. Two major real-life economic scenarios motivate the use of such spaces. They are (i) Consuming commodities at discrete time intervals over an infinite time horizon. Such consumption patterns are modeled by the spaces ℓ_∞ and $L_\infty(\mu)$. The seminal work in this domain has been captured in Bewley [14] & [15], and (ii) Allowing for many different numbers of commodity characteristics. Such consumption patterns are modeled by the space $M(K)$ of (signed)-Borel measures on a compact metric space K . The works of Mas-Colell [62], Jones [55] & [56] put forward the most pioneering ideas for this kind of

scenario. One can extend the club goods framework studied in this thesis to incorporate infinitely many private goods and explore whether the results studied in this thesis go through or not.

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