T 014
19/1/84

RESTRICTED COLLECTION

STUDIES IN BOOLEAN ALGEBRAS AND MEASURE THEORY

INDIAN STATISTICAL INSTITUTE LIBRARY
No. T 014
Date 19 JAN 1984
CALCUTTA

By

K. P. S. BHASKARA RAO

RESTRICTED COLLECTION

A thesis submitted to the Indian Statistical Institute
in partial fulfilment of the requirements for the
degree of Doctor of Philosophy
Calcutta

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

RESTRICTED COLLECTION

ACKNOWLEDGEMENTS

I am grateful to Professor C. R. Rao, F.R.S., Director of the Research and Training School of the Indian Statistical Institute for extending all the research facilities to me to carry out this research.

I am grateful to Professor Ashok Maitra, under whose supervision this work was done, for his constant encouragement, meticulous scrutiny of the first draft of this Thesis and helpful comments.

I thank Mr. M. Bhaskara Rao for his continuous help in preparing this Thesis. Indeed my indebtedness to him is nonmeasurable. I thank him again for giving me permission to include in this Thesis some of the results we jointly obtained.

I thank Professor M. G. Nadkarni for the time he devoted to me in discussing some of the problems treated here.

I take this opportunity to thank all my collegues in Research and Training School and Research Scholars Hostel for creating an ideal and gay atmosphere in which I carried out this research.

Finally I thank Mr. Gour Mohon Das and Mr. Apurba Guha for their efficient typing and cyclostyling respectively.

K. P. S. B. RAO

INDIAN STATISTICAL INSTITUTE LIBRARY
No. T 014
9 JAN 1984

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

RESTRICTED COLLECTION

C O N T E N T S

p.t.o.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Page

PART 2 : MEASURE THEORY

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

INTRODUCTION

This Thesis is divided into six chapters the first three chapters constituting studies in Boolean algebras and the last three chapters constituting studies in measure theory. We give below a sketch of the main problems treated in this Thesis.

CHAPTER 1. A Decomposition theorem due to Sobczyk and Hammer [28] implies that strongly continuous charges on Boolean algebras play a role similar to that of nonatomic measures on Boolean σ-algebras. Rudin [25] and Knowles [14] gave necessary and sufficient conditions for the Boral σ-field of a compact Hausdorff space to admit a nonatomic measure. But there are no necessary and sufficient algebraic conditions for a Boolean σ-algebra to admit a nonatomic measure. In this chapter we solve the analogous problem of existence of strongly continuous charges on Boolean algebras. We also examine the richness of strongly continuous charges in the space of all charges on a Boolean algebra.

CHAPTER 2. Given any charge on a Boolean algebra, equivalently, given any charge space $(\Omega, \underline{\underline{A}}, \mu)$ one associates a natural metric space $(\underline{\underline{A}}(\mu), d_\mu)$ with it. A natural question that arises is: How far the topological properties of the metric space $(\underline{\underline{A}}(\mu), d_\mu)$ reflect on $\underline{\underline{A}}$ and μ? This problem is treated in this chapter. A satisfactory picture emerges when $(\Omega, \underline{\underline{A}}, \mu)$ is a measure space. When $(\Omega, \underline{\underline{A}}, \mu)$ is a charge space the problem is solved partially.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

CHAPTER 3: The contents of this chapter are inspired by a paper of B. V. Rao [23]. For any Boolean algebra $\mathcal{A}$ the class of all subalgebras of $\mathcal{A}$ forms a complete lattice which we call $L_{\mathcal{A}}$. The question of distributivity of $L_{\mathcal{A}}$ is not very interesting because it has a trivial solution. In this chapter we deal with the complementation in $L_{\mathcal{A}}$. A characterisation of $\mathcal{A}$ such that $L_{\mathcal{A}}$ is a complemented lattice is still lacking. In addition to the study of complementation in $L_{\mathcal{A}}$ we also generalize B.V.Rao's results in several directions.

CHAPTER 4: The origin of Chapter 4 is a paper of V. Ficker [7] who attempted to characterise countable chain condition in $\underline{\underline{B}} - \underline{\underline{N}}$ where $(\Omega, \underline{\underline{B}}, \mu)$ is a measure space and $\underline{\underline{N}}$ is the collection of all μ-null sets. We demonstrate that the main theorem of [7] is incorrect and prove a stronger version of Ficker's theorem for certain types of measures.

CHAPTER 5: The problem of this chapter was suggested by Professor M. G. Nadkarni. The problem is : Given two real valued measurable functions f and g defined on a Borel structure $(X, \underline{\underline{B}})$ when does there exist a nonatomic probability measure which makes f and g independent. If nonatomicity is not required the problem has a trivial solution. Here we solve this problem when X is the real line and $\underline{\underline{B}}$ is the Borel σ-algebra of the real line. We consider some extensions also.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

CHAPTER 6: Darst and Goffman [5] gave an example of a Borel subset of $R \times R$ of positive measure which contains no rectangle of positive measure. In this chapter we exhibit a Borel subset of $R \times R$ of full Lebesgue measure which contains no rectangle of positive measure by techniques different to those of Darst and Goffman. Baire category versions of these results were also obtained.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

STUDIES IN BOOLEAN ALGEBRAS AND

MEASURE THEORY

PART 1

BOOLEAN ALGEBRAS

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

CHAPTER 1

NONATOMIC CHARGES ON BOOLEAN ALGEBRAS

1. Introduction This chapter is devoted to a study of decomposition of charges and existence of certain types of charges on Boolean algebras. For notions in Boolean algebras we follow Sikorski [27] and we denote abstract Boolean algebras by Roman capitals A, B, C etc. The operations of join meet, difference, complementation in Boolean algebras are denoted by $\vee$, $\wedge$, $-$, $'$ respectively. The order in Boolean algebras is denoted by $\leq$. Fields and σ-fields of subsets of a fixed set are denoted by $\underline{\underline{F}}$, $\underline{\underline{G}}$ etc. The following are some of the relevant definitions needed in the sequel. For topological notions, we follow Kelley [13] and Kuratowski [15].

Definition 1.1.1. A charge μ on a Boolean algebra B is a real valued nonnegative finitely additive function defined on B vanishing at the zero element, 0, of B.

Definition 1.1.2. A measure λ on a Boolean algebra B is a countably additive charge on B.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Definition 1.1.3. Let μ be a charge on a Boolean algebra B. An element $b \in B$ is said to be a μ- atom if (i) $\mu(b) > 0$ and (ii) $a \in B$, $a \leq b$ implies either $\mu(a) = 0$ or $\mu(a) = \mu(b)$.

Definition 1.1.4. A charge μ on a Boolean algebra B is said to be nonatomic if $b \in B$, $\mu(b) > 0$ implies there exists $a \in B$, $a \leq b$ such that $0 < \mu(a) < \mu(b)$.

Definition 1.1.5. A charge μ on a Boolean algebra B is said to be strongly continuous if given $\epsilon > 0$ there exists a finite partition $b_1, b_2, \ldots, b_n$ of the unit element, 1, of B (i.e., $b_1 \vee b_2 \vee \ldots \vee b_n = 1$ and $b_i \wedge b_j = 0$ for $i \neq j$), such that $\mu(b_i) < \epsilon$ for every i.

Definition 1.1.6. A charge μ on a Boolean algebra B is said to be strongly nonatomic if $b \in B$. $\mu(b) \geq \alpha \geq 0$ implies that there exists $a \in B$, $a \leq b$ such that $\mu(a) = \alpha$.

The following theorem makes clear that the three notions introduced in Definitions 1.1.4, 1.1.5 and 1.1.6 are same for measures defined on Boolean σ-algebras.

Theorem 1.1.7. Let λ be a measure defined on a Boolean σ-algebra A. Then the following statements are equivalent.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

i) λ is strongly nonatomic on Λ.

ii) λ is strongly continuous on Λ.

iii) λ is nonatomic on Λ.

<u>Proof</u>: (i) $\Longrightarrow$ (ii). Let $\epsilon > 0$. If $\lambda(1) < \epsilon$, the partition we give for 1 is the single element 1 itself. If $\lambda(1) \geq \epsilon$, let n be the least positive integer satisfying $n\frac{\epsilon}{2} \leq \lambda(1) < (n+1)\frac{\epsilon}{2}$. We choose $b_1, b_2, \ldots, b_n$ successively as follows. Let $b_1 \in \Lambda$ be such that $\lambda(b_1) = \frac{\epsilon}{2}$. Choose $b_2 \leq 1 - b_1$ satisfying $\lambda(b_2) = \frac{\epsilon}{2}$. At the i^{th} stage choose $b_i \leq 1-(b_1 \vee b_2 \ldots \vee b_{i-1})$ satisfying $\lambda(b_i) = \frac{\epsilon}{2}$. $b_1, b_2, \ldots, b_n, 1-(b_1 \vee b_2 \vee \ldots \vee b_n)$ is a partition of 1 satisfying $\lambda(b_i) < \epsilon$ for every i and $\lambda[1-(b_1 \vee b_2 \vee \ldots \vee b_n)] < \epsilon$.

(ii) $\Longrightarrow$ (iii). Let $b \in \Lambda$ be such that $\lambda(b) = \alpha > 0$. Let $b_1, b_2, \ldots, b_n$ be a partition of 1 satisfying $\lambda(b_i) < \alpha$ for every i. One of the elements $b_1 \wedge b, b_2 \wedge b, \ldots, b_n \wedge b$, say i^{th}, has the property $0 < \lambda(b_i \wedge b) < \lambda(b)$.

(iii) $\Longrightarrow$ (i). This may be proved by using a transfinite exhaustion process. See, for example Halmos [11, p. 174].

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Now we may ask the question whether the preceding theorem is true for charges on Boolean algebras. Observe that in the proof of the implications - (i) ==> (ii) and (ii) ==> (iii), We have not used the information that the charge λ is countably additive and that the Boolean algebra is a σ-algebra. The following examples demonstrate that the converse implications are not true for charges on Boolean algebras.

<u>Example 1.1.8.</u> Example of a strongly continuous charge which is not strongly nonatomic: Let $X = [0, 1)$ and $\underline{\underline{F}}$ the field of subsets of X consisting of sets which are finite disjoint unions of intervals of the form $[a, b)$ where a and b are rational numbers. Let μ be the restriction of Lebesgue measure to $\underline{\underline{F}}$. Then μ is a strongly continuous charge on $\underline{\underline{F}}$ but not strongly nonatomic.

<u>Example 1.1.9.</u> Example of a nonatomic charge which is not strongly continuous: Let $X = [0, 1]$ and let $\underline{\underline{F}}$ be the field of subsets of X generated by the collection of all intervals of the form $(a, b] \subseteq [\frac{1}{4}, \frac{3}{4})$. Let μ be the restriction of Lebesgue measure to $\underline{\underline{F}}$. μ is a nonatomic charge on $\underline{\underline{F}}$ but not strongly continuous. For $\epsilon = \frac{1}{2}$, there is no decomposition of X which satisfies the required properties.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

2. A decomposition theorem. If λ is a measure on a Boolean σ-algebra A, we can write $\lambda = \lambda_1 + \lambda_2$ where λ_1 is a nonatomic measure on A and λ_2 is a completely atomic measure on A, i.e. we can write $\lambda_2 = \sum_{i \geq 1} \alpha_i \mu_i$, where each $\alpha_i \geq 0$ and each μ_i is a 0-1 valued measure on A. The proof is fairly easy if one starts working with all the λ-atoms of A. Sobczyk and Hammer [28, Theorem 4.1, p. 842] gave a similar theorem for charges defined on Boolean algebras for which we give below a simple proof.

Let X be a compact Hausdorff space, $\underline{\underline{B}}$ the σ-field generated by compact G_δ subsets of X and $\underline{\underline{B}}_1$ the σ-field on X generated by compact subsets of X. We call $\underline{\underline{B}}$ and $\underline{\underline{B}}_1$ the Baire and Borel σ-fields on X respectively. If, in addition, X is totally disconnected, then the field of all clopen subsets of X generates $\underline{\underline{B}}$. This result follows from Stone-Weierstrass theorem. The following Lemma is useful in the proof of the decomposition theorem.

Lemma 1.2.1. Let μ be a measure on a σ-field $\underline{\underline{F}}$ of subsets of a set Y and $\underline{\underline{G}}$ a field on Y generating $\underline{\underline{F}}$. μ is strongly continuous on $\underline{\underline{F}}$ if and only if μ is strongly continuous on $\underline{\underline{G}}$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Proof. If part is trivial. Let μ be strongly continuous on $\underline{\underline{F}}$. Let $d > 0$. Let m be a natural number such that $\frac{1}{m} < d$. There exists a partition $F_1, F_2, \ldots, F_n$ of Y in $\underline{\underline{F}}$ such that

$$\mu(F_i) < \left[\frac{m\,\mu(Y)}{m\,\mu(Y) + 1}\right] d \quad \text{for every } i.$$

For each i, we can find $G_i \in \underline{\underline{G}}$ (See Theorem D of Halmos [11, p. 56]) such that

$$\mu(F_i \,\Delta\, G_i) < \frac{\mu(F_i)}{m\mu(Y)}$$

(Assume, without loss of generality, $\mu(F_i) > 0$ for every i.) Since $G_i \subset F_i \cup (F_i \,\Delta\, G_i)$, we have

$$\mu(G_i) < \mu(F_i)\left[1 + \frac{1}{m\,\mu(Y)}\right]$$

$$= \mu(F_i)\left[\frac{m\,\mu(Y) + 1}{m\,\mu(Y)}\right] < d.$$

Further, $\mu\left(Y - \bigcup_{i=1}^{n} G_i\right) \leq \mu\left(\bigcup_{i=1}^{n} F_i - \bigcup_{i=1}^{n} G_i\right)$

$$\leq \mu\left(\bigcup_{i=1}^{n} F_i - G_i\right) \leq \sum_{i=1}^{n} \mu(F_i \,\Delta\, G_i) < \frac{1}{m} < d.$$

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Disjointifying $G_1, G_2, \ldots, G_n$ and $X - \bigcup_{i=1}^{n} G_i$, we get a partition $D_1, D_2, \ldots, D_n$ and D_{n+1} of Y in $\underline{\underline{G}}$ such that $\mu(D_i) < d$ for every i.

Theorem 1.2.2. (Sobczyk and Hammer [28, p. 842]). Let μ be a charge on a Boolean algebra B. Then we can write

$$\mu = \mu_o + \sum_{i \geq 1} \alpha_i \mu_i, \text{ where}$$

i) μ_o is a strongly continuous charge on B,

ii) $\alpha_i \geq 0$ for every i, and

iii) μ_i, $i \geq 1$, is a 0 - 1 valued charge on B.

Proof: Let X be the Stone space of B, $\underline{\underline{C}}$ the field of all clopen subsets of X, $\underline{\underline{B}}$ the Baire σ-field on X and T the isomorphism between B and $\underline{\underline{C}}$. The charge μT^{-1} on $\underline{\underline{C}}$ is indeed a measure on the field $\underline{\underline{C}}$. Since $\underline{\underline{C}}$ is a generator for $\underline{\underline{B}}$, we can extend μT^{-1} from $\underline{\underline{C}}$ to $\underline{\underline{B}}$ as a measure λ on $\underline{\underline{B}}$. Now, we can write $\lambda = \lambda_o + \sum_{i \geq 1} \alpha_i \lambda_i$, where

i) λ_o is a nonatomic measure on $\underline{\underline{B}}$,

ii) $\alpha_i \geq 0$ for every i, and

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

iii) λ_i, $i \geq 1$, is a 0-1 valued measure on $\underline{\underline{B}}$.

Restricting λ_i's to $\underline{\underline{C}}$, and then transferring them from $\underline{\underline{C}}$ to B via T, we get the required decomposition of μ on B. The fact that μ_o is strongly continuous follows from Lemma 1.2.1.

Remark: Uniqueness of such a decomposition can be proved easily.

3. Existence theorems.

Definition 1.3.1. Let B be a Boolean algebra. A collection of nonzero elements $\left\{ b_{i_1,i_2,i_3,\ldots,i_k} : i_1, i_2, \ldots, i_k \text{ is any finite sequence of 0's and 1's, } k \geq 1 \right\}$ in B is said to be a tree in B if

i) $b_o \vee b_1 = 1$,

ii) $b_{i_1,i_2,i_3,\ldots, i_{k-1}, 0} \vee b_{i_1,i_2,i_3,\ldots, i_{k-1}, 1}$

$$= b_{i_1,i_2,\ldots, i_{k-1}}, \text{ and}$$

iii) $b_{i_1,i_2,\ldots,i_{k-1}, 0} \wedge b_{i_1,i_2,\ldots, i_{k-1},1} = 0.$

We use the following theorem in the proof of our main theorem.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Theorem 1.3.2. (Tarski - See Sikorski [27, (b_7), p. 211]).

Let A be a subalgebra of a Boolean algebra B. Then every charge μ_o on A can be extended to a charge μ on B.

Theorem 1.3.3. Let B be a Boolean algebra. The following statements are equivalent.

i) There is a nonzero nonatomic charge on B.

ii) B contains a tree.

iii) There is a nonzero strongly continuous charge on B.

Proof: (i) $\Longrightarrow$ (ii). Let μ be a nonzero nonatomic charge on B. Since $\mu(1) > 0$, we can find b_o and b_1 such that $b_o \vee b_1 = 1$, $b_o \wedge b_1 = 0$ and $0 < \mu(b_o) < \mu(1)$, $0 < \mu(b_1) < \mu(1)$. Applying this technique at every stage, we obtain a tree in B.

(ii) $\Longrightarrow$ (iii). Let $\{b_{i_1, i_2, \ldots, i_k} : k \geq 1$ and $i_1, i_2, \ldots, i_k$ is any finite sequence of 0's and 1's$\}$ be a tree in B. Let A be the subalgebra generated by this tree. In fact, A is precisely the collection of all finite disjoint joins of elements of the tree. Definite $\mu_o(b_{i_1, i_2, \ldots, i_k}) = \frac{1}{2^k}$. μ_o can be extended in the obvious fashion as a probability

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

charge μ_1 on A. Note that μ_1 is strongly continuous on A. By Theorem 1.3.2., there exists a probability charge μ_2 on B which is an extension of μ_1. Obviously, μ_2 is strongly continuous on B.

(iii) $\Longrightarrow$ (i). The proof of this implication is already included in Theorem 1.1.6.

<u>Remark:</u> The equivalence of (i) and (ii) is proved in Bhaskara Rao and Bhaskara Rao [3] using the following Rudin-Knowles Theorem and some difficult arguments. 'Let X be a compact Hausdorff space and $\underline{\underline{B}}_1$ its Borel σ-field. There exists a nonzero regular nonatomic measure on $\underline{\underline{B}}_1$ if and only if X contains a perfect subset' (a subset of X is said to be perfect if every element of the subset is an accumulation point of the subset). See Rudin [25] and Knowles [14, Theorem 1, p. 64]. The proof given here is simple and uses Tarski's Theorem. We obtain below Rudin-Knowles Theorem for compact totally disconnected Hausdorff spaces as a corollary to our Theorem 1.3.3.

<u>Definition 1.3.4.</u> Let B be a Boolean algebra. An element $b \in B$ is an atom of B if $b \neq 0$ and $a \in B$, $a \leq b$ implies either $a = 0$ or $a = b$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Definition 1.3.5. Let B be a Boolean algebra. B is said to be atomic if 1 is the join of all atoms of B.

Definition 1.3.6. B is said to be nonatomic if B has no atoms.

Definition 1.3.7. A Boolean algebra B is said to be superatomic if every subalgebra of B is atomic or equivalently the Stone space X of B is scattered, i.e., no subset of X is perfect. See Sikorski [27, p. 35].

Corollary 1.3.8. A Boolean algebra B is superatomic if and only if B does not contain a tree.

Proof. If B contains a tree, then the subalgebra generated by the tree is nonatomic. Hence B can not be superatomic. If B is not superatomic, then there exists a subalgebra A of B which is not atomic. This implies that there exists a nonzero element $a \in A$ such that a is disjoint from every atom of A. We can find a_o, a_1 in A such that both are nonzero, $a_o \vee a_1 = a$ and $a_o \wedge a_1 = 0$. a_o and a_1 can similarly be decomposed. Thus we obtain a collection of nonzero elements $\{a_{i_1, i_2, \ldots, i_k} : k \geq 1,\ i_1, i_2, \ldots, i_k$ is any finite sequence of 0's and 1's$\}$ in A with the properties

i) $a_o \vee a_1 = a$

INDIAN STATISTICAL INSTITUTE LIBRARY
No. T 014
Date 19 JAN 1984
CALCUTTA

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$$\text{ii)}\quad a_{i_1,i_2,\ldots,i_{k-1},0} \vee a_{i_1,i_2,\ldots,i_{k-1},1}$$

$$= a_{i_1,i_2,\ldots,i_{k-1}},$$

and $$\text{iii)}\quad a_{i_1,i_2,\ldots,i_{k-1},0} \wedge a_{i_1,i_2,\ldots,i_{k-1},1} = 0.$$

Defining $b_{i_1,i_2,\ldots,i_k} \equiv a_{i_1,i_2,\ldots,i_k}$ if

$$i_1,i_2,\ldots,i_k \neq (0,0,0,\ldots,0)$$

$$= a_{i_1,i_2,\ldots,i_k} \vee a' \quad \text{if} \quad (i_1,i_2,\ldots,i_k) = (0,0,\ldots,0)$$

for every finite sequence $i_1, i_2,\ldots,i_k$ of 0's and 1's, we obtain a tree in A and hence in B.

<u>Corollary 1.3.9.</u> (Rudin-Knowles). Let X be a compact totally disconnected Hausfordd space. There exists a regular nonzero nonatomic measure on the Borel σ-field $\underline{\underline{B}}_1$ of X if and only if X contains a perfect subset.

<u>Proof</u>. Let λ be a regular nonzero nonatomic measure on $\underline{\underline{B}}_1$. Then its support $(= X - \cup \{V : V \text{ open and } \lambda(V) = 0\})$ is a perfect subset of X. See Knowles [14, p. 65]. If X contains a perfect subset, then the field $\underline{\underline{C}}$ of all clopen subsets of X is not superatomic. By Theorem 1.3.3 and Corollary 1.3.8,

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

there exists a nonzero strongly continuous charge λ_o on $\underline{\underline{C}}$. Since λ_o is a measure on $\underline{\underline{C}}$, we can extend λ_o as a measure λ_1 to the Baire σ-field $\underline{\underline{B}}$ of X. λ_1 can be extended as a regular measure λ to the Borel σ-field $\underline{\underline{B}}_1$ of X. See Halmos [11, Theorem D, p. 239]. Since λ_o is strongly continuous on $\underline{\underline{C}}$, so is λ on $\underline{\underline{B}}_1$. Hence λ is a nonatomic measure on $\underline{\underline{B}}_1$.

Corollary 1.3.10. Let B be a Boolean algebra. B is superatomic if and only if every charge μ on B is of the form

$\sum_{i \geq 1} a_i \mu_i$, where

i) $a_i \geq 0$ for every i, and

ii) μ_i is a 0-1 valued charge on B for every i.

Proof: This follows from Theorem 1.2.2.

Corollary 1.3.11. Every infinite Boolean σ-algebra B admits a nonzero strongly continuous charge on it.

Proof: We shall exhibit a tree in B. There exists an infinite sequence $a_1, a_2, \ldots,$ of nonzero pairwise disjoint elements in B whose join is 1. Partitioning the given sequence into two disjoint infinite subsequences and taking their join, we obtain a partition b_o, b_1 of 1. Carrying out this procedure at every stage, we obtain a tree in B. Theorem 1.3.3 completes the proof.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

4. <u>Denseness of nonatomic charges</u>. Let B be a Boolean algebra, $\underline{\underline{P}}$ the collection of all probability charges on B, and $\underline{\underline{Q}}$ the collection of all strongly continuous probability charges on B. $\underline{\underline{P}}$ is equipped with a topology by defining convergence as follows: A net μ_α in $\underline{\underline{P}}$ converges to a μ in $\underline{\underline{P}}$ if $\mu_\alpha(b)$ converges to $\mu(b)$ for every $b \in B$. Bhaskara Rao and Bhaskara Rao [3] proved that the collection of all nonatomic probability charges on B is a dense subset of $\underline{\underline{P}}$ if the Stone space, X, of B is perfect. In the proof of this result they make use of the following result due to Knowles.

'Let X be a compact perfect Hausdorff space and $\underline{\underline{B}}_1$ its Borel σ-field. The collection of all nonatomic probability measures is dense in the space of all probability measures on $\underline{\underline{B}}_1$.'

See Knowles [14, Remark, p. 65].

In this section we prove the stronger result that $\underline{\underline{Q}}$ is dense in $\underline{\underline{P}}$ by simple, direct and more transparent methods and we do not use Knowles' result quoted in the previous paragraph in our proof. We can obtain Knowles result for compact totally disconnected perfect Hausdorff spacesas a corollary to our theorem.

<u>Lemma 1.4.1.</u> Let B be a Boolean algebra whose Stone space, X is perfect. Let $b \in B$ be a nonzero element. There exists a

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

strongly continuous probability charge μ on B such that $\mu(b) = 1$.

Proof: Consider the Boolean algebra $A = b \wedge B$

$$= \{b \wedge c \ : c \in B\}.$$

X is perfect is equivalent to the fact that B is nonatomic. It is easy to construct a tree in the Boolean algebra A. There exists a strongly continuous nonatomic probability charge μ_o on A in view of Theorem 1.3.3. μ_o can be extended to a probability charge μ on B by putting $\mu(b') = 0$, i.e., $\mu(c) = \mu_o(b \wedge c)$. Thus μ is a strongly continuous probability charge on B satisfying $\mu(b) = 1$.

Proposition 1.4.2. Let B be a Boolean algebra whose Stone space, X is perfect. For every 0-1 valued charge μ on B there exists a net μ_α of strongly continuous probability charges on B such that μ_α converges to μ in the topology of $\underline{\underline{P}}$.

Proof: The proof of this proposition is essentially contained in the proof of if part of Theorem 1.4.3.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

<u>Theorem 1.4.3</u>. Let B be a Boolean algebra. $\underline{\underline{Q}}$ is dense in $\underline{\underline{P}}$ if and only if the Stone space, X, of B is perfect.

<u>Proof</u>: If part. Let $\mu \in \underline{\underline{P}}$. By Theorem 1.2.3, we can write

$$\mu = \mu_o + \sum_{i \geq 1} a_i \mu_i, \quad \text{where}$$

i) μ_o is a strongly continuous charge on B,

ii) $a_i \geq 0$ for every i and $\mu_o(1) + \sum_{i \geq 1} a_i = 1$, and

iii) μ_i is a 0-1 valued charge on B for every $i \geq 1$.

Let $F_i = \{b \in B : \mu_i(b) = 1\}$, $i \geq 1$. For every $b \in F_i$, fix a strongly continuous probability charge $\mu_b^{(i)}$ on B such that $\mu_b^{(i)}(b) = 1$. Consider the product set $F_1 \times F_2 \times F_3 \times \ldots$ with the following partial order. $(c_1, c_2, \ldots) \geq^* (d_1, d_2, \ldots)$ if $c_i \leq d_i$ for every i. $F_1 \times F_2 \times F_3 \times \ldots$ is a directed set. For every $(b_1, b_2, \ldots) \in F_1 \times F_2 \times \ldots$, let $\mu_{(b_1, b_2, \ldots)} = \mu_o + \sum_{i \geq 1} a_i \mu_{b_i}^{(i)}$. Then, it is easy to verify that $\mu_{(b_1, b_2, \ldots)} \in \underline{\underline{Q}}$.

We claim that the net $\{\mu_{(b_1, b_2, \ldots)} : (b_1, b_2, \ldots) \in F_1 \times F_2 \times \ldots\}$ converges to μ in the topology of $\underline{\underline{P}}$. Let

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$c \in B$. Let $N_1 = \{i \in N : c \in F_i\} = \{i_1, i_2, \ldots\}$, where N is the set of all positive natural numbers. Let $N_2 = N - N_1 = \{j_1, j_2, \ldots\}$. Since each F_i is a maximal filter, $c' \in F_{j_k}$ for every $k \geq 1$. Let $(d_1, d_2, \ldots) \geq^* (c_1^{\partial_1}, c_2^{\partial_2}, c_3^{\partial_3}, \ldots)$,

where $c_i^{\partial_i} = c$ if $i = i_1, i_2, i_3, \ldots$

$c_i^{\partial_i} = c'$ if $i = j_1, j_2, j_3, \ldots$

This implies that $d_i \leq c$ if $i = i_1, i_2, \ldots,$ and $\leq c'$ if $i = j_1, j_2, \ldots$.

Note that

$$\mu_{(d_1, d_2, \ldots)}(c) = \mu_o(c) + \sum_{k \geq 1} a_{i_k} .$$

Hence the net $\{\mu_{(b_1, b_2, \ldots)}(c) : (b_1, b_2, \ldots) \in F_1 \times F_2 \times \ldots\}$ converges to $\mu_o(c) + \sum_{i \geq 1} a_i \, \mu_i(c) = \mu(c)$. This completes the proof of if part of the theorem.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Only if part. Since X is perfect is equivalent to the fact that B is nonatomic, it is enough if we prove that B has no atoms. Suppose $b \in B$ is an atom of B. Take any probability charge μ on B such that $\mu(b) = 1$. By hypothesis there exists a net μ_α in $\underline{\underline{Q}}$ which converges to μ. Since b is an atom, $\mu_\alpha(b) = 0$ for every α whereas $\mu(b) = 1$. This is a contradiction.

<u>Corollary 1.4.4.</u> Let B be a Boolean algebra and $\underline{\underline{Q}}_1$ the collection of all nonatomic probability charges on B. $\underline{\underline{Q}}_1$ is dense in $\underline{\underline{P}}$ if and only if B is nonatomic.

<u>Proof:</u> Note that $\underline{\underline{Q}} \subset \underline{\underline{Q}}_1$.

Combining Theorem 1.4.3 and Corollary 1.4.4, we have the following comprehensive theorem.

<u>Theorem 1.4.5.</u> Let B be a Boolean algebra. Then the following statements are equivalent.

i) B is nonatomic.

ii) The Stone space of B is perfect.

iii) $\underline{\underline{Q}}$ is dense in $\underline{\underline{P}}$.

iv) $\underline{\underline{Q}}_1$ is dense in $\underline{\underline{P}}$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

CHAPTER 2

TOPOLOGICAL PROPERTIES OF CHARGE ALGEBRAS

1. Introduction. In this chapter, we study topological properties of charge algebras associated with charge spaces. The following are some of the definitions needed for the development of this chapter.

Definition 2.1.1. A charge space is a triplet $(\Omega, \underline{\underline{A}}, \mu)$, where Ω is any set, $\underline{\underline{A}}$ a field of subsets of Ω and μ is a charge on $\underline{\underline{A}}$.

Abstract Boolean algebras B with a charge μ defined on B come also under the realm of the above definition. For, by Stone's representation theorem, we may replace (B, μ) by $(\Omega, \underline{\underline{A}}, \mu)$ where Ω is the Stone space of B and $\underline{\underline{A}}$ the field of all clopen subsets of Ω.

With the help of the charge μ defined on the field $\underline{\underline{A}}$, we can make $\underline{\underline{A}}$ a pseudo-metric space by defining the distance function d_μ on $\underline{\underline{A}} \times \underline{\underline{A}}$ as follows. $d_\mu(A, B) = \mu(A \Delta B)$. Let $\underline{\underline{A}}(\mu)$ denote the collection of all equivalence classes of $\underline{\underline{A}}$

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

under the equivalence relation, $A \sim B$ if $\mu(A \Delta B) = 0$.
On $\underline{\underline{A}}(\mu)$, there is a natural metric d_μ (abuse of notation!) defined by $d_\mu([A], [B]) = \mu(A \Delta B)$, where $[A]$ and $[B]$ are equivalence classes of $\underline{\underline{A}}$ containing A and B respectively. We call $(\underline{\underline{A}}(\mu), d_\mu)$ the <u>charge algebra</u> associated with the charge space $(\Omega, \underline{\underline{A}}, \mu)$.

<u>Definition 2.1.2</u>. A measure space is a triplet $(\Omega, \underline{\underline{B}}, \lambda)$, where Ω is any set, $\underline{\underline{B}}$ a σ-field of subsets of Ω and λ is a measure on $\underline{\underline{B}}$.

The pair $(\underline{\underline{B}}(\lambda), d_\lambda)$ is called the <u>measure algebra</u> associated with the measure space $(\Omega, \underline{\underline{B}}, \lambda)$.

The central theme of this chapter is to characterise the topological properties of charge algebra $(\underline{\underline{A}}(\mu), d_\mu)$ in terms of $\underline{\underline{A}}$ and μ. In our treatment of this topic, a satisfactory picture emerges in the case of measure algebras and some partial results are obtained in the case of charge algebras.

The following concepts are used in the characterisation of compactness of charge algebras.

If μ is a two-valued charge on $\underline{\underline{A}}$, then

$\underline{\underline{I}} = \{A \in \underline{\underline{A}} : \mu(A) = 0\}$ is a maximal ideal in $\underline{\underline{A}}$ and

$\underline{\underline{F}} = \{A \in \underline{\underline{A}} : \mu(A) = \mu(\Omega)\}$ is a maximal filter in $\underline{\underline{A}}$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

<u>Definition 2.1.3.</u> Let $\mu_1, \mu_2, \ldots$ be a sequence of two-valued charges on $\mathcal{A}$. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots$ be the corresponding maximal ideals of $\mathcal{A}$ and $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be the corresponding maximal filters of $\mathcal{A}$, as defined in the previous paragraph. The sequence μ_n, $n \geq 1$ is said to be <u>disjoint</u> if

$$\bigcap_{n \geq 1} \mathcal{I}_n^{\partial_n} \neq \emptyset$$

for every sequence $\partial_1, \partial_2, \ldots$ of 0's and 1's, where $\mathcal{I}_n^0 = \mathcal{I}_n$ and $\mathcal{I}_n^1 = \mathcal{F}_n$.

Equivalently, a sequence μ_n, $n \geq 1$ of two-valued charges on $\mathcal{A}$ is said to be <u>disjoint</u> if for every (finite or infinite) sequence $i_1, i_2, \ldots$ of natural numbers, there exists Λ in $\mathcal{A}$ such that $\mu_{i_k}(\Lambda) = 0$ for every $k \geq 1$ and $\mu_j(\Lambda) = \mu_j(\Omega)$ for every $j \neq i_k$ for any $k \geq 1$.

Notice that any finite sequence of distinct two-valued charges on $\mathcal{A}$ are disjoint. Also, any infinite sequence of distinct two-valued measures defined on a σ-field $\mathcal{B}$ of Ω is disjoint. This can be proved as follows. Let $i_1, i_2, \ldots$ be any sequence of natural numbers and $\{j_1, j_2, \ldots\} = N - \{i_1, i_2, \ldots\}$, where N is the set of all natural numbers.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Now, we claim that given any i_k, there exists $A_k \in \underline{\underline{B}}$ such that $\mu_{i_k}(A_k) = 0$ and $\mu_{j_p}(A_k) = \mu_{j_p}(\Omega)$ for every $p \geq 1$. For, since μ_{i_k} and μ_{j_p} are distinct, there exists $B_p \in \underline{\underline{B}}$ such that $\mu_{i_k}(B_p) = 0$ and $\mu_{j_p}(B_p) = \mu_{j_p}(\Omega)$. Take $A_k = \bigcup_{p \geq 1} B_p$. Let $A = \bigcap_{k \geq 1} A_k$. Then $\mu_{i_k}(A) = 0$ for every $k \geq 1$ and $\mu_{j_p}(A) = \mu_{j_p}(\Omega)$ for every $p \geq 1$.

The following is an example of a sequence of distinct 0-1 valued charges defined on a field of sets such that it is not disjoint. Let $\Omega = \{1, 2, 3, \ldots\}$, $\underline{\underline{A}}$ finite cofinite field on Ω and

$$\begin{aligned} \mu_1(A) &= 0 \quad \text{if} \quad A \text{ is finite,} \\ &= 1 \quad \text{if} \quad A \text{ is cofinite, and} \\ \mu_n &= \partial_{n-1}, \text{ the degenerate measure at } n-1. \end{aligned}$$

For the sequence of natural numbers 2,4,6,..., there is no set A in $\underline{\underline{A}}$ for which

$$0 = \mu_2(A) = \mu_4(A) = \mu_6(A) = \ldots \quad \text{and}$$

$$1 = \mu_1(A) = \mu_3(A) = \mu_5(A) = \ldots \; .$$

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

The following is a basic Lemma used inherently in many of our arguments.

<u>Lemma 2.1.4.</u> Let μ be a charge defined on a field $\underline{\underline{A}}$ of subsets of a set Ω. Then $(\underline{\underline{A}}(\mu), d_{\mu})$ is isometrically isomorphic to a dense subalgebra of a measure algebra $(\underline{\underline{B}}(\lambda), d_{\lambda})$. In fact, $\underline{\underline{B}}(\lambda)$ is the metric completion of $\underline{\underline{A}}(\mu)$.

<u>Proof:</u> Let X be the Stone space of $\underline{\underline{A}}(\mu)$, $\underline{\underline{C}}$ the field of all clopen subsets of X, T_1 isomorphism between $\underline{\underline{A}}(\mu)$ and $\underline{\underline{C}}$, and $\underline{\underline{B}}$ the Baire σ-field of X. The charge μT_1^{-1} on $\underline{\underline{C}}$ is strictly positive and countably additive, and so can be extended to $\underline{\underline{B}}$ as a measure, λ. The Boolean algebra $(\underline{\underline{C}}, d_{\mu T_1^{-1}})$ is isometrically isomorphic to a dense subalgebra of $(\underline{\underline{B}}(\lambda), d_{\lambda})$. The isomorphism T_2 is defined by $T_2 C = [C] \in \underline{\underline{B}}(\lambda)$, for C in $\underline{\underline{C}}$, where $[C]$ is the equivalence class containing C. Denseness follows from the fact that given any Baire set B in $\underline{\underline{B}}$ and $\epsilon > 0$, there exists a clopen set C in $\underline{\underline{C}}$ such that $\lambda(B \triangle C) < \epsilon$. Further $d_{\mu T_1^{-1}}(C_1, C_2) = \lambda(C_1 \triangle C_2) =$ $= d_{\lambda}([C_1], [C_2]) = d_{\lambda}(T_2 C_1, T_2 C_2)$. Composing T_2 and T_1 completes the proof of the Lemma.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

2. Compactness.

Lemma 2.2.1. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space, where λ is completely atomic. Then the pseudo-metric space $(\underline{\underline{B}}, d_\lambda)$ is compact, where $d_\lambda(A, B) = \lambda(A \Delta B)$ for A, B in $\underline{\underline{B}}$.

Proof: Since d_λ is a complete pseudo-metric (see Halmos [11, Section 40, Exercise 1, p. 169]), it is enough if we show $(\underline{\underline{B}}, d_\lambda)$ is totally bounded.

Case (i). The number of λ-atoms is finite. Let $A_1, A_2, \ldots A_n$ be the λ-atoms of $\underline{\underline{B}}$. Let $\epsilon > 0$. Let $B_1, B_2, \ldots, B_m$ be the sets of all possible unions of $A_1, A_2, \ldots, A_n$. Then the family of open spheres $S(\emptyset, \epsilon)$, $S(B_i, \epsilon)$: $i = 1$ to m covers $\underline{\underline{B}}$, where $S(B, \epsilon) = \{A \text{ in } \underline{\underline{B}} : d_\lambda(B, A) < \epsilon\}$.

Case (ii). The number of λ-atoms is infinite. Let $A_1, A_2, \ldots$ be the λ-atoms of $\underline{\underline{B}}$. Let $\epsilon > 0$. Let $\lambda(A_i) = a_i > 0$. Let N be such that $a_{N+1} + a_{N+2} + \ldots < \epsilon$. Let $B_1, B_2, \ldots, B_m$ be the collection of all sets obtained by taking all possible unions of the sets $A_1, A_2, \ldots, A_N$ and $\bigcup_{i \geq N+1} A_i$. Then $S(\emptyset, \epsilon)$, $S(B_i, \epsilon)$, $i = 1$ to m is a cover of $\underline{\underline{B}}$. For, every B in $\underline{\underline{B}}$ is essentially a union of λ-atoms.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

<u>Lemma 2.2.2.</u> Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space, where λ is a nonzero nonatomic measure on $\underline{\underline{B}}$. Then the pseudo-metric space $(\underline{\underline{B}}, d_\lambda)$ is not compact.

<u>Proof:</u> It is enough if we show that the space $(\underline{\underline{B}}, d_\lambda)$ is not totally bounded. In fact, we show that $\bigcup_{i=1}^{n} S(A_i, \frac{1}{2}) \neq \underline{\underline{B}}$ for every finite sequence $A_1, A_2, \ldots, A_n$ in $\underline{\underline{B}}$. Let $B_1, B_2, \ldots, B_m$ be the atoms of $A_1, A_2, \ldots, A_n$, i.e., the collection of sets $A_1^{\partial_1} \cap A_2^{\partial_2} \cap \ldots \cap A_n^{\partial_n}$, where each $\partial_i = 0$ or 1 and $A_i^0 = A_i$, $A_i^1 = A_i' = \Omega - A_i$. Let $C_1, C_2, \ldots, C_m$ be such that C_i is contained in B_i and $\lambda(C_i) = \frac{1}{2}\lambda(B_i)$ for every i. Let $C = \bigcup_{i=1}^{m} C_i$. Since every A_i is a union of atoms, a routine calculation yields

$$d_\lambda(A_i, C) = \frac{1}{2} \sum_{i=1}^{m} \lambda(B_i) = \frac{1}{2}\lambda(\Omega) \geq \frac{1}{2}, \quad \text{if } \lambda(\Omega) \geq 1.$$

Consequently, in this case, $C \notin S(A_i, \frac{1}{2})$ for every i. If $\lambda(\Omega) < 1$, by what we have proved above, the pseudo-metric space $(\underline{\underline{B}}, d_\pi)$ is not compact, where $\pi(B) = [1/\lambda(\Omega)]\lambda(B)$ for B in $\underline{\underline{B}}$. Hence $(\underline{\underline{B}}, d_\lambda)$ is not compact.

<u>Lemma 2.2.3.</u> Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space such that in the decomposition $\lambda = \lambda_1 + \lambda_2$, where λ_1 is completely

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

atomic and λ_2 is nonatomic, $\lambda_2(\Omega) > 0$. Then $(\underline{\underline{B}}, d_\lambda)$ is not compact.

Proof: Let d_{λ_2} be the pseudo-metric on $\underline{\underline{B}}$ induced by λ_2. Obviously, $d_{\lambda_2} \leq d_\lambda$. If $(\underline{\underline{B}}, d_\lambda)$ were to be compact, then $(\underline{\underline{B}}, d_{\lambda_2})$ will be compact, a contradiction to Lemma 2.2.2.

Theorem 2.2.4. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space. Then $(\underline{\underline{B}}, d_\lambda)$ is compact if and only if λ is completely atomic.

Proof: Follows from Lemmas 2.2.1, 2.2.2 and 2.2.3.

Lemma 2.2.5. Let (M,d) be a pseudo-metric space and (M^*,d^*) the metric identification of (M,d). Then the natural map from M to M^* is open, closed, continuous and onto.

Proof: See Willard [30, 2C, p. 20].

Corollary 2.2.6. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space. Then the measure algebra $(\underline{\underline{B}}(\lambda), d_\lambda)$ is compact if and only if λ is completely atomic.

Proof: Follows from Theorem 2.2.4 and Lemma 2.2.5.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

<u>Theorem 2.2.7.</u> Let X be a Stone space, i.e., a compact totally disconnected Hausdorff space, $\underline{\underline{C}}$ the field of all clopen subsets of X, and μ a charge on $\underline{\underline{C}}$. The charge algebra $(\underline{\underline{C}}(\mu), d_\mu)$ is compact if and only if

i) $\mu = \sum_{i \geq 1} a_i \mu_i$, for some sequence (finite or infinite) $\mu_n : n \geq 1$ of 0-1 valued charges on $\underline{\underline{C}}$, where a_i's are positive real numbers, and

ii) the family $\mu_n : n \geq 1$ is disjoint.

<u>Proof</u>: Let $\underline{\underline{B}}$ be the Baire σ-field of X. Let λ be the extension of μ to $\underline{\underline{B}}$. By Lemma 2.1.4, $(\underline{\underline{C}}(\mu), d_\mu)$ is isometrically isomorphic to a dense subalgebra of $(\underline{\underline{B}}(\lambda), d_\lambda)$. Suppose $(\underline{\underline{C}}(\mu), d_\mu)$ is compact. Then $(\underline{\underline{B}}(\lambda), d_\lambda)$ is compact. By Corollary 2.2.6, λ is completely atomic. We can write $\lambda = \sum_{i \geq 1} a_i \lambda_i$, where a_i's are positive real numbers and λ_i's are 0-1 valued measures on $\underline{\underline{B}}$. (If in the representation of λ only finitely many 0-1 valued charges are involved, then (ii) is trivially satisfied .) For any 0-1 valued measure π on $\underline{\underline{B}}$, there exists a unique point x in X such that $\pi = \partial_x$, where ∂_x is the Dirac measure concentrated at x, i.e., $\partial_x(B) = 1$ if $x \in B \in \underline{\underline{B}}$, $= 0$ otherwise. Consequently,

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

we can write $\lambda = \sum_{i \geq 1} a_i \partial_{x_i}$ for some sequence $x_n : n \geq 1$ of points in X. Let μ_i be the restriction of λ_i to $\underline{\underline{C}}$. Thus, we have $\mu = \sum_{i \geq 1} a_i \mu_i$ on $\underline{\underline{C}}$. We have to show that the family $\mu_n : n \geq 1$ is disjoint. Let $\partial_1, \partial_2, \ldots$ be a sequence of 0's and 1's. Let $\partial_{i_1}, \partial_{i_2}, \ldots$ be the subsequence of all 1's in the given sequence $\partial_1, \partial_2, \ldots$. There exists a Baire set B containing all the elements of the sequence $x_{i_1}, x_{i_2}, \ldots$ and none of the rest of the elements of the sequence $x_1, x_2, \ldots$. Since the isomorphism between $(\underline{\underline{C}}(\mu), d_\mu)$ and $(\underline{\underline{B}}(\lambda), d_\lambda)$ is onto, there exists a clopen set C in $\underline{\underline{C}}$ such that $\lambda(B \Delta C) = 0$. It is easy to check that $x_{i_n} : n \geq 1$ is contained in C and $x_k \notin C$ if $k \neq i_n$ for any $n \geq 1$. Obviously, $C \in \bigcap_{n \geq 1} \underline{\underline{I}}_n^{\partial_n}$.

Conversely, let $\mu = \sum_{n \geq 1} a_n \mu_n$ with the property that the family $\mu_n : n \geq 1$ is disjoint. (If in the representation of μ only finitely many 0-1 valued charges are involved then $\underline{\underline{C}}(\mu)$ is finite and hence compact.) Let λ and λ_n be the extensions of μ and μ_n respectively to $\underline{\underline{B}}$. It is obvious that λ_n is 0-1 valued. Hence λ is completely atomic on $\underline{\underline{B}}$. So, $(\underline{\underline{B}}(\lambda), d_\lambda)$ is compact by Corollary 2.2.6. In order to show that the isomorphism between $(\underline{\underline{C}}(\mu), d_\mu)$ and

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$(\underline{\underline{B}}(\lambda), d_\lambda)$ is onto, it is enough to show that for every Baire set B in $\underline{\underline{B}}$ there exists a clopen set C in $\underline{\underline{C}}$ subh that $\lambda(B \,\Delta\, C) = 0$. Note that we can write $\mu = \sum_{i \geq 1} a_i \,\partial_{x_i}$ for some sequence $x_1, x_2, \ldots$ of points in X. Let $x_{i_1}, x_{i_2}, \ldots$ be the complete subsequence of $x_1, x_2, \ldots$ available in B. Consider the sequence $\partial_1, \partial_2, \ldots$ where $\partial_{i_n} = 1$ for every $n \geq 1$ and $\partial_k = 0$ if $k \neq i_n$ for any $n \geq 1$. Take any C in $\bigcap_{n \geq 1} \underline{\underline{I}}_n^{\partial_n}$. Then $\lambda(B \,\Delta\, C) = 0$. This completes the proof of the Theorem.

Corollary 2.2.8. Let $(\Omega, \underline{\underline{A}}, \mu)$ be a charge space. Then $(\underline{\underline{A}}(\mu), d_\mu)$ is a compact metric space if and only if

i) $\mu = \sum_{i \geq 1} a_i \mu_i$ for some sequence (finite or infinite) $\mu_i : i \geq 1$ of 0-1 valued charges on $\underline{\underline{A}}$ and a_i's are positive real numbers, and

ii) the family $\mu_i : i \geq 1$ is disjoint.

We characterise Boolean algebras B such that $(B(\mu), d_\mu)$ is compact for every charge μ on B.

Theorem 2.2.9. Let B be a Boolean algebra. $(B(\mu), d_\mu)$ is compact for every charge μ on B if and only if B is finite.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Proof: The if-part is trivial. Since $(B(\mu), d_\mu)$ is compact for every charge μ on B, by Corollary 2.2.8 and Corollary 1.3.10, B is superatomic. If B is infinite, we will exhibit a sequence of 0-1 valued charges on B which is not disjoint. Let X be the Stone space of B. X is an infinite set. In X we can find a convergent sequence of distinct elements. Let X^1 be the set of all accumulation points of X and X^2 be the set of all accumulation points of X^1. Since X is infinite and scattered, $X^1 - X^2 \neq \emptyset$. Let $x_o \in X^1 - X^2$. Since x_o is an isolated point of X^1, there exists a clopen set U_{x_o} containing x_o such that $U_{x_o} \cap X^1 = \{x_o\}$. This is possible since X is totally disconnected. Since $x_o \in X^1$, $U_{x_o} \neq \{x_o\}$ and U_{x_o} is infinite. Let $x_1, x_2, \ldots$ be a sequence of distinct elements in U_{x_o}. We claim that $x_1, x_2, \ldots$ converges to x_o. Suppose not. Then there exists a clopen set V_{x_o} containing x_o and an infinite subsequence $x_{i_1}, x_{i_2}, \ldots$ of $x_1, x_2, \ldots$ such that $\{x_{i_1}, x_{i_2}, \ldots\} \cap V_{x_o} = \emptyset$. Since X is compact, let y be an accumulation point of $x_{i_1}, x_{i_2}, \ldots$. Since $x_{i_1}, x_{i_2}, \ldots \subset U_{x_o}$ and U_{x_o} is closed, $y \in U_{x_o}$. Note

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

that y also $\in X^1$. Hence $y = x_o$. Since $\{x_{i_1}, x_{i_2}, \dots\} \cap V_{x_o} = \emptyset$ and V_{x_o} is closed, $y \notin V_{x_o}$. But this is a contradiction.

Let $Y = \{x_o, x_1, x_2, \dots\}$. The field $\underline{\underline{C}}$ of all clopen subsets of Y is a homomorphic image of B. But $\underline{\underline{C}}$ can be identified as the finite-cofinite field of some countable set. On $\underline{\underline{C}}$ there exists a sequence of $0-1$ valued charges which is not disjoint. See page 22 . By transferring these charges to B, we obtain a sequence of $0-1$ valued charges on B which is not disjoint.

3. <u>Completeness.</u> If $(\Omega, \underline{\underline{B}}, \lambda)$ is a measure space, then the measure algebra $(\underline{\underline{B}}(\lambda), d_\lambda)$ is a complete metric space. In this section, we find necessary and sufficient conditions under which a charge algebra is a complete metric space. If a charge algebra $(\underline{\underline{A}}(\mu), d_\mu)$ is complete metric space, then $\underline{\underline{A}}(\mu)$ is a complete Boolean algebra. For, by Lemma 2.1.4, completeness of the metric space $(\underline{\underline{A}}(\mu), d_\mu)$ ensures isometric ismorphism between $(\underline{\underline{A}}(\mu), d_\mu)$ and $(\underline{\underline{B}}(\lambda), d_\lambda)$. Since $\underline{\underline{B}}(\lambda)$ is a complete Boolean algebra, we find that $\underline{\underline{A}}(\mu)$ is also complete.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

However, the completeness of the Boolean algebra $\underline{\underline{A}}(\mu)$ need not imply the completeness of the metric space $(\underline{\underline{A}}(\mu), d_{\mu})$. The following is a counter example. Let Ω be the set of all natural numbers and $\underline{\underline{A}} = P(\Omega)$, the power set of Ω. Let μ_1 be any 0-1 valued charge on $\underline{\underline{A}}$ such that $\mu_1(\{n\}) = 0$ for every natural number n. Let $\mu_n = \delta_{n-1}$, the degenerate measure at n-1 for $n \geq 2$. Let $\mu = \sum_{i \geq 1} \frac{1}{2^i} \mu_i$. It is easy to verify that the family $\{\mu_n .. n \geq 1\}$ is not disjoint. Note that $\underline{\underline{A}}(\mu) = \underline{\underline{A}}$ is a complete Boolean algebra. Let X be the Stone space of $\underline{\underline{A}}$, $\underline{\underline{C}}$ the collection of all clopen subsets of X and $\underline{\underline{B}}$ the Baire σ-field on X. Assume, without loss of generality, that μ is defined on $\underline{\underline{C}}$. Let λ be the extension of μ from $\underline{\underline{C}}$ to $\underline{\underline{B}}$ as a measure. Observe that λ is completely atomic. Note also that $(\underline{\underline{A}}(\mu), d_{\mu})$ is isometrically isomorp to a dense sub-algebra of $(\underline{\underline{B}}(\lambda), d_{\lambda})$. If $(\underline{\underline{A}}(\mu), d_{\mu})$ is a complete metric space, then the above isomorphism is onto. Since $(\underline{\underline{B}}(\lambda), d_{\lambda})$ is compact, this would imply $(\underline{\underline{A}}(\mu), d_{\mu})$ is compact. But this is a contradiction to Corollary 2.2.8.

The proof of the following theorem is essentially due to Green [9, p. 258]. Green was interested in getting necessary and sufficient conditions for the completeness of the metric

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

spaces $L^p(\Omega, \underline{\underline{A}}, \mu)$ for $1 \leq p < \infty$ for a charge space $(\Omega, \underline{\underline{A}}, \mu)$.

Theorem 2.3.1. Let X be a Stone space, $\underline{\underline{C}}$ the field of all clopen subsets of X, μ a strictly positive charge on $\underline{\underline{C}}$, $\underline{\underline{B}}$ the Baire σ-field on X, and λ the extension of μ from $\underline{\underline{C}}$ to $\underline{\underline{B}}$ as a measure. Then $(\underline{\underline{C}}(\mu), d_\mu)$ is a complete metric space if and only if

i) X is extremally disconnected, i.e., closure of every open subset of X is open, and,

ii) $\lambda(U) = \lambda(\bar{U})$ for every open Baire subset U of X.

Proof: Suppose $(\underline{\underline{C}}(\mu), d_\mu)$ is a complete metric space. From the remarks made at the beginning of this section, it follows that $\underline{\underline{C}}(\mu) = \underline{\underline{C}}$ is a complete Boolean algebra. Consequently, its Stone space X must be extremally disconnected. Since the natural isometric isomorphism between $(\underline{\underline{C}}(\mu), d_\mu)$ and $(\underline{\underline{B}}(\lambda), d_\lambda)$ is onto, given any Baire set B there exists a clopen set C such that $\lambda(B \triangle C) = 0$. Observe that every non-empty open Baire set has positive λ-measure. This follows from the fact that clopen sets form a basis for the topology of X and every clopen set has positive λ-measure. Let U be any

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

open Baire set, and C the clopen set such that $\lambda(U \triangle C) = 0$. Since $U - C$ is an open Baire set and $\lambda(U - C) = 0$, we have $U - C = \emptyset$. This implies $U \subset \bar{U} \subset C$. But $\lambda(C - U) = 0$ which implies $\lambda(\bar{U} - U) = 0$.

Conversely, in order to show that $(\underline{\underline{C}}(\mu), d_\mu)$ is a complete metric space, it is sufficient to exhibit a clopen set C for every Baire set B such that $\lambda(B \triangle C) = 0$. If $\lambda(B) = 0$, empty set, $\emptyset$, would do. Let $\lambda(B) > 0$. Since λ is regular, i.e., the λ-measure of every Baire set is approximable from below by the λ-measure of compact G_δ subsets, we can find a sequence of compact G_δ subsets C_n contained in B such that $\lambda(B - \bigcup_{n \geq 1} C_n) = 0$. By hypothesis, $\lambda(C_n - C_n^o) = 0$, where C_n^o denotes the interior of C_n. Consequently, $\lambda(B - \bigcup_{n \geq 1} C_n^o) = 0$. Let C be the closure of the open set $\bigcup_{n \geq 1} C_n^o$. Then $\lambda(C - \bigcup_{n \geq 1} C_n^o) = 0$. Thus, we find $\lambda(B \triangle C) = 0$ and notice that C is clopen.

<u>Corollary 2.3.2.</u> Let μ be a charge on a field $\underline{\underline{A}}$ of Ω. Then $(\underline{\underline{A}}(\mu), d_\mu)$ is a complete metric space if and only if

i) the Stone space X of $\underline{\underline{A}}(\mu)$ is extremally disconnected, and

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

ii) $\lambda(U) = \lambda(\bar{U})$ for every open Baire subset U of X, where λ is the extension of μ from the field $\underline{\underline{C}}$ of all clopen subsets of X to the Baire σ-field $\underline{\underline{B}}$ of X, (It is assumed that μ is defined on $\underline{\underline{C}}$.) as a measure.

The following theorem gives a set of conditions which are intimately related to the given structure.

<u>Theorem 2.3.3.</u> Let $(\Omega, \underline{\underline{A}}, \mu)$ be a charge space. $(\underline{\underline{A}}(\mu), d_\mu)$ is a complete metric space if and only if

i) $\underline{\underline{A}}(\mu)$ is a complete Boolean algebra, and

ii) μ is a countably additive function on the Boolean algebra $\underline{\underline{A}}(\mu)$.

<u>Proof:</u> Suppose $(\underline{\underline{A}}(\mu), d_\mu)$ is a complete metric space. Proceeding along the lines of Lemma 2.1.4, we conclude that $(\underline{\underline{A}}(\mu), d_\mu)$ and $(\underline{\underline{B}}(\lambda), d_\lambda)$ are isometrically isomorphic. The isomorphism that works between these two spaces also preserves μ and λ. Hence (i) and (ii) follow.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Conversely, let X be the Stone space of $\underline{\underline{A}}(\mu)$, $\underline{\underline{C}}$ the field of all clopen subsets of X and $\underline{\underline{B}}$ the Baire σ-field on X. We can assume that μ is defined on $\underline{\underline{C}}$. Let λ be the extension of μ from $\underline{\underline{C}}$ to $\underline{\underline{B}}$ as a measure. Since $\underline{\underline{A}}(\mu)$ is a complete Boolean algebra, X is extremally disconnected. Countable additivity of μ on the Boolean algebra $\underline{\underline{C}}$ implies that for every disjoint sequence $C_1, C_2, \ldots$ of clopen sets in $\underline{\underline{C}}$, $\mu(\overline{\bigcup_{i\geq 1} C_i}) = \sum_{i\geq 1} \mu(C_i)$. In order to show that $(\underline{\underline{A}}(\mu), d_\mu)$ is a complete metric space, in view of Corollary 2.3.2, it is sufficient if we show that for every open Baire set U, $\lambda(U) = \lambda(\overline{U})$. Since λ is an extension of μ from $\underline{\underline{C}}$ to $\underline{\underline{B}}$, for any natural number n, we can find a disjoint sequence $C_i^n : i \geq 1$ of clopen sets such that $\bigcup_{i\geq 1} C_i^n$ contains U and $\sum_{i\geq 1} \mu(C_i^n) \leq \lambda(U) + \frac{1}{n}$. See Halmos [11, pp. 50 and 54]. Let $C^n = \overline{\bigcup_{i\geq 1} C_i^n}$ and $C = \bigcap_{n\geq 1} C^n$. C^n is clopen and $\mu(C^n) = \lambda(C^n) \leq \lambda(U) + \frac{1}{n}$ for every n. Since C^n contains $\overline{U}$ for every n, C contains $\overline{U}$. Observe that $\lambda(C - U) = 0$ which implies $\lambda(\overline{U}) = \lambda(U)$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

4. Connectedness.

Lemma 2.4.1. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space, where λ is nonatomic. Let $B \in \underline{\underline{B}}$. Then there exists a system $\{B_t \in \underline{\underline{B}} : t \in [0, 1]\}$ with the following properties.

i) $B_o = \emptyset$ and $B_1 = B$,

ii) $B_r \subset B_s$ if $r \leq s$, and

iii) $\lambda(B_r) = r\lambda(B)$ for $0 \leq r \leq 1$.

Proof. If $\lambda(B) = 0$, then we can take $B_r = \emptyset$ if $0 \leq r < 1$ and $B_1 = B$. If $\lambda(B) > 0$, by Liapounov's theorem, there exists $B_{\frac{1}{2}} \in \underline{\underline{B}}$ such that $\lambda(B_{\frac{1}{2}}) = \frac{1}{2}\lambda(B)$. There exist $B_{\frac{1}{2^2}}$ and $B_{\frac{3}{2^2}}$ in $\underline{\underline{B}}$ such that $B_{\frac{1}{2^2}} \subset B_{\frac{1}{2}} \subset B_{\frac{3}{2^2}}$ and $\lambda(B_{\frac{1}{2^2}}) = \frac{1}{2^2}\lambda(B)$, $\lambda(B_{\frac{3}{2^2}}) = \frac{3}{2^2}\lambda(B)$. Continuing this process, we obtain, for every positive dyadic rational $\frac{k}{2^n}(< 1)$ sets $B_{\frac{k}{2^n}}$ such that $\lambda(B_{\frac{k}{2^n}}) = \frac{k}{2^n}\lambda(B)$ and satisfying (ii). For any positive real number $r < 1$, define $B_r = \bigcup_{\frac{k}{2^n} \leq r} B_{\frac{k}{2^n}}$. $B_r \in \underline{\underline{B}}$ and $\lambda(B_r) = r\,\lambda(B)$. Take $B_o = \emptyset$ and $B_1 = B$. Then the system $\{B_r : r \in [0,1]\}$ satisfies the properties (i), (ii) and (iii).

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

<u>Lemma 2.4.2.</u> Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space with λ nonatomic. For every B in $\underline{\underline{B}}$ with $\lambda(B) > 0$, the pseudo-metric space $(\underline{\underline{B}}, d_\lambda)$ contains a homeomorphism of $[0, 1]$ with $\emptyset$ and B as end points.

<u>Proof:</u> By Lemma 2.4.1, we can define a map T from $[0, 1]$ into $\underline{\underline{B}}$ as follows. $T(r) = B_r$. $d_\lambda(T(r), T(s)) = \lambda(B_r \triangle B_s) =$ $= |r - s| \lambda(B)$. Clearly, T is a homeomorphism.

<u>Corollary 2.4.3.</u> Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space with λ nonatomic. Then the pseudo-metric space $(\underline{\underline{B}}, d_\lambda)$ is connected.

<u>Proof:</u> For every B in $\underline{\underline{B}}$, we can find a connected subset of $\underline{\underline{B}}$, by Lemma 2.4.2, containing B and $\emptyset$.

<u>Lemma 2.4.4.</u> Let $(\Omega, \underline{\underline{A}}, \mu)$ be a charge space, and $A \in \underline{\underline{A}}$. Consider the charge space $(A, A \cap \underline{\underline{A}}, \mu_A)$, where $A \cap \underline{\underline{A}}$ is the trace of $\underline{\underline{A}}$ on A and μ_A is the restriction of μ to $A \cap \underline{\underline{A}}$. Then the pseudo-metric space $(A \cap \underline{\underline{A}}, d_{\mu_A})$ is a continuous image of $(\underline{\underline{A}}, d_\mu)$.

<u>Proof:</u> Define $T : \underline{\underline{A}} \longrightarrow A \cap \underline{\underline{A}}$ as follows. $TB = A \cap B$. Note that $d_{\mu_A}(TB_1, TB_2) = \mu(A \cap B_1 \triangle A \cap B_2) \leq \mu(B_1 \triangle B_2) = d_\mu(B_1, B_2)$. Hence T is continuous. It is also onto.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Lemma 2.4.5. Let $(\Omega, \underline{\underline{A}}, \mu)$ be a charge space. Let r be any nonnegative real number. Then $\{A \in \underline{\underline{A}} : \mu(A) \leq r\}$ is a closed subspace of $(\underline{\underline{A}}, d_\mu)$ and $\{A \in \underline{\underline{A}} : \mu(A) < r\}$ is an open subspace of $(\underline{\underline{A}}, d_\mu)$.

Proof: In fact, $\{A \in \underline{\underline{A}} : \mu(A) \leq r\}$ = the closed sphere $\{A \in \underline{\underline{A}} : d_\mu(A, \emptyset) \leq r\}$ and $\{A \in \underline{\underline{A}} : \mu(A) < r\}$ = the open sphere $\{A \in \underline{\underline{A}} : d_\mu(A, \emptyset) < r\}$.

Theorem 2.4.6. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space. Then the pseudo-matric space $(\underline{\underline{B}}, d_\lambda)$ is connected if and only if λ is nonatomic.

Proof: If λ is nonatomic, the connectedness of the space $(\underline{\underline{B}}, d_\lambda)$ is proved in Corollary 2.4.3. Suppose $(\underline{\underline{B}}, d_\lambda)$ is connected. Let $A \in \underline{\underline{B}}$ with $\lambda(A) > 0$. $(A \cap \underline{\underline{B}}, d_{\lambda_A})$ is connected. Let $\lambda(A) > \epsilon > 0$. Then there exists a B in $\underline{\underline{B}}$, B contained in A such that $\lambda(B) = \epsilon$. Suppose not. $\{B \in A \cap \underline{\underline{B}} : \lambda_A(B) < \epsilon\}$ and $\{B \in A \cap \underline{\underline{B}} : \lambda_A(B) > \epsilon\}$ are nonempty disjoint open subsets of $A \cap \underline{\underline{B}}$ whose union is $A \cap \underline{\underline{B}}$. This implies that $(A \cap \underline{\underline{B}}, d_{\lambda_A})$ is disconnected.

Corollary 2.4.7. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space. Then the measure algebra $(\underline{\underline{B}}(\lambda), d_\lambda)$ is connected if and only if λ is nonatomic.

<u>Theorem 2.4.8.</u> Let $(\Omega, \underline{\underline{A}}, \mu)$ be a charge space. If the pseudometric space $(\underline{\underline{A}}, d_\mu)$ is connected, then μ is strongly nonatomic.

<u>Proof</u>. The proof of this theorem is contained in that of Theorem 2.4.6.

<u>Theorem 2.4.9.</u> Let $(\Omega, \underline{\underline{A}}, \mu)$ be a charge space, where $\underline{\underline{A}}$ is a σ-field. The pseudo-metric space $(\underline{\underline{A}}, d_\mu)$ is connected if and only if μ is strongly nonatomic.

<u>Proof:</u> Only if part follows from Theorem 2.4.8. Using the fact that $\underline{\underline{A}}$ is a σ-field, one can prove the if part imitating the steps involved in the proofs of Lemmas 2.4.1 and 2.4.2 and Corollary 2.4.3.

We are unable to obtain necessary and sufficient conditions for the connectedness of the space $(\underline{\underline{A}}, d_\mu)$ associated with a charge space $(\Omega, \underline{\underline{A}}, \mu)$. We do not know whether the condition that $\underline{\underline{A}}$ is a σ-field can be relaxed in Theorem 2.4.9.

5. <u>Total disconnectedness</u>.

A topological space is said to be totally disconnected if it has a base consisting of clopen (open and closed) sets.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Theorem 2.5.1. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space. Then the space $(\underline{\underline{B}}, d_\lambda)$ is totally disconnected if and only if λ is completely atomic.

Proof: Suppose λ is completely atomic. Let $A_1, A_2, \ldots$ be the λ-atoms and $\lambda(A_i) = a_i > 0$. The sets

$$\{B \in \underline{\underline{B}} : A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k} \Subset B\}$$

$$= \{B \in \underline{\underline{B}} : \lambda(B \cap A_{i_1}) = \lambda(A_{i_1}), \ldots, \lambda(B \cap A_{i_n}) = \lambda(A_{i_n})\}$$

$$= \underline{\underline{B}} - \{B \in \underline{\underline{B}} : \lambda(B \cap A_{i_1}) = 0 \text{ or } \ldots \lambda(B \cap A_{i_n}) = 0\},$$

where $\Subset$ denotes essential inclusion and $i_1, i_2, \ldots, i_k$ is any finite set of natural numbers, are clopen and form a basis for the topology of $(\underline{\underline{B}}, d_\lambda)$. Hence $(\underline{\underline{B}}, d_\lambda)$ is totally disconnected. Conversely, let $(\underline{\underline{B}}, d_\lambda)$ be totally disconnected. Suppose λ is not completely atomic. Then there exists A in $\underline{\underline{B}}$ such that $\lambda(A) > 0$ and the restriction λ_A of λ to $A \cap \underline{\underline{B}}$ is non-atomic. We can find B in $\underline{\underline{B}}$, B contained in A such that $\lambda(B) > 0$ and $\lambda(A - B) > 0$. Since $d_\lambda(B, A - B) \geq 0$, we can find a clopen set $\underline{\underline{B}}_1$ such that $B \in \underline{\underline{B}}_1$ and $A - B \notin \underline{\underline{B}}_1$. Let $\underline{\underline{B}}_2 = \underline{\underline{B}} - \underline{\underline{B}}_1$. So, $\{A \cap \underline{\underline{B}}_1, A \cap \underline{\underline{B}}_2\}$ is a nontrivial disconnection of $(A \cap \underline{\underline{B}}, d_{\lambda_A})$. But $(A \cap \underline{\underline{B}}, d_{\lambda_A})$ is connected by Theorem 2.4.6.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

This contradiction shows that λ is completely atomic.

Corollary 2.5.2. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space. The measure algebra $(\underline{\underline{B}}(\lambda), d_\lambda)$ is totally disconnected if and only if it is compact if and only if λ is completely atomic.

Remark. If λ is completely atomic and $A_1, A_2, \ldots$ are the λ-atoms of $\underline{\underline{B}}$ with $\lambda(A_i) = a_i > 0$, then $(\underline{\underline{B}}(\lambda), d_\lambda)$ is homeomorphic to $(\pi_{i \geq 1} X_i, d)$, where $X_i = \{0, 1\}$ for every i, and $d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i \geq 1} d_i(x_i, y_i)$ with $d_i(x_i, y_i) = a_i$ if $x_i \neq y_i$, $= 0$ if $x_i = y_i$.

Now, we turn our attention to charge space.

Theorem 2.5.3. Let $(\Omega, \underline{\underline{A}}, \mu)$ be a charge space. If $\mu = \sum_{i \geq 1} a_i \mu_i$, where a_i's are nonnegative real numbers and μ_i's are distinct 0-1 valued charges on $\underline{\underline{A}}$, then $(\underline{\underline{A}}, d_\mu)$ is totally disconnected.

Proof: Assume, without loss of generally, $\underline{\underline{A}}$ to be the collection of all clopen subsets of a Stone space X. Let $\underline{\underline{B}}$ be the Baire σ-field on X. The extension λ of μ from $\underline{\underline{A}}$ to $\underline{\underline{B}}$ as a measure is completely atomic. Hence $(\underline{\underline{B}}, d_\lambda)$ is totally disconnected. $(\underline{\underline{A}}, d_\mu)$ can be viewed as a subspace of $(\underline{\underline{B}}, d_\lambda)$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

and consequently, it is totally disconnected.

Corollary 2.5.4. Let $(\Omega, \underline{\underline{A}}, \mu)$ be a charge space. If $\mu = \sum_{i \geq 1} a_i \mu_i$, where a_i's are nonnegative real numbers and μ_i's are distinct 0-1 valued charges on $\underline{\underline{A}}$, then $(\underline{\underline{A}}(\mu), d_\mu)$ is totally disconnected.

Remark: The converse of Corollary 2.5.4 is not true. Take any countable algebra admitting a strongly continuous charge. By Theorem 1.2.2 and the uniqueness of the representation, it is not possible to write a strongly continuous charge as a countable sum of two valued charges. The charge algebra of a countable algebra is always totally disconnected since it is a countable metric space.

It will be interesting to characterise total disconnectedness of a charge algebra.

6. Local compactness.

Theorem 2.6.1. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space. Then the pseudo-metric space $(\underline{\underline{B}}, d_\lambda)$ is locally compact if and only if it is compact.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Proof: If $(\underline{\underline{B}}, d_\lambda)$ is locally compact, we will show that λ is completely atomic. There exists a closed sphere $S[\emptyset, r] = \{A \in \underline{\underline{B}} : \lambda(A) \leq r\}$ $(r > 0)$ which is compact. If λ is not completely atomic, find a B in $\underline{\underline{B}}$ such that $0 < \lambda(B) \leq r$ and the restriction λ_B of λ to $B \cap \underline{\underline{B}}$ is nonatomic. Now, $(B \cap \underline{\underline{B}}, d_{\lambda_B})$ is a continuous image of $S[\emptyset, r]$. The map $T : S[\emptyset, r] \longrightarrow B \cap \underline{\underline{B}}$ is defined as follows, $TA = A \cap B$.

$d_{\lambda_B}(TA_1, TA_2) = \lambda(A_1 \cap B \,\triangle\, A_2 \cap B) \leq \lambda(A_1 \triangle A_2) = d_\lambda(A_1, A_2)$.

These relations show that T is continuous. Further T is onto. Consequently, $(B \cap \underline{\underline{B}}, d_{\lambda_B})$ is compact. But this is a contradiction to Lemma 2.2.2. Hence λ is completely atomic.

Corollary 2.6.2. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space. Then the measure algebra $(\underline{\underline{B}}(\lambda), d_\lambda)$ is locally compact if and only if λ is completely atomic if and only if $(\underline{\underline{B}}(\lambda), d_\lambda)$ is compact.

Theorem 2.6.3. Let $(\Omega, \underline{\underline{A}}, \mu)$ be a charge space. The following are equivalent.

i) $(\underline{\underline{A}}(\mu), d_\mu)$ is locally compact.

ii) $(\underline{\underline{A}}(\mu), d_\mu)$ is compact.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Proof. Let X be the Stone space of $\underline{\underline{A}}(\mu)$, $\underline{\underline{C}}$ the field of all clopen subsets of X and $\underline{\underline{B}}$ the Baire σ-field on X. Assume that μ is defined on $\underline{\underline{C}}$. Let λ be the extension of μ from $\underline{\underline{C}}$ to $\underline{\underline{B}}$ as a measure. Note that $(\underline{\underline{B}}(\lambda), d_\lambda)$ is the metric completion of $(\underline{\underline{A}}(\mu), d_\mu)$. In any Hausdorff space, a dense locally compact subspace is open. See Kelley [13, G, p. 163]. Consequently $\underline{\underline{A}}(\mu)$ is a dense open subgroup of $\underline{\underline{B}}(\lambda)$. In any topological group, any open subgroup is closed. This implies that the isometric isomorphism from $\underline{\underline{A}}(\mu)$ to $\underline{\underline{B}}(\lambda)$ is onto. So, $(\underline{\underline{B}}(\lambda), d_\lambda)$ is locally compact. By Corollary 2.6.2, $(\underline{\underline{B}}(\lambda), d_\lambda)$ is compact. This implies that $(\underline{\underline{A}}(\mu), d_\mu)$ is compact.

Remark: The above proof incidentally shows that if $\underline{\underline{A}}(\mu)$ is open in $\underline{\underline{B}}(\lambda)$ then $\underline{\underline{A}}(\mu) = \underline{\underline{B}}(\lambda)$. It will be interesting to know if open can be replaced by G_δ.

7. Perfectness. A nonempty subset of a metric space is said to be perfect if every point of the subset is an accumulation point.

Theorem 2.7.1. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space. The measure algebra $(\underline{\underline{B}}(\lambda), d_\lambda)$ is perfect if and only if the range of λ is infinite.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

<u>Proof:</u> If the range of λ is finite, then $\underline{\underline{B}}(\lambda)$ is a finite metric space and hence is not perfect. Suppose range of λ is infinite. First note that the following two statements are true.

1. If λ is a nonzero nonatomic measure, then $\underline{\underline{B}}(\lambda)$ is perfect. For, by Lemmas 2.4.1 and 2.4.2, $\underline{\underline{B}}(\lambda)$ contains a homeomorphism of $[0, 1]$. If $\underline{\underline{B}}(\lambda)$ is not perfect, then it is a discrete metric space giving rise to a contradiction. (Every non-discrete topological group is perfect.)

2. If λ is completely atomic and the number of λ-atoms is infinite, then $\underline{\underline{B}}(\lambda)$ is perfect. This follows from the Remark made after Corollary 2.5.2.

Now, for any general λ, decompose $\Omega = \Omega_1 \cup \Omega_2$, where λ is completely atomic on Ω_1 and λ is nonatomic on Ω_2. $\Omega_1 \cap \underline{\underline{B}}(\lambda_{\Omega_1})$ is isometrically isomorphic to a closed subspace of $\underline{\underline{B}}(\lambda)$ and $\Omega_2 \cap \underline{\underline{B}}(\lambda_{\Omega_2})$ is isometrically isomorphic to a closed subspace of $\underline{\underline{B}}(\lambda)$. If $\underline{\underline{B}}(\lambda)$ is not perfect, then it is a discrete metric space. In view of the statements 1 and 2, this implies $\lambda(\Omega_2) = 0$ and the number of λ-atoms is finite. Hence range of λ is finite. This is a contradiction.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Theorem 2.7.2. Let $(\Omega, \underline{\underline{A}}, \mu)$ be a charge space. $(\underline{\underline{A}}(\mu), d_\mu)$ is perfect if and only if the range of μ is infinite.

Proof: If the range of μ is finite, then $\underline{\underline{A}}(\mu)$ is a finite metric space and hence is not perfect. Suppose that range of μ is infinite. By Lemma 2.1.4, $(\underline{\underline{A}}(\mu), d_\mu)$ is isometrically isomorphic to a dense subalgebra of measure algebra $(\underline{\underline{B}}(\lambda), d_\lambda)$. Note that the range of λ is infinite. By Theorem 2.7.1, $\underline{\underline{B}}(\lambda)$ is perfect. Note that any dense subset of a perfect metric space is perfect.

8. Dimension. For notions in dimension theory, we refer to Nagata [18].

Theorem 2.8.1. Let $\Omega = [0, 1]$, $\underline{\underline{B}}$ = Borel σ-field on Ω, and λ Lebesgue measure on $\underline{\underline{B}}$. Then $\dim \underline{\underline{B}}(\lambda) = \infty$.

Proof: First, we note that $\underline{\underline{B}}(\lambda)$ contains a homeomorphism of $[0, 1]$. This is clear from Lemmas 2.4.1 and 2.4.2. Since $\dim [0, 1] = 1$, we have $\dim \underline{\underline{B}}(\lambda) \geq 1$. Let $\Omega^2 = \Omega \times \Omega$, $\underline{\underline{B}}^2$ = the product σ-field $\underline{\underline{B}} \times \underline{\underline{B}}$ and λ^2 the product measure $\lambda \times \lambda$ on $\underline{\underline{B}}^2$. We will show that the measure algebra $\underline{\underline{B}}^2(\lambda^2)$ contains a homeomorph[illegible] of Ω^2. The relevant map

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$T : \Omega^2 \longrightarrow \underline{\underline{B}}^2(\lambda^2)$ is defined as follows. $T(\alpha, \beta)$ = the equivalence class containing $[0, \alpha] \times [0, \beta]$. Clearly, T is one-one. Further, $\lambda^2[T(\alpha_1, \beta_1) \triangle T(\alpha_2, \beta_2)] \leq |\alpha_1 - \alpha_2|(\beta_1 + \beta_2) + |\beta_1 - \beta_2|(\alpha_1 + \alpha_2) + |\alpha_1 - \alpha_2|\,|\beta_1 - \beta_2|$. This shows that T is continuous. Hence it is a homeomorphism. Consequently, $\dim \underline{\underline{B}}^2(\lambda^2) \geq 2$. By von Neumann-Halmos theorem (see [11, Theorem C, p. 173]), $\underline{\underline{B}}^2(\lambda^2)$ and $\underline{\underline{B}}(\lambda)$ are isometrically isomorphic. Hence $\dim \underline{\underline{B}}(\lambda) \geq 2$. Repeating this argument, we conclude that $\dim \underline{\underline{B}}(\lambda) \geq n$ for any natural number n. Hence $\dim \underline{\underline{B}}(\lambda) = \infty$.

<u>Theorem 2.8.2.</u> Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space with λ nonatomic. Then $\dim \underline{\underline{B}}(\lambda) = \infty$.

<u>Proof:</u> By a theorem of Bhaskara Rao and Bhaskara Rao [1], there exists a separable sub σ-field $\underline{\underline{D}}$ of $\underline{\underline{B}}$ such that λ is nonatomic on $\underline{\underline{D}}$. Note that $\underline{\underline{D}}(\lambda)$ is isometrically isomorphic to a closed subspace of $\underline{\underline{B}}(\lambda)$. By Theorem 2.8.1 and Von Neumann-Halmos theorem cited earlier, $\dim \underline{\underline{D}}(\lambda) = \infty$. Hence $\dim \underline{\underline{B}}(\lambda) = \infty$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Theorem 2.8.3. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space such that in the decomposition of $\Omega = \Omega_1 \cup \Omega_2$, where λ is completely atomic on Ω_1 and λ is nonatomic on Ω_2, we have $\lambda(\Omega_2) > 0$. Then $\dim \underline{\underline{B}}(\lambda) = \infty$.

Proof: Note that the natural mapping from $(\Omega_2 \cap \underline{\underline{B}})(\lambda_{\Omega_2})$ to $\underline{\underline{B}}(\lambda)$ is an isometry, where λ_{Ω_2} is the restriction of λ to $\Omega_2 \cap \underline{\underline{B}}$. Hence $\dim \underline{\underline{B}}(\lambda) = \infty$.

Theorem 2.8.4. Let $(\Omega, \underline{\underline{B}}, \lambda)$ be a measure space. Then $\dim \underline{\underline{B}}(\lambda) = 0$ or ∞. $\dim \underline{\underline{B}}(\lambda) = 0$ if and only if λ is completely atomic. $\dim \underline{\underline{B}}(\lambda) = \infty$ if and only if the nonatomic part of λ is nonzero.

Proof: Use Theorem 2.8.3 and Corollary 2.5.2.

It will be interesting to classify infinite dimensional measure algebras in the light of transfinite dimension. The reader may note that we have not tackled the dimension problem for charge algebras which seems to be difficult!

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

CHAPTER 3

LATTICE OF BOOLEAN ALGEBRAS

1. Introduction. Let B be a Boolean algebra. We denote by L_B the collection of all subalgebras of B. On L_B we define a partial order $\leq$ as follows. Let C and D $\in L_B$. We say $C \leq D$ if $C \subset D$. With respect to this partial order L_B is a complete lattice with the first element $\{0, 1\}$ and the last element B itself. For any family $\{C_\alpha\} \subset L_B$, sup C_α = the subalgebra of B generated by the family $\{C_\alpha\}$ and is denoted by $\bigvee_\alpha C_\alpha$. Similarly, $\operatorname{Inf}_\alpha C_\alpha = \bigcap_\alpha C_\alpha$ and is denoted by $\bigwedge_\alpha C_\alpha$. The symbols V, Λ as applied to families of subalgebras should not be confused with the same symbols used for the elements of some fixed Boolean algebra. In the context it will be clear in what sense these symbols are used. The primary object of this chapter is to study the Lattice structure of L_B.

Some of the natural questions that arise in the study of the lattice L_B are the following.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

i) Is L_B distributive ? i.e., is it true that $C \wedge (D \vee E) = (C \wedge D) \vee (C \wedge E)$ for every C, D, E in L_B?

ii) Is L_B complemented ? i.e., given any C in L_B does there exist a $D \in L_B$ satisfying $C \vee D = B$ and $C \wedge D = \{0, 1\}$? (we say that D is a complement of C.)

We remark that the lattice L_B is distributive if and only if B consists of four elements. The if part is easy to see. If B consists of more than four elements, take three non-zero disjoint elements a, b, c from B satisfying $a \vee b \vee c = 1$. Let $C = \{0, a \vee b, c, 1\}$, $D = \{0, a \vee c, b, 1\}$ and $E = \{0, b \vee c, a, 1\}$. Note that $C \wedge (D \vee E) \neq (C \wedge D) \vee (C \wedge E)$

The study of the second question is the central theme of this chapter.

2. Is L_B complemented?

We need the following definitions.

Definition 3.2.1. Let B be a Boolean algebra and I be an ideal in B. We have the natural homomorphism $h : B \longrightarrow B/I$ defined by $h(b) = [b]$, where B/I is the quotient Boolean

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

algebra and $[b]$ is the equivalence class containing b. We say that <u>h admits a lifting</u> if there is a subalgebra $C \subset B$ such that h restricted to C is one to one and takes C onto B/I. We call C a lifting of h. (Actually h becomes an isomorphism between C and B/I.)

<u>Definition 3.2.2.</u> A closed subset $Y \subset X$, a totally disconnected compact Hausdorff space is said to be <u>a retract of X</u> if there is a continuous map $f : X \xrightarrow{onto} Y$ such that f on Y is identity. See Sikorski [27, p. 46].

Let I be an ideal of a Boolean algebra B. Let X be the Stone space of B. Then the Stone space of the quotient Boolean algebra B/I can be identified as a closed subset $Y \subset X$. (See Sikorski [27, p. 31 - last paragraph].) The following theorem connects lifting and retract.

<u>Theorem 3.2.3.</u> The following are equivalent.

1) The natural homomorphism $h : B \longrightarrow B/I$ admits a lifting

2) The closed subset Y is a retract of X.

Proof is easy and we omit it.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

<u>Definition 3.2.4:</u> Let B be a Boolean σ-algebra and I be a σ-ideal in B. Then the natural homomorphism $h : B \to B/I$ is said to admit a σ-lifting if there is a subalgebra $C \subset B$ which is a Boolean σ-algebra by itself such that h restricted to C is one to one and h takes C onto B/I.

As a first result regarding complementation we have

<u>Theorem 3.2.5.</u> Any finite subalgebra $A \subset B$ has a complement in L_B.

<u>Proof:</u> Let $a_1, \ldots, a_n$ be all the atoms of A. So $\bigvee_1^n a_i = 1$. We shall straightaway construct a complement of A. Let $F_1, F_2, \ldots, F_n$ be any maximal filters in B containing $a_1, \ldots, a_n$ respectively. Let $F = F_1 \cap \ldots \cap F_n$. Then F is a filter in B. Let C be the Boolean algebra generated by F in B.

$$\text{So} \quad C = \{ b \in B : b \text{ or } b' \in F \}.$$

We claim that

$$A \wedge C = \{0, 1\} \quad - \quad (1)$$

$$A \vee C = B \quad - \quad (2)$$

<u>Proof of (1).</u> Let $b \in A \wedge C$ be such that $b \neq 0$ and $b \neq 1$. $b \in C$ and so we can assume without loss of generality that

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$b \in F$. Since $b \in \mathbb{A}$, $b = a_{i_1} \vee \ldots \vee a_{i_k}$ for some atoms $a_{i_1} \ldots a_{i_k}$ of $\mathbb{A}$. Take a j such that $j \neq i_1, \ldots, i_k$ and $1 \leq j \leq n$. Since $b \in F$, $b \in F_j$ and a_j also $\in F_j$. But $b \wedge a_j = 0$. So $0 = b \wedge a_j \in F_j$, a contradiction.

Proof of (2): It is sufficient to show that every element of B which is $\leq a_1$ belongs to $\mathbb{A} \vee C$. Then, by similar argument, every element of B which is $\leq a_j$ belongs to $\mathbb{A} \vee C$ for all j. Then for any $b \in B$

$$b = (b \wedge a_1) \vee (b \wedge a_2) \vee \ldots \vee (b \wedge a_n) \in \mathbb{A} \vee C$$

Let $c \in B$ be such that $c \leq a_1$. Then either $c \in F_1$ or $a_1 - c \in F_1$. If $c \in F_1$, $c \vee a_2 \ldots \vee a_n \in F$ and hence $a_1 \wedge (c \vee a_2 \ldots \vee a_n) \in \mathbb{A} \vee C$, i.e., $c \in \mathbb{A} \vee C$. If $a_1 - c \in F_1$, by a similar argument, we have $a_1 - c \in \mathbb{A} \vee C$. But $a_1 \in \mathbb{A} \vee C$. Hence $c \in \mathbb{A} \vee C$. This completes the proof of Theorem 3.2.5.

Our next theorem characterises complements of certain sub-algebras of B. We need the following notation and Lemmas.

For an ideal I in a Boolean algebra B, by $B(I)$ we mean the Boolean algebra generated by I in B. So

$$B(I) = \{c \in B : c \text{ or } c' \in I\}.$$

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Lemma 3.2.6: Let C be any subalgebra of B. Then $B(I) \vee C = \{b \in B : b \triangle c \in I \text{ for some } c \in C\}$.

Proof: Let $D = \{b \in B : b \triangle c \in I \text{ for some } c \in C\}$. First we show that D is a Boolean algebra. Let $b_1, b_2 \in D$. So there exist $c_1, c_2 \in C$ such that $b_1 \triangle c_1 \in I$ and $b_2 \triangle c_2 \in I$ Note that $(b_1 \vee b_2) \triangle (c_1 \vee c_2) \leq (b_1 \triangle c_1) \vee (b_2 \triangle c_2)$. Since I is an ideal in B, and C is a subalgebra of B $(b_1 \vee b_2) \triangle (c_1 \vee c_2) \in I$. Hence $b_1 \vee b_2 \in D$.

Next let $b \in D$. Then $b' \triangle c' = b \triangle c \in I$. Since $c' \in$ we have $b' \in D$. Obviously $0 \in D$ and hence D is a subalgebra of B.

Now, note that $C \subset D$ and $I \subset D$. Consequently, $D \supset B(I) \vee C$. On the other hand, let $b \in D$. There exists $c \in C$ such that $b \triangle c = d \in I$. Thus $b = d \triangle c$ with $d \in B(I)$ and $c \in C$. Hence $b \in B(I) \vee C$.

Lemma 3.2.7. Let C be a subalgebra of B such that $B(I) \wedge C = \{0, 1\}$. Then given $b \in B(I) \vee C$, there exists a unique $c \in C$ such that $b \triangle c \in I$.

Proof: Existence of at least one $c \in C$ such that $b \triangle c \in I$ is guaranteed by the previous lemma. Suppose c_1 and $c_2 \in C$

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

and $b \Delta c_1$ and $b \Delta c_2 \in I$. Then $c_1 \Delta c_2 \in I$ and $c_1 \Delta c_2 \in C$. Since $B(I) \wedge C = \{0, 1\}$ we have $c_1 \Delta c_2 = 0$ or 1. Since I is a proper ideal $c_1 \Delta c_2 = 0$ i.e., $c_1 = c_2$.

Theorem 3.2.8. Let B be a Boolean algebra, I an ideal in B and $B(I)$ be the subalgebra of B generated by I. Then the following statements are equivalent.

i) $B(I)$ has a complement in L_B.

ii) The natural homomorphism of B onto B/I admits a lifting.

iii) The Stone space Y of B/I which is a closed subset of X, the Stone space of B, is a retract of X.

Proof: (i) $\longrightarrow$ (ii). Let $h : B \longrightarrow B/I$ be the natural homomorphism, i.e., $h(b) = [b]$ for $b \in B$. Let C be a complement of $B(I)$ in L_B. We shall show that C is a lifting of h. So we have to show that h restricted to C is one to one and onto B/I.

Since $B(I) \vee C = B$, by Lemma 3.2.6, for any $b \in B$ there is a $c \in C$ such that $b \Delta c \in I$. So $h(c) = [c] = [b]$. Hence h restricted to C is onto. Since $B(I) \wedge C = \{0, 1\}$, by

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Lemma 3.2.3, there is a unique $c \in C$ such that $b \Delta c \in I$. Hence h restricted to C is one to one.

ii) $\longrightarrow$ (i). Let C be a lifting of the natural homomorphim $h : B \longrightarrow B/I$. We claim that C is a complement of $B(I)$.

Let $b \in C \wedge B(I)$. Either $b \in I$ or $b' \in I$. Suppose $b \in I$. So $h(b) = [b] = [0]$. Since h restricted to C is one to one and since $h(b) = h(0)$, we have $b = 0$. If $b' \in I$ then $h(b') = [b'] = [0]$. Again since h restricted to C is one to one and since $h(b') = h(0)$ we have $b' = 0$, i.e., $b = 1$. Hence $C \wedge B(I) = \{0, 1\}$.

Now let us prove that $C \vee B(I) = B$. Let $b \in B$. Since h restricted to C is onto B/I there is a $c \in C$ such that $h(c) = [b]$. i.e. $c \Delta b \in I$. By Lemma 3.2.3 $b \in B(I) \vee C$. Hence $B(I) \vee C = B$.

The equivalence of (ii) and (iii) was stated in Theorem 3.2.3.

<u>Remarks:</u> If $B(I)$ has a complement in L_B, any complement of $B(I)$ is isomorphic to B/I.

Hence if $B(I)$ has a complement in L_B, then any two complements of $B(I)$ are isomorphic. This statement is not true for any subalgebra. As an example we have

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$$X = \{1, 2, 3, 4\}, \ B = P(X), \ C = \{\emptyset, X, \{1, 2\}, \{3, 4\}\},$$
$$D = \{\emptyset, X, \{1, 4\}, \{2, 3\}\} \text{ and } E = \{\emptyset, X, \{1\}, \{3\},$$
$$\{2, 4\}, \{1,3\}, \{1,2,4\}, \{3,2,4\}\}.$$

Then D and E both are complements of C. But D and E are not isomorphic.

Now we give an example of a Boolean algebra B and a subalgebra A of B such that A has no complement in L_B.

Theorem 3.2.9. Let N be the set of natural numbers and P(N) the power set of N. Let C be the field of all finite cofinite subsets of N. Then C has no complement in $L_{P(N)}$.

Proof: Observe that C is the field generated by the ideal I of all finite subsets of N. If C were to admit a complement in $L_{P(N)}$ then, by Theorem 3.2.4, the Stone space $\beta N - N$, of the quotient Boolean algebra P(N)/I would be a retract of βN, the Stone space of P(N). But $\beta N - N$ is not a retract of βN. See Gillman and Jerison [8, 6Q, p. 97].

In the next two sections, we study certain classes of Boolean algebras in the light of complementation of Boolean algebras.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

3. C_1 - Boolean algebras.

In this section, we introduce a new class of Boolean algebras.

Definition 3.3.1. A Boolean algebra B is said to be a C_1-Boolean algebra if every subalgebra of B has a complement in L_B

Theorem 3.3.2. Every finite Boolean algebra is a C_1 - Boolean algebra.

Proof: This follows from Theorem 3.2.5.

Theorem 3.3.3. Let B and D be two Boolean algebras such that D is a homomorphic image of B. If B is a C_1 - Boolean algebra so is D.

Proof: Let $h : B \longrightarrow D$ be a homomorphism, mapping B onto D. Let E be a subalgebra of D. Let $F = \{b \in B : h(b) \in E\}$. It is easy to verify that F is a subalgebra of B. Let G be a complement of F in L_B. Now, we claim that h(G) is a complement of E in L_D. Let $d \in E \wedge h(G)$. There exist $b_1 \in$ $b_2 \in G$ such that $h(b_1) = d = h(b_2)$. Clearly, $b_2 \in F$. Since $F \wedge G = \{0, 1\}$, $b_2 = 0$ or 1. Hence $d = 0$ or 1. So,

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

we have $E \wedge h(G) = \{0, 1\}$. Since F and G generate B, $h(F)$ and $h(G)$ generate $h(B)$. But $h(F) = E$, $h(B) = D$. Hence $E \vee h(G) = D$.

Theorem 3.3.4. Let B be any infinite Boolean σ-algebra. Let N be the set of all natural numbers and $P(N)$ the power set of N. Then $P(N)$ is a homomorphic image of B.

Proof: Let $b_1, b_2, \ldots$ be a sequence of nonzero, pairwise disjoint elements in B such that $b_1 \vee b_2 \vee \ldots = 1$. Let $F_1, F_2, \ldots$ be a sequence of maximal filters in B containing $b_1, b_2, \ldots$ respectively. Define $h : B \longrightarrow P(N)$ as follows.
$h(b) = \{n: b \in F_n\}$. Clearly $h(0) = \emptyset$ and $h(1) = N$.

$$h(a_1 \vee a_2) = \{n : a_1 \vee a_2 \in F_n\} = \{n : a_1 \in F_n\} \cup \{n : a_2 \in F_n\}.$$

For, let $n \in$ LHS. $a_1 \vee a_2 \in F_n$. Then either $a_1 \in F_n$ or $a_2 \in F_n$. If not, $a_1 \notin F_n$, $a_2 \notin F_n \implies a_1' \in F_n$, $a_2' \in F_n \implies a_1' \wedge a_2' \in F_n \implies (a_1 \vee a_2) \wedge (a_1' \wedge a_2') = 0 \in F_n$, which is a contradiction. If $n \in$ RHS, it is obvious that $n \in$ LHS. Similarly, we can show that $h(a_1 \wedge a_2) = h(a_1) \cap h(a_2)$. Next, we claim that h is onto. Let $N_1 = \{n_1, n_2, \ldots\}$ be a subset of N. Then, $h(b_{n_1} \vee b_{n_2} \vee \ldots) = N_1$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Corollary 3.3.5: Let B be any infinite Boolean σ-algebra. Then B is not a C_1 - Boolean algebra.

Proof: If B were to be a C_1 - Boolean algebra, then P(N), by Theorem 3.3.3, would be a C_1 - Boolean algebra. By Theorem 3.2.9, P(N) is not a C_1 - Boolean algebra.

Corollary 3.3.6. Let X be any infinite set. Then the power set P(X) of X is not a C_1 - Boolean algebra.

Corollary 3.3.7. Let X be any set. Then P(X) is a C_1 - Boolean algebra if and only if X is a finite set.

The trend of these results creates a gloomy picture concerning the existence of good C_1 - Boolean algebras. The following result somewhat retrieves the situation.

Theorem 3.3.8. Let X be any set and $\underline{\underline{C}}$ the field of all finite co-finite subsets of X. Then $\underline{\underline{C}}$ is a C_1 - Boolean algebra.

Proof: Since $\underline{\underline{C}}$ is a superatomic Booleam algebra, every subalgebra $\underline{\underline{D}}$ of $\underline{\underline{C}}$ is atomic. See Sikorski [27, example D, p.35].

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Case (i) : One of the atoms of $\underline{\underline{D}}$ is cofinite. Then $\underline{\underline{D}}$ is a finite algebra. Hence $\underline{\underline{D}}$ has a complement in $L_{\underline{\underline{C}}}$, by Theorem 3.2.5.

Case (ii) : Every atom of $\underline{\underline{D}}$ is finite. Let $\{D_\alpha : \alpha \in \Gamma\}$ be the collection of all atoms of $\underline{\underline{D}}$. Choose and fix one element $x_\alpha \in D_\alpha$. Let $\underline{\underline{E}}$ be the subfield of $\underline{\underline{C}}$ generated by $\{\{x\} : x \in X \quad x \neq x_\alpha$ for any $\alpha\}$. If M is the set of all x_α's, $\underline{\underline{E}} = \{A \subset X : A \cap M = \emptyset$ and A is finite or $A \supset M$ and A is cofinite$\}$. Now we claim that $\underline{\underline{E}}$ is a complement of $\underline{\underline{D}}$. Let $H \in \underline{\underline{D}} \wedge \underline{\underline{E}}$. H is either finite or cofinite. Assume H is finite. Since $H \in \underline{\underline{D}}$, then $H = D_{\alpha_1} \cup D_{\alpha_2} \cup \ldots \cup D_{\alpha_n}$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n$ in Γ. Therefore, $H = (D_{\alpha_1} - \{x_{\alpha_1}\}) \cup (D_{\alpha_2} - \{x_{\alpha_2}\}) \ldots \cup (D_{\alpha_n} - \{x_{\alpha_n}\}) \cup \{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}\}$. Each $D_{\alpha_i} - \{x_{\alpha_i}\} \in \underline{\underline{E}}$. Since $H \in \underline{\underline{E}}$, we get $\{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}\} \in \underline{\underline{E}}$. This shows that $H = \emptyset$. In the case when H is cofinite, we can prove that, by the same argument given above, $H' = \emptyset$. Hence $H = X$. This, we have $\underline{\underline{D}} \wedge \underline{\underline{E}} = \{\emptyset, X\}$. Now, we claim that $\underline{\underline{D}} \vee \underline{\underline{E}} = \underline{\underline{C}}$. For thus, it is enough if we show that every $\{x\} \in \underline{\underline{D}} \vee \underline{\underline{E}}$. Every $\{x_\alpha\} \in$ $\in \underline{\underline{D}} \vee \underline{\underline{E}}$. For, $D_\alpha \in \underline{\underline{D}}$, $\{x \in D_\alpha : x \neq x_\alpha\} \in \underline{\underline{E}}$, so

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$\{x_\alpha\} \in \underline{\underline{D}} \vee \underline{\underline{E}}$. Every $\{x\}$ such that $x \neq x_\alpha$ for any α belongs to $\underline{\underline{E}}$. Hence $\underline{\underline{D}} \vee \underline{\underline{E}} = \underline{\underline{C}}$.

Remark 1: A similar proof can be given to show that $L_{\underline{\underline{C}}}$ for the above $\underline{\underline{C}}$ is a relatively complemented lattice (i.e., $\underline{\underline{B}}$, $\underline{\underline{D}}$, $\underline{\underline{E}} \in L_{\underline{\underline{C}}}$ such that $\underline{\underline{B}} < \underline{\underline{D}} < \underline{\underline{E}}$ implies there exists $\underline{\underline{F}} \in L_{\underline{\underline{C}}}$ such that $\underline{\underline{D}} \wedge \underline{\underline{F}} = \underline{\underline{B}}$ and $\underline{\underline{D}} \vee \underline{\underline{F}} = \underline{\underline{E}}$).

Remark 2: We do not know if every superatomic Boolean algebra is a C_1-Boolean algebra.

4. C_2 - Boolean algebras

Definition 3.4.1. A Boolean algebra B is said to be a C_2 - Boolean algebra if for every ideal I in B, B(I) has a complement in L_B.

Theorem 3.4.2: Let B be a Boolean algebra and X its Stone space. Then B is a C_2 - Boolean algebra if and only if every closed subset of X is a retract of X.

Proof. This follows from Theorem 3.2.8.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Corollary 3.4.3: Every countable Boolean algebra B is a C_2 - Boolean algebra.

Proof: The Stone space X of B is a compact totally disconnected metric space and for such a space X, every closed subset of X is a retract of X. See the Last paragraph in Sikorski [27, p. 46] or Kelley [13, 0, p. 165]. This Corollary can also be obtained from von Neumann - Stone's Theorem 17 of [29, p. 369] which is stated as Theorem 3.6.7 below.

Corollary 3.4.4. Let B and D be two Boolean algebras such that D is a homomorphic image of B. If B is a C_2 - Boolean algebra so also is D.

Proof. Let X and Y be the Stone spaces of B and D respectively. Y is a closed subspace of X. Since every closed subspace of X is a retract of X, so also every closed subspace of Y is.

Definition 3.4.5. Let B and D be two Boolean algebras. The direct Union B + D of B and D is defined to be the product space B x D with pointwise operations. B+ D is also a Boolean algebra. See Sikorski [27, Section 16, p. 50]. Countable direct Union of Boolean algebras is defined in an analogous way.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

If X and Y are the Stone spaces of B and D respectively, then the Stone space of $B + D$ is the disjoint union $X \cup Y$ of X and Y equipped with union topology.

<u>Corollary 3.4.6.</u> Let B and D be two C_2 - Boolean algebras. Then $B + D$ is also a C_2 - Boolean algebra.

<u>Proof:</u> Let X and Y be the Stone spaces of B and D respectively. If every closed subset of X is a retract of X and every closed subset of Y is a retract of Y, so also every closed subset of $X \cup Y$ is.

<u>Remark:</u> The above Corollary does not extend to countable direct Unions because $P(N)$ is a countable direct Union of two element Boolean algebras and $P(N)$ is not a C_2 - Boolean algebra.

<u>Theorem 3.4.7.</u> No infinite Boolean σ-algebra B is a C_2 -Boolean algebra.

<u>Proof.</u> If B were to be a C_2 - Boolean algebra, then, by Corollary 3.4.4. and Theorem 3.3.4, $P(N)$ would be a C_2 - Boolean algebra.

<u>Corollary 3.4.8.</u> Let X be any infinite set. Then the power

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

set $P(X)$ of X is not a C_2 - Boolean algebra.

Corollary 3.4.9. Let X be any set. Then the power set $P(X)$ of X is a C_2 - Boolean algebra if and only if X is a finite set.

Theorem 3.4.10. Let α be any ordinal number. Let $\underline{\underline{C}}$ be the field of all clopen subsets of $[0, \alpha]$, where $[0, \alpha]$ is equipped with order topology. Then $\underline{\underline{C}}$ is a C_2 - Boolean algebra.

Proof. It is enough if we show that every closed subset H of $[0, \alpha]$ is a retract of $[0, \alpha]$. The idea of the proof is taken from some observations in the proof of Lemma 1 of Bhaskara Rao and Bhaskara Rao [2, p. 195].

We shall denote $[0, \alpha]$ by X. We have to define a map f from X onto H which is identity on H. Let α_o be the first element of H. Since H is closed there exists a last element which we call α_1. For $\beta \in H$ let β' be the first succeeding element of β in H. Since H is closed, for any $x \in [\alpha_o, \alpha_1] - H$ there exists a $\beta \in H$ such that $\beta < x < \beta'$. (Such a β is unique.) We define f as follows

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$$
\begin{aligned}
f(x) &= \alpha_0 && \text{if } 0 \leq x < \alpha_0 \\
&= \alpha_1 && \text{if } \alpha_1 < x \leq \alpha \\
&= x && \text{if } x \in H \\
&= \beta' && \text{if } x \in [\alpha_0, \alpha_1] - H
\end{aligned}
$$

where β is the element of H such that $\beta < x < \beta'$.

To conclude that the f defined above is continuous it is sufficient to show that

for any well ordered transfinite sequence x_i increasing to x_0 $f(x_i)$ converges to x_0.

This we prove as follows.

<u>If $x_0 \leq \alpha_0$,</u> $f(x_i) = \alpha_0 = f(x_0)$ for all x_i. Hence $f(x_i)$ converges to $f(x_0)$.

<u>If $\alpha_1 < x_0$,</u> x_i is eventually greater than α_1 and hence $f(x_i)$ is eventually equal to α_1 which is equal to $f(x_0)$.

<u>If $x_0 \in [\alpha_0, \alpha_1] - H$,</u> there exists $\beta \in H$ such that $\beta < x_0 < \beta'$ and $f(x_0) = \beta'$. Then x_i is eventually in the interval (β, β') and hence $f(x_i)$ is eventually equal to $\beta' = f(x_0)$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

If $x_o \in H$ and if there exists $\beta \in H$ with $\beta' = x_o$, $f(x_o) = \beta'$ and $(\beta, \beta']$ is an open neighbourhood of x_o. Hence eventually $x_i > \beta$. Hence $f(x_i) = \beta'$ eventually.

If $x_o \in H$, $x_o \neq \alpha_o$ and if there does not exist $\beta \in H$ such that $\beta' = x_o$, then for any $\beta \in H$ such that $\beta < x_o$, β' is also $< x_o$. If $x \in X - H$ and $x < x_o$ since H is closed, there exists $\beta \in H$ such that $x < \beta < x_o$. So x_i is eventually in $(\beta, x_o]$. So $f(x_i)$ is eventually in $(\beta, x_o]$. So $f(x_o)$ is eventually in $(x, x_o]$. Hence $f(x_i)$ converges to $f(x_o)$.

Thus f is continuous and identity on H. Hence H is a retract of X.

5. Complementation in general fields.

B. V. Rao [23] considered the following problem. Let X be any arbitrary set. Let $L^{\sigma}_{P(X)}$ be the collection of all sub σ-fields of $P(X)$. $L^{\sigma}_{P(X)}$ is a complete lattice with $\{\emptyset, X\}$ as the first element and $P(X)$ as the last element. For any family $\underline{\underline{C}}_{\alpha}$ of sub σ-fields of $P(X)$, $\bigvee_{\alpha} \underline{\underline{C}}_{\alpha}$ is the sub σ-field of $P(X)$ generated by the family $\underline{\underline{C}}_{\alpha}$, and $\bigwedge_{\alpha} \underline{\underline{C}}_{\alpha} = \bigcap_{\alpha} \underline{\underline{C}}_{\alpha}$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$L^{\sigma}_{P(X)}$ is said to be complemented if for every element $\underline{\underline{B}}$ in $L^{\sigma}_{P(X)}$ there is an element $\underline{\underline{D}}$ in $L^{\sigma}_{P(X)}$ such that $\underline{\underline{B}} \wedge \underline{\underline{D}} = \{\emptyset, X\}$ and $\underline{\underline{B}} \vee \underline{\underline{D}} = P(X)$. B. V. Rao proved that if X is uncountable, then $L^{\sigma}_{P(X)}$ is not complemented. In effect he showed that the countable-cocountable σ-field on X has no complement in $L^{\sigma}_{P(X)}$. In this section, we give two simple proofs of this result, one based on measure theory and the other on set theory.

Let the cardinality of X be $\aleph_1$, Aleph-one. Let $\underline{\underline{C}}$ be the countable cocountable σ-field on X. Suppose $\underline{\underline{C}}$ has a complement $\underline{\underline{D}}$ in $L^{\sigma}_{P(X)}$. Then $\underline{\underline{C}} \vee \underline{\underline{D}} = P(X) = \{Y \subset X : Y \triangle D$ is countable for some $D \in \underline{\underline{D}}\}$. This may be proved along the lines of the proof of Lemma 3.2.6. Moreover, for every $Y \subset X$, there exists unique $D \in \underline{\underline{D}}$ such that $Y \triangle D$ is countable. A proof of this result is similar to that given for Lemma 3.2.7. Let μ be any measure on $\underline{\underline{D}}$. We can define a measure μ on $P(X)$ which is an extension of μ by the following formula. $\mu(Y) = \mu(D)$, where D is the unique set in $\underline{\underline{D}}$ such that $Y \triangle D$ is countable. Observe that μ is always a continuous measure, i.e., vanishes for singleton sets, whatever be the measure μ on $\underline{\underline{D}}$. For, for every $\{x\}$, $x \in X$, empty set $\emptyset$ in $\underline{\underline{D}}$ satisfies the property $\{x\} \triangle \emptyset$ is countable. So,

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$\mu(\{x\}) = \mu(\emptyset) = 0$. So, if we start with a 0-1 valued measure μ on $\underline{\underline{D}}$, μ is a 0-1 valued continuous measure on $P(X)$. This is a contradiction to Ulam's theorem which states that there is no continuous probability measure on $P(X)$. For a new proof of Ulam's theorem, see Bhaskara Rao and Bhaskara Rao [2, p. 196]. In Section 6 we use the above argument to prove a more general theorem.

Set theoretic proof of the above result is included in the more general Theorem 3.5.3 to be proved later in this section.

Let k, λ be any two candinal numbers. Let X be any set of cardinality λ. By L_λ^k we denote the collection of all k-fields on X. A collection $\underline{\underline{D}} \subset P(X)$ is said to be a k-field if $\underline{\underline{D}}$ is closed under complementation and $< k$ many unions. L_λ^k is complete under the following lattice operations. If $\underline{\underline{D}}_\alpha$ is a family of k-fields contained in $P(X)$, $\vee\, \underline{\underline{D}}_\alpha$ is defined to be the smallest sub- k-field of $P(X)$ generated by $\{\underline{\underline{D}}_\alpha\}$ and $\wedge\, \underline{\underline{D}}_\alpha = \bigcap_\alpha \underline{\underline{D}}_\alpha$. In this terminalogy, every σ-field is an $\aleph_1$-field, and every field is an $\aleph_0$-field. Further, $L^\sigma_{P(X)} = L_\lambda^{\aleph_1}$ and $L_{P(X)} = L_\lambda^{\aleph_0}$. Let Y be a set of cardinality k. k is said to be regular if $\aleph_0 \leq k$ and there does not exist a decomposition $\{Y_i, i \in \Gamma\}$ of Y such that cardinality

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

of each Y_i is $< k$ and cardinality of Γ is $< k$. For any cardinal number k, k^+, its succeeding cardinal is regular. This observation and the following Theorem reduces the problem of complementation in L_λ^k for any k to the problem of complementation for the case of regular cardinals, k.

Theorem 3.5.1. If k is not regular, then

$$L_\lambda^k = L_\lambda^{k^+} .$$

Proof. It is clear that $L_\lambda^{k^+} \subset L_\lambda^k$. Let $\underline{\underline{C}}$ be any k-field. We will show that $\underline{\underline{C}}$ is also a k^+ - field. Let $C_i : i \in \Gamma$ be any collection of sets such that the cardinality of $\Gamma < k^+$. It is enough if we treat the case the cardinality of Γ is k to show that $\bigcup_{i \in \Gamma} C_i \in \underline{\underline{C}}$. Since k is not regular, we can write $\Gamma = \bigcup_{i \in J} \Gamma_i$, where cardinality of each Γ_i is less than k, cardinality of J is $< k$ and Γ_i's are disjoint. Now,

$$\bigcup_{i \in \Gamma} C_i = \bigcup_{i \in J} \bigcup_{j \in \Gamma_i} C_j \quad . \quad \text{But} \quad \bigcup_{j \in \Gamma_i} C_j \in \underline{\underline{C}}$$

for every i and cardinality of J is $< k$ implies $\bigcup_{i \in \Gamma} C_i \in \underline{\underline{C}}$. Hence $\underline{\underline{C}}$ is also a k^+-field. Thus, we have $L_\lambda^k = L_\lambda^{k^+}$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Theorem 3.5.2. If $\lambda < k$, then every element $\underline{\underline{C}}$ in L_λ^k has a complement $\underline{\underline{D}}$ in L_λ^k.

Proof. First, observe that every element of L_λ^k is a complete field. Hence every element of L_λ^k is atomic. See Sikorski [27, p. 105]. Let $C_\alpha : \alpha \in \Gamma$ be the collection of all atoms of $\underline{\underline{C}}$. Choose and fix one element $x_\alpha \in C_\alpha$. Let $\underline{\underline{D}}$ be the k-field generated by $\left\{\{x\} : x \neq x_\alpha \text{ for any } \alpha\right\}$. It is easy to verify that $\underline{\underline{D}}$ is a complement of $\underline{\underline{C}}$.

Theorem 3.5.3. Let k be a regular cardinal number. If $\lambda = k$, then there exists an element $\underline{\underline{C}}$ in L_λ^k such that $\underline{\underline{C}}$ has no complement in $L_{P(X)}$.

Proof. Let $\underline{\underline{C}} = \left\{A \subset X : \text{cardinality of } A \text{ is} < k \text{ or cardinality of } A' \text{ is} < k\right\}$. Since k is regular, $\underline{\underline{C}} \in L_\lambda^k$. In proving that $\underline{\underline{C}}$ is closed under $< k$ many unions, we use the regularity of k. Suppose $\underline{\underline{D}}$ is a complement of $\underline{\underline{C}}$ in $L_{P(X)}$. Let $\underline{\underline{I}}$ be the collection of all subsets of X of cardinality $< k$. Then $P(X)/\underline{\underline{I}}$ is isomorphic to $\underline{\underline{D}}$ by Theorem 3.2.8. By the following theorem of Sierpinski (26, Theorem 1, p. 451) -

Let X be a set of cardinality k. Then there exists a family

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$\{X_\alpha : \alpha \in \Gamma\}$ of subsets of X such that cardinality of each X_α is k, cardinality of Γ is $> k$ and cardinality of $X_\alpha \cap X_\beta$ is $< k$ for every $\alpha \neq \beta$.

There are more than k disjoint nonzero elements in $P(X)/\underline{\underline{I}}$, namely the family $\{[X_\alpha] : \alpha \in \Gamma\}$. But, $\underline{\underline{D}}$ being a subfield of $P(X)$, $\underline{\underline{D}}$ can contain atmost k many pairwise nonzero disjoint elements. Hence $\underline{\underline{D}}$ can not be isomorphic to $P(X)/\underline{\underline{I}}$. This contradiction proves the result.

Corollary 3.5.4. Let k be a regular cardinal. If $\lambda = k$, then there exists an element $\underline{\underline{C}}$ in L_λ^k such that $\underline{\underline{C}}$ does not have a complement in L_λ^k.

Proof. Let $\underline{\underline{C}}$ and $\underline{\underline{I}}$ be as defined in the proof of previous theorem. It is clear that $P(X)(\underline{\underline{I}}) = \underline{\underline{C}}$. Suppose $\underline{\underline{C}}$ has a complement $\underline{\underline{D}}$ in L_λ^k, i.e., $\underline{\underline{D}}$ is a k-field, the smallest k-field containing $\underline{\underline{C}}$ and $\underline{\underline{D}}$ is $P(X)$ and $\underline{\underline{C}} \cap \underline{\underline{D}} = \{\emptyset, X\}$. We shall show that the smallest field generated by $\underline{\underline{C}}$ and $\underline{\underline{D}}$ itself is equal to $P(X)$.

Smallest field containing $\underline{\underline{C}}$ and $\underline{\underline{D}} = \underline{\underline{C}} \vee \underline{\underline{D}}$ in $L_{P(X)}$

$$= P(X)(\underline{\underline{I}}) \vee \underline{\underline{D}}$$

$$= \{A \in P(X) : A \triangle B \in \underline{\underline{I}} \text{ for some } B \in \underline{\underline{D}}\},$$

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

by Lemma 3.2.6. Since $\underline{\underline{I}}$ is a k-ideal and $\underline{\underline{D}}$ is a k-field $\{A \in P(X) : A \triangle B \in \underline{\underline{I}}$ for some $B \in \underline{\underline{D}}\}$ is closed under all $< k$ unions. Hence $P(X)(\underline{\underline{I}}) \vee \underline{\underline{D}}$ is a k-field. Hence $\underline{\underline{C}} \vee \underline{\underline{D}}$ in $L_{P(X)} = P(X)$. Hence $\underline{\underline{D}}$ is a complement of $\underline{\underline{C}}$ in $L_{P(X)}$. This gives rise to a contradiction to Theorem 3.5.3.

<u>Theorem 3.5.5.</u> Let k be a regular cardinal number. If $\lambda > k$, then there exists an element $\underline{\underline{C}}$ in L_λ^k which has no complement in L_λ^k.

<u>Proof.</u> Let Y be a set of cardinality k and X be a set of cardinality λ such that $Y \subset X$. By Corollary 3.5.4, $L^k_{P(Y)}$ contains an element which has no complement in $L^k_{P(Y)}$. It is not difficult to see that $L^k_{P(X)}$ contains an element which has no complement in $L^k_{P(X)}$.

Combining the previous theorems, we have the following result.

<u>Theorem 3.5.6.</u> Let k be a regular cardinal number and λ any cardinal number.

i) If $\lambda < k$, then every element in L_λ^k has a complement.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

ii) If $\lambda \geq k$, then there exists an element in L_λ^k which has no complement in L_λ^k.

<u>Remark:</u> (i) When $\lambda = \aleph_1 = k$, $L_\lambda^k = L_{P(X)}^\sigma$ contains an element which has no complement in $L_{P(X)}^\sigma$ furnishing a set theoretic proof of B. V. Rao's result quoted earlier.

(ii) The above results strengthen B. V. Rao's results is several directions.

(iii) The problem considered in Theorem 3.5.6 was suggested by B. V. Rao.

Combining Theorem 3.5.6 and Theorem 3.5.1 we have

<u>Theorem 3.5.7.</u> Let λ and k be any cardinal numbers.

i) If $\lambda < k$, every element in L_λ^k has a complement in L_λ^k.

ii) If $\lambda > k$, there exists an element in L_λ^k which has no complement in L_λ^k.

iii) If $\lambda = k$ and k is not regular, every element in L_λ^k has a complement in L_λ^k.

iv) If $\lambda = k$ and k is regular, there exists an element in L_λ^k which has no complement in L_λ^k.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

6. Some Complements to the Complementation Problem

In this section we consider the complementation problem in the light of sub σ-fields and sub-fields for specific examples.

Firstly, the argument given in second paragraph of section 5 of this chapter gives the following Theorem.

Theorem 3.6.1. Let $\underline{\underline{A}}$ be a σ-field of subsets of a set X containing singletons and satisfying the following property: There is no nonzero continuous 0-1 valued measure on $\underline{\underline{A}}$. Let $\underline{\underline{I}}$ be any σ-ideal in $\underline{\underline{A}}$ which contains all singletons. Then $\underline{\underline{A}}(\underline{\underline{I}})$, the σ-field generated by $\underline{\underline{I}}$ on X, has no complement in $L^{\sigma}_{\underline{\underline{A}}}$, the lattice of all sub σ-fields of $\underline{\underline{A}}$.

Proof. Suppose $\underline{\underline{A}}(\underline{\underline{I}})$ has a complement $\underline{\underline{B}}$ in $L^{\sigma}_{\underline{\underline{A}}}$. Then the σ-field $\underline{\underline{B}}$ is isomorphic to the quotient Boolean σ-algebra $\underline{\underline{A}}/\underline{\underline{I}}$. The argument is similar to the one given in the proof of Theorem 3.2.8. Since $\underline{\underline{B}}$ is a σ-field of sets there is a 0-1 valued measure on $\underline{\underline{B}}$. In fact, any degenerate measure would do. Consequently, there is a nonzero 0-1 valued measure μ on the Boolean σ-algebra $\underline{\underline{A}}/\underline{\underline{I}}$. This measure μ can be lifted as a nonzero measure μ on $\underline{\underline{A}}$. Since $\underline{\underline{I}}$ contains all singletons,

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

μ is continuous. This contradiction shows that $\underline{\underline{A}}(\underline{\underline{I}})$ has no complement in $L^{\sigma}_{\underline{\underline{A}}}$.

From this theorem, we reap a harvest of corollaries.

Corollary 3.6.2. The conclusion of Theorem 3.6.1 is True if one takes $\underline{\underline{A}} = P(X)$, the cardinality of X being non-measurable (i.e., there is no nonzero 0-1 valued continuous measure on $P(X)$).

Corollary 3.6.3. The conclusion of Theorem 3.6.1 is True if one takes $\underline{\underline{A}}$ to be any separable σ-field containing all singletons.

One can easily derive a version of Corollary 3.6.3 for general separable σ- fields with atoms playing the role of singletons.

Corollary 3.6.4: The conclusion of Theorem 3.6.1 is True if one takes X to be the real line R, $\underline{\underline{A}}$ to be the Borel σ-field on R and $\underline{\underline{I}}$ to be the σ-ideal of all Borel null sets (with respect to Lebesgue measure) or the σ-ideal of all Borel first category subsets of R.

Now, we generalise Corollary 3.6.4 to general measure spaces and to general topological spaces with suitable modifications.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Let $(X, \underline{\underline{A}}, \mu)$ be a measure space where μ is σ-finite and $\underline{\underline{A}}$ is complete with respect to μ (i.e., if A is a μ-null set, then any subset of A is in $\underline{\underline{A}}$). Let $\underline{\underline{I}}_{\mu}$ be the ideal of all μ-null sets in $\underline{\underline{A}}$.

<u>Theorem 3.6.5.</u> $\underline{\underline{A}}(\underline{\underline{I}}_{\mu})$, the sub σ-field of $\underline{\underline{A}}$ generated by the σ-ideal $\underline{\underline{I}}_{\mu}$, has a complement in $L^{\sigma}_{\underline{\underline{A}}}$ if and only if μ is completely atomic.

<u>Proof.</u> Suppose μ is completely atomic. Let $A_1, A_2, \ldots$ be a collection of sets in $\underline{\underline{A}}$ with the following properties. 1) A_i's are all pairwise disjoint. 2) $\bigcup_{i \geq 1} A_i = X$. 3) Each A_i is a μ-atom. Since μ is σ-finite $\mu(A_i)$ is finite for each i. Given any $A \in \underline{\underline{A}}$ with $\mu(A) > 0$, we can find unique sub-collection $A_{i_1}, A_{i_2}, \ldots$ such that $\mu(A \,\Delta \bigcup_{k \geq 1} A_{i_k}) = 0$. Let $\underline{\underline{B}}$ be the sub σ-field of $\underline{\underline{A}}$ generated by $A_1, A_2, \ldots$. Then $\underline{\underline{B}}$ is a lifting for the natural homomorphism $h : \underline{\underline{A}} \rightarrow \underline{\underline{A}}/\underline{\underline{I}}_{\mu}$, i.e., $h(A) = [A]_{\mu}$. Hence $\underline{\underline{B}}$ is a complement of $\underline{\underline{A}}(\underline{\underline{I}}_{\mu})$.

Conversely, suppose $\underline{\underline{A}}(\underline{\underline{I}}_{\mu})$ has a complement $\underline{\underline{B}}$ in $L^{\sigma}_{\underline{\underline{A}}}$. Consequently $\underline{\underline{B}}$ is isomorphic to the quotient Boolean σ-algebra $\underline{\underline{A}} / \underline{\underline{I}}_{\mu}$. Since μ is σ-finite, $\underline{\underline{A}}/\underline{\underline{I}}_{\mu}$ satisfies countable

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

chain condition. Hence the σ-field $\underline{\underline{B}}$ also satisfies countable chain condition. So, $\underline{\underline{B}}$ is isomorphic to $P(N)$, where N is set of nonnegative integers. The σ-finite measure μ can be transferred, in a natural way, to $P(N)$ as a strictly positive σ-finite measure. And any such measure on $P(N)$ is completely atomic. See Theorem 4.5.1. This completely proves the Theorem.

The above theorem raises the following natural question. Does $\underline{\underline{A}}(\underline{\underline{I}}_{\mu})$ have a complement in the bigger Lattice $L_{\underline{\underline{A}}}$, the lattice of all sub-fields (same as Boolean sub-algebras) of $\underline{\underline{A}}$? An affirmative answer to this question follows from a theorem of Maharam [16] which we quote below.

<u>Theorem 3.6.6.</u> (Maharam). For any measure space $(X, \underline{\underline{A}}, \mu)$ with μ, a σ-finite complete measure, there exists a field of sets $\underline{\underline{C}} \subset \underline{\underline{A}}$ which is a lifting of the natural homomorphism from $\underline{\underline{A}}$ onto $\underline{\underline{A}}/\underline{\underline{I}}_{\mu}$.

The following question remains open. Let $(X, \underline{\underline{A}}, \mu)$ be a charge space. Does $\underline{\underline{A}}(\underline{\underline{I}}_{\mu})$ have a complement in $L_{\underline{\underline{A}}}$?

Now we examine the problem how far the assumption of completeness is essential in Theorem 3.6.6. Below we shed some light on this aspect. For this we need the following theorem.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

<u>Theorem 3.6.7.</u> (von Neumann and Stone [29, Theorem 17, p. 369] and Theorem 15, p. 367]).
Let $\mathbb{A}$ be a Boolean algebra and let I be an ideal in $\mathbb{A}$ with the following property.

(*) for every $J \subset I$ with cardinality of J < cardinality of $\mathbb{A}/I$, there exists $a \in \mathbb{A}$ which is supremum of all elements in J.

Then we can find a subalgebra B of $\mathbb{A}$ which is a lifting of the natural homomorphism from $\mathbb{A}$ onto $\mathbb{A}/I$.

<u>Remark:</u> In fact, by the above theorem, B is a complement of $\mathbb{A}(I)$, the Boolean subalgebra of $\mathbb{A}$ generated by I, in $L_{\mathbb{A}}$.

<u>Theorem 3.6.8.</u> Assume continuum Hypothesis. Let $\underline{\underline{A}}$ be a countably generated σ-field on a set X. Let $\underline{\underline{I}}$ be any σ-ideal in $\underline{\underline{A}}$. Then $\underline{\underline{A}}(\underline{\underline{I}})$ has a complement in $L_{\underline{\underline{A}}}$.

<u>Proof.</u> Since $\underline{\underline{A}}$ is countably generated, cardinality of $\underline{\underline{A}}$ is less than or equal to c(cardinality of the continuum). Consequently cardinality of $\underline{\underline{A}} / \underline{\underline{I}}$ is $\leq c$. Since $\underline{\underline{I}}$ is a σ-ideal, continuum Hypothesis implies (*) of Theorem 3.6.7 is satisfied. This completes the proof.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Remarks. The proof of the above theorem goes through for any σ-field $\underline{\underline{A}}$ of cardinality $\leq c$. The above theorem covers Borel σ-field of the real line, Borel null sets and Borel σ-field of the real line, Borel first category sets.

Now, we turn our attention to the topological case. Let $(X, \mathcal{T})$ be a topological space. A subset B of X is said to have the property of Baire if we can find an open set A such that $B \,\Delta\, A$ is of first category in X. Let $\underline{\underline{B}}$ be the collection of all subsets of X with the property of Baire. Then $\underline{\underline{B}}$ is a σ-field on X. See Oxtoby [21, Theorem 4.3, p. 19]. Let $\underline{\underline{I}}$ be the σ-ideal of all first category subsets of X. It is clear that $\underline{\underline{B}}$ is complete with respect to $\underline{\underline{I}}$, i.e., $B \in \underline{\underline{I}}$, C is a subset of B, implies $C \in \underline{\underline{B}}$. With this set up the following two problems arise

1) Does $\underline{\underline{B}}(\underline{\underline{I}})$, the sub σ-field of $\underline{\underline{B}}$ generated by $\underline{\underline{I}}$, have a complement in $L^{\sigma}_{\underline{\underline{B}}}$?

We answer this question in the negative.

2) Does $\underline{\underline{B}}(\underline{\underline{I}})$ have a complement in $L_{\underline{\underline{B}}}$?

We answer this question in the affirmative.

Proposition 3.6.9. Let X be the real line equipped with the usual topology. Then $\underline{\underline{B}}(\underline{\underline{I}})$ has no complement in $L^{\sigma}_{\underline{\underline{B}}}$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Proof. Note that since $\underline{\underline{B}}$ contains the Borel σ-field, $\underline{\underline{B}}$ does not support a nonzero continuous 0-1 valued measure. Moreover $\underline{\underline{I}}$ contains singletons. Now the result follows from an application of Theorem 3.6.1.

To answer the second question we need the following theorem

Theorem 3.6.10. (von Neumann-Stone [29, Theorem 18, p. 372 and Theorem 15, p. 367])

Let A be a Boolean algebra and let I be an ideal in A satisfying the following property

(**) For any two nonvoid J_1, $J_2 \subset I$ such that cardinality of J_1 and cardinality of J_2 < cardinality of A/I and such that every element of $J_1 \leq$ every element of J_2, there exists $a \in A$ such that $c \leq a \leq d$ for every $c \in J_1$ and $d \in J_2$. Suppose there is a function $F : A \to A$ satisfying

1) $F(a) \triangle a \in I$ for every $a \in A$,

2) $a \triangle b \in I$ implies $F(a) = F(b)$ for any a and b in A,

3) $F(a \vee b) = F(a) \vee F(b)$ for $a, b \in A$.

Then we can find a subalgebra B of A such that B is a

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

lifting of the natural homomorphism from A onto A/I . The following theorem answers the second equation.

<u>Theorem 3.6.11.</u> $\underline{\underline{B}}(\underline{\underline{I}})$ has a complement in $L_{\underline{\underline{B}}}$.

<u>Proof.</u> We prove this theorem by using the above theorem. Obviously the ideal $\underline{\underline{I}}$ satisfies (**). Now, we define a function $F : \underline{\underline{B}} \longrightarrow \underline{\underline{B}}$ as follows.

For $A \in \underline{\underline{B}}$, $F(A) = \{x \in X$: for every open set V containing x, $V \cap A$ is not of first category in $X\}$.

Then F satisfies the following properties.

0) $F(A) \in \underline{\underline{B}}$ for every A in $\underline{\underline{B}}$

1) $F(A) \triangle A \in \underline{\underline{I}}$ for every A in $\underline{\underline{B}}$

2) $A, B \in \underline{\underline{B}}$, $A \triangle B \in \underline{\underline{I}}$ implies $F(A) = F(B)$

3) $F(A \cup B) = F(A) \cup F(B)$ for $A, B \in \underline{\underline{B}}$.

See [Kuratowski [15, pp. 83-85]].

Invoking Theorem 3.6.10 and Theorem 3.2.8, we get the result.

Finally we make a remark on a problem of B. V. Rao.

In [23, p. 215], B. V. Rao posed the problem of charactering sub σ-fields of the Borel σ-field $\underline{\underline{B}}$ of the real line which have complements in $L^{\sigma}_{\underline{\underline{B}}}$. He gives in [23, Theorem 3, p. 215] a

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

class of countably generated sub σ-fields which have complements in $L^{\sigma}_{\underline{\underline{B}}}$. Here we give another class.

We need the following theorem of Blackwell and Mackey. We shall state it in the form which is needed here. See Blackwell [4, Section 4] and Mackey [17, Section 4].

Let B be a Borel subset of the real line. Any countably generated sub σ-field on B of the Borel σ-field on B which separates points is the Borel σ-field on B.

<u>Theorem 3.6.12:</u> Any countably generated sub σ-field $\underline{\underline{C}}$ of $\underline{\underline{B}}$ in which every atom is countable has a complement in $L^{\sigma}_{\underline{\underline{B}}}$.

<u>Proof</u>. By a Theorem of Lusin (see [20, Section 9. p. 14]), we can get a Borel set B such that $B \cap C$ is a singleton for every atom C of $\underline{\underline{C}}$. Let $\underline{\underline{D}}$ be the σ-field defined by $\underline{\underline{D}} = \{ C \in \underline{\underline{B}} : C \cap B = \emptyset \text{ or } C \supset B \}$. Since B is a Borel set $\underline{\underline{D}}$ is a separable sub σ-field of $\underline{\underline{B}}$.

The σ-field $\underline{\underline{D}} \cap B^{c}$ is countably generated and separates points of B^{c}. By Blackwell-Mackey Theorem $\underline{\underline{D}} \cap B^{c}$ is Borel σ-field of B^{c}. $\underline{\underline{C}} \cap B$ is also countably generated and since B is a selection $\underline{\underline{C}} \cap B$ separates points of B. Again by Blackwell-Mackey theorem $\underline{\underline{C}} \cap B$ is Borel σ-field of B. Since

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

the σ-field generated by $\underline{\underline{D}}$ and $\underline{\underline{C}}$ contains B we conclude that the σ-field generated by $\underline{\underline{D}}$ and $\underline{\underline{C}}$ is the Borel σ-field $\underline{\underline{B}}$.

Since any set in $\underline{\underline{C}}$ which contains B is whole space it follows that $\underline{\underline{D}} \cap \underline{\underline{B}} = \{\emptyset, X\}$. Thus $\underline{\underline{D}}$ is a complement of $\underline{\underline{C}}$ in $L^{\sigma}_{\underline{\underline{B}}}$.

The same proof works to prove the following more general theorem.

<u>Theorem 3.6.13.</u> Any countably generated sub σ-field $\underline{\underline{C}}$ for which there is a Borel selection (i.e., a Borel set B with $B \cap C$ is a singleton for every atom C of $\underline{\underline{C}}$) has a complement in $L^{\sigma}_{\underline{\underline{B}}}$.

7. <u>Ultrastructures.</u>

Let B be a Boolean algebra with the associated lattice L_B of all subalgebras of B. With each element $C \neq B$, $C \in L_B$ we can associate an ideal $I_C = \{ D \in L_B : D \leq C \}$ in the lattice L_B. When is a maximal ideal (proper) in L_B is of the form I_C ? For this we introduce 'Ultrastructures'.

An element $C \neq B$, $C \in L_B$ is said to be an <u>ultrastructure</u>

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

in L_B if for any $D \in L_B$ such that $C \leq D \leq B$ implies either $D = C$ or $D = B$. (B. V. Rao [23] defined ultrastructures in the lattice $L^{\sigma}_{P(X)}$ and gave a characterisation of ultrastructures). It is easy to see that I_C is a maximal ideal in L_B iff C is an ultrastructure. So the problem of characterisation of maximal ideals of the form I_C in L_B boils down to the characterisation of ultrastructures in L_B. We do this in this section.

The following characterisation of ultrastructures in L_B is similar to the one obtained by B. V. Rao [23].

<u>Theorem 3.7.1.</u> Let I and J be two distinct maximal ideals in B. Let

$$\mathcal{A}(I, J) = \{b \in B : b \text{ or } b' \in I \cap J\}.$$

Then $\mathcal{A}(I, J)$ is an ultrastructure in L_B. Conversely, every ultrastructure in L_B is of this form.

<u>Proof.</u> It is clear that $\mathcal{A}(I, J)$ is a subalgebra of B. In fact, $\mathcal{A}(I, J)$ is the subalgebra generated by the ideal $I \cap J$ in B. Since I and J are distinct, $\mathcal{A}(I, J) \neq B$. Let D be any element in L_B which contains $\mathcal{A}(I, J)$ properly. We will show that $D = B$. For this, it is sufficient to show that

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$I \subseteq D$. (Since I is maximal in B, the subalgebra generated by I is B itself.) Let $d \in D$ satisfying $d \notin A(I, J)$. Consequently, d and $d' \notin I \cap J$. Without loss of generality assume that $d \in I$, $d' \in J$. Let $b \in I$. It follows that $b \wedge d \in I \cap J$ and hence $b \wedge d' \in D$. Now, $b \wedge d = [(b \wedge d) \vee d'] \wedge d \in D$. Hence $b \in D$. Consequently $I \subseteq D$.

To prove the converse, we need the following lemma for the case $k = 3$. Since it is interesting by itself we state and prove for the general k.

<u>Lemma 3.7.2</u>. Let $I_1, I_2, \ldots, I_k$ be k distinct maximal ideals in a Boolean algebra B. Let N_1, N_2 be any arbitrary partition of $\{1, 2, 3, \ldots, k\}$. Then there **exists a** $b \in B$ such that $b \in I_i$ for every $i \in N_1$ and $b \notin I_j$ for every $j \in N_2$.

<u>Proof.</u> We prove the Lemma by induction. For two distinct maximal ideals, the result is obvious. Assume the result to be true for any $n-1$ $(n \geq 3)$ distinct maximal ideals. The case when one of the sets in the decomposition of $\{1, 2, \ldots, n\}$ is empty, the result trivially follows. Assume, without loss of generality, $N_1 = \{1, 2, \ldots, \lambda\}$, $N_2 = \{\lambda+1, \lambda+2, \ldots, n\}$ and cardinality of $N_2 \geq 1$. By induction hypothesis, there exists

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$a\, b \in I_i$ for every $1 \leq i \leq \ell$ and $b \notin I_j$ for $\ell + 1 \leq j \leq n-1$, and $a\, c \in I_i$ for every $1 \leq i \leq \ell$ and $c \notin I_j$ for $\ell + 2 \leq j \leq n$. Then $b \vee c \in I_i$ for $1 \leq i \leq \ell$ and $b \vee c \notin I_j$ for $\ell + 1 \leq j \leq n$.

Let B be a Boolean algebra and let D be a subalgebra of B. It is easy to verify that if I is a maximal ideal in B, then $I \cap D$ is a maximal ideal in D. It is also true that if I_1 is a maximal ideal in D, there exists a maximal ideal I in B containing I_1, which we call an extension of I_1.

Lemma 3.7.3. Let B be a Boolean algebra and let D be a Boolean subalgebra of B. If every maximal ideal in D has a unique extension in B, then $D = B$.

Proof. Let X and Y be the Stone spaces of B and D respectively. We identify the Stone spaces as the collection of maximal ideals. We define $f : X \longrightarrow Y$ as follows. $f(I) = D \cap I$. From the hypothesis it follows that f is one-one. Hence the inclusion map $i : D \rightarrow B$ which induces f is onto. See Sikorski [27, first four paragraphs of p. 34]. Hence $D = B$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Now, we prove the converse part of the Theorem. Let C be an ultrastructure in L_B. We claim that there is no maximal ideal in C which admits more than two extensions in B. Suppose not. Let I_1 be a maximal ideal in C and J_1, J_2, J_3 be three distinct extensions of I_1 in B. Now, $I_1 \subset J_1 \cap J_2 \cap J_3 \subset J_1 \cap J_2$. The latter inclusion is strict in view of the Lemma 3.7.2. Now $C = B(I_1)$, the Boolean subalgebra of B generated by I_1, is strictly contained in $B(J_1 \cap J_2) = A(J_1, J_2)$. Hence C is not an ultrastructure.

Now, we claim that there exists atleast one maximal ideal I_1 in C which admits exactly two distinct extensions I and J in B. In view of Lemma 3.7.3, this is obvious.

Now, observe that $C = B(I_1) \subset B(I \cap J)$ (because $I_1 \subset I \cap J$). Since $B(I \cap J) = A(I, J)$ is an ultrastructure $C = B(I \cap J)$, i.e., C is of the form $A(I, J)$ for some maximal ideals I and J in B.

After characterising ultra structures in L_B, we make a remark about the complements of ultrastructures in L_B. Let C

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

be an ultrastructure in L_B. Take any element $b \in B$ such that $b \notin C$. Then the Boolean sub-algebra of B generated by $\{b\}$ is a complement of C. Conversely any complement of the Boolean subalgebra of B generated by an element $b \in B$ such that $b \neq 0$ and $b \neq 1$ is an ultrastructure. The above two statements are not difficult to prove and the proof is omitted.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

STUDIES IN BOOLEAN ALGEBRAS AND MEASURE THEORY

PART 2

MEASURE THEORY

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

CHAPTER 4

COUNTABLE CHAIN CONDITION AND σ-FINITENESS OF MEASURES

1. Introduction. Let $(X, \underline{\underline{A}})$ be a Borel structure. In this chapter we allow measures μ on $\underline{\underline{A}}$ to take the value ∞ also. A measure μ on $\underline{\underline{A}}$ is said to be σ-finite if there exists a sequence of sets A_i, $i \geq 1$ in $\underline{\underline{A}}$ such that $\bigcup_{i \geq 1} A_i = X$ and $\mu(A_i) < \infty$ for every i. Let μ be a σ-finite measure on $\underline{\underline{A}}$. Let $\underline{\underline{N}}$ be the collection of μ-null sets in $\underline{\underline{A}}$, i.e., $\underline{\underline{N}} = \left\{A \in \underline{\underline{A}} : \mu(A) = 0\right\}$. Then $\underline{\underline{A}} - \underline{\underline{N}}$ satisfies countable chain condition. A family of sets $\left\{A_\alpha : \alpha \in \Delta\right\}$ is said to satisfy countable chain condition if any subfamily of pairwise disjoint sets is atmost countable. We denote, hereafter, countable chain condition by C C C.

There is another notion of measure weaker than σ-finiteness. A measure μ on $\underline{\underline{A}}$ is said to be σ-sum measure if we can write $\mu = \sum_{i \geq 1} \mu_i$, where each μ_i is a finite measure on $\underline{\underline{A}}$. Every σ-finite measure μ on $\underline{\underline{A}}$ is a σ-sum measure. For, let

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$A_1, A_2, \ldots$ be a sequence of sets in $\underline{\underline{A}}$ such that (i) $\bigcup_{i\geq 1} A_i = X$ and (ii) $\mu(A_i) < \infty$ for every i. Without loss of generality, we can assume A_i's, $i \geq 1$, to be disjoint. Define $\mu_i : \underline{\underline{A}} \longrightarrow R$ as follows. $\mu_i(B) = \mu(B \cap A_i)$. Clearly, $\mu = \sum_{i\geq 1} \mu_i$. But a σ-sum measure need not be σ-finite. Let X be any uncountable set and $\underline{\underline{A}}$ countable - cocountable σ-field on X. Define λ and μ on $\underline{\underline{A}}$ as follows.

$$\lambda(A) = 0 \quad \text{if } A \text{ is countable,}$$
$$= \infty \quad \text{if } A \text{ is cocountable.}$$
$$\mu(A) = 0 \quad \text{if } A \text{ is countable,}$$
$$= 1 \quad \text{if } A \text{ is cocountable.}$$

λ and μ are measures on $\underline{\underline{A}}$ and λ is not σ-finite. But $\lambda = \sum_{i\geq 1} \mu_i$, where each $\mu_i = \mu$. What we want to emphasize on σ-sum measures is the following. If μ is a σ-sum measure on $\underline{\underline{A}}$ and $\underline{\underline{N}}$ is the collection of all μ-null sets in $\underline{\underline{A}}$, then $\underline{\underline{A}} - \underline{\underline{N}}$ satisfies C C C. The proof of this is not difficult.

Question : Is the converse true ?

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Ficker [7, Section 2,p. 242] came forward with the following theorem.

(*) <u>Theorem.</u> Let μ be a measure on a σ-field $\underline{\underline{A}}$ of X and $\underline{\underline{N}}$ denote the collection of all sets in $\underline{\underline{A}}$ of μ-measure zero. Then $\underline{\underline{A}} - \underline{\underline{N}}$ satisfies C C C if and only if μ is a σ-sum measure.

In Section 2 we give a counter example to show that the 'only if' part of Ficker's theorem is incorrect. In Section 3 post mortem is done on the Ficker's proof of his Theorem (*). In Section 4 we improve the conclusion of Ficker's theorem for a certain class of measures. In Section 5 we study C C C in σ-fields.

2. <u>Example.</u> Let B be a Boolean σ-algebra satisfying C C C such that there is no strictly positive finite measure on B. For example, one can take the Boolean σ-algebra of all Borel subsets of the real line modulo first category Borel sets. See Halmos [10, Lemma 4, p. 68]. Let X be the Stone space of B, $\underline{\underline{B}}$ the Baire σ-field on X and $\underline{\underline{I}}$ the collection of all first category Baire subsets of X. By Loomis' theorem (see, for example, Halmos [10 , p. 102]), the quotient Boolean

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

σ-algebra $\underline{\underline{B}} / \underline{\underline{I}}$ and B are isomorphic. Since $\underline{\underline{I}}$ is a σ-ideal, the function μ defined by the formula,

$$\mu(A) = 0 \quad \text{if } A \in \underline{\underline{I}}, \text{ and}$$

$$= \infty \quad \text{if } A \in \underline{\underline{B}} - \underline{\underline{I}},$$

is a measure on $\underline{\underline{B}}$. Note that $\underline{\underline{B}} / \underline{\underline{I}}$ satisfies C C C and so $\underline{\underline{B}} - \underline{\underline{I}}$ satisfies C C C. If Ficker's theorem (*) were to be true, we can write μ as a countable sum of finite measures on $\underline{\underline{B}}$ which implies that μ is equivalent to a finite measure λ on $\underline{\underline{B}}$. For, if $\mu = \sum_{i \geq 1} \mu_i$, then

$\mu \equiv \lambda = \sum_{i \geq 1} \frac{1}{2^i} \cdot \frac{\mu_i(\cdot)}{\mu_i(X)}$. ('≡' means μ and λ have the same null sets.) Since $\underline{\underline{I}}$ is precisely the collection of all λ null sets, we have a strictly positive finite measure on $\underline{\underline{B}} / \underline{\underline{I}}$. But this is a contradiction.

3. <u>Post mortem.</u> For a measure μ on a σ-field $\underline{\underline{A}}$, there are two definitions of μ-atoms.

(I) A set A in $\underline{\underline{A}}$ is said to be a μ-atom if

(i) $\mu(A) > 0$ and (ii) $B \in \underline{\underline{A}}$, $B \subset A$ implies $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

(II) A set A in $\underline{\underline{A}}$ is said to be a μ-atom if (i) $\mu(A) > 0$ and (ii) $B \in \underline{\underline{A}}$, $B \subset A$ implies $\mu(B) = 0$ or $\mu(A - B) = 0$.

These two definitions are not equivalent. A set $A \in \underline{\underline{A}}$ which is a μ-atom in the sense of definition (II) is also a μ-stom in the sense of definition (I). The following example demonstrates that the converse is not true. Let R be the real line and $\underline{\underline{A}}$ its Borel σ-field. Define

$\mu : \underline{\underline{A}} \longrightarrow [0, \infty]$ as follows.

$\mu(A) = 0$ if Lebesgue measure of $A = 0$,

$= \infty$ if Lebesgue measure of $A > 0$.

μ is a measure on $\underline{\underline{A}}$, and the open interval $(0, 1)$ is a μ-atom in the sense of definition (I) but not in that of (II).

An analysis of Ficker's proof shows that he had tacitly assumed the equivalence of definitions (I) and (II). This is as pointed above incorrect. However some form of Ficker's results can nonetheless be retrieved provided certain conditions are imposed on the measure which will ensure the equivalence of (I) and (II).

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

This last observation leads us to the notion of semifinite measure. A measure μ on $\underline{\underline{A}}$ is said to be semi-finite if $A \in \underline{\underline{A}}$, $\mu(A) = \infty$ implies there exists a $B \subset A$, $B \in \underline{\underline{A}}$ such that $0 < \mu(B) < \infty$. It is easy to verify that the definitions of μ-atom according to (I) and (II) are equivalent for semi-finite measures. In the next section we will prove a stronger version of Theorem (*) directly for semifinite measures.

4. A characterisation of C C C

<u>Theorem 4.4.1.</u> Let μ be a semi-finite measure on a σ-field $\underline{\underline{A}}$ of X. Let $\underline{\underline{N}}$ denote collection of all sets in $\underline{\underline{A}}$ of μ-measure zero. Then $\underline{\underline{A}} - \underline{\underline{N}}$ satisfies C C C if and only if μ is σ-finite.

<u>Proof.</u> It has already been noted that if μ is σ-finite then $\underline{\underline{A}} - \underline{\underline{N}}$ satisfies C C C. Suppose $\underline{\underline{A}} - \underline{\underline{N}}$ satisfies C C C. If $\mu(X) < \infty$, there is nothing to prove. If $\mu(X) = \infty$, choose A_1 ~~$\in \underline{\underline{A}}$~~ such that $0 < \mu(A_1) < \infty$. Choose $A_2 \in \underline{\underline{A}}$ such that $A_2 \subset X - A_1$ and $0 < \mu(A_2) < \infty$. Thus we can find a sequence of disjoint sets $A_1, A_2, \ldots$ in $\underline{\underline{A}}$ such that each $A_i \in \underline{\underline{A}} - \underline{\underline{N}}$ and $0 < \mu(A_i) < \infty$. If $\mu(X - \bigcup_{i \geq 1} A_i) < \infty$,

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

choose $\underline{\underline{A}}_w \in \underline{\underline{A}}$ such that $A_w \subset X - \bigcup_{i \geq 1} A_i$ and $0 < \mu(A_w) < \infty$, where w is the first countable ordinal. Continue this process. Since $\underline{\underline{A}} - \underline{\underline{N}}$ satisfies C C C, there exists a countable ordinal α such that $\mu(X - \bigcup_{\beta < \alpha} A_\beta) < \infty$. This implies that μ is σ-finite.

Remark. Let $\underline{\underline{A}}$ be a σ-field on a set X and let $\underline{\underline{N}}$ be a σ-ideal in $\underline{\underline{A}}$. It is easy to prove that $\underline{\underline{A}} / \underline{\underline{N}}$ satisfies C C C if and only if $\underline{\underline{A}} - \underline{\underline{N}}$ satisfies C C C. When $\underline{\underline{N}}$ is the collection of all μ-null sets of a semi-finite measure μ on a σ-field $\underline{\underline{A}}$, the following statements are equivalent.

i) μ is σ-finite

ii) $\underline{\underline{A}} - \underline{\underline{N}}$ satisfies C C C

iii) $\underline{\underline{A}} / \underline{\underline{N}}$ satisfies C C C.

5. Some characterisations. In this section, we characterise σ-fields on which every measure is a σ-sum measure.

Theorem 4.5.1. Let $\underline{\underline{A}}$ be a σ-field on a set X. The following statements are equivalent.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

i) Every measure μ on $\underline{\underline{A}}$ is a σ-sum measure.

ii) Every measure μ on $\underline{\underline{A}}$ is equivalent to a finite measure λ on $\underline{\underline{A}}$, i.e., the collection of all μ-null sets is same as the collection of all λ-null sets.

iii) There is a strictly positive finite measure on $\underline{\underline{A}}$.

iv) $\underline{\underline{A}}$ satisfies C C C.

v) $\underline{\underline{A}}$ is isomorphic to the power set of some countable set.

Proof. (i) ==> (ii) ==> (iii) ==> (iv) are all easy.

(iv) ==> (v). Since $\underline{\underline{A}}$ satisfies C C C it is a complete field, i.e., closed under arbitrary unions and complementation. See Halmos [10, Corollary, p. 62]. Since $\underline{\underline{A}}$ is a complete field of sets, it is atomic. Hence $\underline{\underline{A}}$ is isomorphic to the power set of some set. See Sikorski [27, 25.1, p. 105]. Since $\underline{\underline{A}}$ satisfies C C C, $\underline{\underline{A}}$ is isomorphic to the power set of some countable set.

(v) ==> (i). Let μ be any measure on the power set, P(N), where N is the set of all natural numbers. Note that, we can write

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$$\mu = \sum_{i \geq 1} \mu_i, \qquad \text{where}$$

$$\mu_i(A) = \mu(\{i\}) \qquad \text{if } i \in A$$

$$= 0 \qquad \text{if } i \notin A.$$

Observe that each μ_i is a measure on $P(N)$. We shall write each μ_i as a countable sum of finite measures. Let i be any natural number. If $\mu(\{i\}) < \infty$, then we represent μ_i by itself. If $\mu(\{i\}) = \infty$. Then we write $\mu_i = \sum_{n \geq 1} \mu_{in}$, where, for every $n \geq 1$,

$$\mu_{in}(A) = 1 \quad \text{if } i \in A,$$

$$= 0 \quad \text{if } i \notin A.$$

The proof is complete.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

CHAPTER 5

EXISTENCE OF A NONATOMIC MEASURE MAKING TWO MEASURABLE FUNCTIONS INDEPENDENT

1. Introduction. Let $(X, \underline{\underline{B}})$ be a Borel structure, i.e., $\underline{\underline{B}}$ is a σ-field of subsets of X. Let μ be a probability measure on $\underline{\underline{B}}$. Two real valued measurable functions f and g defined on X ($f^{-1}(B) \in \underline{\underline{B}}$ for every Borel subset B of the real line) are said to be μ-independent if

$$\mu\left\{x \in X : f(x) \in B_1, g(x) \in B_2\right\} =$$

$$\mu\left\{x \in X : f(x) \in B_1\right\} \mu\left\{x \in X : g(x) \in B_2\right\}$$

for all Borel sets B_1 and B_2 of the real line, R.

Generally independence is discussed after fixing a measure. Here we fix a pair of measurable functions and ask when does there exist a measure μ which makes the two measurable functions independent. If μ is a degenerate measure at a point $x_0 \in X$, i.e., $\mu(A) = 1$ if $x_0 \in A \in \underline{\underline{B}}$, and

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$\mu(A) = 0$ if $x_0 \notin A \in \underline{\underline{B}}$, then any two real valued measurable functions are μ-independent. In this chapter we deal with the problem of existence of a nonatomic probability measure on $\underline{\underline{B}}$ with respect to which the given measurable functions f and g on X are independent. A satisfactory characterisation of the existence of such a measure is obtained in the next section for X - an uncountable Borel subset of a Polish space and $\underline{\underline{B}}$ - its Borel σ-field.

There is another way of looking at the problem posed in the previous paragraph which is more instructive and elegant. Let $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ be two sub σ-fields of $\underline{\underline{B}}$. $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are said to be <u>μ-independent</u> if

$$\mu(B_1 \cap B_2) = \mu(B_1)\, \mu(B_2)$$

for every $B_1 \in \underline{\underline{B}}_1$ and $B_2 \in \underline{\underline{B}}_2$. Any real valued measurable function f on X gives rise to a sub σ-field $\underline{\underline{B}}_f$ of $\underline{\underline{B}}$ defined by $\underline{\underline{B}}_f = \left\{ f^{-1}(B) : B \text{ Borel subset of } R \right\}$. Since the Borel σ-field on R is separable, i.e., has a countable generator, $\underline{\underline{B}}_f$ is a separable sub σ-field of $\underline{\underline{B}}$. The following proposition is easy to prove.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Proposition 5.1.1. Let $(X, \underline{\underline{B}}, \mu)$ be a probability space. Let f and g be two real valued measurable functions defined on X. f and g are μ-independent if and only if $\underline{\underline{B}}_f$ and $\underline{\underline{B}}_g$ are μ-independent.

In view of the above proposition, the problem is reformulated as follows: Given two separable sub σ-fields $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ of $\underline{\underline{B}}$, when does there exist a nonatomic probability measure μ on $\underline{\underline{B}}$ such that $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are μ-independent?

Here, we record some of the facts concerning separable σ-fields which will be needed in the sequel.

Every separable σ-field is atomic. If $\underline{\underline{B}}$ is a separable σ-field on a set X, then its Marczewski function, h, is defined as follows. Fix any countable generator $B_1, B_2, \ldots$ for $\underline{\underline{B}}$. Define $h(x) = \sum_{i \geq 1} \frac{2}{3^i} \; I_{B_i}(x)$ for $x \in X$, where I_{B_i} is the indicator function of the set B_i. See B. V. Rao [24]. Moreover, $\underline{\underline{B}}_h = \underline{\underline{B}}$. There is no 0-1 valued measure on any separable σ-field $\underline{\underline{B}}$ vanishing at the atoms of $\underline{\underline{B}}$.

Since we propose to solve the problem for X uncountable Borel subset of any Polish space, in what follows we assume

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$X = [0, 1]$ and $\mathcal{B}$ its Borel σ-field, unless otherwise specified. This is possible since any uncountable Borel subset of a Polish space is Borel isomorphic to $[0, 1]$. See Parthasarathy [22, Theorem 2.12, p. 14] or Kuratowski [15, Theorem 2 , p.450].

2. Some characterisations.

Theorem 5.2.1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two separable sub σ-fields of $\mathcal{B}$. Then there exists a nonatomic probability measure μ on $\mathcal{B}$ such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are μ-independent if and only if atleast one of the atoms of $\mathcal{B}_1$ or $\mathcal{B}_2$ is uncountable.

Proof. If part. Let $B \in \mathcal{B}_1$ be an uncountable atom of $\mathcal{B}_1$. Consider the Borel structure $(B, B \cap \mathcal{B})$. Let λ be any non-atomic probability measure on $B \cap \mathcal{B}$. Such a λ exists since $(B, B \cap \mathcal{B})$ and $(X, \mathcal{B})$ are Borel isomorphic. Note that $B \in \mathcal{B}$. Lift the measure λ on $B \cap \mathcal{B}$ to a measure μ on $\mathcal{B}$ as follows. $\mu(C) = \lambda(B \cap C)$ for every $C \in \mathcal{B}$. μ is a nonatomic probability measure on $\mathcal{B}$ sitting on B. Let $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. Since B is an atom of $\mathcal{B}_1$, either $B \subset B_1$ or $B \cap B_1 = \emptyset$. In the first case, $\mu(B_1 \cap B_2) = \mu(B \cap B_2) = \mu(B_2)$, and $\mu(B_1)\,\mu(B_2) = 1 \times \mu(B_2) = \mu(B_2)$. In

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

the second case, $\mu(B_1 \cap B_2) = 0$, and $\mu(B_1)\, \mu(B_2) = 0 \times \mu(B_2) = 0$. Hence $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are μ-independent.

Only if part. Suppose none of the atoms of $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ is uncountable, i.e., every atom of $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ is countable. Let f and g be Marczewski functions of $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ respectively. Note that f and g are countable-to-one functions on X, i.e., $f^{-1}\{x\}$ and $g^{-1}\{x\}$ are countable for every $x \in X$. Let $A_1 = f(X)$, and $A_2 = g(X)$. Then $A_1 \subset X$ and $A_2 \subset X$. Further f and g map Borel subsets of X into Borel subsets of X. See Kuratowski [15, Corollary 5, p. 498]. Consequently, A_1 and A_2 are Borel subsets of X. Let $\underline{\underline{C}}_1$ and $\underline{\underline{C}}_2$ be the relatived Borel σ-fields on A_1 and A_2 respectively. Define $T : X \longrightarrow A_1 \times A_2$ in the following way. $Tx = (f(x), g(x))$. $A_1 \times A_2$ is equipped with the product σ-field $\underline{\underline{C}}_1 \times \underline{\underline{C}}_2$. Now, T is $(\underline{\underline{B}}, \underline{\underline{C}}_1 \times \underline{\underline{C}}_2)$ measurable. For, f is $(\underline{\underline{B}}, \underline{\underline{C}}_1)$ measurable and g is $(\underline{\underline{B}}, \underline{\underline{C}}_2)$ measurable. Let λ, λ_1 and λ_2 be the probability measures defined on $\underline{\underline{C}}_1 \times \underline{\underline{C}}_2$, $\underline{\underline{C}}_1$ and $\underline{\underline{C}}_2$ respectively defined by

$$\lambda = \mu T^{-1},$$

$$\lambda_1 = \mu f^{-1}, \text{ and}$$

$$\lambda_2 = \mu g^{-1}.$$

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

Observe that

$\lambda_1(\{x\}) = 0$ for every $x \in \Delta_1$. For, $f^{-1}\{x\}$ is countable for every $x \in \Delta_1$ and μ is nonatomic on $\underline{\underline{B}}$. A similar argument shows that $\lambda_2(\{x\}) = 0$ for every $x \in \Delta_2$. Since $\underline{\underline{C}}_1$ and $\underline{\underline{C}}_2$ are separable, λ_1 and λ_2 are nonatomic probability measures on $\underline{\underline{C}}_1$ and $\underline{\underline{C}}_2$ respectively. Now, we claim that $\lambda = \lambda_1 \times \lambda_2$, the product measure of λ_1 and λ_2 on $\underline{\underline{C}}_1 \times \underline{\underline{C}}_2$. For this, it is enough if we show that

$\lambda(C_1 \times C_2) = \lambda_1(C_1)\, \lambda_2(C_2)$ for every C_1 and C_2 in $\underline{\underline{C}}_1$ and $\underline{\underline{C}}_2$ respectively. $\lambda(C_1 \times C_2) = \mu\ T^{-1}(C_1 \times C_2) =$

$= \mu\,[f^{-1}(C_1) \cap g^{-1}(C_2)] = \mu[f^{-1}(C_1)] \cdot \mu[g^{-1}(C_2)]$, since $f^{-1}(C_1) \in \underline{\underline{B}}_1$, $g^{-1}(C_2) \in \underline{\underline{B}}_2$ and $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are independent. But $\mu\,[f^{-1}(C_1)] \cdot \mu\,[g^{-1}(C_2)] = \lambda_1(C_1)\, \lambda_2(C_2)$. Let $Y = TX$. Now, we note that T is also countable-to-one function from X onto Y. For $Tx = (f(x), g(x)) \in Y$, $T^{-1}\{(f(x), g(x))\} = \{y : (f(y), g(y)) = (f(x), g(x))\}$ is a countable set since f and g are countable-to-one functions. This implies that T maps Borel subsets of X into sets belonging to $\underline{\underline{C}}_1 \times \underline{\underline{C}}_2$. In particular, $Y \in \underline{\underline{C}}_1 \times \underline{\underline{C}}_2$. Note that every section of Y in countable. For, $Y_{f(x)} = \{g(y) \in \Delta_2 : f(x) = f(y)\}$ which is countable since f is countable-to-one. Since λ_2 is

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

nonatomic, $\lambda_2 (Y_{f(x)}) = 0$ for every $x \in X$. Hence, by Fubini's Theorem, $\lambda(Y) = 0$ which is a contradiction to $\lambda (Y) = 1$. Hence atleast one of the atoms of $\underline{\underline{B}}_1$ or $\underline{\underline{B}}_2$ is uncountable.

Corollary 5.2.2. Let f and g be two measurable functions defined on $X = [0, 1]$, where X is equipped with its usual Borel σ-field, $\underline{\underline{B}}$. There exists a nonatomic probability measure on $\underline{\underline{B}}$ with respect to which f and g are independent if and only if either f or g is constant on an uncountable subset of X.

Proof. Observe that the nonempty sets among $\{f^{-1}\{x\}: x \in R\}$ are the atoms of $\underline{\underline{B}}_f$.

Now, we take up the following problem: Given two separable sub σ-fields $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ of $\underline{\underline{B}}$ does there exist a nonatomic probability measure μ on $\underline{\underline{B}}$ such that (i) $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are μ-independent, and (ii) the restrictions of μ to $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are nonatomic on $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ respectively? We give partial answer to this question.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

<u>Theorem 5.2.3.</u> Let $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ be two separable sub σ-fields of $\underline{\underline{B}}$ satisfying

i) exactly one of the atoms, say B_1, of $\underline{\underline{B}}_1$ is uncountable, and

ii) every atom of $\underline{\underline{B}}_2$ is countable.

Let μ be a nonatomic probability measure on $\underline{\underline{B}}$ such that $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are μ-independent. Then $\mu(B_1) = 1$.

<u>Proof.</u> As in the proof of Theorem 5.2.1, let

i) f be a Marczewski function of $\underline{\underline{B}}_1$,

ii) g be a Marczewski function of $\underline{\underline{B}}_2$,

iii) $A_1 = f(X)$,

iv) $A_2 = g(X)$,

v) $T : X \longrightarrow A_1 \times A_2$ defined by $Tx = (f(x), g(x))$, and

vi) $TX = Y$.

Since $\underline{\underline{B}}_1$ contains exactly one uncountable atom B_1, f and T map Borel subsets of X into Borel subsets of A_1 and $A_1 \times A_2$ respectively. Let

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$$\lambda = \mu T^{-1},$$

$$\lambda_1 = \mu f^{-1}, \text{ and}$$

$$\lambda_2 = \mu g^{-1}.$$

We observe that $\lambda = \lambda_1 \times \lambda_2$, and λ_2 nonatomic.

Further,

$1 = \lambda(Y) = \int_{\Delta_1} \lambda_2(Y_x)\, \lambda_1(dx)$. Let $x_o = f(B_1)$. Note that Y_x = x - section of Y for x in Δ_1 is countable for all $x \neq x_o$. Since λ_2 is nonatomic, $1 = \lambda_2(Y_{x_o})\, \lambda_1(\{x_o\})$. This implies that $\lambda_1(\{x_o\}) = 1 = \mu(f^{-1}\{x_o\}) = \mu(B_1)$. This completes the proof of the theorem.

The preceding theorem says that if exactly one of the atoms of $\underline{\underline{B}}_1$ is uncountable and all the atoms of $\underline{\underline{B}}_2$ are countable, then any nonatomic probability measure μ on $\underline{\underline{B}}$ with respect to which $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are independent cannot be nonatomic on $\underline{\underline{B}}_1$. The following theorem says more on this problem than the preceding theorem.

Theorem 5.2.4. Let $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ be two separable sub σ-fields of $\underline{\underline{B}}$, and let μ be a nonatomic probability measure on $\underline{\underline{B}}$ with respect to which $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are independent. Further,

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

assume that μ on $\underline{\underline{B}}_1$ and μ on $\underline{\underline{B}}_2$ are also nonatomic. Then there are uncountably many uncountable atoms in $\underline{\underline{B}}_1$ as well as in $\underline{\underline{B}}_2$.

Proof. Suppose the conclusion of the theorem **is false.** Let f and g be Marczewski functions of $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ respectively. Assume that the number of uncountable atoms in $\underline{\underline{B}}_1$ is countable.

1°. Let $Z = [0, 1] \times [0, 1] = X \times X$ equipped with the product σ-field $\underline{\underline{B}} \times \underline{\underline{B}}$. Let $T : X \longrightarrow Z$ be defined as follows : $Tx = (f(x), g(x))$. Since f and g are $(\underline{\underline{B}}, \underline{\underline{B}})$ measurable, T is $(\underline{\underline{B}}, \underline{\underline{B}} \times \underline{\underline{B}})$ measurable. Let $\lambda = \mu T^{-1}$, $\lambda_1 = \mu f^{-1}$ and $\lambda_2 = \mu g^{-1}$. We note that λ_1 and λ_2 are nonatomic. For this, it is enough if we show that $\lambda_1(\{x\}) = 0 = \lambda_2(\{x\}) = 0$ for every $x \in X$. Since $f^{-1}(\{x\}) = \emptyset$ or an atom of $\underline{\underline{B}}_1$ and μ is nonatomic on $\underline{\underline{B}}_1$, $\lambda_1(\{x\}) = \mu[f^{-1}(\{x\})] = 0$. The same argument applies to λ_2.

2°. Now, we claim that $\lambda = \lambda_1 \times \lambda_2$. For this, it is enough if we verify that

$$\lambda(C \times D) = \lambda_1(C).\lambda_2(D)$$

for every C and D in $\underline{\underline{B}}$.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$$\lambda(C \times D) = \mu[T^{-1}(C \times D)] = \mu[f^{-1}(C) \cap g^{-1}(D)]$$

$$= \mu[f^{-1}(C)].\ \mu[g^{-1}(D)], \text{ since}$$

$f^{-1}(C) \in \underline{\underline{B}}_1$, $g^{-1}(D) \in \underline{\underline{B}}_2$ and $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are μ-independent.

Thus $\lambda(C \times D) = \lambda_1(C) \times \lambda_2(D)$.

3°. Since T is measurable, $TX = Y$ is analytic and so is available in the completion of $\underline{\underline{B}} \times \underline{\underline{B}}$ with respect to λ. This follows from capacitability theorem of Choquet. See, for example, Neveu [19, Exercise 1.5.4, pp. 24-25].

4°. Let $C_1, C_2, \ldots$ be the uncountable atoms of $\underline{\underline{B}}_1$. Let $x_i = f(C_i)$, $i \geq 1$. Now, we observe that

$$Y_x = \emptyset \quad \text{if} \quad x \notin f(X),$$
$$= \text{countable if} \quad x \in (f(X) - \{x_1, x_2, \ldots\})$$

where $Y_x = \{y \in [0, 1] ; (x, y) \in Y\}$. $Y_x = \emptyset$ if $x \notin f(X)$ is clear from the fact that $Y = \{(f(x), g(x): x \in X\}$. Let $x \in (f(X) - \{x_1, x_2, \ldots\})$. There exists $y \in X$ such that $x = f(y)$. $Y_{f(y)} = \{g(z) : f(z) = f(y)\} = g[f^{-1}(f(y))]$. Since $f(y) \neq x_i$ for any i, $f^{-1}(f(y))$ is countable. Hence $g[f^{-1}(f(y))]$ is countable. Thus except for the set

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$$A = \{x_1, x_2, \ldots\}, \quad \lambda_2(Y_x) = 0. \quad \text{But} \quad \lambda_1(A) = 0.$$

Hence Y is a null set. See Oxtoby [21, Theorem 14.2, p. 53]. But $\lambda(Y) = 1$. This contradiction proves the theorem.

Now, we may ask whether the converse of the Theorem 5.2.4 is true. More explicitly, suppose $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are two separable sub σ-fields of $\underline{\underline{B}}$ and there are uncountably many uncountable atoms $\{B_\alpha^{(1)}\}$ and $\{B_\beta^{(2)}\}$ in $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ respectively. Is it possible to find a nonatomic probability measure μ on $\underline{\underline{B}}$ such that (i) $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are μ-independent, and (ii) μ on $\underline{\underline{B}}_1$ and μ on $\underline{\underline{B}}_2$ are nonatomic ? The following example shows that it is not always possible.

3. <u>Counter-Example.</u> Let X_1, X_2, X_3, X_4 be 4 distinct uncountable Polish spaces. Let $Y = X_1 \cup X_2$ and $Z = X_3 \cup X_4$. Let $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ be the Borel σ-fields on Y and Z respectively.

Let

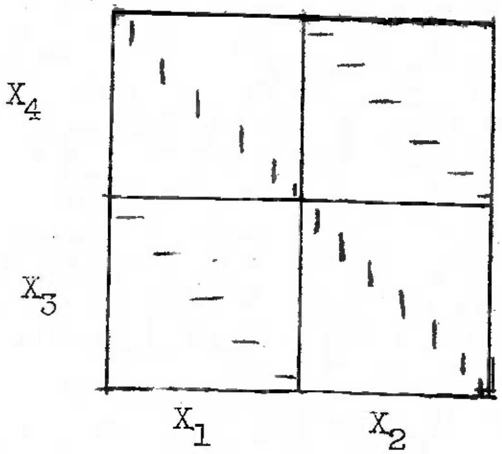
$$e : X_1 \longrightarrow X_3,$$
$$f : X_4 \longrightarrow X_1,$$
$$g : X_3 \longrightarrow X_2, \text{ and}$$
$$h : X_2 \longrightarrow X_4$$

be four measurable

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

functions such that inverse image of every singleton is uncountable. Let X = graph of e U graph of f U graph of g U graph of $h = \{(x_1, e(x_1)) : x_1 \in X_1\} \cup \{(f(x_4), x_4) : x_4 \in X_4\}$

$$\cup \{(g(x_3), x_3) : x_3 \in X_3\} \cup \{(x_2, h(x_2)) : x_2 \in Y_2\}.$$



is a Borel subset of $Y \times Z$, i.e., $X \in \underline{\underline{B}}_1 \times \underline{\underline{B}}_2$. Consider the Borel structure $(X, X \cap \underline{\underline{B}}_1 \times \underline{\underline{B}}_2)$ with the sub σ-fields $X \cap (Y \times \underline{\underline{B}}_2)$ and $X \cap (\underline{\underline{B}}_1 \times Z)$ of $X \cap (\underline{\underline{B}}_1 \times \underline{\underline{B}}_2)$. Note that every atom of $X \cap (Y \times \underline{\underline{B}}_2)$ and every atom of $X \cap (\underline{\underline{B}}_1 \times Z)$ is uncountable. For instance, every atom of $X \cap (Y \times \underline{\underline{B}}_2)$ is either $h^{-1}\{x_4\} \times \{x_4\} \cup f(x_4) \times \{x_4\}$ for some $x_4 \in X_4$, or

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$e^{-1}(\{x_3\}) \times \{x_3\} \cup \{g(x_3)\} \times \{x_3\}$ for some $x_3 \in \mathfrak{X}_3$. Suppose there exists a nonatomic probability measure on $\mathfrak{X} \cap \underline{\underline{B}}_1 \times \underline{\underline{B}}_2$ such that (i) $\mathfrak{X} \cap (Y \times \underline{\underline{B}}_2)$ and $\mathfrak{X} \cap (\underline{\underline{B}}_1 \times Z)$ are μ-independent and (ii) μ on $\mathfrak{X} \cap (Y \cap \underline{\underline{B}}_2)$ and μ on $\mathfrak{X} \cap (\underline{\underline{B}}_1 \times Z)$ are nonatomic. We will show that this leads to a contradiction.

Let λ, λ_2 and λ_1 be the extensions of μ on $\mathfrak{X} \cap (\underline{\underline{B}}_1 \times \underline{\underline{B}}_2)$ to $\underline{\underline{B}}_1 \times \underline{\underline{B}}_2$, μ on $\mathfrak{X} \cap (Y \times \underline{\underline{B}}_2)$ to $Y \times \underline{\underline{B}}_2$ and μ on $\mathfrak{X} \cap (\underline{\underline{B}}_1 \times Z)$ to $\underline{\underline{B}}_1 \times Z$ respectively by putting mass zero on $(Y \times Z) - \mathfrak{X}$. We note that $\underline{\underline{B}}_1 \times Z$ and $Y \times \underline{\underline{B}}_2$ are λ - independent. For, $\lambda(B_1 \times Z \cap Y \times B_2) =$

$$= \mu\,(\mathfrak{X} \cap (B_1 \times Z) \cap \mathfrak{X} \cap (Y \times B_2))$$

$$= \mu\,[\mathfrak{X} \cap (B_1 \times Z)].\ \mu\,[\mathfrak{X} \cap (Y \times B_2)]$$

$$= \lambda\,(B_1 \times Z).\ \lambda(Y \times B_2).$$

Hence λ must be a product measure and in fact, is equal to $\lambda_1 \times \lambda_2$. It is clear that λ_1 and λ_2 are nonatomic. It is easy to verify that $\lambda_1 \times \lambda_2$ ($\mathfrak{X} \cap$ graph of e)

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

$$= 0 = \lambda_1 \times \lambda_2 \text{ (graph of } f) = \lambda_1 \times \lambda_2 (X \cap \text{graph of } g)$$

$$= \lambda_1 \times \lambda_2 \text{ (graph of } h). \text{ Hence } \lambda_1 \times \lambda_2(X) = 0.$$

This is a contradiction.

So in the above example we produced a σ-field $\underline{\underline{B}}$ with sub σ-fields $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ such that $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ have uncountably many (in fact all) uncountable atoms and $\underline{\underline{B}}$ does not admit a nonatomic measure μ which is nonatomic on both $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ and such that $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are independent with respect to μ.

It will be interesting to give necessary and sufficient conditions for the existence of a nonatomic probability measure μ with respect to which the given sub σ-fields $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are independent and further μ on $\underline{\underline{B}}_1$ and μ on $\underline{\underline{B}}_2$ are nonatomic.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

CHAPTER 6

A BOREL SET OF FULL MEASURE WHICH CONTAINS NO RECTANGLE OF POSITIVE MEASURE IN R x R

1. Introduction. Let m stand for the Lebesgue measure on the real line and let m_2 stand for the product Lebesgue measure on $R \times R$. In this chapter we give an example of a Borel set E in $R \times R$ such that $m_2((R \times R) - E) = 0$ and E does not contain any $A \times B$ such that $m(A) > 0$ and $m(B) > 0$. In [5], R. B. Darst and C. Goffman gave an example of a Borel set of positive measure (not full) in $R \times R$ which does not contain any rectangle of positive measure. In Section 2 we generalize a well known result in measure theory which we use to construct an example in Section 3. In Section 4 we get the category analogues of the results of Sections 2 and 3.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

2. Generalization of a result in measure theory.

It is well known that (See Halmos [11] or Oxtoby [21]) for any Borel set $A \subset R$ with $m(A) > 0$, $A - A = \{x - y : x \in A, y \in A\}$ contains an open interval. The following proposition generalizes this result.

Proposition 6.2.1: If $A \subset R$, $B \subset R$ are two Borel sets with $m(A) > 0$ $m(B) > 0$ then $A - B = \{x - y : x \in A, y \in B\}$ contains an open interval.

Proof. It is a well known fact of measure theory (or ergodic theory) that if a Borel set of positive measure is invariant under translations with all rationals, then it is of full measure (i.e., complement is of measure zero). Now $\bigcup_{r \text{ rational}} A + r$ is a Borel set invariant under all rationals. Hence $m\left(\left(\bigcup_{r \text{ rational}} A + r\right) \cap B\right) = m(B)$. Since $m(B) > 0$ there exists a rational r such that $m((A + r) \cap B) > 0$. Let $C = (A + r) \cap B$ Hence $C - C$ contains an interval I. Now

$A - B = [(A + r) - B] - r \supset [C - C] - r \supset I - r$, an interval.

Hence the proof.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

The example given below is a slight modification of Darst-Goffman example. The above proposition simplifies the proof of our example.

3. Example : Let $E=\{(x, y) \in R \times R \mid x-y$ is an irrational $\}$. Let $A, B \subset R$ be Borel sets with $m(A) > 0$ and $m(B) > 0$. Then by our above proposition $A-B$ contains a rational say r. Let $x \in A$, $y \in B$ be such that $x-y = r$. Now (x,y) cannot belong to E. So $A \times B \not\subset E$. Obviously $m_2(R \times R - E) = 0$. Thus E is a Borel set in $R \times R$ with full measure such that E does not contain any rectangle of positive measure.

4. Case of sets with property of Baire.

A set $A \subset R$ is said to have property of Baire if A can be written as $A = G \Delta P$ where G is open and P is of first category. It is known that [See Oxtoby [21]] for any second category set with property of Baire $A - A = \{x-y \mid x, y \in A\}$ contains an interval. The following is category analogue of proposition 6.2.1.

Proposition 6.4.1. If $A, B \subset R$ are second category sets with property of Baire then $A - B = \{x-y \mid x \in A, y \in B\}$ contains

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

an interval.

<u>Proof.</u> Let $A = G_1 \Delta P_1$, $B = G_2 \Delta P_2$ where G_1, G_2 are open sets and P_1 and P_2 are first category sets. Since A, B are of Second category G_1 and G_2 are nonempty. Take an interval I such that $G_1 - G_2 \supset I$, i.e., for $x \in I$, $(G_2 + x) \cap G_1$ is nonempty. The following is easily checked for any x.

$$(B + x) \cap A \supset (G_2 + x) \cap G_1 - (P_1 \cup P_2).$$

For $x \in I$, $(G_2 + x) \cap G_1$ is a nonempty open set and since in R no open set is of first category $(G_2 + x) \cap G_1 - (P_1 \cup P_2)$ is non-empty, i.e., for $x \in I$, $(B + x) \cap A$ is nonempty.

Hence $A - B \supset I$.

Example. Let E be the set constructed in section three. Just as in Section 3 it can be shown easily that E is the compliment of a I category set which does not contain any rectangle $A \times B$ such that A and B are second category Baire sets.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

R E F E R E N C E S

[1] Bhaskara Rao, K.P.S. and Bhaskara Rao, M. : A remark on nonatomic measures, Annals of Mathematical Statistics, 43, (1972), pp. 369-370.

[2] Bhaskara Rao, K.P.S. and Bhaskara Rao, M. : Borel σ-algebra on $[0, \Omega]$, Manuscripta Mathematica, 5 (1971), pp. 195-198.

[3] Bhaskara Rao, K.P.S. and Bhaskara Rao, M.: Existence of nonatomic charges, To appear in J. of Australian Math. Soc.

[4] Blackwell, D. : On a class of probability spaces, Proc. III Berkeley Symp., 2 , Univ. of Calif. Press, (1956), pp. 1-6.

[5] Darst, R. B. and Goffman, C. : A Borel set which contains no rectangles, Amer. Math. Monthly, 77 (1970), pp. 728-729.

[6] Engleking, R. : Outline of general topology, North Holland and PWN, (1968).

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

[7] Ficker, V. : On the equivalence of a countable disjoint class of sets of positive measure and a weaker condition than total σ-finiteness of measures, Bull. Australian Math. Soc. 1, (1969), pp. 237-243.

[8] Gillman, L. and Jerison, M. : Rings of continuous functions, D. Van Nostrand Co., Inc., (1960).

[9] Green, E. : Completeness of L^p-spaces over finitely additive set functions, Coll. Math., 22, (1971), pp. 257-261.

[10] Halmos, P. R. : Lectures on Boolean Algebras, D. Van Nostrand Co. Inc., (1963).

[11] Halmos, P. R. : Measure Theory, D. Von Nostrand Co. Inc., (1950).

[12] Hausdorff, F. : Set Theory, Chelsea, (1957).

[13] Kelley, J. L. : General Topology, D. Von Nostrand Co. Inc., (1955).

[14] Knowles, J. D. : On the existence of nonatomic measures, Mathematika, 14, (1967), pp. 62-67.

[15] Kuratowski, K. : Topology, Volume 1, Academic Press and PWN, (1966).

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

[16] Maharam, D. : On a theorem of von Neumann, Proc. Amer. Math. Soc., 9, (1958), pp. 987-994.

[17] Mackey , G. W.: Borel Structures in Groups and their duals, Trans. Amer. Math. Soc., 85, (1957), pp. 134-165.

[18] Nagata, J. I. : Modern Dimension Theory, North.Holland, (1955).

[19] Neveu, J. : Mathematical foundations of the calculus of probability, Holden-Day, (1965).

[20] Novikoff, P. : Sur les fonctions implicites measurables B, Fund. Math., 17, (1931), pp. 8-26.

[21] Oxtoby, J. C. : Measure and Category. Springer-Verlog, (1971).

[22] Parthasarathy, K. R. : Probability measures on Metric Spaces, Academic Press, (1967).

[23] Rao, B. V. : Lattice of Borel Structures, Coll. Math., 23, (1971), pp. 213-216.

[24] Rao, B. V. : Remarks on Analytic Sets, Fund. Math., 66, (1970), pp. 237-239.

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor

[25] Rudin, W. : Continuous function on compact spaces without perfect subsets, Proc. Amer. Math. Soc., 8, pp. 39-42.

[26] Sierpinski, W. : Cardinal and ordinal numbers, Second edition, PWN, (1965).

[27] Sikorski, R. : Boolean Algebras, Second edition, Springer-Verlag, (1964).

[28] Sobczyk, A. and Hammer, P.C. : A decomposition of additive set functions, Duke Math. J., 11, (1944), pp. 839-846.

[29] von Neumann, J. and Stone, M. H. : The determination of representative elements in the residual classes of a Boolean algebra, Fund. Math., 25, (1935), pp. 353-376.

[30] Willard, S. : General topology, Addison-Wesley, (1970).

PDF compression, OCR, web optimization using a watermarked evaluation copy of CVISION PDFCompressor