Projective corepresentations and cohomology of compact quantum groups

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Dedicated to my parents

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Contents

Notations

$\mathbb{C}$	The set of complex numbers
S^1	The circle group
$M_n(C)$	The set of all $n \times n$ complex matrices
$\mathcal{H}$	Hilbert space
$C_r^*(G)$	Reduced group C^* algebra
$C^*(G)$	full group C^* algebra
id	The identity map
$Sp(A)$	$\mathbb{C}$- linear span of a set A
$[A]$	Closed linear span of a set A
$\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{B}$	Algebraic tensor product of two C^* algebras $\mathcal{A}$ and $\mathcal{B}$
$\mathcal{A} \otimes \mathcal{B}$	Minimal tensor product of two C^* algebras $\mathcal{A}$ and $\mathcal{B}$
$\mathcal{A} \bar{\otimes} \mathcal{B}$	von Neumann algebraic tensor product of two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$
$K(\mathcal{H})$	The set of all compact operators on $\mathcal{H}$.
$B(\mathcal{H})$	The set of all bounded operators on $\mathcal{H}$.
$(\mathcal{Q}, \Delta)$	Compact quantum group
$\hat{\mathcal{Q}}$	Dual discrete quantum group of $\mathcal{Q}$
$L^\infty(\mathcal{Q})$	von Neumann algebraic compact quantum group
Ω	Left/right 2-Cocycle of a compact quantum group

Chapter 0

Introduction

Symmetry plays a major role in both mathematics and physics. This triggered a flurry of research in the theory of groups and their representations, both in analytic and algebraic frameworks. In the formalism of quantum mechanics in terms of operators on Hilbert spaces, it is often natural to consider symmetry as a map on the level of rays, that is on the set of unit vectors up to scalar multiplication. This leads to consideration of projective representation of the symmetry group. Mathematically, the theory of projective representation of group is closely related with group extension and cohomology theory [Bro94].

Quantum groups and more generally the theory of rigid tensor categories have generalized the classical group symmetry in mathematical physiscs. Beginning with the algebraic formulation due to Drinfield and Jimbo [Dri89], [Dri85], [Jim85] which was motivated by questions in physics related to the solution of quantum Yang-Baxter equations, the theory of quantum groups and Hopf algebras have traversed a long way, with various analytic formalism due to a number of mathematician, most notably Woronowicz [Wor87], Vaes-Kusterman [KV99], [KV00], [KV03] Van Daele [VD94], [VDVK94], [VD96], [MVD98], [VDW96]. In physics, there is a lot of interaction between the mathematical theory of quantum groups and tensor categories with the emerging field of topological states of matter and quantum computation [CGLW13, Che16].

This makes it a natural question whether one can extend the classical theory of projective representation of group to the realm of quantum groups. Substantial work in this direction have already been done by Kenny De Commer([DCMN24], [DC11b], [DC11a], [DC09], [DCY15], [DCY12]), Sergey Neshveyev([NT11a]), [NT12]), [NY16]), Lars Tuset([NT11b]), Makoto Yamashita([NY16]). The goal of present thesis is to contribute to understanding of projective (co)representation of compact quantum groups and study the associated cohomology. Some of the main results obtained by us concern extension of projective corepresentations of a given (compact) quantum group to (linear) corepresentations of a bigger quantum group . Indeed, using Tannaka-Krein reconstruction theorem, for a given compact quantum group, we prove existence of a possibly larger compact quantum group such that any unitary projective corepresentation of the original quantum group lifts to a unitary corepresentation of the bigger quantum group. In fact, we carry out such enveloping construction for various types (left, right,bi) of projective corepresentation defined by us. For strongly projective corepresentations defined by us, which in a sense are the closet to the classical case, we can relaize the dual of the original quantum group as a normal quantum subgroup of (dual of) the corresponding envelope. This leads to connection with the the second invariant cohomology of quantum groups in the sense of ([NT]). In the last part of the thesis we compute such cohomology for a few quantum groups explicitly using the techniques of tensor category and fiber functors.

Let us briefly discuss some possible applications of our results to the domain of physics, more precisely topological phases. Some quantum systems permit a gapped spectrum with a single or degenerate vacuum state. In order to specify conditions for the latter, a natural definition of symmetry involves of maps that preserve the transition probability of rays. Changing the point of view from rays to specific states, the symmetry action becomes projective. Specifically, symmetries are operators on the Hilbert space that are either linear and unitary or anti-linear and anti-unitary. The first case can be interpreted as a projective representation of groups. The vacuum state degeneracy is then contigent to the projective phase, whenever the symmetry is preserved by the Hamiltonian in question. The issue, whether a vacuum is degenerate, is then solved by answering whether a phase redefinition on the states can eliminate the projective phase. This problem is related to the field of group cohomology. Different vacuum state are parametrized by different equivalence classes in the second group cohomology, $H^2(G, U(1))$ of the symmetry group G. Symmetry protected topological phase are found to be exactly those vacuum states with non-trivial elements in $H^2(G, U(1))$. When one replaces group symmetry on quantum mechanical system by a natural quantum group symmetry, it is clear that projective corepresentation and cohomology of quantum groups will play a crucial role to label and understand topologiacl phase.

We conclude with a chapter wise summary of the thesis.

Chapter 1

In the first section, we introduce all basic terminology and theory of projective representation of classic group, which are mostly taken from [Bro94], [Che15]. We recall all the theorems which are related to projective representation and central extension of group. At the end, we give brief description of universal central extension and its properties. Section 2 is devoted to basic theorems on C^* algebra and von-Neumann algebras and tensor products of C^* algebras. Section 3 contains all the basic definitions and theorems on Hopf algebras , compact quantum group (CQG) and discrete compact quantum group (DQG). We explicitly give some examples which will be used in later chapters, brief description of corepresentations of compact quantum groups and the dual DQG of a CQG as in Van Daele [VD96] . At the end of this chapter, we define C^* tensor category and state the Woronowicz Tannaka-Krein duality and also an alternative description of fiber functor [BRV06].

Chapter 2

After this, we recall the definitions of Galois co-object of von-Neumann algebraic compact quantum groups and measurable projective corepresentations [DC11b]. Here we prove that any measurable projective corepresentation of a von-Neumann algebraic compact quantum group on a finite dimensional Hilbert space corresponds to a right Galois co-object. From this, we get a left 2-cocycle. Then we define the notion of left/right/bi projective corepresentation of compact quantum group. In the first section of this chapter, we introduce the notions of cohomology of a CQG, the invariant second cohomology. Then we briefly describe invariant 2-cocycles of compact quantum group. We give some examples at the end of this section.

Chapter 3

It is known from [Bar54] that any projective representation of classical compact group lifts to a linear representation of a bigger compact group. In this chapter we prove a similar result for a compact quantum group. In the first section of this chapter, we prove that for any left/right/bi projective corepresentation U of a compact quantum group $\mathcal{Q}$ on $\mathcal{H}$ the contragredient $\bar{U}$ on $\bar{\mathcal{H}}$ is also unitary for a suitable choice of inner product. From that we prove that contragredient of a left/right/bi projective corepresentation is also a right/left/bi projective unitary corepresentation for a suitable Hilbert space. In the second section of this chapter, we construct a universal rigid C^* tensor category and using suitable subcategories we define universal compact quantum groups $\mathcal{Q}_L, \mathcal{Q}_R, \mathcal{Q}_{bi}$ such that any left/right/bi projective corepresentation of $\mathcal{Q}$ corresponds to a unitary corepresentation of $\mathcal{Q}_L, \mathcal{Q}_R, \mathcal{Q}_{bi}$ respectively.

Chapter 4

First we define the notion of normalizer of a tensor category. By applying Tannaka-Krein duality we prove that discrete quantum group corresponding to the normalizer of the category of (finite dimensional) representation of a DQG is a normal discrete quantum subgroup of it. In the next section we define left/right strongly projective corepresentation of $\mathcal{Q}$ and prove that the associated 2-cocycle is invariant. We study the quoient quantum group $\widehat{N(\mathcal{Q})}/\hat{\mathcal{Q}}$ and discuss its relation with cohomology.

Chapter 5

In this chapter we explicitly compute the invariant second cohomology groups

$H^2_{uinv}(\hat{\mathcal{Q}}, S^1)$ and $H^2_{inv}(\hat{\mathcal{Q}}, \mathbb{C}^*)$ for two interesting examples, namely dual of the Kac-Paljutkin Hopf algebra ($\mathcal{Q}_{kp}$) and $\mathbb{C}\Gamma$ for a group of order 32 considered by [Wal47]. We use the description of cohomology in terms of fiber functos as in [NT]. In particuar we obtain $H^2_{uinv}(\mathbb{C}\Gamma, S^1) = Z_2 = H^2_{inv}(\mathbb{C}\Gamma, \mathbb{C}^*)$ and $H^2_{uinv}(\hat{\mathcal{Q}}_{kp}, S^1) = \{1\} = H^2_{inv}(\hat{\mathcal{Q}}_{kp}, \mathbb{C}^*)$.

Chapter 1

Preliminaries

1.1 Projective representation of a group and relation with group cohomology

We review some basic facts on projective representation from [Bro94], without proof.

1.1.1 Projective representation of a topological group

Let G be a group, V is a $\mathbb{C}$-Vector space and $GL(V)$ be the general linear group of V. The projective linear group $PGL(V)$ defined by the quotient $GL(V)/\mathbb{C}^* Id_V$.

Definition 1.1.1. *A projective representation of a group G on a vector space V is a group homomorphism*

$$F : G \to PGL(V).$$

Definition 1.1.2. *A map $\Omega : G \times G \to \mathbb{C} - \{0\}$ is said to be a 2-cocycle If*

$$\Omega(xy, z)\Omega(x, y) = \Omega(x, yz)\Omega(y, z),$$

where $x, y, z \in G$. If it takes value in S^1, then it is said to be a unitary 2 cocycle.

Proposition 1.1.3. *If $F : G \to GL(V)$ is a projective representation then there are set maps $\phi_F : G \to GL(V)$ and $\Omega_F : G \times G \to \mathbb{C} - 0$ such that*

$$\phi_F(g)\phi_F(h) = \Omega_F(g, h)\phi_F(gh). \quad (1.1.1)$$

Conversely, if there are set maps $\phi : G \to GL(V)$ and Ω on $G \times G \to \mathbb{C} - 0$ which satisfying (1.1.1) then there is a unique group homomorphism $F : G \to PGL(V)$ such that $F = \pi \circ \phi$, where $\pi : GL(V) \to PGL(V)$ is the quotient map.

So, If we have two set maps ϕ and Ω on G satisfying (1.1.1), then (ϕ, V, Ω) is said to be a projective representation of G. Any linear representation of G is automatically a projective representation of G.

Definition 1.1.4. *If $(\phi_1, V, \Omega_1), (\phi_2, W, \Omega_2)$ are two projective representations of G then morphisms between them are given by,*

$$\begin{aligned} & Mor((\phi_1, V, \Omega_1), (\phi_2, W, \Omega_2)) \\ & = \{T \in B(V, W) : T\phi_1(g) = \phi_2(g)T, \ for \ all \ g \in G\}. \end{aligned}$$

Two projective representations $(\phi_1, V, \Omega_1), (\phi_2, W, \Omega_2)$ are said to be equivalent if there exists a vector space isomorphism $T : V \to W$ such that $T \in Mor((\phi_1, V, \Omega_1), (\phi_2, W, \Omega_2))$.

Lemma 1.1.5. *If (ϕ, V, Ω) is a projective representation of a group G, then Ω is a 2 cocycle of G.*

Proof. Let $g, h, k \in G$. Now,

$$\begin{aligned} \Omega(g, h)\Omega(gh, k)\phi(ghk) =& \Omega(g, h)\phi(gh)\phi(k) \\ =& \phi(g)\phi(h)\phi(k) \\ =& \phi(g)\Omega(h, k)\phi(hk) \\ =& \Omega(h, k)\Omega(g, hk)\phi(ghk). \end{aligned}$$

From this, we can conclude that Ω is a 2-cocycle of G. □

1.1.2 Group Cohomology

Let G be a group, M be an abelian group. In the theory of group cohomology, there is a more general approach where G acts on M. But we will not consider such action.

Definition 1.1.6. *Let n be a positive integer. An n-cochain of G in M is a map $F : G^n(n \ times \ product) \to M$. Then $C^n(G, M)$ is the group of all $n-$cochains where the multiplication,inverse,identity ,are given by*

$$\begin{aligned} & 1) fg(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n)g(x_1, x_2, ..., x_n) \\ & 2) f^{-1}(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n)^{-1} \\ & 3) 1 : (x_1, x_2, ..., x_n) = 1_M, \end{aligned}$$

where $f, g \in C^n(G, M), x_i \in G$. A 0-cochain is defined to be an element of M.

Definition 1.1.7. *Coboundary of an n-cochain is an $(n + 1)$-cochain $\delta^n f$ given by*

$$(\delta^n f)(x_1, x_2, ..., x_{n+1}) = f(x_2, ..., x_{n+1})(\Pi_{i=1}^n f(x_1, ..x_i x_{i+1}, ..x_n)^{(-1)^i})f(x_1, .., x_n)^{(-1)^{n+1}}.$$

For all n-cochains we have $\delta^{n+1}(\delta^n(f)) = 1$ *and* $\delta^n(fg) = \delta^n(f)\delta^n(g)$. *Coboundary map* $\delta^n : C^n(G, M) \to C^{n+1}(G, M)$ *is a group homomorphism.*

Definition 1.1.8. *Let* $ker\delta^n$ *and* $image(\delta^{n-1})$ *be denoted by* $Z^n(G, M)$, $B^n(G, M)$ *respectively. The* $n-th$ *cohomology of* G *is defined as the group:* $H^n(G, M) = Z^n(G, M)/B^n(G, M)$. *Its elements are called cohomology classes. Two cocycles contained in the same cohomology classes are said to be cohomolgous.*

Example 1.1.9. *(2 nd cohomology group) Let* $\Omega \in Z^2(G, M)$, *which means* $(\delta^2\Omega)(x, y) = 1_M,\ for\ all\ x, y \in G$. *In other words, it satisfies the following equation*

$$\Omega(y, z)\Omega(xy, z)^{-1}\Omega(x, yz)\Omega(x, y)^{-1} = 1_M$$

for all $x, y, z \in G$.
If $\rho \in B^2(G, M)$, *then there exists an 1-cochain* f *such that* $\delta(f) = \rho$. *So, for all* $x, y \in G$

$$\rho(x, y) = f(x)f(xy)^{-1}f(y).$$

Let Ω_1, Ω_2 *be 2-cocycles. They are cohomologous if and only if there exists an 1-cochain* f *such that* $\Omega_1(x, y) = \Omega_2(x, y)f(x)f(xy)^{-1}f(y)$.

1.1.3 Central extension and cohomology class

Definition 1.1.10. *An exact sequence of groups is a sequence of group homomorphisms*

$$1 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} ... \xrightarrow{f_{n-1}} G_n \xrightarrow{f_n} 1$$

such that $Image(f_i) = ker(f_{i+1}), i \in \{1, 2, .., n\}$. *We call this sequence is a short exact sequence when* $n = 2$.

Definition 1.1.11. *An extension of a group* Q *by the group* N *is a short exact sequence*

$$1_N \xrightarrow{i} N \xrightarrow{f} G \xrightarrow{g} Q \xrightarrow{h} 1.$$

If $Image(f)$ *is in the center of* G, *then it is called central extension of group* Q *by* N.

Example 1.1.12. *This exact sequence*

$$1 \to \mathbb{C}^* \to GL(V) \xrightarrow{\pi} PGL(V).$$

is an example of central extension of $PGL(V)$ *by* $\mathbb{C}^*$.

Definition 1.1.13. *A morphism of exact sequences is a commutative diagram of group homomorphisms:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_1 & \longrightarrow & G_1 & \longrightarrow & Q_1 & \longrightarrow & 1 \\ & & \alpha\big\downarrow & & \beta\big\downarrow & & \gamma\big\downarrow & & \\ 1 & \longrightarrow & N_2 & \longrightarrow & G_2 & \longrightarrow & Q_2 & \longrightarrow & 1. \end{array}$$

Definition 1.1.14. *Let* $1 \to N \xrightarrow{f_1} G_1 \xrightarrow{p_1} Q_1 \to 1$ *and* $1 \to N \xrightarrow{f_2} G_2 \xrightarrow{p_2} Q_2 \to 1$ *be two extensions of the group* Q *by* N*. We say both are equivalent if there is a morphism* (Id_N, ϕ, Id_Q) *of exact sequences such that*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{f_1} & G_1 & \xrightarrow{p_1} & Q & \longrightarrow & 1 \\ & & Id_N\big\downarrow & & \phi\big\downarrow & & \big\downarrow Id_Q & & \\ 1 & \longrightarrow & N & \xrightarrow{f_2} & G_2 & \xrightarrow{p_2} & Q & \longrightarrow & 1. \end{array}$$

From five lemma, we can conclude that $\phi : G_1 \to G_2$ *is a group isomorphism.*

Let $P : G \to PGL(V)$ be a projective representation. From lemma (1.1.3), we know that there are set maps $P : G \to GL(V)(\ section\ of\ P), \Omega : G \times G \to \mathbb{C}^*(Schur\ multiplier)$. Then the cohomology class $\bar{\Omega}$ of Ω is called the cohomology class associated with the projective representation P and denoted by C_P.

Lemma 1.1.15. *If* $1 \to N \xrightarrow{i} G \xrightarrow{\rho} Q \to 1$ *is a central extension of the group* Q *by* N *, then for each section(i.e. a map such that* $\rho f = Id_Q$*)* $f : Q \to G$ *of* ρ *such that* $f(1_Q) = 1_G$*, we have* $f(n)f(n^{'})f(nn^{'})^{-1} \in image(i(N)),\ for\ all\ n, n^{'} \in N$ *. Therefore, the set map defined by* $\Omega_f(n, n^{'}) = i^{-1}(f(n)f(n^{'})f(nn^{'})^{-1})$ *is a* N *valued 2-cocycle.*

Theorem 1.1.16. *Two central extensions* $(G_1, \rho_1), (G_2, \rho_2)$ *of* Q *by the abelian group* N *is equivalent if and only if their associated cohomology classes* $C_{G_1,\rho_1}, C_{G_2,\rho_2} \in H^2(Q, N)$ *are equal.*

Now, we briefly discuss the relation between projective representation of a group, its central extensions and lifting of a projective representation to a linear representation.

Lemma 1.1.17. *Consider the commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_1 & \xrightarrow{f_1} & G_1 & \xrightarrow{\rho_1} & Q_1 & \longrightarrow & 1 \\ & & \alpha\big\downarrow & & \beta\big\downarrow & & & & \\ 1 & \longrightarrow & N_2 & \xrightarrow{f_2} & G_2 & \xrightarrow{\rho_2} & Q_2 & \longrightarrow & 1. \end{array}$$

Then there is a unique group homomorphism $\gamma : Q_1 \to Q_2$ *such that* $\gamma\rho_1 = \rho_2\beta$*.*

Definition 1.1.18. *Let $(Q_1, \rho_1), (Q_2, \rho_2)$ be two central extensions. Let $f : Q_1 \to Q_2$ be a group homomorphism. A lifting of f is a morphism of exact sequences (α, g, f) such that:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_1 & \xrightarrow{i_1} & G_1 & \xrightarrow[\rho_1]{} & Q_1 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 1 & \longrightarrow & N_2 & \xrightarrow{i_2} & G_2 & \xrightarrow{\rho_2} & Q_2 & \longrightarrow & 1. \end{array}$$

Lifting of a projective representation:

Let (ϕ, V, Ω) be a projective representation. Now, we consider the set $H = \{(g, A) \in G \times GL(V) : \pi\phi(g) = \pi(A)\}$, where $\pi : GL(V) \to PGL(V)$ is the quotient map. H is a subgroup of $G \times GL(V)$. Let $\psi : H \to G$ given by $\psi(g, A) = g$. Kernel of ψ is $\{(e_G, cId_{GL(V)}) : c \in \mathbb{C}^*\}$ which is in the center of the group $G \times GL(V)$. We can define an ordinary representation of H by setting to, $F(g, A) = A$. The ordinary representation F is the lift of (ϕ, V, Ω) in the sense that:

$$\pi F(g, A) = \pi\phi(g) = \pi\phi(\psi(g, A)).$$

Let $\{(\phi_i, V_i, \Omega_i) : i \in I\}$, where I is an indexed set, be the collection of all projective representations of G. Then we can lift all of them to linear representations of a subgroup $H = \{(h_i)_{i \in I}$: such that $\pi_i(h_i) = \pi_j(h_j)$, for all$i, j \in I\}$ of the group $\Pi_i H_i$. Here H_i is a group to which we can lift the projective representation ϕ_i. We denote by G' the commutator of a group G,which is the subgroup generated by $\{g^{-1}h^{-1}gh; g, h \in G\}$. G is said to be a perfect group if $G = G^{'}$. G is said to be super perfect group if its abelianization and Schur-multiplier both vanish.

1.1.4 Universal central extension

Let $E : 1 \to N \to G \to Q$ and $\bar{E} : 1 \to \bar{N} \to \bar{G} \to Q$ be two central extensions of Q. E is a cover of $\bar{E}$ if there exists a morphism (α, β, Id) of extensions such that,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{f} & G & \xrightarrow{g} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle Id} & & \\ 1 & \longrightarrow & \bar{N} & \xrightarrow{\bar{f}} & \bar{G} & \xrightarrow{\bar{g}} & Q & \longrightarrow & 1. \end{array}$$

E is said to be universal central extension of Q if E uniquely covers every central extension of Q.

Lemma 1.1.19. *If $E : 1 \to ker(\tau) \to G \xrightarrow{\tau} Q \to 1$ and $\bar{E} : 1 \to ker(\bar{\tau}) \to \bar{G} \xrightarrow{\bar{\tau}} Q \to 1$ be two central extensions of Q. Then the following statements hold:*

1) *There is a group isomorphism $\phi : G \to \bar{G}$ such that $\phi(ker(\tau)) = ker(\bar{\tau})$.*

2) If G is a perfect group and E covers $\bar{E}$, then E uniquely cover $\bar{E}$.

Lemma 1.1.20. *i) A universal central extension of a group G exists if and only if G is a perfect group.*

i) If Q is a perfect group then the universal central extension of Q is also a perfect group.

ii) If Q is a perfect group then the universal central extension is a super perfect group.

Example 1.1.21. *Special unitary group $SU(2)$ is a simply connected perfect compact group. The universal continuous central extension of $SU(2)$ is itself. So, every projective representation of $SU(2)$ is projectively equivalent to a linear representation.*

Example 1.1.22. $1 \to Z_2 \to SU(2) \to SO(3)$ *is universal central extension of special real orthogonal matrix group $SO(3)$. Every projective representation of $SO(3)$ is a linear representation of $SU(2)$.*

Theorem 1.1.23 (Theorem from [Rag94])**.** *If G is a connected semi-simple lie group,then $H^2(G, S^1)$ is isomorphic to the Pontryagin dual $\widehat{\pi_1(G)}$ of the fundamental group of $\pi_1(G)$. We can say that $SU(2)$ has no non-trivial projective representation.*

Remark 1.1.24. *From theorem (1.1.25), we can say $SO(3)$ has only one non-trivial projective representation because $\pi_1(SO(3)) = Z_2$.*

1.2 Operator algebra

1.2.1 C^* algebras

A Banach $*$-algebra $\mathcal{A}$ is a C^* algebra if it is satisfies : $\|x^*x\| = \|x\|^2$ for all x in $\mathcal{A}$. If $\mathcal{A}$ has unit then it is said to be unital or it is called non-unital. Every finite dimensional C^* algebra is unital. From Geland-Naimark-Segal theorem, we know any C^* algebra is isometrically isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ (where $B(\mathcal{H})$ denote the set of all bounded operators on a Hilbert space $\mathcal{H}$).

Let x be an element of $\mathcal{A}$, then the spectrum of x, denoted by $\sigma(x)$ is defined as $\{z \in \mathbb{C} : (z1 - x) \in GL(\mathcal{A})^c\}$, where $GL(\mathcal{A})$ is the set of all invertible elemnts of $\mathcal{A}$. x is said to be self adjoint if $x = x^*$, normal if $x^*x = xx^*$, unitary if $x^* = x^{-1}$ and a idempotent if $x^2 = x$ and a projection if it is a selfadjoint and idempotent, positive if $x = y^*y$ for some y in $\mathcal{A}$. For a normal element $x \in \mathcal{A}$, there is a injective ring homomorphism $C(\sigma(x))$ onto $C^*(x)$, which is a isometric *-isomorphism.

A C^* homomorphism between two C^* algebras $\mathcal{A}_1, \mathcal{A}_2$ is a ring homomorphism and a $*$ preserving map. If A linear map between two C^* algebras, send positive elements then it is said to be a positive map. So, any C^* homomorphism is a positive linear map. A state is a positive linear functional ϕ such that $\phi(1) = 1$. A state ϕ is called tracial state if $\phi(ab) = \phi(ba)$ for all a, b in $\mathcal{A}$ and faithful if $\phi(x^*x) = 0$ implies $x = 0$. Given a state ϕ on a C^* algebra $\mathcal{A}$, there exists a triple (called the GNS triple) $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ consisting of a Hilbert space $\mathcal{H}_\phi$, a $*$ representation π_ϕ of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\phi)$ and a vector ξ_ϕ in $\mathcal{H}_\phi$ which is cyclic in the sense that $\{\pi_\phi(x)\xi_\phi : x \in \mathcal{A}\}$ is total in $\mathcal{H}_\phi$ satisfying

$$\phi(x) = \langle \pi_\phi(x)\xi_\phi,\ \xi_\phi \rangle .$$

1.2.2 Multiplier C^* algebra

Definition 1.2.1. *A closed ideal I in a C^* algebra A is said to be essential ideal if for every nonzero closed ideal J of A, $I \cap J$ is a non-zero ideal in A.*

Definition 1.2.2. *A double centralizer of a C^* algebra A is a pair (L, R) of bounded linear map on A such that*

$$aL(b) = R(a)b, \quad \forall a, b \in A.$$

It follows from the definition that if (L, R) is a double centralizer of A then $\|L\| = \|R\|$. The set of all double centralizers of A can be given a C^* algebra structure. This C^* algebra is called the multiplier algebra $M(A)$ of A.

Remark 1.2.3. *A is an essential ideal in $M(A)$. If A is a unital C^* algebra then $A \cong M(A)$.*

Lemma 1.2.4. *Let I be an ideal in a C^* algebra A. If $\pi : I \to B$ is a faithful non-degenerate representation then π can be extended uniquely to a representation of $M(A)$.*

Consider the topology given by the seminorms $\{l_a, r_a : a \in A\}$ where $l_a(x) = ||ax||, r_a(x) = ||xa||$. The resulting topology is called the strict topology on $M(A)$. A is dense in $M(A)$ under this topology.

1.2.3 Universal C^* Algebra

Definition 1.2.5. *Let elements $E = \{x_i : i \in I\}$ be given, where I is some index set.*

a) A noncommutative monomial in E is a word $x_{i_1}x_{i_2}...x_{i_m}$ with $i_1, i_2....i_m \in I$ and $m \in N - \{0\}$.

b) A noncommutative polynomial in E is a formal complex linear combination of noncommutative monomials:$\sum_{k=1}^{n} \alpha_k y_k$ with $n \in N$,$\alpha_k \in \mathbb{C}$ and $y_1, y_2, ..., y_n$ being noncommutative monomials in E.

c) *on noncommutative monomials, we consider the concatenation of words,i.e.*

$$(x_{i_1}x_{i_2}...x_{i_m}).(x_{j_1}x_{j_2}...x_{j_n}) = x_{i_1}x_{i_2}...x_{i_m}x_{j_1}x_{j_2}...x_{j_n},$$

where $x_{i_1}x_{i_2}....x_{i_m}$ *and* $x_{j_1}x_{j_2}...x_{j_n}$ *are two monomials.*

d) *The free algebra on the generator set* E *is given the set of noncommutative polynomials in* E *together with canonical adition and scalar multiplication, and the multiplication of elements given by the concatenation. The elements in* E *are understood being distinct.*

The algebra is free in the sense that the elements x_i satisfy no nontrivial relation, i.e. the only polynomial in the generators which is zero, is the zero polynomial itself. Hence, the free algebra has following universal property: whenever B is some algebra containing elements $\{y_i : \imath \in I\}$, there is a algebra homomorphism from the free algebra to B sending $x_i \to y_i$, for all $i \in I$. Given $E = \{x_i : i \in I\}$, we add another set of generators $E^* = \{x_i^* : i \in I\}$ and we define an involution on the free algebra on $E \cup E^*$ by extending

$$(\alpha x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} ....x_{i_m}^{\epsilon_m})^* = \bar{\alpha} x_{i_m}^{\bar{\epsilon}_m} x_{i_2}^{\bar{\epsilon}_{m-1}} ....x_{i_1}^{\bar{\epsilon}_1}$$

to linear combinations; here $\alpha \in \mathbb{C}, \epsilon_k \in \{1, *\}$ and $\bar{\epsilon}_k = 1$ if $\epsilon_k = *$ and $\bar{\epsilon}_k = *$ if $\epsilon_k = 1$. So, we obtain the free * algebra $P(E)$ that is generated by E.

Definition 1.2.6. *We consider the following data :*

i) *Let* $E = \{x_i : i \in I\}$ *be a set of elements,* I *some index set.*

ii) *Let* R *be a set of polynimals in* $P(E)$.

Let $J(R)$ *be the two sided * ideal generated by* R. *The universal * algebra with generators* E *and relations* R *is defined as the quotient* $A(E|R) = P(E)/J(R)$. *Now, we want to find a* C^* *norm on* $A(E|R)$. *Let, define a seminorm on* $A(E|R)$ *given by*

$$\|x\| = sup\{\|\pi(x)\| : \pi \text{ is a } * \text{ homomorphism from A(E|R)} \to \text{B}(\mathcal{H})\},$$

where $\mathcal{H}$ *is a Hilbert space and the set of all* $*$ *homomorphisms is a nonempty set.*

Definition 1.2.7. *If* E *is the set of generators and* R *is the given relation and for all* $x \in A(E|R)$,$\|x\| < \infty$,*then the universal* C^* *algebra* $C^*(E|R)$ *generated by* E *with respect to relation* R, *defined by the completion of the * algebra* $A(E|R)/\{x \in A(E|R) : \|x\| = 0\}$ *with respect to* $\|.\|$.

Example 1.2.8. *Let $E = \{U, V\}$ and R be the relation given by $UU^* = U^*U = 1 = V^*V = VV^*$ and $UV = e^{2\pi i\theta}VU$. Then the universal C^* algebra (noncommutative 2 torus) $A_\theta = C^*(u, v : u, v \text{ unitaries}, \ uv = e^{2\pi i\theta}vu\}$ for θ belongs to $[0, 1]$.*

Example 1.2.9. *Let G be a locally compact group with left Haar measure μ. One can make $L^1(G)$ into a Banach $*$-algebra by defining*

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)d\mu(s),$$

$$f^*(t) = \Delta(t)^{-1}\overline{f(t^{-1})}.$$

Here $f, \ g$ are in $L^1(G), \ \Delta$ is the modular homomorphism of G.

$L^1(G)$ has a distinguished representation π_{reg} on $L^2(G)$ defined by $\pi_{reg}(f) = \int f(t)\pi(t)d\mu(t)$ where $\pi(t)$ is a unitary operator on $L^2(G)$ defined by $(\pi(t)f)(g) = f(t^{-1}g)$ $(f \in L^2(G), \ t, g \in G)$. The reduced group C^* algebra of G is defined to be $C_r^*(G) := \overline{\pi_{reg}(L^1(G))}^{\mathcal{B}(L^2(G))}$.

*The free or full group C^* algebra $C^*(G)$ is the universal C^* algebra which is generated by the Banach *-algebra $L^1(G)$.*

Remark 1.2.10. *For an abelian group, the following holds: $C_r^*(G) \cong C_0(\widehat{G})$, where $\widehat{G}$ is the group of characters on G.*

1.2.4 Von-Neumann algebra

Definition 1.2.11. *Let, $\mathcal{H}$ be a Hilbert space.*

1) *The strong operator topology on $B(\mathcal{H})$ is given by the family of seminorms $\{P_h : P_h(T) = \|T(h)\|, T \in B(\mathcal{H}), h \in H\}$.*

2) *The weak operator topology on $B(\mathcal{H})$ is given by the family of seminorms $\{P_{h_1,h_2} : P_{h_1,h_2}(T) = |\langle Th_1, h_2\rangle|\}, h_1, h_2 \in \mathcal{H}, T \in B(\mathcal{H})$.*

3) *The strong * topology on $B(\mathcal{H})$ is defined by the seminorms $P_h(T) = \|T(h)\|$ and $P_h^*(T) = \|T^*(h)\|$.*

Let B be a unital C^* subalgebra of $B(\mathcal{H})$. The commutant B' of B is defined by the $\{x \in B(\mathcal{H}) : xT = Tx, \ for \ all \ T \in B(\mathcal{H})\}$. B is said to be a von-Neumann algebra if $B = B''$. From the Double commutant theorem , we know B'' is weak * closure of the C^* subalgebra B.

Lemma 1.2.12. *Let $B \subseteq B(\mathcal{H})$ is a von-Neumann algebra then the followings hold :*

a) *B contains the range projections of all its elements.*

b) B is closed linear span of all its projections.

Theorem 1.2.13. *Let B be a unital $*$ subalgebra of $B(\mathcal{H})$. Then the closure of B under weak operator topology and strong operator topology are equal and it is equal to the double commutant $B^{''}$ of B.*

Definition 1.2.14. *A projection p_0 is said to be a sub-projection of p if $(p - p_0)$ is a positive element in B.*

Murray-von Neumann equivalence and comparison between teo projections

Fix a von-Neumann algebra $B \subseteq B(\mathcal{H})$. Let $P, Q \in B$ be two projections. P, Q are said to be Murray-von-Neumann equivalent (written $P \sim Q$) if there is a partial isometry $u \in B$ s.t. $u^*u = p, uu^* = q$. $\leq$ is a partial order relation on projections of $B(\mathcal{H})$ defined by $p \leq q$, if there is a subprojection q_0 of q s.t. $p \sim q_0$.

Definition 1.2.15. *Let $p \in B \subseteq B(\mathcal{H})$ is a projection.*

i) p is said to be a finite projection if there is no proper sub-projection p_0 of p, which is equivalent to p.

ii) p is said to be an infinite projection if there is a proper sub-projection p_0 of p, which is not equivalent to p.

A von-Neumann algebra B is said to be finite if 1_B is a finite projection.
The center of a von Neumann algebra is denoted by $Z(B)$ and the set of all projections of B is denoted by $P(B)$.

Theorem 1.2.16. *A von-Neumann algebra B is finite if and only if there exists a positive normal bounded linear map $\tau : B \to Z(B)$ such that*

$$
\begin{aligned}
&1)\tau(ab) = \tau(ba),\\
&2)\ if\ a \geq 0, \tau(a) = 0\ then\ a = 0,\\
&3)\tau(c) = c\ for\ c \in z(B),\\
&4)\tau(ca) = c\tau(a)\ for\ c \in z(B).
\end{aligned}
$$

Definition 1.2.17. *An infinite projection p is said to be properly infinite if $P_zP = 0$ or P_zP is an infinite projection for each central projection $P_z \in P(Z(B))$.*

Remark 1.2.18. *A von-Neumann algebra B is finite if and only if any isometry of B is unitary . For any finite-dimensional Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ is a finite von-Neumann algebra.*

1.2.5 Tensor product of C^* algebras

Let us assume that $\mathcal{A}$ and $\mathcal{B}$ are two algebras over $\mathbb{C}$. We will use the notation $\mathcal{A} \otimes \mathcal{B}$ for the algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{A}$ and $\mathcal{B}$ be two C^* algebras, then there is more than one norm on $\mathcal{A} \otimes_{alg} \mathcal{B}$ so that the completion with respect to that norm is a C^* algebra. We will mainly work with the injective tensor product, that is the completion of $\mathcal{A} \otimes_{alg} \mathcal{B}$ with respect to the norm given on $\mathcal{A} \otimes_{alg} \mathcal{B}$ by $\| \sum_{i=1}^{n} a_i \otimes b_i \| = \sup \| \sum_{i=1}^{n} \pi_1(a_i) \otimes \pi_2(b_i) \|_{\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}$, where $\pi_2 : \mathcal{A}_2 \to \mathcal{B}(\mathcal{H}_2)$, $\pi_1 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_1)$ are $*$-homomorphisms, $a_i \in \mathcal{A}, b_i \in \mathcal{B}$ and the supremum runs over all possible choices of $(\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2)$. When $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, $\mathcal{A} \subseteq B(\mathcal{H}), \mathcal{B} \subseteq B(\mathcal{K})$, we also consider the von Neumann algebraic $\mathcal{A} \bar{\otimes} \mathcal{B} \subseteq B(\mathcal{H} \otimes \mathcal{K})$ generated by $\mathcal{A} \otimes_{alg} \mathcal{B}$, that is $(\mathcal{A} \otimes_{alg} \mathcal{B})''$. We also make the following convention (unless otherwise mentioned). When either $\mathcal{A}$ or $\mathcal{B}$ is finite dimensional, we do not always write $\mathcal{A} \otimes_{alg} \mathcal{B}$ but simply write $\mathcal{A} \otimes \mathcal{B}$. However, when both are infinite dimensional, one has to specify the norm and we use $\mathcal{A} \otimes_{alg} \mathcal{B}$ for the algebraic tensor product.

1.3 Quantum group

Here, we briefly discuss about compact quantum group(CQG) and discrete quantum group(DQG), for detail discussion and proof, we refer to [Wan98], [VDW96, VD96], [KV00], [CP95], [Kas98], [KS97], [Maj95].

1.3.1 Hopf algebra

Let A be an unital associative algebra which is a vector space A over $\mathbb{C}$ together with two linear maps $m : A \otimes A \to A$ called the multiplication map and $\eta : \mathbb{C} \to A$ called the unit such that $m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)$ and $m(\eta \otimes \mathrm{id}) = \mathrm{id} = m(\mathrm{id} \otimes \eta)$. After dualizing this, we get the following definition.

Definition 1.3.1. *A coalgebra A is a vector space over $\mathbb{C}$ equipped with two linear maps $\Delta : A \to A \otimes A$ called the comultiplication or coproduct and $\epsilon : A \to \mathbb{C}$ such that*

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta,$$

$$(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon)\Delta.$$

Definition 1.3.2. *If a vector space $\mathcal{A}$ is an algebra and a coalgebra along with the conditions that $\epsilon : \mathcal{A} \to \mathbb{C}$ and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ are algebra homomorphisms then it is said to be a bialgebra.*

Definition 1.3.3. *A bialgebra $\mathcal{A}$ is said to be a Hopf algebra if there exists a linear* *map (antipode) $\kappa : \mathcal{A} \to \mathcal{A}$ such that*

$$m \circ (\kappa \otimes \mathrm{id})\Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes \kappa) \circ \Delta.$$

Definition 1.3.4. *A morphisms between two Hopf-algebras $(A, m_A, \eta_A, \Delta_A, \kappa_A), (B, m_B, \eta_B, \Delta_B, \kappa_B)$ is a unital algebra homomorphism $F : A \to B$ that is compatible with the structure maps in the sense that the followings are hold:*

$$\Delta_B F = (F \otimes F)\Delta_A,$$
$$\epsilon_B F = \epsilon_A,$$
$$\kappa_B F = F\kappa_A.$$

Let σ be the flip map on $A \otimes_{alg} A$ defined by $\sigma(a \otimes b) = b \otimes a$. A Hopf algebra $\mathcal{A}$ is said to be cocommutative if $\sigma\Delta = \Delta$.

Example 1.3.5. *The universal enveloping Lie algebra $U(\mathcal{G})$ of a Lie algebra $\mathcal{G}$ is the universal unital algebra generated by elements of $\mathcal{G}$, satisfying the relation $[x,\ y] = xy - yx$, for all $x, y \in \mathcal{G}$. $U(\mathcal{G})$ has the universal property:*

For a algebra $\mathcal{A}$ and every linear map $F : \mathcal{G} \to A$ that holds $F([x, y]) = F(x)F(y) - F(y)F(x)$ for all $x, y \in \mathcal{G}$, then there exists a unique unital algebra homomorphism $U(\mathcal{G}) \to A$ that extends F. From universal property of universal enveloping Lie algebra, The linear maps given by,

$$\Delta : \mathcal{G} \to U(\mathcal{G}) \otimes_{alg} U(\mathcal{G}) \quad \text{s.t. } \Delta(x) = x \otimes 1 + 1 \otimes x$$
$$\epsilon : \mathcal{G} \to \mathbb{C} \ s.t. \ \epsilon(x) = 0$$
$$\kappa : \mathcal{G} \to g \ s.t. \ \kappa(x) = -x$$

extends to a unital algebra homomorphisms $\Delta : U(\mathcal{G}) \to U(\mathcal{G}) \otimes_{alg} U(\mathcal{G}), \epsilon : U(\mathcal{G}) \to \mathbb{C}, \kappa : U(\mathcal{G}) \to U(\mathcal{G})$. One can easily check that it is a cocommutative Hopf algebra.

Definition 1.3.6. *A Hopf algebra ($\mathcal{A}, \Delta, \kappa, \epsilon$) is said to be a Hopf $*$-algebra if there is an involution map $*$ which maps a to an element denoted by a^* satisfying the following properties:*

1. For all a, b in $\mathcal{A}$, α, β in $\mathbb{C}, (a^*)^* = a,\ (\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*,\ \ (a.b)^* = b^* a^*$.

2. $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a $*$-homomorphism where the involution on $\mathcal{A} \otimes \mathcal{A}$ is defined by $(a \otimes b)^* = a^* \otimes b^*$.

1.3.2 Dual Hopf-algebra

Let $\mathcal{A}$ be a finite dimensional Hopf * algebra. Let $\mathcal{A}'$ be the dual of $\mathcal{A}$ in the category of complex Vector spaces. As $\mathcal{A}$ is a finite-dimensional vector space we have $(\mathcal{A} \otimes \mathcal{A})' =$

$\mathcal{A}^{'} \otimes \mathcal{A}^{'}$. The algebra structure on $\mathcal{A}^{'}$ given by $(fg)(a) = (f \otimes g)\Delta(a)$ and the coalgebra structure on $\mathcal{A}^{'}$ defined by $\Delta_{\mathcal{A}^{'}}(f)(a \otimes b) = f(ab)$, where $f, g \in \mathcal{A}^{'}$ and $a, b \in \mathcal{A}$. Antipode on $\mathcal{A}^{'}$ given by $\kappa^{'}_{\mathcal{A}}$ which is dual to the antipode of $\mathcal{A}$. Dualizing the unital map and counital map of $\mathcal{A}$, gives the counital and unital map $\mathcal{A}^{'}$. Involution given by $f^*(a) = \overline{f(\kappa_{\mathcal{A}}(a)^*)}$. Now, one can prove that it is a Hopf * algebra.

1.3.3 Compact quantum group

Definition 1.3.7. *Let $\mathcal{Q}$ be a unital C^* algebra and Δ be a unital C^* homomorphism from $\mathcal{Q}$ to $\mathcal{Q} \otimes \mathcal{Q}$. Then $(\mathcal{Q}, \Delta)$ is said to be a compact quantum group if it satisfying the following properties:*

(i) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$,

(ii) Each of the linear subspaces generate by $[\Delta(\mathcal{Q})(1 \otimes \mathcal{Q})]$ and $\Delta(\mathcal{Q}))(\mathcal{Q} \otimes 1)$ is norm dense in $\mathcal{Q} \otimes \mathcal{Q}$.

Example 1.3.8. *Let G be a compact group and $C(G)$ be the C^* algebra of $\mathbb{C}$-valued continuous functions on G. If the coproduct Δ on $C(G)$ given by $\Delta(f)(g, h) = f(g.h)$ for all f in $C(G)$, g, h in G. Then $(C(G), \Delta)$ is a compact quantum group.*

Remark 1.3.9. *Let $(\mathcal{Q}, \Delta)$ be a commutative CQG. Let $\sigma(\mathcal{Q})$ denote the Gelfand spectrum of $\mathcal{Q}$. The product structure on $\sigma(\mathcal{Q})$ given by $\chi\chi' = (\chi \otimes \chi')\Delta$ where χ, χ' are in $\sigma(\mathcal{Q})$. $\sigma(\mathcal{Q})$ is a compact set under weak $*$-topology inside dual $S^{'}$ of S. $\sigma(\mathcal{Q})$ is a semigroup and it satisfies the cancellation properties. So, It is a compact group.*

Definition 1.3.10. *The Haar state h on a CQG $\mathcal{Q}$ is the unique state on $(\mathcal{Q},\Delta)$ which satisfies the following conditions:*

$$(\mathrm{id} \otimes h)(\Delta(a)) = (h \otimes \mathrm{id})\Delta(a) = h(a)1_{\mathcal{Q}}.$$

Remark 1.3.11. *Let G be a compact group with a normalized Haar measure μ. Then μ induces a linear functional on the compact quantum group $C(G)$ which satisfies the following equations:*

$$\int_G f(xy)d\mu(y) = \int_G f(y)d\mu(y) = \int_G f(yx)d\mu(y).$$

The functional $\mu : C(G) \to \mathbb{C}$ defined by $\mu(f) = \int_G f(x)d\mu(x)$ is the Haar state for the CQG $C(G)$. The Haar state on an arbitrary CQG $\mathcal{Q}$ is a non-commutative analogue of Haar measure on G.

Definition 1.3.12. *A CQG morphism among two CQG* $(\mathcal{Q}_1, \Delta_1)$ *and* $(\mathcal{Q}_2, \Delta_2)$ *is a unital* $*$ *algebra Homomorphism* $\phi : \mathcal{Q}_1 \to \mathcal{Q}_2$ *such that*

$$(\Delta_2 \otimes \Delta_2) \circ \phi = (\phi \otimes \phi) \circ \Delta_1.$$

Remark 1.3.13. *If* h_1, h_2 *are Haar states on* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *respectively and* $\phi : \mathcal{Q}_1 \to \mathcal{Q}_2$ *is an injective CQG morphism then* $h_2 \circ \phi = h_1$*.*

Definition 1.3.14. *A Woronowicz* C^**-subalgebra* $\mathcal{Q}_2$ *of a compact quantum group* $(\mathcal{Q}_1, \Delta)$ *is a* C^* *subalgebra of* $\mathcal{Q}_1$ *such that* $\Delta(Q_2) \subseteq Q_2 \otimes Q_2$ *and the inclusion map* $i : \mathcal{Q}_2 \to \mathcal{Q}_1$ *is a CQG morphism.*

Definition 1.3.15. *A* C^* *ideal* J *of a CQG* $(\mathcal{Q}, \Delta)$ *is called a Woronowicz* C^**-ideal of* $(\mathcal{Q}, \Delta)$ *if* $\Delta(J) \subseteq \mathrm{Ker}(\pi \otimes \pi)$*, where* π *is the quotient map from* $\mathcal{Q}$ *to* $\mathcal{Q}/J$*.*

Remark 1.3.16. *kernel of a CQG morphism is a Woronowicz* C^**-ideal.*

Definition 1.3.17. *A CQG* $(\mathcal{Q}_1, \Delta_1)$ *is said to be a* **compact quantum subgroup** *of* $(\mathcal{Q}_2, \Delta_2)$ *if there exists a surjective CQG morphism from* $\mathcal{Q}_2$ *to* $\mathcal{Q}_1$*.*

Example 1.3.18. *Let* q *belongs to* $[-1, 1]$*. The* C^* *algebra* $SU_q(2)$ *is the universal unital* C^* *algebra generated by* α, γ *satisfying:*

$$\alpha\alpha^* + q^2\gamma\gamma^* = 1, \tag{1.3.1}$$

$$\alpha^*\alpha + \gamma^*\gamma = 1, \tag{1.3.2}$$

$$\gamma\gamma^* = \gamma^*\gamma, \tag{1.3.3}$$

$$q\gamma^*\alpha = \alpha\gamma^*, \tag{1.3.4}$$

$$q\gamma\alpha = \alpha\gamma. \tag{1.3.5}$$

Coproduct Δ *of* $SU_q(2)$ *is given by :*

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$$

which makes it into a CQG. Let h *denote the Haar state and* $\mathcal{H} = L^2(SU_q(2))$ *be the corresponding G.N.S space. We now look at the Haar state on* $SU_q(2)$*.*

For all $m \geq 1, n, l, k \geq 0, k' \neq k''$,

$$h((\gamma^*\gamma)^k) = \frac{1-q^2}{1-q^{2k+2}}, \; h(\alpha^m\gamma^{*n}\gamma^l) = 0, \; h(\alpha^{*m}\gamma^{*n}\gamma^l) = 0, \; h(\gamma^{*k'}\gamma^{*k''}) = 0. \tag{1.3.6}$$

Example 1.3.19. *Let* G *be a discrete group. We already know* $C_r^*(G)$ *is a* C^* *algebra. Here, the Haar state comes from the counting measure. Coproduct on* $C_r^*(G)$ *is given* *by:*

$$\Delta(\delta_\gamma) = \delta_\gamma \otimes \delta_\gamma.$$

Haar state:

$$h(\delta_\gamma) = 1 \; if \; \gamma = id_G$$
$$0 \; if \; \gamma \neq Id_G, for \; \gamma \in G.$$

The comultiplication is cocommutative. Similarly, We can conclude $C^*(G)$ is a CQG.

Example 1.3.20. *The* **quantum permutation group** S_n^+ *is the universal C^* algebra, generated by a_{ij} ($i, j = 1, 2, ...n$) and satisfying the following relations:*

$$a_{ij}^2 = a_{ij} = a_{ij}^*, \; i, j = 1, 2, ...n,$$

$$\sum_{i=1}^{n} a_{ij} = 1, \; i = 1, 2, ...n,$$

$$\sum_{j=1}^{n} a_{ij} = 1, \; i = 1, 2, ...n.$$

and the coproduct is given by $\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$. We refer the reader to [Wan98], where this compact quantum group was first introduced and also references [Ban05], [BBC07], [BB07] for more detailed study of quantum permutation group and quantum automorphism groups.

Example 1.3.21. Kac-Paljutkin Algbera:

Kac-Paljutkin algebra $(\mathcal{Q}_{kp}, \Delta)$, is a noncommutative , noncocommutative Hopf algebra and a eight dimensional Hopf algebra. It is given by

$$\mathcal{Q}_{kp} = \mathbb{C} \cdot \epsilon \oplus \mathbb{C} \cdot \alpha \oplus \mathbb{C} \cdot \beta \oplus \mathbb{C} \cdot \gamma \oplus M_2(\mathbb{C}),$$

as an $*$-algebra. The comultiplication $\Delta\colon \mathcal{Q}_{kp} \to \mathcal{Q}_{kp} \otimes \mathcal{Q}_{kp}$ is defined by

$$\Delta(x) = \epsilon \otimes x + \alpha \otimes u_\alpha x u_\alpha^* + \beta \otimes u_\beta x u_\beta^* + \gamma \otimes u_\gamma x u_\gamma^*$$
$$+ x \otimes \epsilon + \overline{u}_\alpha x \overline{u}_\alpha^* \otimes \alpha + \overline{u}_\beta x \overline{u}_\beta^* \otimes \beta + \overline{u}_\gamma x \overline{u}_\gamma^* \otimes \gamma,$$
$$\Delta(\epsilon) = \epsilon \otimes \epsilon + \alpha \otimes \alpha + \beta \otimes \beta + \gamma \otimes \gamma + \frac{1}{2} \sum_{1 \le i,j \le 2} e_{ij} \otimes e_{ij},$$
$$\Delta(\alpha) = \epsilon \otimes \alpha + \alpha \otimes \epsilon + \beta \otimes \gamma + \gamma \otimes \beta$$
$$+ \frac{1}{2}(e_{11} \otimes e_{22} + ie_{12} \otimes e_{21} - ie_{21} \otimes e_{12} + e_{22} \otimes e_{11}),$$
$$\Delta(\beta) = \epsilon \otimes \beta + \beta \otimes \epsilon + \alpha \otimes \gamma + \gamma \otimes \alpha$$
$$+ \frac{1}{2}(e_{11} \otimes e_{22} - ie_{12} \otimes e_{21} + ie_{21} \otimes e_{12} + e_{22} \otimes e_{11}),$$
$$\Delta(\gamma) = \epsilon \otimes \gamma + \gamma \otimes \epsilon + \alpha \otimes \beta + \beta \otimes \alpha$$
$$+ \frac{1}{2}(e_{11} \otimes e_{11} - e_{12} \otimes e_{12} - e_{21} \otimes e_{21} + e_{22} \otimes e_{22}),$$

where $\epsilon, \alpha, \beta, \gamma$ are projections, $x \in M_2(\mathbb{C})$ and e_{ij} are the matrix units in $M_2(\mathbb{C})$ and

$$u_\alpha = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \ u_\beta = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}, \ u_\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

1.3.4 Corepresentation of CQG

Definition 1.3.22. *A unitary corepresentation of $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$ is a unitary element $U \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{Q})$ such that $(Id \otimes \Delta)(U) = U_{12}U_{13}$. Here, $U_{12} = U \otimes 1_{\mathcal{Q}}$ and U_{13} is the image of U under the algebra homomorphism $a \otimes b \to a \otimes 1_{\mathcal{Q}} \otimes b$. If $\mathcal{H}$ is a finite-dimensional Hilbert space then $U \in B(\mathcal{H}) \otimes \mathcal{Q}$ is said to be a finite dimensional unitary corepresentation.*

Example 1.3.23. *For a classical compact group G, There is a one-to-one correspondence with finite dimensional unitary representations of G and the finite dimensional unitary corepresentations of $(C(G), \Delta)$. This correspondence is given by: $U = (u_g) \to U \in B(\mathcal{H}) \otimes C(G) \equiv C(G, B(\mathcal{H})$, defined by $U(g) = u_g$ for all $g \in G$, where $u_g \in U(\mathcal{H})$ is a representation of G.*

Definition 1.3.24. *A unitary corepresentation U of $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$ is said to be irreducible if $End(U) = \{T \in B(\mathcal{H}) : (T \otimes 1)U = U(T \otimes 1)\} = \mathbb{C}Id_{\mathcal{H}}$*

Definition 1.3.25. *If $U \in \mathcal{M}(\mathcal{K}(\mathcal{H}_U) \otimes \mathcal{Q})$ and $V \in \mathcal{M}(\mathcal{K}(\mathcal{H}_V) \otimes \mathcal{Q})$ are two corepresentations then the intertwiners between them denoted by $Mor(U, V)$ and defined by $\{T \in B(\mathcal{H}_U, \mathcal{H}_V) : V(T \otimes 1) = (T \otimes 1)U\}$.*

Definition 1.3.26. *Tensor product of two corepresentations U, V defined by $U \otimes V = U_{12}V_{13}$.*

Right regular corepresentation of CQG

Let $\mathcal{H}$ be the Hilbert space that came from the G.N.S representation of $\mathcal{Q}$ associated with Haar state h and ϵ_0 is the cyclic vector of this G.N.S representation. Let $\mathcal{K}$ be the another Hilbert space on which $\mathcal{Q}$ acts non-degenerately and faithfully . Then the unitary operator U on $\mathcal{H} \otimes \mathcal{K}$ defined by $U(a\xi_0 \otimes \eta) = \Delta(a)(\xi_0 \otimes \eta)$ is called right regular corepresentation, where a is in $\mathcal{Q}$, η is in $\mathcal{K}$. U is an element of multiplier of $\mathcal{K}(\mathcal{H}) \otimes \mathcal{Q}$ and said to be the right regular corepresentation of $\mathcal{Q}$ and $\Delta(a) = U(a \otimes 1)U^*$.

Remark 1.3.27. *Let G be a discrete group. Haar state on $C_r^*(G)$ is a faithful state. So, $C_r^*(G)$ acts faithfully and non-degenrately on the Hilbert space $\mathcal{H}$ which came from the G.N.S representation of $C_r^*(G)$ associated with Haar state h. The right regular representation V of $C_r^*(G)$ given by*

$$V : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \ s.t.$$
$$V(a\epsilon_0 \otimes b\epsilon_0) = a\epsilon_0 \otimes ab\epsilon_0,$$

where ϵ_0 is the cyclic vector of the G.N.S representation and $a, b \in C_r^*(G)$.

Proposition 1.3.28. *(1) Let $\{u^\alpha : \alpha \in I\}$ be a complete set of mutually inequivalent, irreducible unitary corepresentations of a CQG $(\mathcal{Q}, \Delta)$. We will denote the associated Hilbert space and dimension of u^α by $\mathcal{H}_\alpha$ and $n(\alpha)$ respectively.*

Then the Schur's orthogonality relation takes the following form:

For any α in I, there exists a positive invertible linear map F^α, acting on $\mathcal{H}_\alpha$ such that for any α, β in I and $1 \leq i, p \leq n(\beta)$ $1 \leq j, q \leq n(\alpha)$

$$h((u_{ip}^\beta)^* u_{jq}^\alpha) = \delta_{\alpha\beta}\delta_{pq}F_{ij}^\alpha.$$

(2) Let us denote the linear span of $\{u_{pq}^\alpha : \alpha \in I, 1 \leq p, q \leq n(\alpha)\}$ by $Pol(\mathcal{Q})$. It is a dense $*$-subalgebra of $\mathcal{Q}$.

(3) Moreover, Δ maps $\mathrm{Pol}(\mathcal{Q})$ *into* $\mathrm{Pol}(\mathcal{Q}) \otimes_{alg} \mathrm{Pol}(\mathcal{Q})$. *In fact, Δ is given by $\Delta(u_{pq}^\alpha) = \sum_{k=1}^{n_\alpha} u_{pk}^\alpha \otimes u_{kq}^\alpha$. A counit and an antipode are defined on* $\mathrm{Pol}(\mathcal{Q})$ *respectively by the formulae,*

$$\epsilon(u_{pq}^\alpha) = \delta_{pq}, \ \kappa(u_{pq}^\alpha) = (u_{qp}^\alpha)^*.$$

It follows that $\mathrm{Pol}(\mathcal{Q})$ *becomes a Hopf $*$-algebra.*

1.3.5 Coaction of CQG

Definition 1.3.29. *[Pod95]*

*Let A be a C^* algebra. A right coaction of a CQG $\mathcal{Q}$ on A is a *-homomorphism δ:$A \rightarrow A \otimes \mathcal{Q}$, which satisfies the following properties:*

(i) δ intertwines the co-multiplication, meaning that $(\delta \otimes Id)\delta=(Id\otimes\Delta)\delta$.

(ii) δ satisfy the density conditions $[\delta(A)(1\otimes \mathcal{Q})]=A\otimes\mathcal{Q}$.

If the right coaction is injective, then it is called $\mathcal{Q} - C^*$ algebra.

Remark 1.3.30. *If A is a finite dimensional algebra then $\delta(A) \subseteq A \otimes \mathrm{Pol}(\mathcal{Q})$.*

Example 1.3.31. *The comultiplication Δ is an right coaction of $(\mathcal{Q},\Delta)$ on itself. It is called the right regular coaction on $\mathcal{Q}$.*
Similarly, we can define the left coaction of the CQG $(\mathcal{Q}, \Delta)$ on a C^* algebra A.

Example 1.3.32. *Let $X_n = \{x_1, x_2, ..., x_n\}$ be a set of n distinct elements. The Commutative C^* algebra $C(X_n)$(set of all complex-valued functions on X_n) is isomorphic to the universal C^* algebra $C^*\{e_i : e_i^2 = e_i = e_i^*,\ \sum_{r=1}^{n} e_r = 1, i = 1, 2, ..., n\}$.*

Then S_n^+ has a C^* coaction on $C(X_n)$ given by:

$$\alpha(e_j) = \sum_{i=1}^{n} e_i \otimes a_{ij}, j = 1, 2, ...n.$$

Definition 1.3.33. *Let $(\mathcal{Q}, \delta)$ has a C^* coaction δ on the C^* algebra A. Then δ is said to be a faithful coaction if there is no proper Woronowicz C^*-subalgebra $\mathcal{Q}_1$ of $\mathcal{Q}$ such that δ is a C^* coaction of $\mathcal{Q}_1$ on A.*

Definition 1.3.34. *A right coaction δ of $\mathcal{Q}$ on A is said to be ergodic if $A^\delta=\{\delta(a)=a\otimes 1\}$ is isomorphic to the $\mathbb{C}$, that means the fixed point set of δ is a one-dimensional algebra.*

Proposition 1.3.35. *Let $\mathcal{Q}$ be a CQG. Let (A,δ) be a unital $\mathcal{Q}$-C^* algebra. If δ is ergodic then there exist a unitary corepresentation $V_A \in M(K(L^2(A)\otimes \mathcal{Q})$ of $\mathcal{Q}$ such that $\delta(a)=V_A(a\otimes 1)V_A^*$.*

A similar result can obtained when δ is not a Ergodic coaction. For that we have to use the conditional expectation E:$A\rightarrow A^\delta$ given by $E(a)=(\mathrm{id}\otimes h)\delta(\mathrm{a})$ for all a in A, to make a Hilbert A^δ module $L^2(A,E_\delta)$.

1.3.6 Discrete quantum group

Let A be an algebra over $\mathbb{C}$. A is said to be a nondegenrate algebra if $aA = 0$ implies $a = 0$ and $Aa = 0$ implies $a = 0$ for all a in A.

Definition 1.3.36. *An algebraic multiplier $\rho = (\rho_1, \rho_2)$ of the algebra A is a pair of linear mappings in $End_k(A)$ such that $\rho_2(a)b = a\rho_1(b)$ for all $a, b \in A$. The set of algebraic multipliers of A will be denoted by $M_{alg}(A)$.*

Remark 1.3.37. *It is a unital algebra which contains A as essential ideal through the embedding $a \hookrightarrow (a., .a)$. Hence $\rho \cdot a = (\rho_1(a)\cdot, \cdot\rho_1(a)) \equiv \rho_1(a)$ and $a \cdot \rho = (\rho_2(a)\cdot, \cdot\rho_2(a)) \equiv \rho_2(a)$ for all $\rho \in (A)$ and $a \in A$. If A is unital then $A = M(A)$. If A is a $*$-algebra then $M_{alg}(A)$ is a $*$-algebra through $\rho^* = (\rho_2^*, \rho_1^*)$ where $\psi^*(a) := \psi(a^*)^*$ for any $a \in A, \psi \in End_k(A)$. Since the multiplication of A is supposed to be non-degenerate an algebraic multiplier $\rho = (\rho_1, \rho_2)$ of A is uniquely determined by its first or second component. For a tensor product of two algebras A and B one obtains the canonical algebra embeddings*

$$A \otimes_{alg} B \hookrightarrow M_{alg}(A) \otimes_{alg} M_{alg}(B) \hookrightarrow M_{alg}(A \otimes_{alg} B)$$

Definition 1.3.38. *Let A be an algebra. An algebra morphism $\Delta : A \to M(A \otimes_{alg} A)$ is called a comultiplication on A if for all $a, a' \in A$. The linear maps $T_1, T_2 : A \otimes_{alg} A \to A \otimes_{alg} A$ as defined below*

$$\left.\begin{array}{l} T_1(a \otimes a') := \Delta(a)(1 \otimes a') \\ T_2(a \otimes a') := (a \otimes 1)\Delta(a') \end{array}\right\} \in A \otimes_{alg} A$$

satisfying the relation:

$$(T_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes T_1) = (\mathrm{id} \otimes T_1) \circ (T_2 \otimes \mathrm{id})\,.$$

If T_1 and T_2 are bijective then the pair (A, Δ) is called a multiplier Hopf algebra.

If A is a $*$-algebra and the comultiplication map Δ to be a $*$-algebra homomorphism, then A is said to be a multiplier Hopf $*$ algebra. The multiplier Hopf algebra (A, Δ) is called regular if in addition (A, Δ^{op}) is a multiplier Hopf algebra, where Δ^{op} is the opposite comultiplication, $\Delta^{op}(a)(b \otimes c) = \sigma(\Delta(a)(c \otimes b))$ for $a, b, c \in A$, and henceforth $\sigma : A \otimes_{alg} A \to A \otimes_{alg} A$ is the usual tensor transposition.

Definition 1.3.39. *A discrete quantum group (DQG) is a pair (A, Δ), where A is C_0 direct sum of full matrix algebra say $\overline{\bigoplus_{i \in I\ C_o} M_{n_i}(\mathbb{C})}$, where I is some index set and Δ is a comultiplication such that $(A_0, \Delta/_{A_0})$ becomes a multiplier Hopf $*$- algebra, where $A_0 = \oplus_{i \in I} M_{n_i}(\mathbb{C})$ (algebraic finite sum) without C^* closure.*

Remark 1.3.40. *If A is a DQG then there exists a unique counit ϵ which is a $*$ homomorphism from A to $\mathbb{C}$. It will satisfy*

$$(\epsilon \otimes Id)\Delta(a) = (Id \otimes \epsilon)\Delta(a) = a,$$

for all $a \in A$.

Dual DQG of a CQG

There exists a canonical duality between CQG and DQG in a way such that for a CQG $\mathcal{Q}$ with complete enumeration of mutually inequivalent unitary corepresentations $\{U_i : i \in I\}$, where I is an indexed set and $U_i \in B(\mathcal{H}_i) \otimes \mathcal{Q}$. Then the corresponding dual DQG is given by $\hat{\mathcal{Q}} = \overline{\oplus_{i \in I} B(\mathcal{H}_i)}$, where we take the C_0 closure.

Remark 1.3.41. *1) If A is a finite dimensional DQG, then A is also a compact quantum group.*

2) If A is a finite dimensional DQG then $A = \oplus_{i \in I} M_{n_i}(\mathbb{C})$, where I is a finite index set. The counit ϵ of A is a non-zero homomorphism but $\epsilon(M_{n_i}) = 0$ for $n_i > 1$. Hence there exist $i_0 \in I$ such that $n_{i_0} = 1$.

1.4 Normal subgroup of DQG

We recall the notion of normal quantum subgroups of a DQG and quotient with respect to them from [VV02], [BCV20] . Let $\hat{\mathcal{Q}}$ be a DQG given by C_0 direct sum of $\mathcal{B}(\mathcal{H}_\alpha), \alpha \in I$, where $d_\alpha = \dim(\mathcal{H}_\alpha)$ and let $\mathcal{Q}$ be the dual CQG with the mutually inequivalent irreducible corepresentations u^α corresponding to $\alpha \in I$, given by $((u^\alpha_{ij})) \in \mathcal{B}(\mathcal{H}_\alpha) \otimes \mathcal{Q}$. Let $W = \sum_{\alpha,i,j} (u^\alpha_{ji})^* \otimes e^\alpha_{ij} \in (\hat{\mathcal{Q}} \otimes \mathcal{Q})''$ be the so-called left regular corepresentation, where $e^\alpha_{ij}, i, j = 1, \ldots, d_\alpha$ are the 'matrix units' of $\mathcal{B}(\mathcal{H}_\alpha)$.

A quantum subgroup of $\hat{\mathcal{Q}}$ is given by a both sided weakly closed Hopf ideal. Any such ideals of a C_0 direct sum of matrix algebras is again one such a C_0 direct sum, say $\oplus_{\alpha \in J} \mathcal{B}(\mathcal{H}_\alpha)$ for some subset J of I. Then the quotient is again a C_0 direct sum, over $H := J^{(c)}$ (complement of J). We write $\hat{\mathcal{Q}_J}$ and $\hat{\mathcal{Q}_H}$ for the C_0 direct sum over the index sets J and H respectively and denote by P_J, P_H the corresponding central projections in $\hat{\mathcal{Q}}$ onto $\hat{\mathcal{Q}_J}$, $\hat{\mathcal{Q}_H}$ respectively. The co-ideal property of $\hat{\mathcal{Q}_J}$ implies $(P_H \otimes P_H)\Delta(a) = 0$ for all $a \in \hat{\mathcal{Q}_J}$. This implies that for $\alpha, \beta \in H$ and for any $\gamma \in I$ such that the irreducible index by γ is a direct summand of the tensor product of the irreducible correpreentations corresponding to α and β, we must also have $\gamma \in H$ too.

Note that $\hat{\mathcal{Q}_H}$ has the coproduct given by $\Delta_H := (P_H \otimes P_H) \circ \Delta$ restricted to $\hat{\mathcal{Q}_H}$. Indeed, it is clear that Δ_H maps $\hat{\mathcal{Q}_H}$ to $\hat{\mathcal{Q}_H} \otimes \hat{\mathcal{Q}_H}$ and the coassociativity of the map follows from the coassociativity of Δ combined with the fact that $\Delta_H((1 - P_H)(a)) = 0$, implying $\Delta_H(P_H a) = \Delta_H(a)$ for all a. There is a surjective quantum group morphism from $\mathcal{Q}$ to $\mathcal{Q}_H$ which sends a to $P_H(a)$.

Definition 1.4.1. *The quantum subgroup $\hat{\mathcal{Q}}_H$ is said to be normal if $W(\mathcal{Q}_H \otimes 1)W^* \subseteq \mathcal{Q}_H \otimes \mathcal{B}(\mathcal{H})$, or equivalently, $W(\mathcal{Q}_H \otimes 1) \subseteq (\mathcal{Q}_H \otimes \mathcal{B}(\mathcal{H}))W$. Here, $\mathcal{H}$ denotes the Hilbert space which is isomorphic with the GNS space of $\mathcal{Q}$ w.r.t. its Haar state.*

It is well-known (see [BCV20], [VV02]) and references therein that

Proposition 1.4.2. *$\hat{\mathcal{Q}}_H$ is said to be normal subgroup of $\hat{\mathcal{Q}}$ if and only if the left-invariant subalgebra $\{a \in \hat{\mathcal{Q}} : \ ((P_H \otimes \mathrm{id}) \circ \Delta)(a) = 1_H \otimes a\}$ coincides with the right-invariant subalgebra $\{a \in \mathcal{Q} : \ ((\mathrm{id} \otimes P_H) \circ \Delta)(a) = a \otimes 1_H\}$.*

Corollary 1.4.3. *If $\hat{\mathcal{Q}}_H$ is a normal subgroup of $\hat{\mathcal{Q}}$ then the left/right invariant subalgebra is a subset of $ker(P_H) \cup \{\mathbb{C}1_{\mathcal{Q}_H}\}$.*

If $\hat{\mathcal{Q}_H}$ is a normal subgroup, the corresponding quotient quantum group, say $\hat{\mathcal{Q}}/\hat{\mathcal{Q}_H}$, is given by the left (equivalently right) invariant subalgebra mentione above, which also coincides with the C^* algebra generated by the image of the operator-valued weight T_H given by $T(\cdot) = (\tau_H \otimes \mathrm{id}) \circ \Delta$ on $\mathcal{Q} \subseteq \mathcal{B}(\mathcal{H})$ where τ_H denotes the restriction of τ to $\mathcal{Q}_H$, which is a semifinite operator valued weight with all $\mathcal{Q}_\alpha$'s in the domain in particular.

Lemma 1.4.4. *Suppose for each $\alpha \in I$ and $\beta \in H$, there is some $V^{\alpha,\beta} \in \mathcal{Q}_H \otimes \mathcal{B}(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$ such that $(u^\alpha)_{(12)}(u^\beta)_{(13)} = V^{\alpha,\beta}(u^\alpha)_{(12)}$. Then $\hat{\mathcal{Q}_H}$ is a normal quantum subgroup of $\mathcal{Q}$.*

Proof. It suffices to verify the conditions of Definition 1.4.1 for the generators u^β_{kl} of $\mathcal{Q}_H$, for $\beta \in H$. But from assumption, we have $W_{(12)}(u^\beta)_{(13)} = VW_{(12)}$, with $V := \bigoplus_{\alpha \in I} V^{\alpha,\beta} \in \hat{\mathcal{Q}_H} \otimes \mathcal{B}(\mathcal{H}_\alpha) \otimes \mathcal{B}(\mathcal{H}_\beta)$. Taking ϕ to be the functional on $\mathcal{B}(\mathcal{H}_\beta)$ which is 1 on e^β_{kl} and 0 on other matrix elements of $\mathcal{B}(\mathcal{H}_\beta)$ and applying $(\mathrm{id}_{(12)} \otimes \phi)$ on both sides of the above equation, the proof of the lemma is complete. □

Let us now recall the framework of unitary tensor category (UTC for short) or C^* tensor category, from which we refer the reader to [NT] and the references therein. Consider a normal quantum subgroup $\hat{\mathcal{Q}_H}$ of $\hat{\mathcal{Q}}$ as in the preceeding discussion, with $\mathcal{C} = \hat{\mathcal{Q}}/\hat{\mathcal{Q}}_H$ being the quotient DQG, identified as a Woronowicz subalgebra of $\hat{\mathcal{Q}}$. For a DQG $\mathcal{D}$ let Rep($\mathcal{D}$) denote the UTC of finite dimensional $*$-representations of $\mathcal{D}$, which is also equivalent to the category Corep($\hat{\mathcal{D}}$) of finite dimensional corepresentations of the dual CQG $\hat{\mathcal{D}}$. Recall the set I that labels the irreducibles of Corep($\mathcal{Q}$). Let e denote the unit object, which corresponds to the counit ϵ of the DQG $\hat{\mathcal{Q}}$. The counit of $\mathcal{C}$ is the restriction of e to $\mathcal{C}$ and we denote the unit object of Rep($\mathcal{C}$) by $e_{\mathcal{C}}$. We define a relation $\alpha \sim \beta$ for $\alpha, \beta \in$ Rep($\hat{\mathcal{Q}}$) by:

Definition 1.4.5. $\alpha \sim \beta$ *if there is some* $h \in H$ *such that* $\mathrm{Mor}(\alpha, \beta \otimes h) \neq \{0\}$

Lemma 1.4.6. *(i)* $\sim$ *is an equivalence relation.*
(ii) for $\alpha, \beta \in I = \mathrm{Irr}(\hat{\mathcal{Q}})$*,* $\alpha \sim \beta$ *if and only if there is some* $h \in H$ *such that* $\alpha \leq \beta \otimes h$*.*

The above lemma follow from the 4 th section of [DCKSSt18].

There are canonical functors $F_1 : \mathrm{Rep}(\hat{\mathcal{Q}_H}) \to \mathrm{Rep}(\hat{\mathcal{Q}})$ and $F_2 : \mathrm{Rep}(\hat{\mathcal{Q}}) \to \mathrm{Rep}(\mathcal{C})$ given by the following:

$$F_1((\rho, \mathcal{H})) = (\rho \circ P_H, \mathcal{H}), \quad F_2((\theta, \mathcal{K})) = (\theta|_{\mathcal{C}}, \mathcal{K}),$$

where $P_H : \hat{\mathcal{Q}} \to \hat{\mathcal{Q}_H}$ the surjective map and $\theta|_{\mathcal{C}}$ denotes the restriction of θ to the subalgebra $\mathcal{C}$ of $\hat{\mathcal{Q}}$. Both F_1 and F_2 are identity on morphisms. It is easy to verify that F_1 and F_2 are tensor functors.

Lemma 1.4.7. *Suppose for some object* x *of* $\mathrm{Rep}(\hat{\mathcal{Q}})$*,* $e_{\mathcal{C}} \leq F_2(x)$*. Then* $x \in \mathrm{Rep}(\hat{\mathcal{Q}_H})$*.*

Proof. Decomposing x into irreducibles and using the fact that F_2 is a tensor functor, we can assume that there is some irreducible $\alpha \leq x$ such that $e_{\mathcal{C}} \in F_2(\alpha)$. Let α correspond to $(\rho, \mathcal{H})$ where $\mathcal{H}$ is a finite dimensional Hilbert space of dimension d_α and ρ is unitarily conjugate to the projection $P_\alpha : \hat{\mathcal{Q}} \to \hat{\mathcal{Q}_\alpha} \cong \mathcal{B}(\mathbb{C}^{d_\alpha})$. The assumption $e_{\mathcal{C}} \leq F_2(\alpha)$ means there is some one dimensional subspace, say spanned by a nonzero vector ξ, of $\mathcal{H}$ such that

$$\rho(a)\xi = \epsilon(a)\xi \tag{1.4.1}$$

for $a \in \mathcal{C}$. Recall the $\mathcal{C}$-valued weight T_H which projects from $\mathcal{Q}$ to $\mathcal{C}$, given by $T_H(\cdot) = (\tau_H \otimes \mathrm{id}) \circ \Delta(\cdot)$. Take some positive nonzero element $b \in \mathcal{Q}_h$ for some $h \in H$ and let $a = T_H(b) \in \mathcal{C}$. Equation (1.4.1) implies

$$(\tau_H \otimes \rho)(\Delta(b))\xi = \tau_H(b)\xi. \tag{1.4.2}$$

As $\tau_H(b)$ is nonzero, left hand side of Equation (1.4.2) is also nonzero, which is possible only if there is some $h' \in H$ such that $(P_{h'} \otimes P_\alpha)(\Delta(b))$ is nonzero for all nonzero $b \in \mathcal{Q}_h$. But this means $h \leq h' \otimes \alpha$, hence $h \sim \alpha$ or $\alpha \sim h$, i.e. $\alpha \leq h \otimes h_1$ for some $h_1 \in H$, so in particular $\alpha \in H$. □

Let us define an equivalence relation $\sim'$ on $\mathrm{Rep}(\mathcal{C})$ by the following

Definition 1.4.8. *For* $x, y \in \mathrm{Rep}(\mathcal{C})$*, define* $x \sim' y$ *if there are positive integers* m, n *such that* $x \leq my$ *and* $y \leq nx$*, where for any object* z *we denote the direct sum of* k *copies of* z *by* kz*.*

Remark 1.4.9. *It is easy to verify that* $\sim'$ *is an equivalence relation and* $x \sim' y$ *if and only if* x *and* y *are direct sums of the same set of irreducibles, possibly with different multiplicities. In case* x, y *are irreducible objects,* $x \sim' y$ *if and only if they are isomorphic.*

Corollary 1.4.10. $F_2(\alpha) \sim' F_2(\beta)$ *for some* $\alpha, \beta \in \mathrm{Rep}(\hat{\mathcal{Q}})$ *if and only if* $\alpha \sim \beta$.

Proof:
We have $e_{\mathcal{C}} \leq F_2(\alpha) \otimes \overline{F_2(\alpha)} \leq F_2(\alpha) \otimes \overline{F_2(n\beta)} = nF_2(\alpha \otimes \overline{\beta})$ for some positive integer n. As $e_{\mathcal{C}}$ is irreducible, this means $e_{\mathcal{C}} \leq F_2(\alpha \otimes \overline{\beta})$, hence by Lemma 1.4.7 we have $\alpha \otimes \overline{\beta} \in \mathrm{Rep}(\hat{\mathcal{Q}_H})$, say $\alpha \otimes \overline{\beta} \leq h_1 \oplus \ldots \oplus h_k$ for some $h_1, \ldots, h_k \in H$. This implies, $\mathrm{Mor}(\alpha, \oplus_{i=1}^k h_i \otimes \beta) \neq \{0\}$, hence there is some i such that $\mathrm{Mor}(\alpha h_i \otimes \beta) \neq \{0\}$, i.e.$\alpha \sim \beta$.

For the converse, it is enough to note that $F_2(h) = e_{\mathcal{C}}$ for all $h \in H$. □

For $\alpha \in I$ let

$$K_\alpha := \{x \in \mathrm{Irr}(\mathcal{C}) : \ x \leq F_2(\alpha)\}.$$

Lemma 1.4.11. *For any two* $\alpha, \beta \in I$*, the following are equivalent:*
(i) $K_\alpha \bigcap K_\beta$ *is nonempty.*
(ii) $\alpha \sim \beta$.
(iii) $K_\alpha = K_\beta$.

Proof:
(i) implies (ii): let $x \in K_\alpha \bigcap K_\beta$. Then $e_{\mathcal{C}} \subseteq \overline{x} \otimes x \leq \overline{F_2(\alpha)} \otimes F_2(\beta) = F_2(\overline{\alpha} \otimes \beta)$. By Lemma 1.4.7, $\overline{\alpha} \otimes \beta \in \mathrm{Rep}(\mathcal{Q}_H)$, hence $\alpha \sim \beta$.
(ii) implies (iii) : follows from Corollary 1.4.10 by noting that $F_2(\alpha) \sim' F_2(\beta)$ if and only if $K_\alpha = K_\beta$.
(iii) implies (i) is obvious. □

As F_2 is surjective on objects, we get a partition of $\mathrm{Irr}(\mathcal{C})$ in terms of K_α's labelled by equivalence classes $[\alpha] \in I/\sim$:

$$\mathrm{Irr}(\mathcal{C}) = \bigcup_{[\alpha] \in I/\sim} K_\alpha.$$

1.5 Tensor category

1.5.1 C^* tensor category

Definition 1.5.1. *A category* $\mathcal{C}$ *is called a* C^* *category if*

i) For all objects U *and* V, $Mor(U, V)$ *is a Banach space and the map*

$$Mor(V, W) \times Mor(U, V) \to Mor(U, W), \ \textit{given by} \ (S, T) \to ST,$$

is a bilinear map and the following holds: $||ST|| \leq ||S|| \, ||T||$;

ii) There exists an antilinear contravariant functor $* : \mathcal{C} \to \mathcal{C}$ *which maps object to the same object and for a* $T \in Mor(U,V), T^* \in Mor(V,U)$ *and the followings hod :*

a) $T^{**} = T$, *for* $T \in Mor(U,V)$*;*

b) $||T^*T|| = \big||T|\big|^2$. $End(U) = \ Mor(U,U)$ *is a unital* C^* *algebra.*

c) For any $T \in Mor(U,V)$*, the element* $T^*T \in End(U)$ *is a positive element.*

Definition 1.5.2. *A* C^**- category is said to be* $\mathcal{C}^*$*-tensor category if the following conditions holds:*

1) $\mathcal{C}$ *is a small category;*

2) There is a bilinear bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *and unitary natural isomorphisms* $\alpha_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$*, which is called associativity morphisms and a unit object* 1 *and also natural unitary isomorphisms* $\lambda_U : 1 \otimes U \to U, \rho_U : U \otimes 1 \to U$

3) Associativity morphism $\alpha_{U,V,W}$ *satisfies the following pentagonal diagram:*

$$\begin{array}{ccccc} & & ((U \otimes V) \otimes W) \otimes X & & \\ & \swarrow^{\alpha_{U,V,W} \otimes i} & & \searrow^{\alpha_{U \otimes V,W,X}} & \\ (U \otimes (V \otimes W) \otimes X & & & & (U \otimes V) \otimes (W \otimes X) \\ \downarrow{\scriptstyle \alpha_{U,V \otimes W,X}} & & & & \downarrow{\scriptstyle \alpha_{U,V,W \otimes X}} \\ U \otimes ((V \otimes W) \otimes X) & & \xrightarrow{\quad i \otimes \alpha_{V,W,X} \quad} & & U \otimes (V \otimes (W \otimes X)) \end{array}$$

4) $\lambda_1 = \rho_1$ *and this diagram*

$$\begin{array}{ccc} (U \otimes 1) \otimes V & \xrightarrow{\quad \alpha_{U,1,V} \quad} & U \otimes (1 \otimes V) \\ & {\scriptstyle \rho \otimes id} \searrow \quad \swarrow {\scriptstyle id \otimes \lambda} & \\ & U \otimes V & \end{array}$$

commutes;

5) $\mathcal{C}$ *is closed under finite direct sums. If we choose any two objects* U, V*, then there exists an object* W *and isometries* $u \in Mor(U,W)$ *and* $v \in Mor(U,W)$ *such that* $uu^* + vv^* = 1$*;*

6) *If* $U \in obj\mathcal{C}$ *and* $P \in End(U)$ *is a projection then there exists an object* V *and isometry* $v \in Mor(U,V)$ *such that* $vv^* = P$*. The zero object is the corresponding subobject for the zero projection in* $End(U)$*.*

7) *for the unit object* $End(1) = \mathbb{C}1$*.*

Remark 1.5.3. *Every* C^* *tensor category is equivalent to a strict* C^* *tensor category. From now on, usually we will consider strict* C^* *categories only.*

Example 1.5.4. *The category of finite dimensional Hilbert space, denoted by* $Hilb_f$*, which has finite dimensional Hilbert spaces as the set of objects, morphisms given by linear maps from* $\mathcal{H}$ *to* $\mathcal{K}$*.* $\mathcal{H} \otimes \mathcal{K}$ *is the usual tensor product between two Hilbert spaces. Associative morphism* $\alpha_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}$ *is the identity map from* $(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3$ *to* $\mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes \mathcal{H}_3)$*.* $1_{\mathbb{C}}$ *is unit object of this category* $Hilb_f$*.* $Hilb_f$ *is a strict* C^* *tensor category.*

Example 1.5.5. *Representation category of a compact group is denoted by* $Rep(G)$*. Objects of this category are denoted by* $(\pi, \mathcal{H}_\pi)$*, where* $\pi : G \to U(\mathcal{H}_\pi)$ *is a finite dimensional unitary representation of* G *on* $\mathcal{H}_\pi$*. Morphism between two objects* $(\pi_1, \mathcal{H}_{\pi_1})$ $(\pi_2, \mathcal{H}_{\pi_2})$ *is defined by*

$$Mor((\pi_1, \mathcal{H}_{\pi_1}), (\pi_2, \mathcal{H}_{\pi_2})) = \{T \in \mathcal{B}(\mathcal{H}_{\pi_1}, \mathcal{H}_{\pi_2}) : T\pi_1(g) = \pi_2 T(g), \ for \ all \ g \in G\}.$$

Tensor product between two objects $(\pi_1, \mathcal{H}_{\pi_1})$ $(\pi_2, \mathcal{H}_{\pi_2})$ *is defined by* $(\pi_1 \otimes \pi_2, \mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2})$*, where* $\pi_1 \otimes \pi_2$ *is the usual tensor product of two group representations and* $\mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2}$ *is the usual tensor product between two Hilbert spaces. Associativity morphism is the identity map and unit object is the trivial representation on* $\mathbb{C}$*.* $Rep(G)$ *is a strict* C^* *tensor category.*

Example 1.5.6. *Unitary corepresentation category of a CQG denoted by* $Corep(\mathcal{Q})$*. objects are given by* $(U, \mathcal{H}_U)$*, where* $U \in B(\mathcal{H}_U) \otimes \mathcal{Q}$ *is a finite dimensional unitary corepresentation of* $\mathcal{Q}$ *on* $\mathcal{H}_U$*. Morphism between two objects* $(U, \mathcal{H}_U), (V, \mathcal{H}_V)$ *is given by*

$$Mor((U, \mathcal{H}_U), (V, \mathcal{H}_V)) = \{T \in B(\mathcal{H}_U, \mathcal{H}_V) : (T \otimes 1)U = V(T \otimes 1)\}$$

Tensor product of two objects $(U, \mathcal{H}_U), (V, \mathcal{H}_V)$ *defined by* $(U_{13}V_{23}, \mathcal{H}_U \otimes \mathcal{H}_V)$*. Associativity morphism is the identity map and unit object is* $Id_{\mathbb{C}} \otimes 1_{\mathcal{Q}}$*. For a projection* $P \in End(U, \mathcal{H}_U)$*,* $((P \otimes 1)U, P\mathcal{H}_U)$ *is a subobject of* $(U, \mathcal{H}_U)$*.* $Corep(\mathcal{Q})$ *is a strict* C^* *tensor category.*

Definition 1.5.7. *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be* C^* *tensor categories. A tensor functor* $\mathcal{C} \to \mathcal{C}'$ *is a functor* $F : \mathcal{C} \to \mathcal{C}'$ *that is linear on morphisms together with an isomorphism*

$F_0 : 1^{'} \to F(1)$ in $\mathcal{C}^{'}$ and natural isomorphisms

$$F_2(U,V) : F(U) \otimes F(V) \to F(U \otimes V),$$

such that the following diagrams commute:

$$\begin{array}{ccccc} (F(U)\otimes F(V))\otimes F(W) & \xrightarrow{F_2(U,V)\otimes i} & F(U\otimes V)\otimes F(W) & \xrightarrow{F_2(U\otimes V,W)} & F((U\otimes V)\otimes W) \\ \downarrow{\scriptstyle \alpha_{F(U),F(V),F(W)}} & & & & {\scriptstyle F(\alpha_{U,V,W})}\downarrow \\ F(U)\otimes(F(V)\otimes F(W)) & \xrightarrow{id\otimes F_2(V,W)} & F(U)\otimes F(V\otimes W) & \xrightarrow{F_2(U,V\otimes W)} & F(U\otimes(V\otimes W)) \end{array}$$

$$\begin{array}{ccc} F(1)\otimes F(U) & \xrightarrow{F_2(1,U)} & F(1\otimes U) \\ \downarrow{\scriptstyle F_0\otimes i} & & {\scriptstyle F(\lambda)}\downarrow \\ 1^{'}\otimes F(U) & \xrightarrow{\lambda^{'}} & F(U) \end{array} \qquad \begin{array}{ccc} F(U)\otimes F(1) & \xrightarrow[F_2(U,1)]{} & F(U\otimes 1) \\ \downarrow{\scriptstyle i\otimes F_0} & & {\scriptstyle F(\rho)}\downarrow \\ F(U)\otimes 1^{'} & \xrightarrow{\rho^{'}} & F(U) \end{array}$$

F is called a unitary tensor functor (UTF) if $F(T^*) = F(T)^*$ for $T \in Mor(U,V)$ and $F_2(U,V), F_0$ are unitary maps.

Definition 1.5.8. *Let $F, G : \mathcal{C} \to \mathcal{C}^{'}$ be two tensor functors between $\mathcal{C}, \mathcal{C}^{'}$. A natural isomorphisom $\eta : F \to G$ is called monoidal if the diagrams below commute,*

$$\begin{array}{ccc} F(U)\otimes F(V) & \xrightarrow{F_2(U,V)} & F(U\otimes V) \\ \downarrow{\scriptstyle \eta_U\otimes\eta_V} & & \downarrow{\scriptstyle \eta_{U\otimes V}} \\ G(U)\otimes G(V) & \xrightarrow{G_2(U,V)} & G(U\otimes V) \end{array} \qquad \begin{array}{ccccc} & & 1^{'} & & \\ & {\scriptstyle F_0}\swarrow & & \searrow{\scriptstyle G_0} & \\ F(1) & & \xrightarrow{\eta_1} & & G(1) \end{array}$$

where U, V be two objects of the category $\mathcal{C}$ and $1^{'}$ is the unit object of this category G.

We will mainly deal with semisimple categories.

Definition 1.5.9. *Let $\mathcal{C}$ and $\mathcal{C}^{'}$ be two C^* tensor categories. They are said to be monoidally equivalent if there exist tensor functors $F_1 : \mathcal{C} \to \mathcal{C}^{'}$ and $F_2 : \mathcal{C}^{'} \to \mathcal{C}$ such that $F_1F_2 \cong Id_{\mathcal{C}^{'}}, F_2F_1 \cong Id_{\mathcal{C}}$. If F_1, F_2 are unitary functors and the natural isomorphisms $F_1F_2 \cong Id_{\mathcal{C}^{'}}, F_2F_1 \cong Id_{\mathcal{C}}$ are unitary natural isomorphisms then $\mathcal{C}$ and $\mathcal{C}^{'}$ are called unitary monoidally equivalent.*

Let $\{U_\alpha, \alpha \in I\}$ (I some index set), be a collection of all pairwise nonequivalents simple objects of C. If $\mathcal{C}$, $\mathcal{C}^{'}$ (two strict C^* tensor category) are monoidally equivalent, then there exists a tensor functor $F : \mathcal{C} \to \mathcal{C}^{'}$ such that $F(U_\alpha)$ are irreducible, pairwise nonequivalent,and any irreducible object in $\mathcal{C}^{'}$ is isomorphic to some $F(U_\alpha)$.

We will assume that every C^* tensor category $\mathcal{C}$ is a strict C^* tensor category as any general C^* tensor category is monoidally equivalent to a strict C^* tensor category.

Definition 1.5.10. *Let U be an object of $\mathcal{C}$. An object $\bar{U}$ of the category C is called a conjugate of U in $\mathcal{C}$ if there exist morphisms*

$$R : 1 \rightarrow \bar{U} \otimes U$$
$$\bar{R} : 1 \rightarrow U \otimes \bar{U}$$

such that

$$U \xrightarrow{i\otimes R} U \otimes \bar{U} \otimes U \xrightarrow{\bar{R}^*\otimes i} U \qquad \bar{U} \xrightarrow{i\otimes \bar{R}} \bar{U} \otimes U \otimes \bar{U} \xrightarrow{R^*\otimes i} \bar{U}$$

are the identity morphisms. $(\bar{R}^* \otimes i)(i \otimes R) = i_U$ and $(R^* \otimes i)(i \otimes \bar{R}) = i_{\bar{U}}$ are called the conjugate equations. If every object of $\mathcal{C}$ has a conjugate object then $\mathcal{C}$ is said to be rigid C^* tensor category. If any object of a strict C^* tensor category has a conjugate object then it is unique up to isomorphism.

Example 1.5.11. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\{e_i : i \in I\}$ is an orthonormal basis of $\mathcal{H}$. Then $\overline{\mathcal{H}} = \{\bar{a} : a \in \mathcal{H}\}$ is a vector space, where $\bar{a} + \bar{b} = \overline{a+b}, c\bar{a} = \overline{\bar{c}a}, c \in \mathbb{C}$. $\{\overline{e_i}\}$ is an orthonormal basis for the vector space $\overline{\mathcal{H}}$. Inner product is defined by $< \bar{a}, \bar{b} >=< b, a >$. Then $\bar{\mathcal{H}}$ is a Hilbert space. If $\mathcal{H}$ is an object of the category $Hilb_f$ then $R : \mathbb{C} \rightarrow \bar{\mathcal{H}} \otimes \mathcal{H}$ is defined by $R(1) = \sum \bar{e}_i \otimes e_i$ and $\bar{R} : \mathbb{C} \rightarrow \mathcal{H} \otimes \bar{\mathcal{H}}$ defined by $\bar{R}(1) = \sum e_i \otimes \bar{e}_i$, are the conjugate equations for $\mathcal{H}$, hence $\bar{\mathcal{H}}$ is a conjugate object of $\mathcal{H}$ and $Hilb_f$ is a rigid strict C^* tensor category.*

Let $j : B(\mathcal{H}) \rightarrow B(\bar{\mathcal{H}})$ be a linear map given by $j(T)(\bar{h}) = \overline{T^*}(h)$. j is a $*$-anti-homomorphism and $j^2 = Id_{\mathcal{H}}$.

Definition 1.5.12. *Let $U \in B(\mathcal{H}) \otimes \mathcal{Q}$ is an invertible element, where $\mathcal{H}$ is a finite dimensional Hilbert space, $\mathcal{Q}$ is a CQG. The contragredient of U is defined by*

$$U^c = (j \otimes id)(U^{-1}) \in B(\bar{\mathcal{H}}) \otimes \mathcal{Q}. \tag{1.5.1}$$

Example 1.5.13. *Consider the category $Corep(\mathcal{Q})$. Let $U \in B(\mathcal{H}) \otimes \mathcal{Q}$ be a finite dimensional unitary corepresentation of $\mathcal{Q}$. From proposition (1.4.14) of [NT], we know that there exists a positive invertible operator $\rho_U \in B(\mathcal{H}_U)$ such that $(\rho_U \otimes 1)U = U^{cc}(\rho_U \otimes 1)$. $\bar{U}$ defined by $(j(\rho_U)^{1/2} \otimes 1)U^c(j(\rho_U)^{-1/2} \otimes 1)$. The operators $\bar{R_U} = (1 \otimes j(\rho_U^{1/2})\bar{R} \in Mor(1_{\mathbb{C}}, U \otimes \bar{U})$ and $R_U = (1 \otimes j(\rho_{\bar{U}}^{1/2})R \in Mor(1_{\mathbb{C}}, \bar{U} \otimes U)$ satisfy the conjugate equations for U, where $R(1) = \sum \bar{e}_i \otimes e_i$ and $\bar{R}(1) = \sum e_i \otimes \bar{e}_i$. $\bar{U}$ is a conjugate object of U. Hence, $Corep(\mathcal{Q})$ is a rigid strict C^* tensor category.*

Theorem 1.5.14. (**Frobenius reciprocity**) *If any object U of $\mathcal{C}$ has a conjugate object then the following holds:*

1) $Mor(U \otimes V, W) \cong Mor(V, \bar{U} \otimes W)$.

2) $Mor(V \otimes U, W) \cong Mor(V, W \otimes \bar{U})$.

1.5.2 Woronowicz's Tannaka-Krein duality

Definition 1.5.15. *A fiber functor $F : \mathcal{C} \to Hilb_f$ is a tensor functor which is faithful and exact .*

Example 1.5.16. *Let $F^{Nat} : Corep(\mathcal{Q}) \to Hilb_f$ is a fiber functor defined by $F^{Nat}(U, \mathcal{H}_U) = \mathcal{H}_U$, where $(U, \mathcal{H}_U)$ is an object of this category , identity on the morphisms and also $F_2(U, V), F_0$ are both identity maps.*

We note a standard fact without proof:

Lemma 1.5.17. *A linear functor F is faithful if and only if the image of a every simple object is nonzero.*

The above lemma is true for any unitary tensor functor because unit object 1 is a subobject of $F(U \otimes \bar{U}) \cong F(U) \otimes F(\bar{U})$. If for any simple object U, $F(U) = 0$, then $F(U \otimes \bar{U}) = 0$. So, it is not possible to embed the unit object 1 inside $F(U \otimes \bar{U})$. Any linear functor is exact. Therefore a fiber functor is simply a tensor functor $F : \mathcal{C} \to Hilb_f$.

Theorem 1.5.18 (Woronowicz's Tannaka-Krein duality [Wor88, Wor98])**.** *Let C be a a strict rigid C^* tensor category and F be a unitary fiber functor on C. Then there exist a CQG $\mathcal{Q}$ and unitary monoidal equivalence $E : C \to Corep(\mathcal{Q})$ such that F is naturally unitary monidally isomorphic to the composition of the canonical fiber functor $Corep(\mathcal{Q}) \to Hilb_f$ with E.*

Example 1.5.19. *Tambara–Yamagami tensor categories [TY98] is equivalent to the the category of representations of the Kac–Paljutkin Hopf algebra [TY98], which is arising from the Klein 4-group $K_4 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Elements of $K_4 = \{e, s, t, st\}$ satisfies the relations $s^2 = t^2 = (st)^2 = e$. $\chi = \chi_c$ is a nondegenerate symmetric bicharacter of K_4 which is given by*

$$\chi_c(a, a) = \chi_c(b, b) = -1, \ \chi_c(a, b) = 1,$$

and considering the parameter $\tau = \frac{1}{2}$ .Let $\mathcal{C}(\chi, \tau)$ be a category and Its objects are finite direct sums of elements in $S = K_4 \cup \{\rho\}$. Sets of morphisms between elements in S are *given by*

$$Mor(s,s') = \begin{cases} \mathbb{C} & s = s', \\ 0 & s \neq s', \end{cases}$$

so S is the set of irreducible classes of $\mathcal{C}(\chi,\tau)$. Tensor products of elements in S are given by

$$s \otimes \rho = \rho = \rho \otimes s, \ \rho \otimes \rho = \bigoplus_{s \in K_4} s, s \otimes t = st, \ (s,t \in K_4)$$

and the unit object is e. Associativities φ are given by

$$\begin{aligned} &\varphi_{s,t,u} = id_{stu}, && \varphi_{s,t,\rho} = \varphi_{\rho,s,t} = id_\rho, \\ &\varphi_{s,\rho,t} = \chi_c(s,t) id_t, && \varphi_{s,\rho,\rho} = \varphi_{\rho,\rho,s} = \bigoplus_{k \in K_4} id_k, \\ &\varphi_{\rho,s,\rho} = \bigoplus_{k \in K_4} \chi_c(s,t) id_k, && \varphi_{\rho,\rho,\rho} = \left(\frac{1}{2}\chi_c(k,l)^{-1} id_\rho\right)_{k,l} : \bigoplus_{k \in K_4} \rho \to \bigoplus_{l \in K_4} \rho, \end{aligned}$$

for $s,t,u \in K_4$. Now, if we choose the natural fiber functor of this category then this category is identified with the corepresentation category of Kac–Paljutkin quantum group $\mathcal{Q}_{kp}$, that is

$$\mathcal{C}\left(\chi_c, \frac{1}{2}\right) \simeq Rep(\mathcal{Q}_{kp}) \simeq Corep(\hat{\mathcal{Q}_{kp}})$$

as tensor categories.

Chapter 2

Projective corepresentation and cohomology of CQG

2.1 Cohomology of quantum groups

In this section, we mainly focus on the second cohomology of compact quantum group. We begin with Hopf *-algebras. In this section, let us denote by $\otimes$ the algebraic tensor product for Hopf *-algebra being considered. Let $(\mathcal{Q}, \Delta)$ be a Hopf *-algebra and

$$\Delta_i : \mathcal{Q}^{\otimes n} \to \mathcal{Q}^{\otimes(n+1)} \textit{ such that}$$
$$\Delta_i = id \otimes ... \otimes \Delta \otimes ... \otimes id,$$

Δ is in the i-th position for $i \in \{1, 2, ., n\}$ and we define $\Delta_0(x) = 1 \otimes x$ *and* $\Delta_{n+1}(x) = x \otimes 1$. So that Δ_i defined for $i = 0, 1, .., n+1$. A left $n-$ cochain χ is an invertible element of $H^{\otimes n}$. Coboundary of a left n-cochain χ is a $(n+1)$-cochain

$$\delta_\chi = (\Pi_{i=0}^{even} \Delta_i(\chi))(\Pi_{i=1}^{odd} \Delta_i(\chi^{-1}))$$

Definition 2.1.1. *A left n-cochain $\chi \in \mathcal{Q}^{\otimes n}$ is said to be a n-cocycle if $\delta_\chi = 1$. A left n-cocycle is said to be counital if $\epsilon_i \chi = 1$, where $\epsilon_i = id \otimes ...\epsilon \otimes ...id$, ϵ at i-th position.*

Example 2.1.2. *A left 1-cocycle χ is an invertible element in $\mathcal{Q}$ such that $\Delta(\chi) = \chi \otimes \chi$. and it is automatically counital and any left 2-cocycle $\chi \in \mathcal{Q}^{\otimes 2}$ is satisfy the equation*

$$(1 \otimes \chi)(id \otimes \Delta)(\chi) = (\chi \otimes 1)(\Delta \otimes id)(\chi)$$

and it is counital if $(\epsilon \otimes id)(\chi) = 1$.

Lemma 2.1.3. *Let G be a finite group then any left 1-cocycle χ for the ring of continuous function $C(G)$ is a group homomorphism $\chi : G \to \mathbb{C} - \{0\}$. A counital left 2-cocycle*

χ of $C(G)$ is a normalized complex valued 2-cocycle of G.

Definition 2.1.4. *Let Ω_1, Ω_2 be left 2-cocycles of $\mathcal{Q}$. Ω_1, Ω_2 are said to be cohomologous if there exists an invertible element h in $\mathcal{Q}$ such that $\Omega_2 = (h \otimes h)\Omega_1 \Delta(h^{-1})$.*

$\mathbf{H^2}(\mathcal{Q}, \mathbb{C}^*)$ be the set of cohomology classes of 2-cocycles. It does not necessarily form a group.

Definition 2.1.5. *A left 2-cocycle Ω is said to be unitary 2-cocycle if and only if Ω is a unitary element of $\mathcal{Q} \otimes \mathcal{Q}$ and two unitary left 2-cocycles are said to be unitarily cohomologous if and only if there exists a unitary element u such that $\Omega_2 = (u \otimes u)\Omega_1 \Delta(u^{-1})$.*

$\mathbf{H^2}(\mathcal{Q}, \mathbf{S^1})$ be the set of unitary cohomology classes of unitary left 2-cocycles.

We can similar define right $n-$cochain and $n-$cocycle. As we need only 1 and 2 right cocycle, let us define them below.

Definition 2.1.6. *A right (unitary) 1-cocycle χ' is an invertible (a unitary) element in $\mathcal{Q}$ such that $\Delta(\chi') = \chi' \otimes \chi'$ and an invertible (a unitary) element $\chi' \in \mathcal{Q} \otimes \mathcal{Q}$, is said to be a (unitary) right 2-cocycle if it satisfying the equation*

$$(id \otimes \Delta)(\chi')(1 \otimes \chi') = (\Delta \otimes id)(\chi')(\chi' \otimes 1).$$

Remark 2.1.7. *Ω is left 2-cocycle if and only if Ω^* is right 2-cocycle.*

Definition 2.1.8. *An element of $\mathcal{Q}^{\otimes n}$ is said to be invariant if it commutes with the elements in the image of $\Delta^{n-1} : \mathcal{Q} \to \mathcal{Q}^{\otimes n}$, where Δ^{n-1} is defined inductively as follows:$\Delta^1 = \Delta$, and Δ^k is obtained by applying Δ to any of the factors of Δ^{k-1}.*

Remark 2.1.9. *An invariant element of $\mathcal{Q} \otimes \mathcal{Q}$ is a left cochain/cocycle if and only if it is also a right cochain/cocycle. Hence we will simply call them invariant cochain/cocycle.*

Lemma 2.1.10. *If $\Omega_1, \Omega_2 \in \mathcal{Q}^{\otimes 2}$ are invariant unitary 2-cocycles, then $\Omega_1 \Omega_2$ is a unitary 2-cocycle and Ω_1^*,Ω_2^* are both unitary 2-cocycles.*

Let $A^1(\mathcal{Q})$ be a set of all central invertible elements of $\mathcal{Q}$ and $A^2(\mathcal{Q})$ be a set of all invariant invertible 2-cocycles of $\mathcal{Q}$.

Lemma 2.1.11. *$\delta : A^1(\mathcal{Q}) \to A^2(\mathcal{Q})$ is a group homomorphism and image δ is a central subgroup of $A^2(\mathcal{Q})$.*

Definition 2.1.12. *Invariant 1-cohomology and 2-cohomology of $\mathcal{Q}$ is given by*

$$H^1_{inv}(\mathcal{Q}, \mathbb{C} - \{0\}) = ker(\delta : A^1(\mathcal{Q}) \to A^2(\mathcal{Q})),$$
$$H^2_{inv}(\mathcal{Q}, \mathbb{C} - \{0\}) = A^2(\mathcal{Q})/Image(\delta).$$

Example 2.1.13 (Theorem 7.1 of [GK10]). $H^2_{inv}(\mathcal{Q}, \mathbb{C} - \{0\}) = 1$ *if* $\mathcal{Q} = \mathbb{C}^*(G)$ *and* G *belongs to the following list of finite groups:*

i) the simple groups,
(ii) the symmetric groups S_n,
(iii) the groups $SL_n(F_q)$,
(iv) the groups $GL_n(F_q)$ *when* n *is coprime to* $q - 1$.

Example 2.1.14 (proposition 7.7 of [GK10]). $H^2_{inv}(\mathcal{Q}, \mathbb{C} - \{0\}) = Z_2$, *if* $\mathcal{Q} = \mathbb{C}^*(A_4)$, *where* A_4 *is the alternating group.*

Lemma 2.1.15 (Theorem 7.4 of [EG20]). *If* G *is a connected affine algebraic group then* $H^2_{inv}(\mathbb{C}^*(G), \mathbb{C}^*)$ *is a commutative group.*

Remark 2.1.16. *For a finite group* G, $H^2_{inv}(\mathbb{C}^*(G), \mathbb{C} - \{0\})$ *can be a noncommutative group.*

Let $A^1_u(\mathcal{Q})$ be a set of all central unitary elements of $\mathcal{Q}$ and $A^2_u(\mathcal{Q})$ be a set of all invariant unitary 2-cocycles of $\mathcal{Q}$. Similarly, we can prove that $\delta : A^1_u \to A^2_u$ is a group homomorphism and $\delta(A^1_u)$ is a central subgroup of $A^2_u(\mathcal{Q})$.

Definition 2.1.17. *Unitary Invariant 1-cohomology and 2-cohomology of $\mathcal{Q}$ is given by*

$$H^1_{uinv}(\mathcal{Q}, s^1) = ker(\delta : A^1_u(\mathcal{Q}) \to A^2_u(\mathcal{Q}),$$
$$H^2_{uinv}(\mathcal{Q}, s^1) = A^2_u(\mathcal{Q})/\delta(A^1_u(\mathcal{Q})).$$

The following lemma is easily follows from lemma (3.1.5) of [NT].

Lemma 2.1.18. *There exists an injective group homomorphism* θ , *is given by*

$$\theta : H^2_{uinv}(\mathcal{Q}, s^1) \to H^2_{inv}(\mathcal{Q}, \mathbb{C} - \{0\})$$
$$\theta([\Omega]) = [\Omega],$$

where Ω *is an invariant unitary 2-cocycle in* $\mathcal{Q}$.

Example 2.1.19. *From example (2.1.13), it is easily follows that* $H^2_{uinv}(\mathcal{Q}, S^1) = 1$ *if* $\mathcal{Q} = \mathbb{C}^*(G)$ *and* G *belongs to the following list of finite groups:*

(i) symmetric groups S_n,
(ii) $SL_n(F_q)$,
(iii)$GL_n(F_q)$ when n is coprime to $q-1$,
(iv) the simple groups

We can easily extend the notion of invertible and unitary left/right/invariant cocycles to both CQG and DQG and also von Neumann bialgebra, by replacing the algebraic tensor product by suitable C^* or von Neumann algebraic tensor product, and also replaces $\mathcal{Q} \otimes \mathcal{Q}$ by $M(\mathcal{Q} \otimes \mathcal{Q})$ for a DQG $\mathcal{Q}$. Counital cocycles will make sense under some conditions, we have the counit is bounded e.g for DQG.

2.2 Alternative description of fiber functor

Here we will give an alternative definition of a fiber functor on the strict rigid C^* tensor category $Corep(\mathcal{Q})$.

Let $(\mathcal{Q}, \Delta)$ be a CQG. I be a collection of pairwise non-equivalent irreducible corepresentations of $(\mathcal{Q}, \Delta)$. The dual DQG $\hat{\mathcal{Q}}$ is a C_0 direct sum of $B(\mathcal{H}_U)$, where$(U, \mathcal{H}_U) \in I$ or in general $\hat{\mathcal{Q}} = \overline{\bigoplus_{C_0} B(\mathcal{H}_U)}$, where $(U, \mathcal{H}_U) \in I$.

Proposition 2.2.1. *A unitary tensor functor (UTF) ϕ on $Corep(\mathcal{Q})$ is determined by the association $(U, \mathcal{H}_U) \mapsto \mathcal{H}_{\phi_U} = \phi(U, \mathcal{H}_U)$, which is a finite dimensional Hilbert space and linear maps*

$$\phi : Mor(U_1 \otimes \cdots \otimes U_r, V_1 \otimes \cdots \otimes V_k) \to B(\mathcal{H}_{\phi(U_1)} \otimes \cdots \otimes \mathcal{H}_{\phi(U_r)}, \mathcal{H}_{\phi(V_1)} \otimes \cdots \otimes \mathcal{H}_{\phi(V_k)})$$

satisfying equations

$$\begin{aligned} \phi(1) &= 1 \qquad & \phi(S \otimes T) &= \phi(S) \otimes \phi(T) \\ \phi(S^*) &= \phi(S)^* \qquad & \phi(ST) &= \phi(S)\phi(T). \end{aligned} \tag{2.2.1}$$

Remark 2.2.2. *If ϕ is a unitary fiber functor, then there exists a faithful linear map $\phi : Mor(U, V \otimes W) \to B(\mathcal{H}_{\phi(U)}, \mathcal{H}_{\phi(V)} \otimes \mathcal{H}_{\phi(W)})$, where U, V, W are simple objects. If we choose orthonormal bases for $Mor(U, V \otimes W)$, we can write two orthonormal basis for $Mor(U, X \otimes Y \otimes Z)$. One of the basis consists of morphisms of the form $(S \otimes id)T$ and the other basis consists of $(Id \otimes S')T'$ where X, Y, Z are irreducible objects and $S \in Mor(V, X \otimes Y), T \in Mor(U, V \otimes Z), S' \in Mor(V', Y \otimes Z), T' \in Mor(U, X \otimes V')$, where V is a subobject of $X \otimes Y$ and V' is a subobject of $Y \otimes Z$. The coefficients of transition unitary between both orthonormal bases are called 6j symbols.*

Definition 2.2.3. *Two unitary fiber functors* ϕ_1 *and* ϕ_2 *on* $Corep(\mathcal{Q})$ *are said to be* isomorphic *if there exist unitaries* $u_x \in B(\mathcal{H}_{\phi_1(x)}, \mathcal{H}_{\phi_2(x)})$ *satisfying*

$$\phi_2(S) = (u_{y_1} \otimes \cdots \otimes u_{y_k})\phi_1(S)(u^*_{x_1} \otimes \cdots \otimes u^*_{x_r})$$

for all $S \in Mor(x_1 \otimes \cdots \otimes x_r, y_1 \otimes \cdots \otimes y_k), x_1 \cdots x_r, y_1 \cdots y_k \in I.$

Definition 2.2.4. *An invertible (unitary) left/right 2-cocycle* $\Omega \in M(\hat{\mathcal{Q}} \otimes \hat{\mathcal{Q}})$ *is said to be an invertible (unitary) normalized 2-cocyle if* $(\hat{\epsilon} \otimes 1)\Omega = \hat{\epsilon} \otimes 1$ *and* $(1 \otimes \hat{\epsilon})\Omega = (1 \otimes \hat{\epsilon})$*., where* $\hat{\epsilon}$ *is the counit of* $\hat{\mathcal{Q}}$*.*

Let Ω be a normalized 2-cocycle, and let $\Omega_2 := (\hat{\Delta} \otimes 1)(\Omega)(\Omega \otimes 1) = (1 \otimes \hat{\Omega})(\Omega)(1 \otimes \Omega)$. We define a unique unitary fiber functor ϕ_Ω on $Corep(\mathcal{Q})$ such that

$$H_{\phi_\Omega(U)} = H_U, \quad \phi_\Omega(S) = \Omega S \quad \phi_\Omega(T) = \Omega_2 T, \tag{2.2.2}$$

where $a, x, y, z \in I$ and for all $S \in Mor(X, Y \otimes Z), T \in Mor(a, x \otimes y \otimes z)$. From Proposition (3.12) of [BRV06] , we get a CQG $(\mathcal{Q}_\Omega, \Delta_\Omega)$ whose dual is given by $\hat{\mathcal{Q}_\Omega} = \hat{\mathcal{Q}}$ and $\hat{\Delta}_\Omega(a)\phi_\Omega(S) = \phi_\Omega(S)a$ for all $a \in B(\mathcal{H}_x), S \in Mor(x, y \otimes z), x, y, z \in I$. ϕ_Ω is a monoidal equivalence between $Corep(\mathcal{Q}_\Omega)$ *and* $Corep(\mathcal{Q})$. Coproduct $\hat{\Delta}_\Omega(a) = \Omega^* \hat{\Delta}(a)\Omega$, for all $a \in \hat{\mathcal{Q}}$.

Proposition 2.2.5 (Proposition (4.5) of [BRV06])**.** *:*

Let ϕ *be a unitary fiber functor on* $Corep(\mathcal{Q})$ *and* $dim(\mathcal{H}_{\phi(x)}) = \mathcal{H}_x$*, for all irreducible objects* $x \in Corep(\mathcal{Q}, \Delta)$*. Then there exists a normalized 2 cocycle* Ω *on* $(\hat{\mathcal{Q}}, \hat{\Delta})$ *determined up to coboundary such that* ϕ *is isomorphic to the fiber functor* F^{Nat}_Ω*.*

2.3 Projective corepresentation

We will first discuss little bit about measurable projective corepresentations. However, in this thesis we will mainly concerned with continuous projective corepresentations. Sometimes we drop the word continuous.

Definition 2.3.1. *([DC11b]) A von Neumann bialgebra* (M, Δ_M) *consists of a von Neumann algebra* M *and a faithful normal unital* **-homomorphism* $\Delta_M : M \to M\bar{\otimes}M$*, satisfying the coassociativity condition*

$$(\Delta_M \otimes \iota)\Delta_M = (\iota \otimes \Delta_M)\Delta_M.$$

A von Neumann bialgebra (M, Δ_M) *is called a* compact Woronowicz algebra *if there* *exists a faithful normal state* φ_M *on* M *which is* Δ_M-invariant*:*

$$(\varphi_M \otimes \iota)\Delta_M(x) = (\iota \otimes \varphi_M)\Delta_M(x) = \varphi_M(x)1 \qquad \textit{for all } x \in M.$$

Definition 2.3.2. *([DC11b]) Let* (M, Δ_M) *be a compact Woronowicz algebra. A right* Galois co-object *for* (M, Δ_M) *consists of a Hilbert space* $\mathcal{L}^2(N)$*, a* σ*-weakly closed linear space* $N \subseteq B(\mathcal{L}^2(M), \mathcal{L}^2(N))$ *and a normal linear map* $\Delta_N : N \to N\bar{\otimes}N$*, such that the following properties hold: with* N^{op} *denoting the set*

$$N^{op} := \{x^* \mid x \in N\} \subseteq B(\mathcal{L}^2(N), \mathcal{L}^2(M)),$$

1. $N \cdot \mathcal{L}^2(M)$ *is norm-dense in* $\mathcal{L}^2(N)$*, and* $N^{op} \cdot \mathcal{L}^2(N)$ *is norm-dense in* $\mathcal{L}^2(M)$*,*
2. *the space* N *is a right* M*-module ,*
3. *for each* $x, y \in N$*, we have* $x^*y \in M$*,*
4. $\Delta_N(xy) = \Delta_N(x)\Delta_M(y)$ *for all* $x \in N$ *and* $y \in M$*,*
5. $\Delta_N(x)^*\Delta_N(y) = \Delta_M(x^*y)$ *for all* $x, y \in N$*,*
6. Δ_N *is coassociative:* $(\Delta_N \otimes \iota)\Delta_N = (\iota \otimes \Delta_N)\Delta_N$*, and*
7. *the linear span of the set* $\{\Delta_N(x)(y \otimes z) \mid x \in N, y, z \in M\}$ *is* σ*-weakly dense in* $N\bar{\otimes}N$*.*

If (N_1, Δ_{N_1}) *and* (N_2, Δ_{N_2}) *are two Galois co-objects for a von Neumann bialgebra* (M, Δ_M)*, we call them* isomorphic *if there exists a unitary* $u : \mathcal{L}^2(N_1) \to \mathcal{L}^2(N_2)$ *such that* $uN_1 = N_2$ *and*

$$\Delta_{N_2}(ux) = (u \otimes u)\Delta_{N_1}(x) \qquad \textit{for all } x \in N_1.$$

Example 2.3.3. *If* (M, Δ_M) *is a compact worinowicz algebra, then* (M, Δ_M) *is a right Galois co-object for* (M, Δ_M)*.*

Example 2.3.4. *Let* (M, Δ_M) *be a compact Woronowicz algebra, and* Ω *a unitary 2-cocycle for* (M, Δ_M)*. Now, if we choose* $\mathcal{L}^2(N) = \mathcal{L}^2(M)$*,* $N = M$ *and*

$$\Delta_N(x) = \Omega\Delta_M(x), \qquad \textit{for all } x \in M,$$

then (N, Δ_N) *is a Galois co-object for* (M, Δ_M)*, called the* Galois co-object associated to Ω

Definition 2.3.5. *Let (M, Δ_M) be a compact Woronowicz algebra, (N, Δ_N) is a right Galois co-object for (M, Δ_M). A* (left) *(N, Δ_N)*-corepresentation *of (M, Δ_M) on a Hilbert space $\mathcal{H}$ consists of a unitary map $\mathcal{G} \in N \bar{\otimes} B(\mathcal{H})$ such that*

$$(\Delta_N \otimes \iota)\mathcal{G} = \mathcal{G}_{13}\mathcal{G}_{23}.$$

Definition 2.3.6. *Let (M, Δ_M) be a compact Woronowicz algebra. A measurable left* projective *corepresentation of (M, Δ_M) on a Hilbert space $\mathcal{H}$ consists of a left coaction α of (M, Δ_M) on $B(\mathcal{H})$,*

$$\alpha : B(\mathcal{H}) \to M \bar{\otimes} B(\mathcal{H}).$$

Similarly, a measurable right projective corepresentation of (M, Δ_M) on a Hilbert space H consists of a right coaction β of (M, Δ_M) on $B(H)$

$$\beta : B(\mathcal{H}) \to B(\mathcal{H}) \bar{\otimes} M.$$

We will denote by ${}_{\Omega}\Delta(x) := \Omega\Delta(x)$ *and* $\Delta_{\Omega^*}(x) := \Delta(x)\Omega^*$ for all $x \in \mathcal{Q}$.

Definition 2.3.7. *Let $(\mathcal{Q}, \Delta_{\mathcal{Q}})$ be a C^* algebraic CQG. A continous left projective corepresentation of $(\mathcal{Q}, \Delta_{\mathcal{Q}})$ on a Hilbert space $\mathcal{H}$ consists of a left coaction α of $(\mathcal{Q}, \Delta_{\mathcal{Q}})$ on $\mathcal{K}(\mathcal{H})$,*

$$\alpha : \mathcal{K}(H) \to M(\mathcal{Q} \otimes K(\mathcal{H})).$$

As it is a non-degenrate $*$ homomorphism, we can extend this coaction on $M(K(\mathcal{H})) = B(\mathcal{H})$. Similarly, a continuous right projective corepresentation of $(\mathcal{Q}, \Delta_{\mathcal{Q}})$ on a Hilbert space $\mathcal{H}$ consists of a right coaction β of $(\mathcal{Q}, \Delta_{\mathcal{Q}})$ on $B(\mathcal{H})$.

In this thesis, we will mainly focus on continuous left/right corepresentation of a CQG $(\mathcal{Q}, \Delta_{\mathcal{Q}})$.

Theorem 2.3.8. *Let $(L^\infty(\mathcal{Q}), \Delta)$ be a von-Neumann algebraic CQG and α be a measurable left projective corepresentation of $(L^\infty(\mathcal{Q}), \Delta)$ on a finite dimensional Hilbert space $\mathcal{H}$. Then it corresponds to a Galois co-object (N, Δ_N) where $N = L^\infty(\mathcal{Q})$, $\Delta_N = \Omega\Delta_M$ and Ω is a left 2 cocyle in $L^\infty(\mathcal{Q}) \bar{\otimes} L^\infty(\mathcal{Q})$.*

Proof. Let us assume that $\{e_i : i \in I\}$ is an orthonormal basis for the finite dimensional Hilbert space $\mathcal{H}$ and $\{e_{ij} : i, j \in I\}$ is a set of matrix units of $B(\mathcal{H})$. First we will prove that $\alpha(e_{11})$ and $1 \otimes e_{11}$ are Murray-von Neumann equivalent.
Suppose $\alpha(e_{11})$ is an infinite projection in $L^\infty(\mathcal{Q}) \bar{\otimes} B(\mathcal{H})$. There exists a central projection $Z_0 \in P(Z(L^\infty(\mathcal{Q}) \bar{\otimes} B(\mathcal{H})))$ such that $Z_0\alpha(e_{11})$ is properly infinite and $(1 - Z_0)\alpha(e_{11})$ is a finite projection. Now consider the von Neumann subalgebra

$(1 - Z_0)(L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H}))$ which is a finite von Neumann algebra as $(1 - Z_0)e_{ii}$ is a finite projection and $\sum_i (1 - Z_0)\alpha(e_{ii}) = 1 - Z_0$ is a finite projection. We know from theorem 1.2.16 that there exists a center valued tracial conditional expectation

$$\tau : (1 - Z_0)(L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H})) \to Z((L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H}))(1 - Z_0)).$$

By comparability theorem, there exists a central projection $W \in (1-Z_0)(L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H}))$ such that $W(1 - Z_0)\alpha(e_{11}) \sim P_1 \leq W(1 - Z_0)(1 \otimes e_{11})$ and $(1 - Z_0 - W)(1 \otimes e_{11}) \sim P_1' \leq (1 - Z_0 - W)\alpha(e_{11})$ where P_1, P_1' are projections in $(1 - Z_0)(L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H}))$. As α is a unital $*$ homomorphism, $\alpha(e_{11}) \sim \alpha(e_{ii})$ for every i. Hence there exists $P_i \in P((1 - Z_0)(L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H}))$ such that $W(1 - Z_0)\alpha(e_{ii}) \sim P_i \leq W(1 - Z_0)(1 \otimes e_{ii})$. Now we can conclude $\sum_{i=1}^n \tau(W(1 - Z_0)\alpha(e_{ii})) = \tau(W(1 - Z_0)) = n\tau(W(1 - Z_0)\alpha(e_{11})) \leq n\tau(W(1 - Z_0)(1 \otimes e_{11})) = \tau(W(1 - Z_0))$. So $\tau(W(1 - Z_0)\alpha(e_{11}) = \tau(W(1 - Z_0)(1 \otimes e_{11})$ and similarly $\tau((1 - Z_0 - W)(1 - Z_0)(1 \otimes e_{11})) = \tau((1 - Z_0 - W)\alpha(e_{11}))$. Hence $\tau((1 - Z_0)\alpha(e_{11}) = \tau((1 - Z_0)(1 \otimes e_{11}))$. From this, we can conclude that $(1 - Z_0)\alpha(e_{11}) \sim (1 - Z_0)(1 \otimes e_{11})$.

Now, we will prove that $Z_0\alpha(e_{11}) \sim Z_0(1 \otimes e_{11})$. If we choose the von Neumann subalgebra $Z_0(L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H}))$, then $Z_0\alpha(e_{ii})$ is a properly infinite projection. From Halving lemma, we get $Z_0\alpha(e_{11}) \sim Z_0\alpha(e_{11}) + Z_0\alpha(e_{22})$ that implies $Z_0\alpha(e_{nn}) \sim \sum_i Z_0\alpha(e_{ii}) = Z_0$. Now, we will prove $Z_0(1 \otimes e_{11})$ is a properly infinite projection. Suppose there exists a $Z_1 \in P(Z(Z_0(L^\infty(\mathcal{Q})\bar{\otimes}B(H)))$ such that $Z_1Z_0(1 \otimes e_{ii}) \neq 0$. If $Z_1Z_0(1 \otimes e_{11})$ is finite then $Z_1Z_0(1 \otimes e_{ii})$ is finite as $Z_1Z_0(1 \otimes e_{ii}) \sim Z_1Z_0(1 \otimes e_{11})$. Hence, $Z_1Z_0 = Z_1 = \sum Z_1Z_0(1 \otimes e_{ii})$ is finite projection. As $Z_1Z_0(\alpha(e_{ii})) \leq Z_1Z_0$, $Z_1Z_0(\alpha(e_{ii})$ is a finite projection. But $Z_0(\alpha(e_{ii}))$ is a properly infinite projection, therefore $Z_1Z_0\alpha(e_{ii}) = 0$ or an infinite projection. So, we can conclude that if $Z_1Z_0(1 \otimes e_{ii})$ is a finite projection then $Z_1Z_0\alpha(e_{ii}) = 0$, which implies $Z_1Z_0 = 0$. But we assume that $Z_1Z_0(1 \otimes e_{ii}) \neq 0$. So, $Z_1Z_0(1 \otimes e_{ii})$ is an infinite projection. Hence, $Z_1Z_0(1 \otimes e_{ii})$ is a properly infinite projection. Similarly we can prove that $Z_0(1 \otimes e_{ii}) \sim Z_0$. As $Z_0\alpha(e_{11}) \sim Z_0(1 \otimes e_{11})$ and $(1 - Z_0)\alpha(e_{11}) \sim (1 - Z_0)(1 \otimes e_{11})$, then we can conclude that $\alpha(e_{11}) \sim (1 \otimes e_{11})$.

Let, u be a partial isometry such that $uu^* = (1 \otimes e_{11})$ and $u^*u = \alpha(e_{11})$. Let

$U = \sum_{i=1}^{n}(1 \otimes e_{i1})u\alpha(e_{1i})$, which is a unitary in $L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H})$.

$$\begin{aligned} U^*(1 \otimes e_{i_0 j_0})U =& \sum_{i,j} \alpha(e_{i1})u^*(1 \otimes e_{1i})(1 \otimes e_{i_0 j_0})(1 \otimes e_{j1})u\alpha(e_{1j}) \\ =& \sum_{i,j} \alpha(e_{i1})u^*(1 \otimes e_{11})u\alpha(e_{1j})\delta_{i,i_0}\delta_{j,j_0} \\ =& \alpha(e_{i_0 1})u^*(1 \otimes e_{11})u\alpha(e_{1j_0}) \\ =& \alpha(e_{i_0 1})u^* u u^* u\alpha(e_{1j_0}) \\ =& \alpha(e_{i_0 1})u^* u\alpha(e_{1j_0}) \\ =& \alpha(e_{i_0 j_0}). \end{aligned}$$

Hence a measurable left projective corepresentation of $(L^\infty(\mathcal{Q}), \Delta)$ on a finite dimensional Hilbert space $\mathcal{H}$ is implemented by a unitary U. From Theorem (3.1.5) of [DCMN24], we know that there is a 2-cocycle Ω such that $({}_\Omega\Delta \otimes id)(U) = U_{12}U_{13}$. So, any left measurable measurable projective corepresentation of $(L^\infty(\mathcal{Q}), \Delta)$ on a finite dimensional Hilbert space $\mathcal{H}$ corresponds to a right Galois-coobject $(L^\infty(\mathcal{Q}),_\Omega \Delta)$. □

Definition 2.3.9. *Let δ be a measurable left (right) projective corepresentation. We say that δ is* cleft *if there exists a unitary $U \in L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H})$ $(B(\mathcal{H})\bar{\otimes}L^\infty(\mathcal{Q}))$ such that $\delta(a) = u^*(1 \otimes a)u$ $(\delta(a) = u(a \otimes 1)u^*)$.*

Similarly, a continuous left /right projective corepresentation is said to be cleft if it is implemented by a unitary.

Definition 2.3.10. *A unitary $U \in B(\mathcal{H})\bar{\otimes}L^\infty(\mathcal{Q})$ is said to be a measurable right projective corepresentation if there exists a right 2-cocycle $\Omega^* \in L^\infty(\mathcal{Q})\bar{\otimes}L^\infty(\mathcal{Q})$ such that $(Id \otimes \Delta_{\Omega^*})(U) = U_{12}U_{13}$ and a measurable left projective corepresentation if there exists a left 2-cocycle $\Omega \in L^\infty(\mathcal{Q})\bar{\otimes}L^\infty(\mathcal{Q})$ such that $(Id \otimes_\Omega \Delta)(U) = U_{12}U_{13}$.*

A unitary $U \in M(K(\mathcal{H})\otimes\mathcal{Q})$ is said to be a continuous right projective corepresentation if $(Id\otimes\Delta_{\Omega^*})(U) = U_{12}U_{13}$ for a right 2-cocycle $\Omega^* \in (\mathcal{Q}\otimes\mathcal{Q})$ and a continuous left projective Ω corepresentation if $(Id\otimes_\Omega\Delta)(U) = U_{12}U_{13}$ for a left 2-cocycle $\Omega \in (\mathcal{Q}\otimes\mathcal{Q})$.

Proposition 2.3.11. *(proposition 3.1.9 of [DCMN24]) Let $\delta : B(\mathcal{H}) \to B(\mathcal{H})\overline{\otimes}L^\infty(\mathcal{Q})$ be a cleft right measurable projective corepresentation. Then there exists a right 2-cocycle $\Omega \in L^\infty(\mathcal{Q})\overline{\otimes}L^\infty(\mathcal{Q})$ and a unitary measurable right projective corepresentation $U \in B(\mathcal{H})\overline{\otimes}L^\infty(\mathcal{Q})$ such that $\delta(a) = U(a \otimes 1)U^*$.*

Remark 2.3.12. *If $\delta : B(\mathcal{H}) \to L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H})$ be a cleft left measurable projective corepresentation if and only if there is a right 2-cocycle $\Omega^* \in L^\infty(\mathcal{Q})\bar{\otimes}L^\infty(\mathcal{Q})$ and a measurable left projective corepresentation $U \in B(\mathcal{H})\bar{\otimes}L^\infty(\mathcal{Q})$ such that $\delta(a) = \sigma(U^*(a \otimes 1)U)$, where σ is the flip map from $B(\mathcal{H})\bar{\otimes}L^\infty(\mathcal{Q})$ to $L^\infty(\mathcal{Q})\bar{\otimes}B(\mathcal{H})$.*

Proposition 2.3.13. *(Theorem 3.1.12 of [DCMN24]) Any continuous left/right projective corepresentation of* $(\mathcal{Q}, \Delta)$ *is cleft, which means any right projective corepresentation* α *corresponds to a unitary right projective corepresentation* $U \in M(K(\mathcal{H}) \otimes \mathcal{Q})$ *such that* $\alpha = Ad_U$ *and any left projective corepresentation* β *corresponds to a left projective corepresentation* $V \in M(K(\mathcal{H}) \otimes \mathcal{Q})$ *such that* $\beta = \sigma(Ad_{V^*})$.

Remark 2.3.14. *Note the slight confusion in the terminology as a projective left/right unitary corepresentation can mean either a homomorphism* δ *or a corresponding unitary. We often call a unitary to be a corepresentation without explicitly mentioning the cocycle* Ω *if it is understood from the context. However, we will mostly use unitary picture of projective corepresentation. For a classical compact group* G*, left/right projective representation are the same thing. However, for a general CQG this is not so. Thus, it is natural to consider special class left/right/biprojective projective corepresentations.*

Definition 2.3.15. *A unitary* $U \in M(K(\mathcal{H}) \otimes \mathcal{Q})$ *is said to be a biprojective unitary corepresentation if it is a both left and right projective corepresentation (possibly with different 2-cocycles) of* $\mathcal{Q}$.

Definition 2.3.16. *Let* $U \in M(K(\mathcal{H}_u) \otimes \mathcal{Q}), V \in M(K(\mathcal{H}_v \otimes \mathcal{Q})$ *be two right/left projective corepresentations of* $\mathcal{Q}$*. Morphism between* U, V *are given by*

$$Mor(U, V) = \{T \in B(\mathcal{H}_u, \mathcal{H}_v) : (T \otimes 1)U = V(T \otimes 1)\}.$$

$(U, \mathcal{H})$ *is called irreducible if* $Mor(U, V) = \mathbb{C}Id_{\mathcal{H}_U}$.

Lemma 2.3.17. *If* $(U, \mathcal{H}_u)$ *and* $(V, \mathcal{H}_v)$ *are two irreducible left/right projective corepresentations of* $\mathcal{Q}$*, then either* U *is not unitary equivalent to* V *and* $Mor(U, V) = (0)$*; or* U *is unitary equivalent to* V *and* $Mor(U, V)$ *is a* 1*-dimensional subspace of* $B(\mathcal{H}_u, \mathcal{H}_v)$.

Lemma 2.3.18. *(Lemma (3.2.6) 0f [DCMN24]) Every left/right projective corepresentation* U *of* $\mathcal{Q}$ *decomposes into a direct sum of irreducible left/right projective corepresentations.*

Proposition 2.3.19. *Let* $\mathcal{Q}$ *be a compact quantum group and* Ω *a left* 2*-cocycle on* $\mathcal{Q}$*. Defining* $V^{\Omega} = \Omega V_{\mathcal{Q}}$*, where* $V_{\mathcal{Q}}$ *is right regular corepresentation of* $\mathcal{Q}$*, the following properties hold:*

i) For all $x \in \mathcal{Q}$ *and* $\xi \in H_u$ *we have* $V^{\Omega}(\Lambda(x) \otimes \xi) = {}_{\Omega}\Delta(x)(\xi_{\mathcal{Q}} \otimes \xi)$.

ii) For all $x \in \mathcal{Q}$ *we have* ${}_{\Omega}\Delta(x) = V^{\Omega}(x \otimes 1)V_{\mathcal{Q}}^*$.

iii) The following identity holds: $(id \otimes {}_{\Omega}\Delta)(V^{\Omega}) = V_{12}^{\Omega} V_{13}^{\Omega}$*, so* $V^{\Omega} \in \mathcal{B}(L^2(\mathcal{Q})) \otimes \mathcal{Q}$ *is an* Ω*-representation.*

iv) *The following pentagonal equation holds:* $V^{\Omega}_{12}V^{\Omega}_{13}(V_{\mathcal{Q}})_{23} = V^{\Omega}_{23}V^{\Omega}_{12}$.

The unitary V^{Ω} *is a* left regular projective corepresentation of $\mathcal{Q}$ on $L^2(\mathcal{Q})$ with respect to Ω *or* left regular Ω corepresentation of $\mathcal{Q}$ *on* $L^2(\mathcal{Q})$.

Remark 2.3.20. *Similarly, defining* $W^{\Omega} = W_{\mathcal{Q}}\Omega^*$*, we have that* $(W^{\Omega})^*(\xi \otimes \Lambda(x)) = {}_{\Omega}\Delta(x)(\xi \otimes \xi_{\mathcal{Q}})$*, for all* $x \in \mathcal{Q}$ *and* $\xi \in L^2(\mathcal{Q})$*. For all* $x \in \mathcal{Q}$*, we have* ${}_{\Omega}\Delta(x) = (W^{\Omega})^*(1 \otimes x)W_{\mathbb{G}}$ *and the pentagonal equation:* $(W_{\mathbb{G}})_{12}W^{\Omega}_{13}W^{\Omega}_{23} = W^{\Omega}_{23}W^{\Omega}_{12}$ *and the following identity holds:* $(\Delta_{\Omega^*} \otimes id)(W^{\Omega}) = W^{\Omega}_{13}W^{\Omega}_{23}$*, so* $\Sigma W^{\Omega}\Sigma$ *is an* Ω^**-projective corepresentation.*

The unitary W^{Ω} *is called* right projective regular Ω corepresentation of $\mathcal{Q}$ on $L^2(\mathcal{Q})$ with respect to Ω *or simply* right regular Ω corepresentation of $\mathcal{Q}$ on $L^2(\mathcal{Q})$.

Proposition 2.3.21 (Twisted Peter-Weyl theorem I)**.** *(Theorem (3.2.12) of [DCMN24]) Let* $\mathcal{Q}$ *be a CQG and* Ω *a 2-cocycle. The right projective regular corepresentation* $(V^{\Omega}, L^2(\mathcal{Q})$ *contains all irreducible* Ω*-representations of* $\mathcal{Q}$ *in its direct sum decomposition.*

Proposition 2.3.22 (Twisted Schur's orthogonality relations)**.** *(Theorem (3.2.13 of [DCMN24]) Let* $\mathcal{Q}$ *be a CQG and* Ω *a left 2-cocycle on* $\mathcal{Q}$*. Let* $\{u^x\}_{x\in Irr(\mathcal{Q},\Omega)}$ *be a complete set of mutually inequivalent, irreducible left projective* Ω*-corepresentations, with fixed bases for the associated Hilbert spaces* $\mathcal{H}_x$*. For each* $x \in Irr(\mathbb{G}, \Omega)$ *there exists a positive trace class operator* $F^x \in \mathcal{B}(\mathcal{H}_x)$ *with zero kernel such that the following orthogonality relations hold:*

$$h_{\mathcal{Q}}\big((u^y_{kl})^* u^x_{ij}\big) = \delta_{xy}\delta_{lj}F^x_{ik},$$

for every $x, y \in Irr(\mathcal{Q}, \Omega)$*,* $i, j = 1, \ldots, n_x$ *and* $k, l = 1, \ldots, n_y$*,* $h_{\mathcal{Q}}$ *is the Haar state of* $(\mathcal{Q}, \Delta)$

Theorem 2.3.23 (Twisted Peter-Weyl theorem II)**.** *(Theorem (3.2.14) of [DCMN24]) Let* $\mathcal{Q}$ *be a CQG and* Ω *a left 2-cocycle on* $\mathcal{Q}$*. We have a unitary transformation* $L^2(\mathcal{Q}) \cong \bigoplus_{x\in Irr(\mathcal{Q},\Omega)} \mathcal{H}_x \otimes \overline{\mathcal{H}_x}$ *such that* $\Lambda(u^x_{ij}) \mapsto \sqrt{F^x_i}\ \xi^x_i \otimes \overline{\xi^x_j}$*, for all* $j = 1, \ldots, n_x$*,* $x \in Irr(\mathcal{Q}, \Omega)$*. Similar statement hold for right projective corepresentation of* $\mathcal{Q}$ *for a corresponding right 2-cocycle* Ω^**.*

Proposition 2.3.24 (Twisted Maschke's theorem)**.** *Let* $\mathcal{Q}$ *be a compact quantum group and* Ω *a 2-cocycle on* $\mathcal{Q}$*. Let* $(U, \mathcal{H}_u)$ *and* $(V, \mathcal{H}_v)$ *be two right projective* Ω*-corepresentations of* $\mathcal{Q}$*.*

i) *If* $T : \mathcal{H}_u \longrightarrow \mathcal{H}_v$ *is a linear compact operator, then the average intertwiner* T' *with respect to* T *is again compact.*

ii) The C^*-algebra $\mathcal{D}_u = K(\mathcal{H}_u)^{\delta_u}$ acts non-degenerately on $\mathcal{H}_u$, that is, $[\mathcal{D}_u\mathcal{H}_u] = \mathcal{H}_u$.

iii) If U is an irreducible object, then U is finite dimensional.

Chapter 3

Projective envelope of a compact quantum group

Our main goal of this chapter is to prove that left/right/bi projective corepresentation of a CQG can be lifted to usual (linear) corepresentation of a bigger CQG.

3.1 Unitarity of U^c for a projective corepresentation U

The key step of this construction is the fact that for any unitary projective corepresentation U on a finite-dimensional Hilbert space, U^c is unitary for a suitable choice of inner product. This will be proved in this section.

We recall from chapter 2 the definitions and results about left/right measurable and continuous projective corepresentation. In this chapter, we will deal with continuous projective representations only. It has already been seen in chapter 2 that any continuous projective corepresentation can be expressed as a direct sum of irreducible projective corepresentations and any irreducible projective corepresentation is always finite dimensional. Therefore it is natural to study finite-dimensional projective corepresentations. For rest of this chapter we will assume that any projective corepresentation considered by us if finite dimensional.

Fix an orthonormal basis $\{e_i : i = 1 \cdots \mathrm{n}\}$ of a finite-dimensional Hilbert space $\mathcal{H}$. Hence $\{\bar{e_i}{:}i{=}1 \ \cdots \ n\}$ is an orthonormal basis for the Hilbert space $\bar{\mathcal{H}}$, where the inner product $<,>_{\overline{\mathcal{H}}}$ defined by $< \bar{a}, \bar{b} >_{\overline{\mathcal{H}}} =< b, a >_{\mathcal{H}}$.

Let $\phi{:}B(\mathcal{H}) \rightarrow \mathcal{H} \otimes \overline{\mathcal{H}}$ be the linear map defined by $\phi(e_{ij}){=}e_i{\otimes}\overline{e_j}$, where e_{ij} are the matrix units of $B(\mathcal{H})$. It is a vector space isomorphism map. e_{ij} given by $e_{ij}(e_k) = \delta_{jk} e_i$. Similarly, let $\overline{e_{ij}}$ be the matrix units of $B(\bar{\mathcal{H}})$ given by $\bar{e_{ij}}(\bar{e_k}) = \bar{e_i}\delta_{kj}$.

Let $\mathcal{Q}$ be a CQG and $U \in B(\mathcal{H}) \otimes \mathcal{Q}$ be a unitary element. $U^c = (j \otimes id)(U^*)$

from the definition 1.5.12 of 1 st chapter and $(U \otimes U^c) = U_{13}U^c_{23} \in B(\mathcal{H} \otimes \bar{\mathcal{H}}) \otimes \mathcal{Q} \subseteq B(\mathcal{H} \otimes \bar{\mathcal{H}} \otimes L^2(\mathcal{Q}))$, where $L^2(\mathcal{Q})$ is the GNS space for the Haar state (say h) on $\mathcal{Q}$.

Lemma 3.1.1.

$$Ad_U(a) = (\phi^{-1} \otimes id_{\mathcal{Q}})(U \otimes U^c)(\phi \otimes Id_{\mathcal{Q}})(a \otimes 1)$$

where $a \in B(H)$ and 1 is the identity element of $\mathcal{Q}$ viewed as a cyclic vector of $L^2(\mathcal{Q})$.

Proof. Assume that $U = \sum e_{i,j} \otimes u_{i,j}$,where e_{ij} are the matrix units of $B(\mathcal{H})$ and $u_{i,j} \in \mathcal{Q}$. For $a = e_{i_0,j_0}$,

$$\begin{aligned} Ad_U(e_{i_0j_0}) =& U(e_{i_0j_0} \otimes 1)U^* \\ =& \sum_{i_1,j_1,i_2,j_2} (e_{i_1j_1} \otimes u_{i_1j_1})(e_{i_0j_0} \otimes 1)(e_{j_2i_2} \otimes u^*_{i_2j_2}) \\ =& \sum e_{i_1j_1}e_{i_0j_0}e_{j_2i_2} \otimes u_{i_1j_1}u^*_{i_2j_2} \\ =& \sum_{i_1,i_2} e_{i_1i_2} \otimes u_{i_1i_0}u^*_{i_2j_0}. \end{aligned}$$

We know $U \otimes U^c = U_{13}U^c_{23} = \sum e_{i_1j_1} \otimes \bar{e}_{i_2j_2} \otimes u_{i_1j_1}u^*_{i_2j_2}$.

Now,

$$\begin{aligned} & (\phi^{-1} \otimes Id_{\mathcal{Q}})(U \otimes U^c)(\phi \otimes Id_{\mathcal{Q}})(e_{i_0j_0} \otimes 1) \\ =& (\phi^{-1} \otimes Id_{\mathcal{Q}})(U \otimes U^c)(\phi(e_{i_0j_0}) \otimes 1) \\ =& (\phi^{-1} \otimes Id_{\mathcal{Q}})(U \otimes U^c)(e_{i_0} \otimes \bar{e}_{j_0} \otimes 1) \\ =& (\phi^{-1} \otimes Id_{\mathcal{Q}})(\sum e_{i_1j_1}(e_{i_0}) \otimes \bar{e}_{i_2j_2}(\bar{e}_{j_0}) \otimes u_{i_1j_1}u^*_{i_2j_2}) \\ =& (\phi^{-1} \otimes Id_{\mathcal{Q}})(\sum e_{i_1}\delta_{i_0j_1} \otimes \bar{e_{i_2}}\delta_{j_0j_2} \otimes u_{i_1j_1}u^*_{i_2j_2}) \\ =& (\phi^{-1} \otimes Id_{\mathcal{Q}})(\sum e_{i_1} \otimes \bar{e_{i_2}} \otimes u_{i_1i_0}u^*_{i_2j_0}) \\ =& \sum \phi^{-1}(e_{i_1} \otimes \bar{e_{i_2}}) \otimes u_{i_1i_0}u^*_{i_2j_0} \\ =& \sum e_{i_1i_2} \otimes u_{i_1i_0}u^*_{i_2j_0}. \end{aligned}$$

As the matrix units $e_{i_0j_0}$ form a basis of $B(\mathcal{H})$, it follows that $Ad_U(a) = (\phi^{-1} \otimes Id_{\mathcal{Q}})(u \otimes u^c)(\phi \otimes Id_{\mathcal{Q}})(a \otimes 1)$ for any $a \in B(\mathcal{H})$. ☐

Let δ be a right coaction of $\mathcal{Q}$ on $B(\mathcal{H})$ that corresponds to a right projective corepresentation $U \in B(\mathcal{H}) \otimes \mathcal{Q}$. We can write $\delta = Ad_U$.

Let $\psi : B(\mathcal{H}) \to \mathbb{C}$ be the linear function defined by $\psi(a) = (Tr \otimes h)\delta(a)$.

Lemma 3.1.2. *Then the following identity hold:*

$$(\psi \otimes Id_{\mathcal{Q}})\delta(a) = \psi(a)1_{\mathcal{Q}},$$

where $a \in B(\mathcal{H})$.

Proof. For $a \in B(\mathcal{H})$, we have

$$\begin{aligned}
&(\psi \otimes Id_{\mathcal{Q}})\delta(a)\\
=&((Tr \otimes h)\delta \otimes Id_{\mathcal{Q}})\delta(a)\\
=&(Tr \otimes h \otimes Id_{\mathcal{Q}})(\delta \otimes Id_{\mathcal{Q}})\delta(a)\\
=&(Tr \otimes h \otimes Id_{\mathcal{Q}})(Id_{B(H)} \otimes \Delta)\delta(a)\\
=&(Tr \otimes (h \otimes Id_{\mathcal{Q}})\Delta)\delta(a)\\
=&(Tr \otimes h.1_{\mathcal{Q}})\delta(a)\\
=&(Tr \otimes h)\delta(a)1_{\mathcal{Q}}\\
=&\psi(a)1_{\mathcal{Q}}.
\end{aligned}$$

So, $(\psi \otimes Id_{\mathcal{Q}})\delta(a) = \psi(a)1_{\mathcal{Q}}$. □

Remark 3.1.3. *As ψ is a faithful state on $B(\mathcal{H})$ there exists a positive invertible linear operator $R \in B(\mathcal{H})$ such that $\psi(a) = Tr(Ra)$. Now, we define an inner product $< .,. >_R$ on $\bar{\mathcal{H}}$ st. $< \bar{a}, \bar{b} >_R =< \overline{R^{1/2}(a)}, \overline{R^{1/2}(b)} >$. Let us denote this Hilbert space by $\bar{\mathcal{H}}_R$ and we can give a new inner product on $B(\mathcal{H})$ by using this positive operator R and for any two operators $T, S \in B(\mathcal{H})$,the new inner product is $< S, T >_{new} = \psi(T^*S) = Tr(RT^*S)$. It is easy to check that under this new inner product $(B(\mathcal{H}), < S, T >_{new})$ is isometrically isomorphic to $\mathcal{H} \otimes \bar{\mathcal{H}}_R$.*

Theorem 3.1.4. *Define the linear map $\delta_U : \mathcal{H} \otimes \bar{\mathcal{H}}_R \to \mathcal{H} \otimes \bar{\mathcal{H}}_R \otimes \mathcal{Q}$ given by*

$$\delta_U = (U \otimes U^c)$$

then δ_U is a unitary corepresentation of the CQG $(\mathcal{Q}, \Delta)$.

Proof. As R is a positive matrix in $B(\mathcal{H})$ hence R is diagonalizable. Without loss of generality, we assume that e_i are orthonormal eigenvectors of R. Therefore, $\{e_i \otimes \bar{e_j} : i, j = 1 \cdots n\}$ is an orthonormal basis for $H \otimes \bar{H}_R$.

We will prove that $< \delta(e_{i_0} \otimes \bar{e_{j_0}}), \delta(e_{l_0} \otimes \bar{e_{m_0}}) >=< e_{i_0}) \otimes \bar{e_{j_0}}, e_{l_0} \otimes \bar{e_{m_0}} >$.

$$
\begin{aligned}
&< \delta_u(e_{i_0} \otimes \bar{e_{j_0}}), \delta_u(e_{l_0} \otimes \bar{e_{m_0}} > \\
&=< (u \otimes u^c)(e_{i_0} \otimes \bar{e_{j_0}} \otimes 1), (u \otimes u^c)(e_{l_0} \otimes \bar{e_{m_0}} \otimes 1) \\
&=< \sum_{i_1,i_2} e_{i_1} \otimes \bar{e_{i_2}} \otimes u_{i_1 i_0} u^*_{i_2 j_0}, \sum_{l_1,l_2} e_{l_1} \otimes \bar{e_{l_2}} \otimes u_{l_1 l_0} u^*_{l_2 m_0} > \\
&= \sum_{i_1,i_2,l_1,l_2} < e_{i_1} \otimes \bar{e_{i_2}}, e_{l_1} \otimes \bar{e_{l_2}} > (u_{i_1 i_0} u^*_{i_2 j_0})^* u_{l_1 l_0} u^*_{l_2 m_0} \\
&= \sum < e_{i_1}, e_{l_1} >< \bar{e_{i_2}}, \bar{e_{l_2}} >_{\bar{H_R}} u_{i_2 j_0} u^*_{i_1 i_0} u_{l_1 l_0} u^*_{l_2 m_0} \\
&= \sum \delta_{i_1 l_1} < Re_{l_2}, e_{i_2} > u_{i_2 j_0} u^*_{i_1 i_0} u_{l_1 l_0} u^*_{l_2 m_0} \\
&= \sum_{j_2,l_2} < Re_{l_2}, e_{i_2} > u_{i_2 j_0} (\sum u^*_{i_1 i_0} u_{i_1 l_0}) u^*_{l_2 m_0} \\
&= \sum \delta_{l_0 i_0} < Re_{l_2}, e_{i_2} > u_{i_2 j_0} u^*_{l_2 m_0}.
\end{aligned}
$$

From Lemma (3.1.2), we know that $(\psi \otimes Id)\delta(a) = \psi(a)$.
For $a = e_{m_0 j_0}$, we have

$$
\begin{aligned}
&(\psi \otimes Id)\delta(e_{m_0 j_0}) \\
&=(\psi \otimes Id) Ad_U(e_{m_0 j_0}) \\
&=(\psi \otimes Id)(\sum e_{i_1 i_2} \otimes u_{i_1 m_0} u^*_{i_2 j_0}) \\
&= \sum \psi(e_{i_1 i_2}) u_{i_1 m_0} u^*_{i_2 j_0} \\
&= \sum < Re_{i_1}, e_{i_2} > u_{i_1 m_0} u^*_{i_2 j_0} = \psi(e_{m_0 j_0}) 1_{\mathcal{Q}}.
\end{aligned}
$$

Taking adjoint on both side of the equation we will get, $\sum < Re_{i_1}, e_{i_2} > u_{i_2 j_0} u^*_{i_1 m_0} = \psi(e_{m_0 j_0}) 1_{\mathcal{Q}}$.
It follows that,

$$
\begin{aligned}
&\sum \delta_{l_0 i_0} < Re_{l_2}, e_{i_2} > u_{i_2 j_0} u^*_{l_2 m_0} \\
&= \delta_{l_0 i_0} \psi(e_{m_0 j_0}) 1_{\mathcal{Q}} \\
&=< e_{i_0}, e_{l_0} >< R^{1/2} e_{m_0}, R^{1/2} e_{j_0} > 1_{\mathcal{Q}} \\
&=< e_{i_0}, e_{l_0} >< \bar{e_{j_0}}, \bar{e_{m_0}} >_{\bar{H_R}} 1_{\mathcal{Q}} \\
&=< e_{i_0} \otimes \bar{e_{j_0}}, e_{l_0} \otimes \bar{e_{m_0}} >_{H \otimes \bar{H_R}} 1_{\mathcal{Q}}.
\end{aligned}
$$

Thus $< \delta(e_{i_0} \otimes \bar{e_{j_0}}), \delta(e_{l_0} \otimes \bar{e_{m_0}}) >=< e_{i_0} \otimes \bar{e_{j_0}}, e_{l_0} \otimes \bar{e_{m_0}} >$ and using the fact that matrix units form a basis of $\mathcal{H} \otimes \bar{\mathcal{H}}_R$ we get $< \delta(a), \delta(b) >=< a, b > 1_{\mathcal{Q}}$, for any $a, b \in \mathcal{H} \otimes \bar{\mathcal{H}}_R$.

Now, we will prove $(\delta_U \otimes id)\delta_U = (Id \otimes \Delta)\delta_U$.

we know from Lemma (3.1.1), $Ad_U(\phi^{-1}(a)) = (\phi^{-1} \otimes Id)(U \otimes U^c)(a \otimes 1)$, where $a \in \mathcal{H} \otimes \bar{\mathcal{H}}$, which gives:

$$\begin{aligned}(Ad_U \otimes Id)Ad_U(\phi^{-1}(a)) =& (id \otimes \Delta)Ad_U(\phi^{-1}(a))\\ =& (id \otimes \Delta)(\phi^{-1} \otimes Id_{\mathcal{Q}})\delta_U(a)\\ =& (\phi^{-1} \otimes Id_{\mathcal{Q}} \otimes Id_{\mathcal{Q}})(Id \otimes \Delta)\delta_U(a).\end{aligned}$$

Then, it is easily follows that

$$\begin{aligned}&(Id \otimes \Delta)\delta_U(a)\\ =&(\phi \otimes Id_{\mathcal{Q}} \otimes Id_{\mathcal{Q}})(Ad_U \otimes Id)Ad_U(\phi^{-1}(a))\\ =&(\phi \otimes Id_{\mathcal{Q}} \otimes Id_{\mathcal{Q}})(Ad_U \otimes Id)(\phi^{-1} \otimes id)\delta_U(a)\\ =&((\phi \otimes Id_{\mathcal{Q}}) \otimes Id_{\mathcal{Q}})(Ad_U\phi^{-1} \otimes Id)\delta_U(a)\\ =&((\phi \otimes Id_{\mathcal{Q}}) \otimes Id_{\mathcal{Q}})((\phi^{-1} \otimes Id)\delta_U \otimes Id_{\mathcal{Q}})\delta_U(a) = (\delta_U \otimes Id_{\mathcal{Q}})\delta_U(a).\end{aligned}$$

Note that $\delta = Ad_U$ is a coaction of CQG $(\mathcal{Q}, \Delta)$ on the finite-dimensional space $B(\mathcal{H})$, hence from the general theory of CQG coaction, span$\{Ad_U(x)(1 \otimes q) : x \in B(\mathcal{H}), q \in \mathcal{Q}\} = B(\mathcal{H}) \otimes \mathcal{Q}$. As ϕ^{-1} is a vector space isomorphism, we have span$\{Ad_U(\phi^{-1})(a)(1 \otimes q) : a \in \mathcal{H} \otimes \bar{\mathcal{H}}_R, q \in \mathcal{Q}\} = B(\mathcal{H}) \otimes \mathcal{Q}$. This proves that $\delta_U(\mathcal{H} \otimes \bar{\mathcal{H}})(1 \otimes \mathcal{Q})$ is total in $\mathcal{H} \otimes \bar{\mathcal{H}}_R \otimes \mathcal{Q}$. Hence, δ_U is a unitary corepresentation. □

Corollary 3.1.5. *U^c is a unitary element of $B(\bar{\mathcal{H}}_R) \otimes \mathcal{Q}$.*

Proof. This follows from the fact that $U^*_{13}(U \otimes U^c) = U^c_{23}$ and U^c_{23} is a unitary. Hence, $(1 \otimes U^c) = U_{12}^{-1}U_{13}U^c_{23}$, which is unitary. □

Now, we define a linear map $T : \bar{\mathcal{H}} \to \bar{\mathcal{H}}$ such that $T(\bar{x}) = \overline{R^{(-1/2)}(x)}$. If both the domain and range have the same old inner product structure, then T is a positive invertible map. If the domain space has the original inner product structure of $\bar{\mathcal{H}}$ and the range space has the inner product $<,>_{\bar{\mathcal{H}}_R}$ as in remark (3.1.3), which is induced from the positive matrix R, then T is an isometry from $\mathcal{H}$ to $\bar{\mathcal{H}}_R$ because $< T(\bar{x}), \bar{y} >_{\bar{\mathcal{H}}_R} =< \overline{R^{-(1/2)}(x)}, \bar{y} >_{\bar{\mathcal{H}}_R} =< \bar{x}, \overline{R^{1/2}(y)} >$ implies that $T^*(y) = \overline{R^{1/2}(y)}$ and $T^*T = Id_{\bar{H}}$. T is an isometry from $(\bar{\mathcal{H}}, <,>)$ to $(\bar{\mathcal{H}}, <,>_{\bar{\mathcal{H}}_R})$ and $TT^* = Id_{\bar{\mathcal{H}}}$.

Lemma 3.1.6. *$(T^{-1} \otimes Id_{\mathcal{Q}})U^c(T \otimes Id_{\mathcal{Q}})$ is a unitary element of $B(\mathcal{H}) \otimes \mathcal{Q}$, where the inner product structure in $\bar{\mathcal{H}}$ is the natural inner product structure.*

Proof. U^c is a unitary element in $B(\bar{\mathcal{H}}_R) \otimes \mathcal{Q})$. T is an isometry from $\bar{\mathcal{H}}$ to $\bar{\mathcal{H}}_R$ and an invertible map. By using these three facts, we can conclude that $(T^{-1} \otimes Id_{\mathcal{Q}})U^c(T \otimes Id_{\mathcal{Q}})$ is a unitary element. □

We assume that $T^{-1} = \rho^{1/2}$, where $\rho(\bar{x}) = \overline{R(x)}$.

Lemma 3.1.7. *If U is a unitary right projective corepresentation of $\mathcal{Q}$ on $\mathcal{H}$, then $(\rho^{1/2} \otimes 1)U^c(\rho^{-1/2} \otimes 1)$ is a unitary left projective corepresentation on $\bar{\mathcal{H}}$.*

Proof. If U is a unitary right projective corepresentation with the associated right 2-cocycle Ω then it follows that $(Id \otimes \Delta_{\Omega^*})(U) = U_{12}U_{13}$. Now, one can easily conclude that $(Id \otimes {}_{\Omega}\Delta)(U^*) = U_{13}^* U_{12}^*$.
We know $U^c = (j \otimes Id)(U^*)$, where j is the conjugation map from $B(\mathcal{H})$ to $B(\bar{\mathcal{H}})$).
This implies,

$$\begin{aligned}(Id \otimes {}_{\Omega}\Delta)(U^c) =&(Id \otimes {}_{\Omega}\Delta)(j \otimes Id)(U^*)\\ =&(j \otimes Id \otimes Id)(Id \otimes {}_{\Omega}\Delta)(U^*)\\ =&(j \otimes Id \otimes Id)(U_{13}^* U_{12}^*)\\ =&(j \otimes Id \otimes Id)(U_{12}^*)(j \otimes Id \otimes Id)(U_{13}^*)\\ =&U_{12}^c U_{13}^c.\end{aligned}$$

From this we conclude that

$$\begin{aligned}(Id \otimes {}_{\Omega}\Delta)((p^{1/2} \otimes 1)u^c(p^{-1/2} \otimes 1)) =&(p^{1/2} \otimes 1 \otimes 1)u_{12}^c u_{13}^c(p^{-1/2} \otimes 1 \otimes 1)\\ =&(p^{1/2} \otimes 1 \otimes 1)u_{12}^c(p^{-1/2} \otimes 1 \otimes 1)(p^{1/2} \otimes 1 \otimes 1)u_{13}^c(p^{-1/2} \otimes 1 \otimes 1)\\ =&((\rho^{1/2} \otimes 1)u^c(\rho^{-1/2} \otimes 1))_{12}((\rho^{1/2} \otimes 1)u^c(\rho^{-1/2} \otimes 1))_{13},\end{aligned}$$

where $(\rho^{1/2} \otimes 1)u^c(\rho^{-1/2} \otimes 1)$ is a unitary left projective corepresentation on $\bar{\mathcal{H}}$. □

Lemma 3.1.8. *Let $U \in B(H) \otimes \mathcal{Q}$ is a unitary.Then, followings are equivalent:*

1) If U is a right projective corepresentation.

2) There exists a positive matrix $\rho \in B(\bar{H})$ st. $U \otimes ((\rho^{1/2} \otimes 1)U^c(\rho^{-1/2} \otimes 1))$ is a unitary corepresentation of $(\mathcal{Q}, \Delta)$.

3) $U \otimes U^c$ is an invertible corepresentation on $H \otimes \bar{H}$.

Proof. $(1 \Rightarrow 2)$

Assume that U is a right projective corepresentation with corresponding 2-cocycle Ω. From the lemma (3.1.7) , we know that there exists a positive matrix ρ such that $\bar{U_r} := (\rho^{1/2} \otimes 1)U^c(\rho^{-1/2} \otimes 1)$ is a left projective corepresentation. Now,

$$\begin{aligned}(Id \otimes \Delta)(U \otimes \bar{U_r}) =&(Id \otimes \Delta)(U_{13}\bar{U}_{r23})\\ =&(Id \otimes \Delta)(U_{13})(Id \otimes \Delta)(\bar{U}_{r23})\\ =&U_{13}U_{14}\Omega.\Omega^*\bar{U}_{r23}\bar{U}_{r24}\\ =&U_{13}\bar{U}_{r23}U_{14}\bar{U}_{r24}\\ =&(U \otimes \bar{U_r})_{12}(U \otimes \bar{U_r})_{13}.\end{aligned}$$

This proves that $(U \otimes \bar{U_r})$ is a unitary corepresentation of $(\mathcal{Q}, \Delta)$.

$(2 \Rightarrow 3)$

$(U \otimes U^c) = (1 \otimes (\rho^{1/2} \otimes 1_{\mathcal{Q}})(U \otimes \bar{U_r})(1 \otimes (\rho^{-1/2} \otimes 1_{\mathcal{Q}})$. As $U \otimes \bar{U_r}$ is a unitary corepresentation therefore $U \otimes U^c$ is an invertible corepresentation of $(\mathcal{Q}, \Delta)$.

$(3 \Rightarrow 1)$

Let $U \otimes U^c$ be an invertible corepresentation of $\mathcal{Q}$. As we observe, $Ad_U = (\phi^{-1} \otimes 1_{\mathcal{Q}})(U \otimes U^c)(\phi \otimes 1_{\mathcal{Q}})$(Proposition 3.1.1). Now, the proof of coassociativity in Theorem 3.1.4 applies verbatim in this case as well. Hence Ad_U is a coaction of $\mathcal{Q}$ on $B(\mathcal{H})$. U is a right projective corepresentation corresponding to a 2-cocycle Ω of $\mathcal{Q}$. □

All the above results can be easily extended to left coaction/left projective corepresentation. In particular, we have

Lemma 3.1.9. *Let $U \in B(H) \otimes \mathcal{Q}$ is a unitary. Then followings are equivalent:*

1) *If U is a left projective corepresentation.*
2) *There exists a positive matrix $\rho \in B(\bar{\mathcal{H}})$ such that $((\rho^{1/2} \otimes 1)U^c(\rho^{-1/2} \otimes 1)) \otimes U$ is a unitary corepresentation of $(\mathcal{Q}, \Delta)$.*
3) *$U^c \otimes U$ is an invertible corepresentation on $\bar{\mathcal{H}} \otimes \mathcal{H}$.*

Proof. The proof of this is easy consequence of the lemma (3.1.8), hence ommited. □

Lemma 3.1.10. *Let $U \in B(\mathcal{H}) \otimes \mathcal{Q}$ be a unitary, then followings are equivalent:*

1) *U is a bi projective corepresentation of $\mathcal{Q}$.*
2) *There exists a positive matrices $\rho_2 \in B(\bar{\mathcal{H}})$ such that $((\rho_2^{1/2} \otimes 1)U^c(\rho_2^{-1/2} \otimes 1)) \otimes U, U \otimes ((\rho_2^{1/2} \otimes 1)U^c(\rho_2^{-1/2} \otimes 1))$ are unitary corepresentations of $(\mathcal{Q}, \Delta)$.*
3) *$U \otimes U^c, U^c \otimes U$ are invertible corepresentations of $\mathcal{Q}$.*

Proof of this lemma is ommited.

3.2 Construction of a univeral C^* tensor category for defining projective envelope

Now, we define a category α where objects are $(U,\mathcal{H})$ and $\mathcal{H}$ is a finite-dimensional Hilbert space. Note that the construction of the category, the associated CQG or DQG, corepresentation $\mathcal{U}$ and the homomorphism Π in the proof of theorem (3.2.4) are essentially taken from [GS25]. The basic idea behind the construction of the category was motivated by Stefan Vaes through private communication to the first author of [GS25].

The objects of the category α are $(U,\mathcal{H})$ where $\mathcal{H}$ is a finite dimensional Hilbert space and the following conditions hold;

1) $U \in B(\mathcal{H}) \otimes \mathcal{Q}$ is a unitary element.
2) There exists a unitary element $\bar{U} \in B(\bar{\mathcal{H}}) \otimes \mathcal{Q}$,that satisfies the following properties:
    i) There exists linear maps $R_{S_U} : \mathbb{C} \to \bar{\mathcal{H}} \otimes \mathcal{H}$ and $R_{t_U} : \mathbb{C} \to \mathcal{H} \otimes \bar{\mathcal{H}}$ such that

$$(R_{S_U} \otimes Id)(1_c \otimes Id) = (\bar{U} \otimes U)(R_{S_U} \otimes 1) \quad (3.2.1)$$
$$(R_{t_U} \otimes Id)(1_c \otimes Id) = (U \otimes \bar{U})(R_{t_U} \otimes 1) \quad (3.2.2)$$

also, it satisfies the following equations

$$(R^*_{S_U} \otimes Id_{\bar{\mathcal{H}}})(Id_{\bar{\mathcal{H}}} \otimes R_{t_U}) = Id_{\bar{\mathcal{H}}} \quad (R^*_{t_U} \otimes Id_{\mathcal{H}})(Id_{\mathcal{H}} \otimes R_{S_U}) = Id_{\mathcal{H}}. \quad (3.2.3)$$

For $s \in \bar{\mathcal{H}} \otimes \mathcal{H}$,$R_s : \mathbb{C} \to \bar{\mathcal{H}} \otimes \mathcal{H}$ defined by $R_s(x) = xs$. Similarly, we can define the linear map R_t.
For any two objects $(U,\mathcal{H})$, $(V,\mathcal{K})$ of this category, the morphism space defined by

$$\begin{aligned} &Mor((U,\mathcal{H}),(V,\mathcal{K}))) \\ :=&\{T \in B(\mathcal{H},\mathcal{K}) : V(T \otimes 1) = (T \otimes 1)U \quad ,V \in B(\mathcal{K}) \otimes \mathcal{Q}, U \in B(\mathcal{H}) \otimes \mathcal{Q}\}. \end{aligned}$$

Theorem 3.2.1. *α is a Rigid C^* tensor category.*

Proof. Mor($(U,\mathcal{H})$,$(V,\mathcal{K})$) is a finite-dimensional vector space for any two objects $(U,\mathcal{H}), (V,\mathcal{K})$ of α. So, it is a Banach space. It is routine work to check that α is a C* category.
Let R_{S_U} and R_{t_U} satisfies the following equation (3.2.1) and (3.2.2) and also satifies the equation (3.2.3) for the object $(U,\mathcal{H})$. Similarly, R_{S_V} and R_{t_V} satisfies the following equations (3.2.1) and (3.2.2) and also hold the equations (3.2.3) for the object $(V,\mathcal{K})$.

One can easily check that $(Id_{\bar{v}} \otimes R_{S_U} \otimes Id_v)\ R_{S_V}$ and $(Id_u \otimes R_{t_V} \otimes Id_{\bar{u}})\ R_{t_U}$ satisfies the equation (3.2.1) and (3.2.2) and also (3.2.3) for the object $(U \otimes V, \mathcal{H} \otimes \mathcal{K}) = (U_{12}V_{23}, \mathcal{H} \otimes \mathcal{K})$. Hence, $(U_{13}V_{23}, \mathcal{H} \otimes \mathcal{K})$ is the tensor product of $(U, \mathcal{H})$ *and* $(V, \mathcal{K})$).

Now, we define a bilinear bifunctor $\otimes : \alpha \times \alpha \to \alpha$ such that $((U, \mathcal{H}), (V, \mathcal{K})) \to (U_{12}V_{23}, \mathcal{H} \otimes \mathcal{K})$ and the associative morphism is the Identity map. This associative morphism satisfies the pentagonal diagram and also commutes with the triangular diagram. So, it is a tensor category. $Id_{\mathbb{C}} \otimes \mathcal{Q}$ is unit object of this category. Hence, α is a strict tensor category.

Direct sum:

Let assume that $s_{U\oplus V} = (s_U, 0, 0, s_V)$, $t_{U\oplus V} = (t_U, 0, 0, t_V)$. Then $R_{S_{U\oplus V}}$ satisfy the equation (3.2.1) and $R_{t_{U\oplus V}}$ satisfy the equation (3.2.2) . We will prove that $R_{S_{U\oplus V}}$,$R_{t_{U\oplus V}}$ satisfies the equations:

$$(R^*_{S_{U\oplus V}} \otimes Id_{\bar{\mathcal{H}}+\bar{\mathcal{K}}})(Id_{\bar{\mathcal{H}}+\bar{\mathcal{K}}} \otimes R_{t_{U\oplus V}}) = Id_{\bar{\mathcal{H}}+\bar{\mathcal{K}}},$$
$$(R^*_{t_{U\oplus V}} \otimes Id_{\mathcal{H}+\mathcal{K}})(Id_{\mathcal{H}+\mathcal{K}} \otimes R_{S_{U\oplus V}}) = Id_{\mathcal{H}+\mathcal{K}}.$$

.

First, we will check

$$(R^*_{S_{U\oplus V}} \otimes Id_{\bar{\mathcal{H}}\oplus\bar{\mathcal{K}}})(Id_{\bar{\mathcal{H}}\oplus\bar{\mathcal{K}}} \otimes R_{t_{U\oplus V}}) = Id_{\bar{\mathcal{H}}\oplus\bar{\mathcal{K}}}.$$

$$\begin{aligned}(R^*_{S_{U\oplus V}} \otimes Id_{\bar{\mathcal{H}}+\bar{\mathcal{K}}})(Id_{\bar{\mathcal{H}}\oplus\bar{\mathcal{K}}} \otimes R_{t_{U\oplus V}}) &= (R^*_{S_U} \oplus 0_{\bar{\mathcal{H}}\otimes\mathcal{K}} \oplus 0_{\bar{\mathcal{K}}\otimes\mathcal{H}} \oplus R^*_{S_V}) \otimes Id_{\bar{\mathcal{H}}\oplus\bar{\mathcal{K}}}) \\ &(Id_{\bar{\mathcal{H}}\oplus\bar{\mathcal{K}}} \otimes (R_{t_U} \oplus 0_{\mathcal{H}\otimes\bar{\mathcal{K}}} \oplus 0_{\mathcal{K}\otimes\bar{\mathcal{H}}} \oplus R_{t_V}) \\ &= (R^*_{S_U} \otimes Id_{\bar{\mathcal{H}}})(Id_{\bar{\mathcal{H}}} \otimes R_{t_U}) \oplus (R^*_{S_V} \otimes Id_{\bar{\mathcal{K}}}) \\ &(Id_{\bar{\mathcal{K}}} \otimes R_{t_V}) \\ &= Id_{\bar{\mathcal{H}}} \oplus Id_{\bar{\mathcal{K}}} = Id_{\bar{\mathcal{H}}+\bar{\mathcal{K}}}\end{aligned}$$

Similarly, we will get

$$(R^*_{t_{U\oplus V}} \otimes Id_{\mathcal{H}\oplus\mathcal{K}})(Id_{\mathcal{H}\oplus\mathcal{K}} \otimes R_{t_{U\oplus V}}) = Id_{\mathcal{H}\oplus\mathcal{K}}.$$

Hence, $(U, \mathcal{H}) \oplus (V, \mathcal{K}) = (U \oplus V, \mathcal{H} \oplus \mathcal{K})$ is an object α.

It follows from the property (2) α that each object has a conjugate object.

subobject: First, we will prove a lemma which will help us to prove α is closed under subjects.

Lemma 3.2.2. *Let $(U, \mathcal{H})$ be an object of this category and P be a projection in $End(U, \mathcal{H})$, then there exists a projection $\bar{P}$ in $End(\bar{U}, \bar{\mathcal{H}})$.*

Proof. If $T \in Mor((U,\mathcal{H}),(V,\mathcal{K}))$ then there exists a $T^{'} \in Mor((\bar{V},\bar{\mathcal{K}}),(\bar{U},\bar{\mathcal{H}})$ which satisfy the equation

$$(T^{'} \otimes Id_V)R_{S_U} = (Id_{\bar{U}} \otimes T)R_{S_U}.$$

Let P be a projection in $\text{End}(U,\mathcal{H})$ which corresponds to an operator, denoted by $P^{'} \in End(\bar{U},\bar{\mathcal{H}})$. It holds the following equation: $(P^{'} \otimes Id_U)R_{S_U} = (Id_{\bar{U}} \otimes P)R_{S_U}$.

Now,

$$\begin{aligned}(P'^2 \otimes Id_U)R_{S_U} &= (P' \otimes Id_U)(P' \otimes Id_U)R_{S_U}\\ &= (P' \otimes Id_U)(Id_{\bar{U}} \otimes P)R_{S_U}\\ &= (Id_{\bar{U}} \otimes P)(P' \otimes Id_U)R_{S_U}\\ &= (Id_{\bar{U}} \otimes P^2)R_{S_U}\\ &= (Id_{\bar{U}} \otimes P)R_{S_U}\\ &= (P' \otimes Id_U)R_{S_U}.\end{aligned}$$

From this, we can conclude that

$$(P'^2 - P') \otimes Id_U)R_{S_U} = 0$$

and P' is an idempotent operator because

$$(R^*_{S_U}((P'^2 - P') \otimes Id_U) \otimes Id_{\bar{U}})(Id_{\bar{U}} \otimes R_{t_U}) = P'^2 - P' = 0.$$

$\text{End}((\bar{U},\bar{\mathcal{H}}))$ is a finite-dimensional C* algebra. So, it is a von Neumann algebra. Therefore,there exist a projection $\bar{P} \in End(\bar{U},\bar{\mathcal{H}})$ such that $\bar{P}P'{=}P'$ and $P'\bar{P} = \bar{P}$. □

Now, we will use this to prove that if $P \in End((U,\mathcal{H}))$ then $((P \otimes 1)U, P\mathcal{H})$ is a subobject of $(U,\mathcal{H})$.

First, we choose the vectors $s_{P_u} = (\bar{P} \otimes P)s_U$ and $t_{P_u} = (P \otimes \bar{P})t_U$. It is straightforward to check $R_{s_{pu}} \in Mor((1_{\mathbb{C}} \otimes 1_A, \mathbb{C}), \bar{P}(\bar{\mathcal{H}}) \otimes P\mathcal{H})), R_{t_{P_u}} \in Mor((1_{\mathbb{C}} \otimes 1_A, \mathbb{C}), P\mathcal{H} \otimes \bar{P}(\bar{\mathcal{H}})))$. We will prove that

$$\begin{aligned}&(R^*_{s_{pu}} \otimes Id_{\bar{P}(\bar{\mathcal{H}})})(Id_{\bar{P}(\bar{\mathcal{H}})} \otimes R_{t_{P_u}}) = Id_{\bar{P}(\bar{\mathcal{H}})},\\ &(R^*_{t_{pu}} \otimes Id_{P\mathcal{H}})(Id_{P\mathcal{H}} \otimes R_{s_{P_u}}) = Id_{P\mathcal{H}}.\end{aligned}$$

Now,

$$\begin{aligned}(R^*_{s_{p_u}}\otimes Id_{\bar{P}(\bar{\mathcal{H}})})(Id_{\bar{P}(\bar{\mathcal{H}})}\otimes R_{t_{P_u}}) =&(R^*_{s_{p_u}}\otimes Id_{\bar{P}\bar{\mathcal{H}}})(\bar{P}\otimes P\otimes Id_{\bar{P}(\bar{\mathcal{H}})})(Id_{\bar{P}(\bar{\mathcal{H}})}\otimes P\otimes \bar{P})\\ &(Id_{\bar{P}(\bar{\mathcal{H}})}\otimes R_t)\\ =&(R^*_{s_{p_u}}\otimes Id_{\bar{P}(\bar{\mathcal{H}})})(\bar{P}\otimes P\otimes \bar{P})(Id_{\bar{P}(\bar{\mathcal{H}})}\otimes R_{t_{p_u}})\\ =&(R^*_{s_{p_u}}\otimes Id_{\bar{\mathcal{H}}})(Id_{\bar{\mathcal{H}}}\otimes P\otimes Id_{\bar{\mathcal{H}}})(\bar{P}\otimes Id_{\mathcal{H}}\otimes \bar{P})\\ &(Id_{\bar{\mathcal{H}}}\otimes R_{t_{p_U}})\\ =&(R^*_{s_{p_u}}\otimes Id_{\bar{\mathcal{H}}})(\bar{P}^*\otimes Id_{\mathcal{H}}\otimes Id_{\bar{\mathcal{H}}})(\bar{P}\otimes Id_{\mathcal{H}}\otimes \bar{P})\\ &(Id_{\bar{\mathcal{H}}}\otimes R_{t_{p_u}})\\ =&(R^*_{s_{p_u}}\otimes Id_{\bar{\mathcal{H}}})(P'^*\otimes Id_{\mathcal{H}}\otimes \bar{P})(Id_{\bar{\mathcal{H}}}\otimes R_{t_{p_u}})\\ =&\bar{P}(R^*_{s_{p_u}}\otimes Id_{\bar{\mathcal{H}}})(Id_{\bar{\mathcal{H}}}\otimes R_{t_{p_u}})P'^*\\ =&\bar{P}P'^*\\ =&\bar{P}\end{aligned}$$

and

$$\begin{aligned}(R^*_{t_{p_u}}\otimes Id_{P\mathcal{H}})(Id_{P\mathcal{H}}\otimes R_{s_{p_u}}) =&(R^*_{t_{p_u}}\otimes Id_{P\mathcal{H}})(P\otimes \bar{P}\otimes Id_{P\mathcal{H}})(Id_{P\mathcal{H}}\otimes \bar{P}\otimes P)\\ &(Id_{P\mathcal{H}}\otimes R_{s_{p_u}})\\ =&(R^*_{t_{P_u}}\otimes Id_{P\mathcal{H}})(P\otimes \bar{P}\otimes P)(Id_{P\mathcal{H}}\otimes R_{s_{P_U}})\\ =&(R^*_{t_{p_u}}\otimes Id_{\mathcal{H}})(P\otimes (\bar{P}\otimes P)R_{s_{p_U}})\\ =&(R^*_{t_{p_u}}\otimes Id_{\mathcal{H}})(P\otimes (\bar{P}P'\otimes Id_{\mathcal{H}})R_{s_{p_u}})\\ =&(R^*_{t_{p_u}}\otimes Id_{\mathcal{H}})(P\otimes (P'\otimes Id_{\mathcal{H}})R_{s_{p_u}})\\ =&(R^*_{t_{p_u}}\otimes Id_{\mathcal{H}})(P\otimes (Id_{\bar{\mathcal{H}}}\otimes P)R_{s_{p_u}})\\ =&P(R^*_{t_U}\otimes Id_{\mathcal{H}})(Id_{\mathcal{H}}\otimes R_{S_U})P\\ =&P.\end{aligned}$$

If $P\in End(U,\mathcal{H})$ is a projection then $((P\otimes 1)U,\mathcal{H})$ is a subobject of $(U,\mathcal{H})$. So, α is a rigid strict C* tensor category. □

We have a canonical fiber functor

$$F:\alpha\to Hilb_f \text{ given by}$$
$$F(U,\mathcal{H})=\mathcal{H}, F(T)=T, \forall T\in Mor((U,\mathcal{H}),(V,\mathcal{K}))$$

By Tannaka-Krein duality, there is a CQG Q_{univ} or equivalently the dual DQG $\hat{\mathcal{Q}}_{univ}$ such that each $(U, \mathcal{H}) \in obj(\alpha)$ is a unitary corepresentation of $\mathcal{Q}_{univ}$.

Lemma 3.2.3. *let $U \in B(\mathcal{H}) \otimes \mathcal{Q}$ is a unitary such that there exists a positive invertible matrix $\rho \in B(\bar{\mathcal{H}})$ for which $(\rho \otimes 1)U^c(\rho^{-1} \otimes 1)$ is a unitary. Then $(U, \mathcal{H}_U)$ is an object of α.*

Proof. If $(U, \mathcal{H})$ is an object of this category α then $(\rho \otimes 1)U^c(\rho^{-1} \otimes 1)$ is a unitary, is already discussed above. Let assume that, $(\rho \otimes 1)U^c(\rho^{-1} \otimes 1)$ is a unitary in $B(\bar{\mathcal{H}}) \otimes \mathcal{Q}$. Now, defined the maps $R_s : \mathbb{C} \to \bar{\mathcal{H}} \otimes \mathcal{H}$ and $R_t : \mathbb{C} \to \mathcal{H} \otimes \bar{\mathcal{H}}$ given by

$$R_s(1) = \sum_i \rho^{-1}(\bar{e_i}) \otimes e_i$$
$$R_t(1) = \sum_i e_i \otimes \rho(\bar{e_i}),$$

where $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$ and $\{\bar{e_i}\}$ is the corresponding orthonormal basis for $\bar{\mathcal{H}}$. Now, it is easy to prove it satisfies the following relations (3.2.1),(3.2.2) and (3.2.3). Hence $(U, \mathcal{H})$ is an object of this category α. □

In our work, we will often deal with suitable C^* subcategories of α. To make it convenient for later use, let us give a general construction of such subcategories from a given family of unitary operators. Let C_α be a subset of $Obj(\alpha)$ and the objects of C_α are of the form $(U, \mathcal{H})$. Let $\bar{C}_\alpha$ the smallest subfamily of $Obj(\alpha)$ containing C_α which is closed under direct sums and tensor products and subobjects.

Let us consider the following choices of C_α:

1) $C^L_\alpha = \{(U, \mathcal{H}) :$ U is a left projective corepresentation of $\mathcal{Q}$ on $\mathcal{H}\}$.

2) $C^R_\alpha = \{(U, \mathcal{H}) :$ U is a right projective corepresentation of $\mathcal{Q}$ on $\mathcal{H}\}$.

3) $C^{bi}_\alpha = \{(U, \mathcal{H}) :$ U is a bi projective corepresentation of $\mathcal{Q}$ on $\mathcal{H}\}$.

Apply the above construction, we will get the categories $\bar{C}^L_\alpha \equiv \bar{C}^R_\alpha, \bar{C}^{bi}_\alpha$.

Theorem 3.2.4. *Let $(\mathcal{Q}, \Delta)$ be a CQG. Then there exist CQGs $(\widetilde{\mathcal{Q}_L}, \Delta_L) \equiv (\widetilde{\mathcal{Q}_R}, \Delta_R)$, $(\widetilde{\mathcal{Q}}_{bi}, \Delta_{bi})$ along with injective CQG morphisms*

1) $i_L : \mathcal{Q} \to \widetilde{\mathcal{Q}_L}$,

2) $i_R : \mathcal{Q} \to \widetilde{\mathcal{Q}_R}$,

3) $i_{bi} : \mathcal{Q} \to \widetilde{\mathcal{Q}_{bi}}$

such that identifying $\mathcal{Q}$ as a Woronowicz's subalgebra of each of these CQGs $\widetilde{\mathcal{Q}}_L \equiv \widetilde{\mathcal{Q}}_R, \widetilde{\mathcal{Q}}_{bi}$, the following properties hold:

i) For every left projective unitary corperesentation $(U, \mathcal{H})$ of $\mathcal{Q}$ there exists a unitary (linear) corepresentation $(\tilde{U}_L, \mathcal{H})$ of $\widetilde{\mathcal{Q}}_L$ such that $Ad_{\tilde{U}_L}$ is isomorphic to Ad_U. A similar statement holds for right/bi projective unitary corepresentation of $\mathcal{Q}$.

*ii) The *-algebra generated by the matrix coefficients of $\tilde{U}_L$ is same as the Hopf *-algebra $Pol(\widetilde{\mathcal{Q}}_L)$, where U varies over the set of all left projective unitary corepresentations of $\mathcal{Q}$. A similar statement holds for right/bi projective unitary corepresentation of $\mathcal{Q}$.*

Proof. We prove the statements (i) and (ii) only for $\bar{C}^L_\alpha$, as the proof the other case are very similar. Let $\mathcal{Q} \subseteq B(\mathcal{H}_0)$, where $\mathcal{H}_0$ is the GNS space for the Haar state on $\mathcal{Q}$. Recall the Tannaka-Krein construction, which involves the $*$-algebra $Nat(F^L)$, where F^L is the restriction of the canonical fiber functor F to the subcategory $\bar{C}^L_\alpha$. Any element T of $Nat(F^L)$ is given by a collection $\{T_{(U,\mathcal{H}_U)} : T_{(U,\mathcal{H}_U)} \in B(\mathcal{H}_U)\}$.

The von-Neumann algebraic DQG M of $\widetilde{\mathcal{Q}_L}$ is isomorphic with

$$M = \{T \in Nat(F^L) : sup_{(U,H)\in \mathrm{Obj}(\bar{C}^L_\alpha)} \|T_{(U,H_U)}\| < \infty\} \tag{3.2.4}$$

For $\xi, \eta \in \mathcal{H}_0$, consider $\mathcal{U}_{\xi,\eta} \in \mathrm{Nat}(F^L)$ given by

$$(\mathcal{U}_{\xi,\eta})_{(U,H_U)} = U_{\xi,\eta}$$

where $U_{\xi,\eta} \in \mathcal{Q}$ is the contraction of $U \in \mathcal{B}(\mathcal{H}_U) \otimes \mathcal{Q}$, i.e.

$$\langle \theta_1, U_{\xi,\eta}(\theta_2)\rangle = \langle \theta_1 \otimes \xi, U(\eta \otimes \theta_2)\rangle \text{ for all } \theta_1, \theta_2 \in \mathcal{H}_0.$$

As each U is unitary, observe that $\sup_{(U,\mathcal{H}_U)\in \mathrm{Obj}(\tilde{C}^L_\alpha)} \|U_{\xi,\eta}\| < \|\xi\|\|\eta\|$, which proves that $\mathcal{U}_{\xi,\eta} \in \mathcal{M}$.

Let $\{(U_i, \mathcal{H}_i) : i \in I\}$ be an enumeration of all the irreducible objects of the category $\bar{C}^L_\alpha$. Then $\mathcal{M} \cong \prod_{i\in I} \mathcal{B}(\mathcal{H}_{U_i})$ and $\mathcal{M} \subseteq \mathcal{B}(\oplus_{i\in I}\mathcal{H}_{U_i})$, so that $\mathcal{M} \otimes \mathcal{B}(\mathcal{H}_0) \subseteq \mathcal{B}(\oplus_{i\in I}(\mathcal{H}_{U_i} \otimes \mathcal{H}_0)$.

Viewing $\mathcal{U}$ in this picture we see that $\mathcal{U} = \prod_{i\in I} U_i$, where $U_i \in \mathcal{B}(\mathcal{H}_{U_i}) \otimes \mathcal{Q} \subseteq \mathcal{B}(\mathcal{H}_{U_i} \otimes \mathcal{H}_0)$. As each U_i is unitary, so is $\mathcal{U}$. Next define $\Pi := \Pi_{\mathcal{U}}$ from the Hopf *-algebra $(\widetilde{\mathcal{Q}^L})_0$, spanned by the matrix elements of irreducible corepresentations of $\widetilde{\mathcal{Q}_L}$, to $\mathcal{Q} \subseteq B(\mathcal{H}_0)$ given by

$$\Pi(\omega) = (\omega \otimes Id)(\mathcal{U}), \tag{3.2.5}$$

where $\omega \in (\widetilde{\mathcal{Q}_L})_0$. This is well defined by the general theory of CQG/DQG. Π is a $*$-homomorphism. For $i \in I$, let $\widetilde{U_{i,L}}$ be the irreducible unitary corepresentation of $\widetilde{\mathcal{Q}_L}$ on $\mathcal{H}_{U_i}$. It is easy to see that $(Id \otimes \Pi)(\widetilde{U_{i,L}}) = U_i$.

In fact in a similar way we can show that $(Id \otimes \Pi)(\widetilde{U_L}) = U$, for all objects $(U, \mathcal{H}_U)$, where $\widetilde{U_L}$ is the corresponding unitary corepresentation of $\widetilde{\mathcal{Q}_L}$ on $\mathcal{H}_U$.

Now, note that if $(W, \mathcal{H}_w) \in obj(Corep(\mathcal{Q}))$ that is W is actually a unitary corepresentation of $\mathcal{Q}$ on $\mathcal{H}_W$, then $\Pi(\omega) = \omega$ for any matrix coefficient ω of W. In other words $(Id \otimes \Pi)(W) = W$. One can also obeserve that

$$\Pi(\widetilde{U_L^c}) = (\Pi(\widetilde{U_L}))^c = U^c$$

, for all objects $(U, \mathcal{H}_U)$.

By construction, for all objects $(U, \mathcal{H}_U)$, $(U^c \otimes U, \bar{\mathcal{H}} \otimes \mathcal{H})$ is a corepresentation of $\mathcal{Q}$. But the unitary corepresentation to the object $(U^c \otimes U, \bar{\mathcal{H}} \otimes \mathcal{H})$ is nothing but $\widetilde{U_L^c} \otimes \widetilde{U_L}$. Hence by our earlier observation

$$(Id \otimes \Pi)(\widetilde{U_L^c} \otimes \widetilde{U_L}) = (\widetilde{U_L^c} \otimes \widetilde{U_L}) \tag{3.2.6}$$

so $U^c \otimes U = \widetilde{U_L^c} \otimes \widetilde{U_L}$ which implies that $\widetilde{U_L}$ is a left projective corepresentation on $\mathcal{H}_U$. This proves (i).

(ii) Follows from the construction and the general theory of CQG. $\square$

Corollary 3.2.5. *let $U \in B(\mathcal{H}) \otimes \mathcal{Q}$ is a unitary. $(U, \mathcal{H})$ is an object of the category α if and only if there exists a positive invertible matrix $\rho \in B(\bar{\mathcal{H}})$ such that $(\rho \otimes 1)U^c(\rho^{-1} \otimes 1)$ is a unitary.*

Now we compute the projective envelope for the classical group $SO(3)$, whose universal cover is $SU(2)$. It is well known that there is exactly one irreducible representation $SU(2)$ in each dimension. When the dimension is odd, the representation descends to an ordinary representation of $SO(3)$. When the dimension is even , the representation does not descend to an ordinary representation of $SO(3)$, but it does descend to a projective representation of $SO(3)$. Before that we will prove the following lemma:

Lemma 3.2.6. *For a classical compact group G, the projective envelope of $C(G)$ is a commutative CQG.*

Proof. Let $U \in B(H_u) \otimes C(G), V \in B(H_v) \otimes C(G)$ be two irreducible inequivalent projective corepresentations of $C(G)$. Without loss of generality ,we assume that $U = \sum_{i,j} e_{ij} \otimes u_{ij}, V = \sum_{k,l} f_{kl} \otimes v_{kl}$. Let $T : H_u \otimes H_v \to H_v \otimes H_u$ be the flip operator. For

$e_{i_0} \in H_u, f_{k_0} \in H_v$

$$\begin{aligned}(T\otimes 1)(U\otimes V)(e_{i_0}\otimes f_{k_0}) =&(T\otimes 1)(\sum_{i,j,k,l} e_{ij}(e_{i_0})\otimes f_{kl}(f_{k_0})\otimes u_{ij}v_{kl})\\ =&(T\otimes 1)(\sum_{i,k} e_i\otimes f_k\otimes u_{ii_0}v_{kk_0})\\ =&\sum_{i,k} f_k\otimes e_i\otimes u_{ii_0}v_{kk_0}.\end{aligned}$$

and

$$\begin{aligned}(V\otimes U)(T\otimes 1)(e_{i_0}\otimes f_{k_0}\otimes 1) =&\sum_{i,j,k,l} f_{kl}(f_{k_0})\otimes e_{ij}(e_{i_0})\otimes v_{kl}u_{ij}\\ =&\sum_{k,i} f_k\otimes e_i\otimes v_{kk_0}u_{ii_0}\\ =&\sum_{i,k} f_k\otimes e_i\otimes u_{ii_0}v_{kk_0}.\end{aligned}$$

So, the unitary operator $T \in Mor(U \otimes V, V \otimes U)$. From Tannaka-Krein duality, we can say that $T \in Mor(X^{U\otimes V}, X^{V\otimes U})$. $F_2(U \otimes V)$ is the identity operator that's why $X^{U\otimes V} = X^U \otimes X^V$. Now, we can easily conclude that $x^u_{ii_0}x^v_{kk_0} = x^v_{kk_0}x^u_{ii_0}$, where i_0, k_0, u, v are arbitrary. As matrix coefficients of inequivalent irreducible corepresentations is a basis of the Hopf $*$-algebra $\widetilde{C(G)}$ that's why $\widetilde{C(G)}$ is a commutative compact quantum group. □

Theorem 3.2.7. *$\widetilde{C(SO(3))} \cong C(SU(2) \times \widehat{U}(C(SO(3))))$, where $U(C(SO(3))$ is the unitary group of $C(SO(3))$.*

Proof. It is already known from lemma 3.2.6 that $\widetilde{C(SO(3))}$ is a commutative compact qunatum group. For each dimension n, we denote U_n as the corresponding irreducible projective corepresentation of $C(SO(3))$. And the corresponding cocycle is trivial when n is odd and for even n it corresponds to the nontrivial cocycle Ω of $SO(3)$. Any Irreducible unitary projective corepresentation U of dimension n can be written as a $U = U_n(Id\otimes u)$, where u is a unitary element of $C(SO(3))$. It follows from theorem 3.2.4 that projective envelope of $C(SO(3))$ is generated by the matrix coefficients of U_n and the one dimensional corepresentations $Id_{\mathbb{C}} \otimes X^u$, where $u \in U(C(SO(3)))$. Matrix coefficients of U_n generated the Hopf *-algebra $Pol(SU(2))$. $\{Id_{\mathbb{C}} \otimes X^u : u \in U(C(SO(3)))\}$ generated the Hopf *-algebra $Pol(C^*(U(C(SO(3))))$, where $U(C(SO(3)))$ considered as a discrete group. If we take the universal completion of this Hopf * -algebra which is generated by the matrix coefficients of U_n and $Id_{\mathbb{C}} \otimes X^u$ then we will get the compact quantum group

$C(SU(2) \otimes_{max} C^*(U(C(SO(3)))) \cong C(SU(2)) \otimes_{min} C(\widehat{U}(C(SO(3)))) \cong C(SU(2) \times \widehat{U}(C(SO(3))))$. $\square$

Chapter 4

Strongly projective corepresentation and cohomology of compact quantum group

4.1 Normalizer of a tensor category

As before $(\mathcal{Q}, \Delta)$ be a CQG with the (mutually inequivalent) irreducible uitary corepresentation $U^\alpha = \sum_{i,j=1}^{d_\alpha} u_{ij}^\alpha \otimes e_{ij}^\alpha$, where $\alpha \in L$ (L is an indexed set). Let $H \subseteq L$ and $\mathcal{Q}_H, \widehat{\mathcal{Q}_H}$ be as in chapter (1.4).

Definition 4.1.1. *$\widehat{\mathcal{Q}_H}$ is said to be normal quantum subgroup of $\widehat{\mathcal{Q}}$ if $W_\mathcal{Q}(\mathcal{Q}_H \otimes Id)W_\mathcal{Q}^* \subset \mathcal{Q}_H'' \bar{\otimes} B(L^2(\mathcal{Q}))$, where $L^2(\mathcal{Q})$ denotes the GNS space of $\mathcal{Q}$ with respect to the Haar state and $W_\mathcal{Q}$ denotes the left regular representation of $\mathcal{Q}$, which is given by $W_\mathcal{Q} = \sum_{\alpha,i,j} u_{ij}^\alpha \otimes e_{ij}^\alpha$.*

Lemma 4.1.2. *$\widehat{\mathcal{Q}_H}$ is normal if and only if for any $x \in \mathcal{Q}_H$ and $\alpha \in L, 1 \leq i, i_1 \leq d_\alpha$, $v_{ii_1}^{\alpha,x} = \sum_j u_{ij}^\alpha x u_{i_1 j}^{*\alpha}$ is in $\mathcal{Q}_H$*

Proof. We note that for any x in $\mathcal{Q}_H$,

$$\begin{aligned} W_\mathcal{Q}(x \otimes 1)W_\mathcal{Q}^* =& (\sum_\alpha (\sum_{i,j} u_{ij}^\alpha \otimes e_{ij}^\alpha))(x \otimes 1)(\sum_\alpha (\sum_{i_1,j_1} u_{i_1 j_1}^{*\alpha} \otimes e_{i_1 j_1})) \\ =& \sum_\alpha (\sum_{i_1,j_1,i,j} u_{ij}^\alpha x u_{i_1 j_1}^{*\alpha} \otimes e_{ij}^\alpha e_{j_1 i_1}^\alpha \\ =& \sum_\alpha (\sum_{i,i_1} u_{ij}^\alpha x u_{i_1 j}^{*\alpha} \otimes e_{ii_1}^\alpha) \\ =& \sum_\alpha \sum_{i,i_1} v_{ii_1}^{\alpha,x} \otimes e_{ii_1}^\alpha. \end{aligned}$$

Let us assume that $\widehat{\mathcal{Q}_H}$ is normal quantum subgroup of $\widehat{\mathcal{Q}}$. Then $\sum_{i,i_1} v^{\alpha,x}_{ii_1} \otimes e^{\alpha}_{ii_1}$ is in $\mathcal{Q}''_H \bar{\otimes} B(L^2(\mathcal{Q}))$, if we take the normal linear functional ϕ^α on $B(L^2(\mathcal{Q}))$ such that $\phi^\alpha(e^{\alpha}_{i_0j_0})$=1 and $\phi^\alpha(e^{\alpha}_{i_0j_0})$=0 if $(i,j) \neq (i_0, j_0)$. Then (Id$\otimes$ ϕ^α) $(W_{\mathcal{Q}}(x \otimes 1)W^*_{\mathcal{Q}})$=$v^{\alpha,x}_{i_0j_0} \in \mathcal{Q}_H$. Now, for $x \in Pol(\mathcal{Q}_H)$, clearly $V^{\alpha,x}_{i_0j_0} \in Pol(\mathcal{Q})$. Hence it is in $Pol(\mathcal{Q}) \cap \mathcal{Q}''_H = Pol(\mathcal{Q}_H)$. In general approximating $x \in \mathcal{Q}_H$ by a sequence from $Pol(\mathcal{Q}_H)$, we get $v^{\alpha,x}_{ii_1} \in \mathcal{Q}_H$.

The converse also follows from the lemma (4.1.2) as the sum is a strongly convergent series and $\mathcal{Q}''_H \bar{\otimes} B(L^2(\mathcal{Q}))$ is a von-Neumann algebra. □

Remark 4.1.3. *It is clear from the proof of lemma (4.1.2) that it is enough to verify the condition of the lemma for any subset $K \subseteq L$ which is generating in the sense that every object in L is a subobject of finite direct sums of tensor products of objects from the subset K.*

Now prove a result in the categorical setting, which will be useful for our later discussion. Let us assume that C is a UTC and F is a fiber functor on C. Suppose also that C_0 is an embeded sub UTC of C and F_0 is the restriction of F on C_0. Let $\{x_\alpha : \alpha \in I\}$ is collection of irreducible objects of C and $\{x_\alpha : \alpha \in I_0\}$ be the collection of irreducible objects of C_0.
Let A_J=$\{x \in obj(C) : x \otimes y \otimes \bar{x}, \bar{x} \otimes y \otimes x \in obj(C_0), \forall y \in obj(C_0) \}$.
Now, we define a subcategory C_J by $obj(C_J) = \{\bigoplus_{i=1}^n x_i : x_i \in A_J, n \geq 1\}$.

Lemma 4.1.4. *C_J is a UTC and C_0 is a sub UTC of C_J.*

Proof. Let $\bigoplus x_i, \bigoplus y_j \in obj(C_J)$ (finite direct sums). Now $(\bigoplus x_i) \otimes (\bigoplus y_j)$ is isomorphic to $\bigoplus(x_i \otimes y_j)$. For any object z of C_0, $x_i \otimes y_j \otimes z \otimes \bar{y}_j \otimes \bar{x}_i$ is an object of C_0 because $y_j \otimes z \otimes \bar{y}_j \in obj(C_0)$ and hence $x_i \otimes (y_j \otimes z \otimes \bar{y}_j) \otimes \bar{x}_i \in Obj(C_0)$ too. This means $x_i \otimes y_i \in A_J$, hence their finite sum is also an object of C_J.

Next we will prove it is closed under taking subobject. Let x_p be a subobject of x, where $x \in obj(C_J)$. We have $x \otimes y \otimes \bar{x}, \bar{x} \otimes y \otimes x \in obj(C_0)$. So, there exist an isometry $V_p \in Mor(x_p, x)$ and $V_p V_p^* = P \in End(x)$. We know if x_p is a subobject of x then $\bar{x}_p$ is a subobject of $\bar{x}$. Similarly, we can say that there exist an isometry $\bar{V_p} \in Mor(\bar{x}_p, \bar{x})$ and $\bar{V_p}\bar{V_p^*} = \bar{P} \in End(x)$. Hence it follows that $(V_p \otimes Id_y \otimes \bar{V_p}) \in Mor(x_p \otimes y \otimes \bar{x}_p, x \otimes y \otimes \bar{x})$ and it is an isometry and $V_p V_p^* \otimes Id \otimes \bar{V_p}\bar{V_p^*} = (p \otimes Id \otimes \bar{p}) \in End(x \otimes y \otimes \bar{x})$. So, $x_p \otimes y \otimes \bar{x_p}$ is a subobject of $(x \otimes y \otimes \bar{x}) \in obj(C_0)$ and as C_0 is closed under taking subobjects, $x_p \otimes y \otimes \bar{x}_p \in obj(C_0)$. Now, if we choose an object $x = \bigoplus_{i=1}^n x_i$ then any subobject is of the form $\bigoplus x_{p_i}$,where x_{p_i} is a subobject of x_i as we know $x_{p_i} \otimes y \otimes \bar{x}_{p_i}, \bar{x}_{p_i} \otimes y \otimes x_{p_i} \in obj(C_0)$. This implies $x \in obj(C_0)$. It follows easily that C_J is also closed under taking conjugate. Finally, we note that $obj(C_0) \subseteq obj(C)$ from the definition of C_J and the fact that C_0 is an embeded sub UTC. □

Definition 4.1.5. *We call C_J the normalizer of C_0 in C and define it by $N(C, C_0)$.*

Applying Tannaka-krein duality on C_J and C_0 with the fiber functor F and F_0 respectively, we get CQG $\mathcal{Q}_J$ and $\mathcal{Q}_0$ such that $\mathcal{Q}_0$ is a Woronowicz subalgebra of $\mathcal{Q}_J$.

The following theorem justify the terminology of definition (4.1.4).

Theorem 4.1.6. *$\widehat{\mathcal{Q}_0}$ is a quantum normal subgroup of $\widehat{\mathcal{Q}_J}$.*

Proof. We apply lemma (4.1.2), as before, for an irreducible object z in A_J, choose and fix a unitary corepresentation U^z. By remark (4.1.3) it is enough to verify the condition of lemma (4.1.2) for the generating subset A_J. If x_α is an irreducible object in A_J and x_β is an irreducible object of C_0 then $x_\alpha \otimes x_\beta \otimes \bar{x}_\alpha$ is an object of C_0. So, there exists a unitary operator T such that $(1_\mathcal{Q} \otimes T)(U^{x_\alpha} \otimes U^{x_\beta} \otimes U^{x_\alpha})(1_\mathcal{Q} \otimes T^*)$ is a unitary coreprsentation of $\mathcal{Q}_0$. Write $U^{x_\alpha} = \sum_{i,j} u^\alpha_{ij} \otimes e^\alpha_{ij}$, $U^{x_\beta} = \sum_{k,l} v^\beta_{kl} \otimes f^\beta_{kl}$,$U^{\bar{x}_\alpha} = \sum_{m,n} u^{*\alpha}_{mn} \otimes R^{1/2} e^{\bar{\alpha}}_{nm} R^{-1/2}$ for suitable matrix units $e^\alpha_{ij}, f^\beta_{kl}$ and positive invertible operator $R \in B(\bar{\mathcal{H}_\alpha})$.
Let us assume that $(U^{x_\alpha} \otimes U^{x_\beta} \otimes U^{\bar{x}_\alpha}) = V^{\alpha,\beta,\bar{\alpha}} = \sum u^\alpha_{ij} v^\beta_{kl} u^{*\alpha}_{mn} \otimes e^\alpha_{ij} \otimes f^\beta_{kl} \otimes R^{1/2} e^{\bar{\alpha}}_{nm} R^{-1/2}$.
Now, we take the linear functional ϕ^α on $B(\mathcal{H}_\alpha)$ satisfy $\phi^\alpha(e^\alpha_{i_0 j_0})$=1 and $\phi^\alpha(e^\alpha_{ij}) = 0$ if $(i,j) \neq (i_0, j_0)$. Similarly, we choose the linear functionals ϕ^β and $\phi^{\bar{\alpha}}$ on $B(\mathcal{H}_\beta)$,$B(\bar{\mathcal{H}_\alpha})$ given by $\phi^\beta(e^\beta_{k_0 l_0}) = 1$, $\phi^\beta(e^\beta_{kl}) = 0$ if $(k,l) \neq (k_0, l_0)$ and $\phi^{\bar{\alpha}}(R^{1/2} e^{\bar{\alpha}}_{n_0 m_0} R^{-1/2}) = 1$ and $\phi^{\bar{\alpha}}(R^{1/2} e^{\bar{\alpha}}_{nm} R^{-1/2}) = 0$ if $(n,m) \neq (n_0, m_0)$.
Then,$(Id \otimes \phi^\alpha \otimes \phi^\beta \otimes \phi^{\bar{\alpha}})(V^{\alpha,\beta,\bar{\alpha}}) = u^\alpha_{i_0 j_0} v^\beta_{k_0 l_0} v^{*\alpha}_{m_0 n_0}$ is in $Pol(\mathcal{Q}_0)$. This proves that $\sum_j u^\alpha_{ij} v^\beta_{k_0 l_0} v^{*\alpha}_{i_1 j} \in Pol(\mathcal{Q}_0)$. Finally, approximating a general element $x \in \mathcal{Q}_0$ by a sequence from $Pol(\mathcal{Q}_0)$.

4.2 Strongly projective corepresentation

Let U be an object of the category α.

Definition 4.2.1. *U is said to be a strongly right projective corepresentation if $U \otimes V \otimes \bar{U}$ is a corepresentation of $\mathcal{Q}$ for any corepresentation V of $\mathcal{Q}$.*

Lemma 4.2.2. *If U is a strongly right projective corepresentation then it is a right projective corepresentation of $\mathcal{Q}$.*

Proof. By definition, $U \otimes V \otimes \bar{U}$ is a corepresentation of $\mathcal{Q}$ for any corepresentation V of $\mathcal{Q}$. For $V = Id_\mathbb{C} \otimes 1_\mathcal{Q}$, $U \otimes \bar{U}$ is a corepresentation. So it is a right projective corepresentation of $\mathcal{Q}$. □

Remark 4.2.3. *Let X, Y be two strongly right projective corepresentations of $\mathcal{Q}$ then $X \otimes Y$ is a right projective corepresentation of $\mathcal{Q}$ because if we put $V = Id_c \otimes 1_{\mathcal{Q}}$ then $X \otimes Y \otimes V \otimes \bar{Y} \otimes \bar{X} = X \otimes Y \otimes \bar{Y} \otimes \bar{X}$ is a corepresentation of $\mathcal{Q}$.*

Lemma 4.2.4. *U is a strongly right projective corepresentation and Ω is a corresponding right 2-cocycle of $\mathcal{Q}$ if and only if right 2-cocycle Ω commutes with $\Delta(x)$ for any $x \in \mathcal{Q}$.*

Proof. If U is a right projective corepresentation and $(Id \otimes \Delta_{\Omega^*})(U) = U_{12}U_{13}$ then $\bar{U}$ is a left projective corepresentation and correspond to left 2-cocycle Ω. By definition it follows that for any corepresentation V of $\mathcal{Q}$, $U \otimes V \otimes \bar{U}$ is a corepresentation of $\mathcal{Q}$. So

$$\begin{aligned}(Id \otimes Id \otimes Id \otimes \Delta)(U \otimes V \otimes \bar{U}) =&(U \otimes V \otimes \bar{U})_{1234}(U \otimes V \otimes \bar{U})_{1235}\\ =&(U_{14}V_{24}\bar{U_{34}} \otimes 1_{\mathcal{Q}})(U_{15}V_{25}\bar{U_{35}})\\ =&U_{14}U_{15}V_{24}V_{25}\bar{U_{34}}\bar{U_{35}}.\end{aligned}$$

and

$$(Id \otimes Id \otimes Id \otimes \Delta)(U \otimes V \otimes \bar{U})$$

$$= (Id \otimes Id \otimes Id \otimes \Delta)(U_{14}V_{24}\bar{U_{34}})$$

$$= (Id \otimes Id \otimes Id \otimes \Delta)(U_{14}V_{24}\bar{U_{34}})$$

$$= (Id \otimes Id \otimes Id \otimes \Delta)(U_{14})(Id \otimes Id \otimes Id \otimes \Delta)(V_{24})(Id \otimes Id \otimes Id \otimes \Delta)(\bar{U}_{34})$$

$$= U_{14}U_{15}(Id \otimes Id \otimes Id \otimes \Omega)V_{24}V_{25}(Id \otimes Id \otimes Id \otimes \Omega^*)\bar{V_{34}}\bar{V_{35}}.$$

From those two equations, one can conclude that

$$V_{24}V_{25} =(Id \otimes Id \otimes Id \otimes \Omega)V_{24}V_{25}(Id \otimes Id \otimes Id \otimes \Omega^*).$$

Hence,

$$(Id \otimes \Omega)(Id \otimes \Delta)(V) = (Id \otimes \Delta)(V)(Id \otimes \Omega).$$

For any $x \in pol(\mathcal{Q})$, $\Omega\Delta(x) = \Delta(x)\Omega$. Converse part follows easily. □

Lemma 4.2.5. *If Ω is an invariant right 2-cocycle of $\mathcal{Q}$ then*
1) Ω^* is a invariant right 2 cocycle of $\mathcal{Q}$.
2) If U is a right projective corepresentation and Ω is corresponding 2-cocycle if and only if U is a left projective corepresentation .

Proof. 1) As $\Delta(x)$ commutes with Ω for all $x \in \mathcal{Q}$, $(\Delta \otimes id)(y)$ commutes with $(\Omega \otimes 1)$ for all $y \in \mathcal{Q} \otimes \mathcal{Q}$. Now taking $*$ on both side of the relation $(\Omega \otimes 1)(\Delta \otimes id)(\Omega) = (1 \otimes \Omega)(1 \otimes \Delta)(\Omega)$, is easily follows

$$\begin{aligned}(\Delta \otimes id)(\Omega^*)(\Omega^* \otimes 1)) =&((\Delta \otimes id)(\Omega))^*(\Omega^* \otimes 1))\\ =&((\Omega \otimes 1)(\Delta \otimes 1)(\Omega))^*\\ =&((\Delta \otimes Id)(\Omega)(\Omega \otimes 1))^*\end{aligned}$$

Similarly one can prove that $(1 \otimes \Delta)(\Omega^*)(1 \otimes \Omega^*) = (1 \otimes \Omega^*)(1 \otimes \Delta)(\Omega^*)$. So Ω^* is a right 2-Cocycle, which is invariant as Ω^* and $\Delta(x)$ commutes with $\forall x \in \mathcal{Q}$.

2) Note that by (1), $\Delta_{\Omega^*}(x) = {}_{\Omega}{}^*\Delta(x)$. If U is a right projective corepresentations then

$$(Id \otimes \Delta_{\Omega^*})(U) = (Id \otimes_{\Omega^*} \Delta)(U) = U_{12}U_{13}.$$

Hence U is also left projective corepresentation. □

Corollary 4.2.6. *A unitary $\Omega \in \mathcal{Q} \otimes \mathcal{Q}$ is an invariant right 2-cocycle if and only if it is an invariant left 2-cocycle.*

Proof. Clearly Ω is an invariant 2-cocycle, which commutes with $\Delta(x)$ for all $x \in \mathcal{Q}$ if and only if Ω^* does so. Moreover by lemma (4.2.5) we see that Ω invariant right 2-cocycle implies that Ω^* is an invariant right 2-cocycle,hence Ω is an invariant left 2-cocycle.

Similarly, we can prove that any left 2-cocycle Ω is also invariant right 2-cocycle. □

Remark 4.2.7. *In view of the above, we will call a left/right invariant 2-cocycle simply an invariant 2-cocycle.*

Definition 4.2.8. *If $Y \in B(H_y) \otimes \mathcal{Q}$ is a unitary element and for any corepresentation V of $(\mathcal{Q}, \Delta)$, $\bar{Y} \otimes V \otimes Y$ is a corepresentation of $\mathcal{Q}$ then Y is a strongly left projective corepresentation of $\mathcal{Q}$.*

Remark 4.2.9. *One can easily conclude that any strongly left projective corepresentation of $\mathcal{Q}$ is a left projective corepresentation of $\mathcal{Q}$.*

Lemma 4.2.10. *If Ω is a left 2-cocycle and U is a strongly left projective corepresentation with corresponding left 2-cocycle Ω if and only if Ω commutes with $\Delta(x)$, for any $x \in q$.*

Proof. Proof of this lemma similar to the proof of the lemma (4.2.4) . Hence omitted. □

Lemma 4.2.11. *If U is a unitary in $B(H_u) \otimes \mathcal{Q}$ then the following are equivalent:*

1. *U is a strongly left projective corepresentation.*
2. *U is a strongly right projective corepresentation.*

Proof. Let U be a strongly right projective corepresentation with corresponding invariant 2-cocycle Ω_u. $\bar{U}$ is a left projective corepresentation with corresponding 2-cocycle Ω_u^*. Hence by lemma (4.2.5), it is also a right projective corepresentation with corresponding 2-cocycle Ω_u^*. Hence, for every corepresentation V of $\mathcal{Q}$, $\bar{U} \otimes V \otimes U$ corresponds to the cocycle $\Omega_u^* . \Omega_u = 1$ (as $(Id \otimes \Omega)$ commutes with $((Id \otimes \Delta)(V))$, that is $\bar{U} \otimes V \otimes U$ is a corepresentation of $\mathcal{Q}$. This proves $\bar{U}$ is a strongly right projective corepresentation, so U is strongly left projective corepresentation of $\mathcal{Q}$.

The other way is similar to prove. □

In the view of above, we make the following definition

Definition 4.2.12. *A unitary $U \in B(\mathcal{H}_u) \otimes \mathcal{Q}$ a strongly projective corepresentation if it satisfying any of two equivalent conditions:*

1) $U \otimes V \otimes \bar{U}$ is corepresentation for all corepresentation V of $\mathcal{Q}$.

2) $\bar{U} \otimes V \otimes U$ is corepresentation for all corepresentation V of $\mathcal{Q}$.

Lemma 4.2.13. *Let U, W be two strongly projective Ω_u, Ω_w corepresentations,*

1) $U \otimes W$ is a strongly projective $\Omega_u \Omega_w$ corepresentations.

2) U is a subobject of W then $\Omega_u = \Omega_w$.

Proof. 1) It is already known that $(Id \otimes \Delta_{\Omega_u})(U) = U_{12}U_{13}$ and $(Id \otimes \Delta_{\Omega_w})(W) = W_{12}W_{13}$. Now,

$$\begin{aligned}(Id \otimes \Delta_{\Omega_u\Omega_w})(U \otimes W) =& (Id \otimes Id \otimes \Delta_{\Omega_u})(U_{13})(Id \otimes Id \otimes \Delta_{\Omega_W})(W_{23}) \\ &= U_{13}U_{14}W_{23}W_{24} \\ &= U_{13}W_{23}U_{14}W_{24} \\ &= (U \otimes W)_{12}(U \otimes W)_{13}.\end{aligned}$$

Hence, $(U \otimes W)$ is a strongly projective $\Omega_u\Omega_w$ corepresentation.

2) Let P is a projection in $Mor(W,W)$ such that $(P \otimes 1)W = U = W(P \otimes 1)$.

$$\begin{aligned}(Id \otimes \Delta_{\Omega_w^*})((P \otimes 1)W) =&(P \otimes 1)((Id \otimes \Delta_{\Omega_w^*})(W)\\ =&(P \otimes 1)W_{12}W_{13}\\ =&((P \otimes 1)W)_{12}((P \otimes 1)W)_{13}\\ =&U_{12}U_{13}\\ =&(Id \otimes \Delta_{\Omega_u^*})(U).\end{aligned}$$

Hence $\Omega_u = \Omega_w$. □

Remark 4.2.14. *When $\mathcal{Q} = C(G)$ i.e. commutative as a C^* algebra, any left/right projective corepresentation is automatically strongly projective corepresentation.For a compact group G, there is a one-one correspondence between finite dimensional projective corepresentations of $(C(G), \Delta)$ and continuous projective representations of G. Let ϕ be a continuous projective representation of G over V (Hilbert space) and ω_ϕ be the corresponding 2-cocycle. We can write $\phi(g) = \sum c_{ij}(g)e_{ij}$, where e_{ij} are the matrix units of $B(V)$. Then $u_\phi := \sum e_{ij} \otimes c_{ij}$ is a unitary element of $B(V) \otimes \mathcal{Q}$. One can easily conclude that $(Id \otimes \Delta_{\omega_\phi^*})(u_\phi) = (u_\phi)_{12}(u_\phi)_{13}$. So, $\phi \mapsto u_\phi$ is a one-one correspondence between projective corepresentations of $(C(G), \Delta)$ and continuous projective representations of G on a finite dimensional Hilbert space .*

Recall the category α constructed in the chapter 3, section (3.2), consider the subfamily of $obj(\alpha)$ given by $(U, \mathcal{H})$, where U is a strongly projective corepresentation of $\mathcal{Q}$. The corresponding C^* tensor category generated by this family , in a similar way as in theorem (3.2.4) , will be denoted by C_{stp} and let F_{stp} be the restriction of F on C_{stp}. Now, if we apply the Tannaka-Krein duality on (C_{stp}, F_{stp}), we will get $\mathcal{Q}_{stp}$ which is called the strongly projective envelope of $\mathcal{Q}$. Clearly $\mathcal{Q} \subseteq \mathcal{Q}_{stp}$ is a Woronowicz subalgebra.

Let $\mathcal{G}$ be the UTC generated by corepresentation of $\mathcal{Q}$ and $(Id_{\mathbb{C}} \otimes q)$,where $(Id_{\mathbb{C}} \otimes q)$ is an object of the UTC C_{stp}.

Lemma 4.2.15. *If $(q, \mathbb{C})$ is an object of this category C_{stp}, where q is a unitary in $\mathcal{Q}$ if and only if $(q^* \otimes q^*)\Delta(q)$ commutes with $\Delta(x)$ for all $x \in \mathcal{Q}$. Moreover , in this case $\Delta(q)$ commutes with $(q^* \otimes q^*)$.*

Proof. $(Id_{\mathbb{C}} \otimes q)$ is an object of C_{stp} that means that $(Id_{\mathbb{C}} \otimes q)V(Id_{\mathbb{C}} \otimes q^*)$ is a corepresentation of $\mathcal{Q}$, for any corepresentation V of $\mathcal{Q}$. Hence $(Id \otimes q)$ is a strongly projective corepresentation of $\mathcal{Q}$ and $(q^* \otimes q^*)\Delta(q)$ corresponding 2-coycle . We conclude from lemma (4.2.4) that $(q^* \otimes q^*)\Delta(q)$ commutes with $\Delta(x)$ for any $x \in \mathcal{Q}$. Taking $x \in \mathcal{Q}$, we get

$$(q^* \otimes q^*) = (q^* \otimes q^*)\Delta(q)\Delta(q^*) = \Delta(q^*)(q^* \otimes q^*)\Delta(q)$$

Hence $\Delta(q)(q^* \otimes q^*) = (q^* \otimes q^*)\Delta(q)$. □

Let $\mathcal{Q}_{\mathcal{G}}$ be the CQG obtained by Tannaka-Krein reconstruction by restricting F to the subcategory $\mathcal{G}$. We have the normalizer $N(C_{stp}, \mathcal{G})$ and let the CQG $N(\mathcal{Q}_{\mathcal{G}})$ be the one obtained by restriction of F to $N(C_{stp}, \mathcal{G})$. Here we recall the equivalence relation $\sim$ of lemma (1.4.6) on $Rep(\hat{\mathcal{Q}})$ associated with the discrete normal subgroup $\hat{\mathcal{Q}}_{\mathcal{G}}$ of $\widehat{N(\mathcal{Q}_{\mathcal{G}})}$ and also the fiber functors

$$F_1 : Rep(\hat{\mathcal{Q}}_{\mathcal{G}}) \to Rep(\widehat{N(\mathcal{Q}_{\mathcal{G}})}),$$
$$F_2 : Rep(\widehat{N(\mathcal{Q}_{\mathcal{G}})}) \to Rep(C)$$

where $C = \widehat{N(\mathcal{Q}_{\mathcal{G}})}/\hat{\mathcal{Q}}_{\mathcal{G}}$.

Lemma 4.2.16. *For an irreducible object $(U, \mathcal{H}) \in obj(N(C_{stp}, \mathcal{G}))$, the equivalence class K_U contains exatly one (irreducibe) 1-dimensional representation, say ρ_U of the quotient DQG C.*

Proof. Denote by $\Pi_U : \widehat{N(\mathcal{Q}_{\mathcal{G}})} \to B(\mathcal{H})$ the representation dual to the corepresentation U. Note that $U \otimes \bar{U}$ a corepresentation of $\mathcal{Q}$, so in particular an object of $\mathcal{G}$. Hence $F_2(\pi_U \otimes \bar{\pi}_U)$ must be n^2 copies of the trivial representation of e of $\widehat{N(\mathcal{Q}_{\mathcal{G}})}/\hat{\mathcal{Q}}_{\mathcal{G}}$, where $n = dim(\mathcal{H})$. Now suppose $F_2(\pi_U) \cong \oplus_{i=1}^{k} \pi_i$, each π_i is a irreducible representation of $\widehat{N(\mathcal{Q}_{\mathcal{G}})}/\hat{\mathcal{Q}}_{\mathcal{G}}$. As $F_2(\pi_U) \otimes \overline{F_2(\pi_U)}$ is direct sum of copies of e only. But this is clearly possible if and only if each $\pi_i \otimes \bar{\pi}_j$ is isomorphic with e, hence in particular 1-dimensional. This further implies π_i to be 1-dimensional and $\pi_i \cong \pi_j$ as well, which completes the proof. □

Lemma 4.2.17. *If two irreducible strongly projective corepresentations $(U, \mathcal{H}), (W, \mathcal{K}) \in obj(N(C_{stp}, \mathcal{G}))$, we have $K_U = K_W$ if and only if the corresponding invariant 2-cocycles Ω_U, Ω_W are related by :*

$$\Omega_w = \Delta(q)\Omega_u(q^* \otimes q^*) \tag{4.2.1}$$

for some unitary q such that $(q, \mathbb{C}) \in obj(C_{stp})$.

Proof. If $K_U = K_W$ if and only if $U \sim W$, which is equivalent to U being a subobject of $W \otimes V$, where V is an object of C_{stp}. By lemma (4.2.13) cocycle of $W \otimes V$ is $\Omega_w \Omega_v$.

We can replace $V \in obj(C_{stp})$ by any suitable generating subset . In particular V may be either a corepresentation or a unitary element q such that $(q, \mathbb{C}) \in obj(C_{stp})$. If V is a corepresentation then $\Omega_u = \Omega_w$ and the other case $\Omega_u = \Omega w(q \otimes q)\Delta(q^*)$.

Conversely, suppose $\Omega_w = \Delta(q)\Omega_u(q^* \otimes q^*)$. For $U(Id_{\mathbb{C}} \otimes q)$ and W they have the same invariant 2-cocycle Ω_w. Hence $V = \bar{W} \otimes U(Id_{\mathbb{C}} \otimes q)$ is a corepresentation of $\mathcal{Q}$. Now $U(id_{\mathbb{C}} \otimes q) \leq W \otimes \bar{W} \otimes U(id_{\mathbb{C}} \otimes q) = W \otimes V$ where $\leq$ means subobject. Hence U is a subobject of $W \otimes V \otimes (Id_{\mathbb{C}} \otimes q)$ which completes the proof. □

Theorem 4.2.18. *$\widehat{N(Q_{\mathcal{G}})}/\hat{Q}_{\mathcal{G}}$ is isomorphic to $C_0(\Gamma)$ for some discrete group Γ.*

Proof. It follows from lemma (4.2.16) that every irreducible representation of the DQG $\widehat{N(Q_{\mathcal{G}})}/\hat{Q}_{\mathcal{G}}$, is 1-dimensional . Hence it is a commutative as a C^* algebra ,which proves the theorem. □

Theorem 4.2.19. *For a compact group G with $\mathcal{Q} = C(G)$, the group Γ of theorem is isomorphic with the group $H^2_{uinv}(C(G), S^1)$.*

Proof. Let $(U, \mathcal{H}), (Id_{\mathbb{C}} \otimes q)$ be two strongly projective corepresentations of $C(G)$. Then $U \otimes (Id_{\mathbb{C}} \otimes q) \otimes \bar{U}$ is isomorphic with $U \otimes \bar{U} \otimes (Id_{\mathbb{C}} \otimes q) \in obj(C_{\mathcal{G}})$. From this we can conclude that C_{stp} is a subcategory of $N(C_{stp}, C_{\mathcal{G}})$ and by our construction of normalizer $N(C_{stp}, C_{\mathcal{G}})$ is a subcategory of C_{stp}. Hence $C_{stp} = N(C_{stp}, C_{\mathcal{G}})$ that implies $\widehat{N(\mathcal{Q}_{\mathcal{G}})} = \hat{\mathcal{Q}}_{stp}$. We already observed that the set of 1-dimensional irreducible representation of $C_0(\Gamma)$, which are the points of Γ, are in 1-1 correspondence with the equivalence classes K_U. Moreover by lemma (4.2.17), $K_U = K_W$ if and only if Ω_u *and* Ω_w represents the same classes at $H^2_{uinv}(C(G), s^1)$. As $C(G)$ is a commutative, it is equal to its center, so that every $q \in C(G)$ is central and $(q \otimes q)\Delta(q^*)$ is an invariant 2-cocycle. Thus, there is a 1-1 correspondence between $H^2_{uinv}(C(G), S^1)$ *and* ρ_U which is clearly also a group homomorphism, as $\Omega_{u \otimes w} = \Omega_u \Omega_w$. □

Remark 4.2.20. *For a general CQG, which is not commutative as a C^* algebra it is interesting to ask how the group Γ obtained in theorem (4.2.18) is related to $H^2_{uinv}(\mathcal{Q}, S^1)$.*

Remark 4.2.21. *We will see in the next chapter an example where Γ can be different from $H^2_{uinv}(\mathcal{Q}, S^1)$.*

Chapter 5

Explicit computation of the second invariant cohomology of certain finite dimensional compact quantum groups

In this chapter, we will explicitly compute $H^2_{uinv}(\cdot, S^1)$ and $H^2_{inv}(\cdot, \mathbb{C} - \{0\})$ for two interesting finite dimensional Hopf algebras namely the group ring of $G := Z_8 \rtimes Aut(Z_8)$ and the dual of Kac-Paljutkin algebra. We use categorical description of these cohomologies which will be explained below. Let us mention that in [GK10], partial results towards computing $H^2_{uinv}(C^*(G), S^1)$ were obtained by quite tedious calculation using other methods and also using computer. In that paper, it could only be proved that the order of the cohomology group has either 2 or 4, without a definite conclusion. On the other hand, our categorical approach seems to simplify the computations to some extent and we are able to conclusively identify the group as Z_2, without computer aided calculations.

Let $(\mathcal{Q}, \Delta)$ be a CQG, I be the set of mutually inequivalent irreducible corepresentations. Let us recall the fiber functor ϕ_Ω on $Corep(\mathcal{Q})$ such that

$$H_{\phi_\Omega(x)} = H_x, \quad \phi_\Omega(S) = \Omega^{-1} S \quad \phi_\Omega(T) = \Omega_2^{-1} T, \tag{5.0.1}$$

where Ω is an invertible 2-cocycle of dual DQG of $\mathcal{Q}$, $, x, y, z \in I$ and for all $S \in Mor(x, y \otimes z), T \in Mor(a, x \otimes y \otimes z)$. ϕ_Ω defines a new DQG $\widehat{\mathcal{Q}_\Omega}$, where the * algebra $End(\phi_\Omega) = End(F_{Nat}) = \Pi_{x \in I} B(\mathcal{H}_x)$ and coproduct is given by $\hat{\Delta}_\Omega(a) = \Omega \hat{\Delta} \Omega^*$. ϕ_Ω is a unitary monoidal equivalence between $Corep(\mathcal{Q})$ and $Corep(\mathcal{Q}_\Omega)$.

If Ω is an invariant 2-cocycle then $\hat{\Delta}_\Omega(a) = \hat{\Delta}(a)$ for $a \in \Pi_{x \in I} B(\mathcal{H}_x)$. Hence the algebra structure of $\mathcal{Q}_\Omega$ same as $\mathcal{Q}$. Now one can conclude that $(\mathcal{Q}_\Omega, \Delta) = (\mathcal{Q}, \Delta)$.

Lemma 5.0.1. *ϕ_Ω is an a monoidal autoquivalence of $Corep(\mathcal{Q})$, where Ω is an invertible invariant 2-cocycle. In case Ω is unitary, this gives unitary monoidal autoequivalence.*

Proof. Ω^{-1} is also a unitary invariant 2-cocycle of dual DQG of $\mathcal{Q}$. Now we choosing the fiber functor $\phi_{\Omega^{-1}}$, it follows that $\phi_\Omega \phi_{\Omega^{-1}} = \phi_{\Omega^{-1}} \phi_\Omega = Id_{corep(\mathcal{Q})}$. □

Our approach will be based on the following result, contained in chapter (3.1) of [NT]. For the sake of completeness we give an outline of the proof here as well.

Theorem 5.0.2. *$H^2_{uinv}(\widehat{\mathcal{Q}}, \hat{\Delta})$ is isomorphic with the group of unitary isomorphism class of all unitary monoidal autoequivalences of $Corep(\mathcal{Q}) \cong Rep(\hat{\mathcal{Q}})$ which are naturally isomorphic to the identity functor. A similar statement holds for $H^2_{inv}(\hat{\mathcal{Q}}, \mathbb{C} - \{0\})$ without the requirement of unitary.*

Proof. Let us proof the case of unitary cohomology only, as the other case is very similar. Given a unitary invariant 2-cocycle Ω, the UTF ϕ_Ω clearly gives a unitary monoidal autoequivalence of $Corep(\mathcal{Q})$, identifying $Corep(\mathcal{Q}_\Omega)$ with $Corep(\mathcal{Q})$.

To prove the converse, assume that ϕ is an autoequivalence with the stated properties. As $\phi(x) \cong x$ for every object x of $Corep(\mathcal{Q})$, ϕ is dimension preserving, hence it must be isomorphic with one of the form ϕ_Ω for some unitary right 2-cocycle. As ϕ_Ω is isomorphic with the identity functor, for each irreducible object x, y, z, we have unitary morphisms $\eta_x, \eta_{y\otimes z}$ such that $\phi_\Omega(t) = \eta^{-1}_{y\otimes z} t \eta_x$ for all $t \in Mor(x, y \otimes z)$. Moreover, as x is irreducible, η_x is a nonzero constant multiple of identity, say $c_x I_x$. This implies that the component $\Omega_{y\otimes z} \in Mor(y \otimes z, y \otimes z)$ is given by $\eta_{y\otimes z} \cdot c_x^{-1}$, so it commutes with $\hat{\Delta}(\Theta)_{y,z} = \Theta_{y\otimes z}$ for any Θ in $End(F_{Nat})$. This proves that Ω is an invariant 2-cocycle. □

Remark 5.0.3. *Note that any functor $\phi : Corep(\mathcal{Q}) \to Corep(\mathcal{Q})$ is natuarally isomorphic with the identity functor if and only if $\phi(x) \cong x$ for all objects x of $Corep(\mathcal{Q})$.*

5.1 Cohomology of the group ring $Z_8 \rtimes Aut(Z_8)$

The group $G := Z_8 \rtimes Aut(Z_8)$ was considered by G. E. Wall in his paper [Wal47], $Z_8 \rtimes Aut(Z_8)$ is generated by s, t, u, where s, t, u satisfies the relations

$$s^2 = t^2 = u^8 = 1, \quad st = ts, \quad sus^{-1} = u^3, \quad tut^{-1} = u^5 \ . \tag{5.1.1}$$

Let χ_{ijk} be characters of G, defined by

$$\chi_{ijk}(u) = (-1)^k, \quad \chi_{ijk}(s) = (-1)^i, \quad \chi_{ijk}(t) = (-1)^j \quad i, j, k \in Z_2. \tag{5.1.2}$$

Let π_2 and $\pi_2^{'}$ be irreducible representations of G on $\mathcal{H}_{\pi_2}$ and $\mathcal{H}_{\pi_2^{'}}$, where $\{e_1, e_2\}$ is an orthonormal basis for $\mathcal{H}_{\pi_2}$ and $\{f_1, f_2\}$ is an orthonormal basis for $\mathcal{H}_{\pi_2^{'}}$. π_2 is given by

$$\pi_2(u) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^6 \end{pmatrix}, \pi_2(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pi_2(t) = Id_{\mathcal{H}_{\pi_2}}, \tag{5.1.3}$$

where $\omega = e^{2\pi i/8}$.

$\pi_2^{'}$ is given by

$$\pi_2^{'}(u) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^6 \end{pmatrix}, \pi_2^{'}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pi_2^{'}(t) = -Id_{\mathcal{H}_{\pi_2^{'}}}. \tag{5.1.4}$$

Let π_4 be an irreducible representation of $Z_8 \rtimes Aut(Z_8)$ on $\mathcal{H}_{\pi_4}$ and assume that $\{x_1, x_2, x_3, x_4\}$ is an orthonormal basis for $\mathcal{H}_{\pi_4}$. π_4 is given by

$$\pi_4(u) = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^3 & 0 & 0 \\ 0 & 0 & \omega^5 & 0 \\ 0 & 0 & 0 & \omega^7 \end{pmatrix}, \pi_4(s) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \pi_4(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5.1.5}$$

By an easy calculation one can verify that all the above are mutually inequivalent irreducible representations of G and looking at the dimension of $C^*(G)$, we deduce that they exhaust the all irreducible representations of G.

Let $I := \{\chi_{ijk}, \pi_2, \pi_2^{'}, \pi_4\}$ be the collection of all pairwise non-equivalent irreducible representations of G.

Lemma 5.1.1. *The following fusion rules hold:*

1) $\chi_{i_1 j_1 k_1} \otimes \chi_{i_2 j_2 k_2} = \chi_{(i_1+i_2)(j_1+j_2)(k_1+k_2)}$.

2) $\pi_2 \otimes \pi_2 = \bigoplus \chi_{i0k}$.

3) $\pi_2^{'} \otimes \pi_2^{'} = \bigoplus \chi_{i0k}$.

4) $\pi_2 \otimes \pi_2^{'} = \bigoplus \chi_{i1k}$.

Proof. Proof is omitted, as it is a straightforward verification. □

Let ϕ be a unitary autoequivalence on $Corep(C(G)) \cong Rep(G)$, where $C(G)$ is the CQG of all continuous functions of G. Then ϕ is automatically a dimension-preserving fiber functor.

Let $Mor(a, b \otimes c \otimes d) := \Upsilon_a^{b\otimes c\otimes d}$ define a basis of $Mor(a, b \otimes c \otimes d)$ of unit norm, where $a, b, c, d \in \{\chi_{ijk}, \pi_2, \pi_2'\}$. Assume that $\phi(Mor(a, b \otimes c \otimes d)) := \widetilde{\Upsilon}_a^{b\otimes c\otimes d}$.

Now, we will introduce some notation to simplify our mathematical expressions.

$$\widetilde{\Upsilon}^{\chi_1\otimes\chi_2}_{\chi_1\chi_2} = \sigma(\chi_1, \chi_2)\Upsilon^{\chi_1\otimes\chi_2}_{\chi_1\chi_2}, \text{ where } \chi_1, \chi_2 \text{ are characters.} \tag{5.1.6}$$

$$\widetilde{\Upsilon}^{\chi\otimes\pi_2}_{\pi_2} = c_\chi \Upsilon^{\chi\otimes\pi_2}_{\pi_2}, \text{ where } \chi \in \{\chi_{i0k}\}. \tag{5.1.7}$$

$$\widetilde{\Upsilon}^{\chi\otimes\pi_2'}_{\pi_2'} = c_\chi' \Upsilon^{\chi\otimes\pi_2'}_{\pi_2'}, \text{ where } \chi \in \{\chi_{i0k}\}. \tag{5.1.8}$$

$$\widetilde{\Upsilon}^{\chi'\otimes\pi_2}_{\pi_2'} = d_{\chi'} \Upsilon^{\chi'\otimes\pi_2}_{\pi_2'}, \text{ where } \chi' \in \{\chi_{i1k}\}. \tag{5.1.9}$$

$$\widetilde{\Upsilon}^{\chi'\otimes\pi_2'}_{\pi_2} = e_{\chi'} \Upsilon^{\chi'\otimes\pi_2'}_{\pi_2}, \text{ where } \chi' \in \{\chi_{i1k}\}. \tag{5.1.10}$$

$$\widetilde{\Upsilon}^{\pi_2\otimes\pi_2}_{\chi} = \lambda_\chi \Upsilon^{\pi_2\otimes\pi_2}_{\chi}, \text{ where } \chi \in \{\chi_{i0k}\}. \tag{5.1.11}$$

$$\widetilde{\Upsilon}^{\pi_2'\otimes\pi_2'}_{\chi} = \lambda_\chi' \Upsilon^{\pi_2'\otimes\pi_2'}_{\chi}, \text{ where } \chi \in \{\chi_{i0k}\}. \tag{5.1.12}$$

$$\widetilde{\Upsilon}^{\pi_2\otimes\pi_2'}_{\chi'} = \eta_{\chi'} \Upsilon^{\pi_2\otimes\pi_2'}_{\chi'}, \text{ where } \chi' \in \{\chi_{i1k}\}. \tag{5.1.13}$$

Also assume that for $\chi = \chi_{i0k}$, $\chi' = \chi_{i1k}$.

By our defining notations, we can say that for any $T \in Mor(a, b \otimes c)$, $\widetilde{T} = c_0 T$ (for a suitable choice of $c_0 \in \mathbb{C}$ that satisfies our predefined notations). From now on, we only write $\widetilde{\Upsilon}_a^{b\otimes c} = \widetilde{T} = c_0$ instead of $c_0 T$, where we fix a $T \in Mor(a, b \otimes c)$, *and* $a, b, c \in \{\chi_{ijk}, \pi_2, \pi_2'\}$.

Lemma 5.1.2. *σ is a 2-cycle on the group $\{\chi_{ijk}\}$.*

Proof. Let χ_1, χ_2, χ_3 be characters of $Z_8 \rtimes Aut(Z_8)$. From this diagram

$$\begin{array}{ccc} \chi_1 \otimes \chi_2 \otimes \chi_3 & \xrightarrow{\sigma^*(\chi_1,\chi_2)\otimes Id_{\chi_3}} & \chi_1\chi_2 \otimes \chi_3 \\ {\scriptstyle id_{\chi_1}\otimes\sigma^*(\chi_2,\chi_3)}\downarrow & & \downarrow{\scriptstyle \sigma^*(\chi_1\chi_2,\chi_3)} \\ \chi_1 \otimes \chi_2\chi_3 & \xrightarrow{\sigma^*(\chi_1,\chi_2\chi_3)} & \chi_1\chi_2\chi_3 \end{array}$$

we can say that

$$\sigma(\chi_1, \chi_2\chi_3)\sigma(\chi_2, \chi_3) = \sigma(\chi_1\chi_2, \chi_3)\sigma(\chi_1, \chi_2).$$

So, σ is a 2-cocycle on the group $\{\chi_{ijk}\} \cong Z_2 \times Z_2 \times Z_2$. Without loss of generality, we assume that σ is a normalized 2-cocycle. □

Lemma 5.1.3. *$c_\chi^2 = (c'_\chi)^2 = 1$ and $c_{\chi_{000}} = c'_{\chi_{000}} = 1$.*

Proof. For any character χ, $\chi \otimes \chi \equiv \chi_{000}$ and $Mor(\pi_2, \chi \otimes \chi \otimes \pi_2) \equiv \mathbb{C}$. If we choose a $T \in Mor(\pi_2, \chi \otimes \pi_2)$ then $(Id_\chi \otimes T)T \in Mor(\pi_2, \chi \otimes \chi \otimes \pi_2)$. From(5.1.7), we can say that $\widetilde{T} = c_\chi T$. Now, we can observe that $(Id_\chi \otimes \widetilde{T})\widetilde{T} = c_\chi^2 (Id_\chi \otimes T)T$. Hence $c_\chi^2 = 1$ as $\sigma(\chi, \chi) = 1$ and also it is easy to observe that $c_{\chi_{000}} = 1$.

Similarly, we can conclude that $(c'_\chi)^2 = 1$ and $(c'_\chi)^2 = 1$. □

Lemma 5.1.4. *$c'_\chi = \psi(\chi) c_\chi$, where $\chi \in \{\chi_{i0k}\}$ and ψ is a character on $\{\chi_{i0k}\}$.*

Proof. Let us assume that $\chi_a, \chi_b \in \{\chi_{i0k}\}$. Dimension of $Mor(\pi_2, \pi_2 \otimes \chi_a \otimes \chi_b) = 1$. From this commutative diagram below

$$\begin{array}{ccc} \pi_2 \otimes \chi_a \otimes \chi_b & \xrightarrow{\sigma^*(\chi_a,\chi_b)} & \pi_2 \otimes \chi_a\chi_b \\ {\scriptstyle c^*_{\chi_a}}\downarrow & & \downarrow{\scriptstyle c^*_{ab}} \\ \pi_2 \otimes \chi_b & \xrightarrow[c^*_{\chi_b}]{} & \pi_2, \end{array}$$

we can conclude that

$$\sigma(\chi_a, \chi_b) = \frac{c_{\chi_a} c_{\chi_b}}{c_{\chi_a\chi_b}}. \tag{5.1.14}$$

From the following commutative diagram,

$$\begin{array}{ccc} \chi_a \otimes \chi_b \otimes \pi'_2 & \xrightarrow{(\sigma(\chi_a,\chi_b))^*} & \chi_a\chi_b \otimes \pi'_2 \\ {\scriptstyle (c'_{\chi_b})^*}\downarrow & & \downarrow{\scriptstyle (c'_{\chi_a\chi_b})^*} \\ \chi_a \otimes \pi'_2 & \xrightarrow[(c'_{\chi_a})^*]{} & \pi'_2, \end{array}$$

we will get

$$\sigma(\chi_a, \chi_b) = \frac{c'_{\chi_a} c'_{\chi_b}}{c'_{\chi_a\chi_b}}. \tag{5.1.15}$$

So, $\sigma(\chi_a, \chi_b) = \frac{c'_{\chi_a} c'_{\chi_b}}{c'_{\chi_a\chi_b}} = \frac{c_{\chi_a} c_{\chi_b}}{c_{\chi_a\chi_b}}$ and this implies that $c'_{\chi_a} = \psi'(\chi_a) c_{\chi_a}$, where ψ' is a character on the group $\{\chi_{iok}\} \cong Z_2 \times Z_2$. □

Lemma 5.1.5. *$e_{\chi'} d_{\chi'} = 1$, where $\chi' \in \{\chi_{i1k}\}$.*

Proof. Let us assume that $\chi_a^{'}, \chi_b^{'} \in \{\chi_{i1k}\}$. Now, from the diagram below,

$$\begin{CD}
\chi_a^{'} \otimes \chi_b^{'} \otimes \pi_2 @>{(\sigma(\chi_a^{'},\chi_b^{'}))^*}>> \chi_a\chi_b \otimes \pi_2\\
@V{(d_{\chi_b^{'}})^*}VV @VV{(c_{\chi_a\chi_b})^*}V\\
\chi_a^{'} \otimes \pi_2^{'} @>>{(e_{\chi_a^{'}})^*}> \pi_2
\end{CD}$$

we will get

$$\sigma(\chi_a^{'}, \chi_b^{'}) = \frac{d_{\chi_b^{'}} e_{\chi_a^{'}}}{c_{\chi_a\chi_b}}. \tag{5.1.16}$$

Taking $\chi_a^{'} = \chi_b^{'}$ we have $d_{\chi_a^{'}} e_{\chi_a^{'}} = 1$. □

Lemma 5.1.6. $e^2_{\chi_a^{'}} = e^2_{\chi_{010}} \psi^{'}(\chi_a)$,*where* $\chi_a^{'} \in \{\chi_{i1k}\}$ *and assume* $\chi_a^{'} = \chi_{i1k}$ *when* $\chi_a = \chi_{i0k}$.

Proof. We can obtain

$$\sigma(\chi_a^{'}, \chi_b^{'}) = \frac{e_{\chi_b^{'}} d_{\chi_a^{'}}}{c^{'}_{\chi_a\chi_b}} \tag{5.1.17}$$

from the diagram below

$$\begin{CD}
\chi_a^{'} \otimes \chi_b^{'} \otimes \pi_2^{'} @>{(\sigma(\chi_a^{'},\chi_b^{'}))^*}>> \chi_a\chi_b \otimes \pi_2^{'}\\
@V{(e_{\chi_b^{'}})^*}VV @VV{(c^{'}_{\chi_a\chi_b})^*}V\\
\chi_a^{'} \otimes \pi_2 @>>{(d_{\chi_a^{'}})^*}> \pi_2^{'}.
\end{CD}$$

Comparing equations (5.1.16) and (5.1.17), we get

$$\frac{e_{\chi_b^{'}} d_{\chi_a^{'}}}{c^{'}_{\chi_a\chi_b}} = \frac{e_{\chi_b^{'}} d_{\chi_a^{'}}}{\psi^{'}(\chi_a\chi_b) c_{\chi_a\chi_b}} = \sigma(\chi_a^{'}, \chi_b^{'}) = \frac{d_{\chi_b^{'}} e_{\chi_a^{'}}}{c_{\chi_a\chi_b}}, \tag{5.1.18}$$

$$d_{\chi_b^{'}} e_{\chi_a^{'}} = \psi^{'}(\chi_a\chi_b) e_{\chi_b^{'}} d_{\chi_a^{'}}. \tag{5.1.19}$$

If $\chi_b^{'} = \chi_{010}$, we have $e^2_{\chi_a^{'}} = e^2_{\chi_{010}} \psi^{'}(\chi_a)$. □

Lemma 5.1.7. $\lambda^{'}_{\chi_b} = \psi^{'}(\chi_b) c_{\chi_b} \lambda^{'}_{\chi_{000}}$.

Proof.

$$\begin{array}{ccc}
\chi_b \otimes \pi_2' \otimes \pi_2' & \xrightarrow{(c_{\chi_b}')^*} & \pi_2' \otimes \pi_2' \\
{\scriptstyle (\lambda_{\chi_a}')^*}\big\downarrow & & \big\downarrow{\scriptstyle (\lambda_{\chi_a\chi_b}')^*} \\
\chi_b \otimes \chi_a & \xrightarrow[(\sigma(\chi_a,\chi_b))^*]{} & \chi_a\chi_b
\end{array}$$

From this diagram, we observe that

$$\sigma(\chi_a, \chi_b) = \frac{c_{\chi_b}' \lambda_{\chi_b\chi_a}'}{\lambda_{\chi_a}'} = \frac{\psi'(\chi_b) c_{\chi_b} \lambda_{\chi_b\chi_a}'}{\lambda_{\chi_a}'}. \tag{5.1.20}$$

After comparing equations (5.1.14) and (5.1.20), the following relation will occur

$$\lambda_{\chi_{ab}}' c_{\chi_a\chi_b} = \psi'(\chi_b) c_{\chi_a} \lambda_{\chi_a}'. \tag{5.1.21}$$

Let us assume that $\chi_a = \chi_{000}$. From equation (5.1.21), we will get

$$\lambda_{\chi_b}' = \psi'(\chi_b) c_{\chi_b} \lambda_{\chi_{000}}'. \tag{5.1.22}$$

□

Lemma 5.1.8. $\lambda_\chi = c_\chi \lambda_{\chi_{000}}$.

Proof. From the diagram below

$$\begin{array}{ccc}
\chi_a \otimes \pi_2 \otimes \pi_2 & \xrightarrow{(\lambda_{\chi_b})^*} & \chi_a \otimes \chi_b \\
{\scriptstyle (c_{\chi_a})^*}\big\downarrow & & \big\downarrow{\scriptstyle (\sigma(\chi_a,\chi_b))^*} \\
\pi_2 \otimes \pi_2 & \xrightarrow[\lambda_{\chi_a\chi_b}^*]{} & \chi_a\chi_b.
\end{array}$$

one can conclude that

$$\sigma(\chi_a, \chi_b) = \frac{c_{\chi_a} \lambda_{\chi_b\chi_a}}{\lambda_{\chi_b}}. \tag{5.1.23}$$

From equation (5.1.14) , we will get

$$\sigma(\chi_a, \chi_b) = \frac{c_{\chi_a} \lambda_{\chi_b\chi_a}}{\lambda_{\chi_b}} = \frac{c_{\chi_a} c_{\chi_b}}{c_{\chi_a\chi_b}}. \tag{5.1.24}$$

So, $\lambda_{\chi_a\chi_b} c_{\chi_a\chi_b} = \lambda_{\chi_b} c_{\chi_b}$. If we assume $\chi_b = \chi_{000}$ then

$$c_{\chi_a} \lambda_{\chi_a} = \lambda_{\chi_{000}}. \tag{5.1.25}$$

□

Lemma 5.1.9. *$\sigma(\chi_a, \chi'_b) = \frac{c_{\chi_a} e_{\chi'_b}}{e_{\chi_a \chi'_b}}$ and $\sigma(\chi'_a, \chi_b) = \frac{c'_{\chi_b} e_{\chi'_a}}{e_{\chi'_a \chi_b}}$, where $\chi_a \in \{\chi_{i0k}\}, \chi'_b \in \{\chi_{i1k}\}$.*

Proof.

$$\sigma(\chi_a, \chi'_b) = \frac{c_{\chi_a} e_{\chi'_b}}{e_{\chi_a \chi'_b}} \tag{5.1.26}$$

follows from this commutative diagram

$$\begin{array}{ccc} \chi_a \otimes \chi_b^{'} \otimes \pi_2^{'} & \xrightarrow{(e_{\chi'_b})^*} & \chi_a \otimes \pi_2 \\ \Big\downarrow {\scriptstyle (\sigma(\chi_a, \chi_b^{'}))^*} & & \Big\downarrow {\scriptstyle (c_{\chi_a})^*} \\ \chi_a \chi_b^{'} \otimes \pi_2^{'} & \xrightarrow[(e_{\chi_a \chi'_b})^*]{} & \pi_2. \end{array}$$

One can easily observe that

$$\sigma(\chi'_a, \chi_b) = \frac{c'_{\chi_b} e_{\chi'_a}}{e_{\chi'_a \chi_b}} \tag{5.1.27}$$

from

$$\begin{array}{ccc} \chi'_a \otimes \chi_b \otimes \pi'_2 & \xrightarrow{(c'_{\chi_2})^*} & \chi'_a \otimes \pi'_2 \\ \Big\downarrow {\scriptstyle (\sigma(\chi'_a, \chi_b))^*} & & \Big\downarrow {\scriptstyle (e_{\chi'_a})^*} \\ \chi'_a \chi_b \otimes \pi'_2 & \xrightarrow{(e_{\chi'_a \chi_b})^*} & \pi_2. \end{array}$$

□

Lemma 5.1.10. *$e^2_{\chi'}$ is a constant.*

Proof. For a fix $\chi \in \{\chi_{i0k}\}$, we denote $A_\chi := (\Upsilon_\chi^{\pi \otimes \pi} \otimes \pi')\Upsilon_{\pi'}^{\chi \otimes \pi'}$. Similarly, for a fixed $\chi' \in \{\chi_{i1k}\}$, denoted $B_{\chi'} := (id_\pi \otimes \Upsilon_{\chi'}^{\pi \otimes \pi'})\Upsilon_{\pi'}^{\pi \otimes \chi'}$. One can easily check that $\{B_{\chi_{i1k}} : i, k = 0, 1\}$ is a basis of $Mor(\pi'_2, \pi_2 \otimes \pi_2 \otimes \pi'_2)$.

Assume that $\chi_i \in \{\chi_{m0n} : m, n \in \{0, 1\}\}$. We can write $A_{\chi_i} = \sum_j c_{ij} B_{\chi'_j}$, where $\chi'_j \in \{\chi_{m1n}\}, c_{ij} \in \mathbb{C}$. From this, we will get

$$\widetilde{A}_{\chi_i} = \lambda_{\chi_i} c'_{\chi_i} A_{\chi_i} = \sum_j c_{ij} d_{\chi'_j} \eta_{\chi'_j} B_{\chi'_j} = \sum_j c_{ij} \widetilde{B}_{\chi'_j}. \tag{5.1.28}$$

Now, we can conclude that

$$\lambda_{\chi_i} c'_{\chi_i} = d_{\chi'_j}\eta_{\chi'_j} = a_0 \ for\ all\ \chi_i \in \{\chi_{mon}\},\ \chi'_j \in \{\chi_{m1n}\}, \tag{5.1.29}$$

where a_0 is a constant.

Let $P_{\chi'} := (\Upsilon^{\pi_2\otimes\pi'_2}_{\chi'} \otimes Id_{\pi'_2})\Upsilon^{\chi'\otimes\pi'_2}_{\pi_2}$ and $Q_\chi := (Id_{\pi_2} \otimes \Upsilon^{\pi'_2\otimes\pi'_2}_{\chi})\Upsilon^{\pi_2\otimes\chi}_{\pi_2}$. $\{Q_{\chi_{iok}}\}$ is a basis of $Mor(\pi_2, \pi_2 \otimes \pi'_2 \otimes \pi'_2)$. Let us assume that $P_{\chi'_i} = \sum_j d_{ij} Q_{\chi_j}$. Now,

$$\widetilde{P_{\chi'_i}} = \eta_{\chi'_i} e_{\chi'_i} P_{\chi'_i} = \sum_j d_{ij} c_{\chi_j} \lambda'_{\chi_j} Q_{\chi_j} = \sum_j d_{ij} \widetilde{Q_{\chi_j}}, \tag{5.1.30}$$

From which, one can easily observe that for all $\chi'_i \in \{\chi_{m1n}\}, \chi_j \in \{\chi_{m0n}\}$

$$\eta_{\chi'_i} e_{\chi'_i} = c_{\chi_j} \lambda'_{\chi_j} = a'_0, \tag{5.1.31}$$

where a'_0 is a constant.

From equations (5.1.29) and (5.1.31), we can conclude that

$$d^2_{\chi'_i} = \frac{d_{\chi'_j}\eta_{\chi'_j}}{\eta_{\chi'_i} e_{\chi'_i}} = \frac{a_0}{a'_0}. \tag{5.1.32}$$

Hence $e^2_{\chi'}$ is a constant. □

Lemma 5.1.11. $\psi' \equiv 1$ *on* $\{\chi_{i0k}\}$.

Proof. From Lemma (5.1.6), it folllows that $\psi'(\chi_a) = 1$, for all $\chi_a \in \{\chi_{i0k}\}$. □

Let $V_\chi : \mathbb{C}_\chi \to \mathbb{C}_\chi$, $V_{\chi'} : \mathbb{C}_{\chi'} \to \mathbb{C}_{\chi'}$, $V_{\pi_2} : \mathcal{H}_{\pi_2} \to \mathcal{H}_{\pi_2}$ and $V_{\pi'_2} : \mathcal{H}_{\pi'_2} \to \mathcal{H}_{\pi'_2}$ be unitary linear maps defined by

$$V_\chi(1_{\mathbb{C}_\chi}) = c_\chi 1_{\mathbb{C}_\chi},\ \ V_{\chi'}(1_{\mathbb{C}_{\chi'}}) = \frac{a_0^{1/2}}{a_0'^{1/2}} e_{\chi'} 1_{\mathbb{C}_{\chi'}},\ \ V_{\pi_2} = a_0^{1/2} Id_{\mathcal{H}_{\pi_2}},\ \ V_{\pi'_2} = a_0'^{1/2} Id_{\mathcal{H}_{\pi'_2}}. \tag{5.1.33}$$

.

Lemma 5.1.12. *σ is equivalent to the trivial 2-cycle of the group* $\{\chi_{ijk}\}$.

Proof. For any two $\chi_1, \chi_2 \in \{\chi_{i0k}\}$, we have

$$\sigma(\chi_1, \chi_2) = (V_{\chi_1} \otimes V_{\chi_2}) V^*_{\chi_1\chi_2} \tag{5.1.34}$$

$$= c_{\chi_1} c_{\chi_2} (c_{\chi_1\chi_2})^{-1}. \tag{5.1.35}$$

Assume that $\chi_1', \chi_2' \in \{\chi_{i1k}\}$.

$$\begin{aligned}\sigma(\chi_1,\chi_2') =&(V_{\chi_1}\otimes V_{\chi_2'})V^*_{\chi_1\chi_2'}\\ =&c_{\chi_1}\frac{a_0^{1/2}}{a_0'^{1/2}}e_{\chi_2'}(e_{\chi_1\chi_2'})^{-1}\frac{a_0'^{1/2}}{a_0^{1/2}}\\ =&\frac{c_{\chi_1}e_{\chi_2'}}{e_{\chi_1\chi_2'}},\end{aligned}$$

and

$$\begin{aligned}\sigma(\chi_1',\chi_2') =&(V_{\chi_1'}\otimes V_{\chi_2'})V^*_{\chi_1'\chi_2'}\\ =&(e_{\chi_1'}e_{\chi_2'}\frac{a_0}{a_0'})(c_{\chi_1'\chi_2'})^{-1}\\ =&(\frac{e_{\chi_1'}}{c_{\chi_1'\chi_2'}})(e_{\chi_2'}\frac{a_0}{a_0'})\\ =&\frac{e_{\chi_1'}}{c_{\chi_1'\chi_2'}}\frac{1}{e_{\chi_2'}} \qquad [\; from \;\; e^2_{\chi_2'}=\frac{a_0'}{a_0}]\\ =&\frac{e_{\chi_1'}d_{\chi_2'}}{c_{\chi_1'\chi_2'}}.\end{aligned}$$

$$\sigma(\chi_1',\chi_2) =(V_{\chi_1'}\otimes V_{\chi_2})V^*_{\chi_1'\chi_2} \tag{5.1.36}$$
$$=e_{\chi_1'}c_{\chi_2}(e_{\chi_1'\chi_2})^{-1} \tag{5.1.37}$$
$$=\frac{e_{\chi_1'}c_{\chi_2}}{e_{\chi_1'\chi_2}} \tag{5.1.38}$$
$$=\frac{e_{\chi_1'}c'_{\chi_2}}{e_{\chi_1'\chi_2}}. \quad [from \;\; c'_\chi = c_\chi] \tag{5.1.39}$$

Hence, σ is equivalent to the trivial 2-cycle. □

Without loss of generality assume that σ is a trivial 2-cycle.
Assume that $\phi(\Upsilon^{\chi\otimes\pi_4}_{\pi_4}) = \widetilde{\Upsilon}^{\chi\otimes\pi_4}_{\pi_4} = \tau(\chi)\Upsilon^{\chi\otimes\pi_4}_{\pi_4}$.

Lemma 5.1.13. *τ is a character on the group $\{\chi_{ijk}\}$.*

Proof. For any $\chi_1, \chi_2 \in \{\chi_{ijk}\}$, we will get

$$(\Upsilon^{\chi_1\otimes\chi_2}_{\chi_1\chi_2}\otimes Id_{\pi_4})\Upsilon^{\chi_1\chi_2\otimes\pi_4}_{\pi_4} = (Id_{\chi_1}\otimes\Upsilon^{\chi_2\otimes\pi_4}_{\pi_4})\Upsilon^{\chi_1\otimes\pi_4}_{\pi_4}. \tag{5.1.40}$$

If we apply ϕ on both sides of the equation then

$$\tau(\chi_1\chi_2)(\Upsilon^{\chi_1\otimes\chi_2}_{\chi_1\chi_2} \otimes Id_{\pi_4})\Upsilon^{\chi_1\chi_2\otimes\pi_4}_{\pi_4} = \tau(\chi_1)\tau(\chi_2)(Id_{\chi_1} \otimes \Upsilon^{\chi_2\otimes\pi_4}_{\pi_4})\Upsilon^{\chi_1\otimes\pi_4}_{\pi_4}. \tag{5.1.41}$$

Hence τ is a character on $\{\chi_{ijk}\}$. □

Let us assume that $\widetilde{\Upsilon}^{\pi_4\otimes\pi_4}_{\chi} := m_\chi \Upsilon^{\pi_4\otimes\pi_4}_{\chi}$. Then the following identity holds,

$$(Id_{\chi_1} \otimes \Upsilon^{\pi_4\otimes\pi_4}_{\chi_2})\Upsilon^{\chi_1\otimes\chi_2}_{\chi_1\chi_2} = (\Upsilon^{\chi_1\otimes\pi_4}_{\pi_4} \otimes Id_{\pi_4})\Upsilon^{\pi_4\otimes\pi_4}_{\chi_1\chi_2}.$$

Lemma 5.1.14. $m_{\chi_1} = \tau(\chi_1)m_{\chi_{000}}$.

Proof. Observe that

$$\phi((Id_{\chi_1} \otimes \Upsilon^{\pi_4\otimes\pi_4}_{\chi_2})\Upsilon^{\chi_1\otimes\chi_2}_{\chi_1\chi_2}) = (Id_{\chi_1} \otimes \widetilde{\Upsilon}^{\pi_4\otimes\pi_4}_{\chi_2})\widetilde{\Upsilon}^{\chi_1\otimes\chi_2}_{\chi_1\chi_2} \tag{5.1.42}$$
$$= m_{\chi_2}(Id_{\chi_1} \otimes \Upsilon^{\pi_4\otimes\pi_4}_{\chi_2})\Upsilon^{\chi_1\otimes\chi_2}_{\chi_1\chi_2}), \tag{5.1.43}$$

and

$$\phi((\Upsilon^{\chi_1\otimes\pi_4}_{\pi_4} \otimes Id_{\pi_4})\Upsilon^{\pi_4\otimes\pi_4}_{\chi_1\chi_2})) = (\widetilde{\Upsilon}^{\chi_1\otimes\pi_4}_{\pi_4} \otimes Id_{\pi_4})\widetilde{\Upsilon}^{\pi_4\otimes\pi_4}_{\chi_1\chi_2} \tag{5.1.44}$$
$$= \tau(\chi_1)m_{\chi_1\chi_2}((\Upsilon^{\chi_1\otimes\pi_4}_{\pi_4} \otimes Id_{\pi_4})\Upsilon^{\pi_4\otimes\pi_4}_{\chi_1\chi_2}). \tag{5.1.45}$$

We can conclude that $m_{\chi_2} = \tau(\chi_1)m_{\chi_1\chi_2}$ from equations (5.1.43) and (5.1.45). If we choose $\chi_2 = \chi_{000}$ then $m_{\chi_1} = \tau(\chi_1)m_{\chi_{000}}$. Assume that $\mu = m_{\chi_{000}}$. □

Lemma 5.1.15. $c_\chi = \tau(\chi)\psi(\chi)$ *for a character* ψ *on* $\{\chi_{i0k}\}$.

Proof. From the commutative diagram

$$\begin{array}{ccc} \chi_1 \otimes \chi_2 \otimes \pi_4 & \xrightarrow{\tau^*(\chi_2)} & \chi_1 \otimes \pi_4 \\ \sigma^*(\chi_1,\chi_2)\downarrow & & \downarrow\tau^*(\chi_1) \\ \chi_1\chi_2 \otimes \pi_4 & \xrightarrow{\tau^*(\chi_1\chi_2)} & \pi_4 \end{array}$$

we have $\sigma(\chi_1, \chi_2) = \frac{\tau(\chi_1)\tau(\chi_2)}{\tau(\chi_1\chi_2)} = 1$. From (5.1.14),

$$\frac{c_{\chi_1}c_{\chi_2}}{c_{\chi_1\chi_2}} = \frac{\tau(\chi_1)\tau(\chi_2)}{\tau(\chi_1\chi_2)}.$$

Hence there exists a character ψ on $\{\chi_{i0k}\}$ such that $c_\chi = \tau(\chi)\psi(\chi)$ for $\chi \in \{\chi_{i0k}\}$. □

Let $\{T_1, T_2\}$ be a basis of $Mor(\pi_4, \pi_2 \otimes \pi_4)$. T_1, T_2 are defined by,

$$T_1(x_1) = e_1 \otimes x_4, T_1(x_2) = e_2 \otimes x_3, T_1(x_3) = e_1 \otimes x_2, T_1(x_4) = e_2 \otimes x_1$$
$$T_2(x_1) = e_2 \otimes x_2, T_2(x_2) = e_1 \otimes x_1, T_2(x_3) = e_2 \otimes x_4, T_2(x_4) = e_1 \otimes x_3.$$

Let $E_i^\chi := (Id_\chi \otimes T_i)\Upsilon_{\pi_4}^{\chi\otimes\pi_4} \in Mor(\pi_4, \chi \otimes \pi_2 \otimes \pi_4)$ and $F_j^\chi := (\Upsilon_{\pi_2}^{\chi\otimes\pi_2} \otimes Id_{\pi_4})T_j \in Mor(\pi_4, \chi \otimes \pi_2 \otimes \pi_4)$. Now, one can easily check that the following equations hold,

$$E_1^{\chi 000} = F_1^{\chi 000}, \quad E_2^{\chi 000} = F_2^{\chi 000}, \tag{5.1.46}$$
$$E_1^{\chi 001} = F_2^{\chi 001}, \quad E_2^{\chi 001} = F_1^{\chi 001}, \tag{5.1.47}$$
$$E_1^{\chi 101} = -F_2^{\chi 101}, \quad E_2^{\chi 101} = -F_1^{\chi 101}, \tag{5.1.48}$$
$$E_1^{\chi 100} = F_1^{\chi 100}, \quad E_2^{\chi 100} = -F_2^{\chi 100}. \tag{5.1.49}$$

Let us assume that $\widetilde{T_1} = \sum_{i=1}^2 \omega_{1k} T_k, \widetilde{T_2} = \sum_{i=1}^2 \omega_{2k} T_k$, where $\omega_{ij} \in \mathbb{C}, i, j \in \{1, 2\}$ and $E_i^\chi = \sum_k a_{ik}^\chi F_k^\chi$.

Lemma 5.1.16. $\tau(\chi)\omega a^\chi = c_\chi a^\chi \omega$.

Proof.

$$\widetilde{F_j^\chi} = c_\chi (\Upsilon_{\pi_2}^{\chi\otimes\pi_2} \otimes Id_{\pi_4})\widetilde{T_j} \tag{5.1.50}$$
$$= c_\chi \sum_k \omega_{jk} (\Upsilon_{\pi_2}^{\chi\otimes\pi_2} \otimes Id_{\pi_4}) T_k \tag{5.1.51}$$
$$= c_\chi \sum_k \omega_{jk} F_k^\chi. \tag{5.1.52}$$

Now,

$$\phi(E_i^\chi) = \widetilde{E_i^\chi} = \sum_k a_{ik}^\chi \widetilde{F_k^\chi} \tag{5.1.53}$$
$$= c_\chi \sum_{k,p} a_{ik}^\chi \omega_{kp} F_p^\chi \tag{5.1.54}$$

and

$$\begin{aligned}\widetilde{E_j^\chi} =&(Id_\chi \otimes \widetilde{T}_j)\tau(\chi)\Upsilon_{\pi_4}^{\chi\otimes\pi_4} && (5.1.55)\\ =&\sum_k \omega_{jk}(Id_\chi \otimes T_k)\tau(\chi)\Upsilon_{\pi_4}^{\chi\otimes\pi_4} && (5.1.56)\\ =&\sum_k \omega_{jk}\tau(\chi)E_k^\chi && (5.1.57)\\ =&\sum_{k,l} \omega_{jk}\tau(\chi)a_{kl}^\chi F_l^\chi && (5.1.58)\\ =&\tau(\chi)\sum_{k,l} \omega_{jk}a_{kl}^\chi F_l^\chi. && (5.1.59)\end{aligned}$$

From (5.1.54) and (5.1.59), we will get

$$\tau(\chi)\omega a^\chi = c_\chi a^\chi \omega, \tag{5.1.60}$$

where $\omega = (\omega_{ij}), a^\chi = (a_{ij})$ are $2\times$ 2 matrices. □

From (5.1.46),(5.1.47),(5.1.48) and (5.1.49),

$$a^{\chi_{000}} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, a^{\chi_{001}} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, a^{\chi_{101}} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, a^{\chi_{100}} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{5.1.61}$$

Lemma 5.1.17. *There are only two choices of ψ, which are $\psi \equiv 1$ and $\psi(\chi_{100}) = 1, \psi(\chi_{001}) = -1$ and the followings hold*

1) If $\psi \equiv 1$ then $\omega = \lambda Id$, where $\lambda \in \mathbb{C} - \{0\}$.

2) If $\psi(\chi_{100}) = 1, \psi(\chi_{001}) = -1$ then $\omega = diag(\lambda, -\lambda)$ for a $\lambda \in \mathbb{C} - \{0\}$.

Proof. From the equation (5.1.60), one can observe that

$$\begin{aligned}\tau(\chi)wa^\chi =&c_\chi a^\chi w\\ =&\tau(\chi)\psi(\chi)a^\chi w.\end{aligned}$$

So, it is easily follows $\psi(\chi)wa^\chi = a^\chi w$.

1) For $\psi \equiv 1$, w commutes with the matrices $a^{\chi_{000}}, a^{\chi_{101}}, a^{\chi_{101}}, a^{\chi_{100}}$. From $wa^{\chi_{100}} = a^{\chi_{100}}w$ we can conclude $w = diag(a, b)$ and from $wa^{\chi_{001}} = a^{\chi_{001}}w$ one can check that $a = b$. Hence $w = diag(\lambda, \lambda)$ for a non-zero complex number λ.

2) If $\psi(\chi_{100}) = 1, \psi(\chi_{001}) = -1$ then $w = diag(a, b)$ from previous observation. From $(-1)wa^{\chi_{001}} = a^{\chi_{001}}w$ one can prove that $b = -a$ and it satisfies $(-1)diag(a, -a)a^{\chi_{101}} = a^{\chi_{101}}diag(a, -a)$. So $w = diag(\lambda, -\lambda)$ for a $\lambda \in \mathbb{C} - \{0\}$.

3) If $\psi(\chi_{100}) = -1, \psi(\chi_{001}) = 1$ then $w = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ from the equation $\psi(\chi_{100}) w a^{\chi_{100}} = a^{\chi_{100}} w$. It is easy to observe $w = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$ from equation $\phi(\chi_{001}) \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} a^{\chi_{001}} = a^{\chi_{001}} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. Now, we can conclude $b = 0$ from equation $\psi(\chi_{101}) \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} a^{\chi_{101}} = a^{\chi_{101}} \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$. So there is no such matrix w for $\psi(\chi_{100}) = -1, \psi(\chi_{001}) = 1$.

4) Similarly we can prove that there is no such matrix w for $\psi(\chi_{100}) = -1, \psi(\chi_{001}) = -1$.

□

Let $T'_1, T'_2 \in Mor(\pi_4, \pi'_2 \otimes \pi_4)$ be defined by,

$$T'_1(x_1) = f_1 \otimes x_4, T'_1(x_2) = f_2 \otimes x_3, T'_1(x_3) = -f_1 \otimes x_2, T_1(x_4) = -f_2 \otimes x_1$$
$$T'_2(x_1) = f_2 \otimes x_2, T'_2(x_2) = f_1 \otimes x_1, T'_2(x_3) = -f_2 \otimes x_4, T'_2(x_4) = -f_1 \otimes x_3.$$

Then $\{T'_1, T'_2\}$ is a basis of $Mor(\pi_4, \pi'_2 \otimes \pi_4)$. Let $E_i^{\chi'} := (Id_{\chi'} \otimes T_i) \Upsilon_{\pi_4}^{\chi' \otimes \pi_4} \in Mor(\pi_4, \chi \otimes \pi_2 \otimes \pi_4)$ and $F_j^{\chi'} := (\Upsilon_{\pi'_2}^{\chi' \otimes \pi_2} \otimes Id_{\pi_4}) T'_j \in Mor(\pi_4, \chi \otimes \pi_2 \otimes \pi_4)$. Now, one can easily check that the following equations hold,

$$E_1^{\chi_{010}} = F_1^{\chi_{010}}, \quad E_2^{\chi_{010}} = F_2^{\chi_{010}}, \tag{5.1.62}$$
$$E_1^{\chi_{011}} = -F_2^{\chi_{011}}, \quad E_2^{\chi_{011}} = -F_1^{\chi_{011}}, \tag{5.1.63}$$
$$E_1^{\chi_{111}} = -F_2^{\chi_{111}}, \quad E_2^{\chi_{111}} = F_1^{\chi_{111}}, \tag{5.1.64}$$
$$E_1^{\chi_{110}} = F_1^{\chi_{110}}, \quad E_2^{\chi_{110}} = -F_2^{\chi_{110}}. \tag{5.1.65}$$

Assume that $\widetilde{T'_1} = \sum_{i=1}^{2} \omega'_{1k} T'_k, \widetilde{T'_2} = \sum_{i=1}^{2} \omega'_{2k} T'_k$, where $\omega'_{ij} \in \mathbb{C}, i, j \in \{1, 2\}$ and $E_i^{\chi'} = \sum_k a_{ik}^{\chi'} F_k^{\chi'}$.

Lemma 5.1.18. $\tau(\chi') \omega a^{\chi'} = d_{\chi'} a^{\chi'} \omega'$.

Proof.

$$\phi(F_j^{\chi'}) = \widetilde{F_j^{\chi'}} = d_{\chi'} (\Upsilon_{\pi'_2}^{\chi' \otimes \pi_2} \otimes Id_{\pi_4}) \widetilde{T'_j} \tag{5.1.66}$$
$$= d_{\chi'} \sum_k \omega'_{jk} (\Upsilon_{\pi'_2}^{\chi' \otimes \pi_2} \otimes Id_{\pi_4}) T'_k \tag{5.1.67}$$
$$= d_{\chi'} \sum_k \omega'_{jk} F_k^{\chi'}. \tag{5.1.68}$$

$$\phi(E_i^{\chi'}) = \widetilde{E_i^{\chi'}} = \sum_k a_{ik}^{\chi'} \widetilde{F_k^{\chi'}} \tag{5.1.69}$$
$$= d_{\chi'} \sum_{k,p} a_{ik}^{\chi'} \omega'_{kp} F_p^{\chi'}, \tag{5.1.70}$$

and

$$\widetilde{E_j^{\chi'}} = (Id_{\chi'} \otimes \widetilde{T}_j)\tau(\chi')\Upsilon_{\pi_4}^{\chi' \otimes \pi_4} \tag{5.1.71}$$
$$= \sum_k \omega_{jk}(Id_{\chi'} \otimes T_k)\tau(\chi')\Upsilon_{\pi_4}^{\chi' \otimes \pi_4} \tag{5.1.72}$$
$$= \sum_k \omega_{jk}\tau(\chi')E_k^{\chi'} \tag{5.1.73}$$
$$= \sum_{k,l} \omega_{jk}\tau(\chi')a_{kl}^{\chi'}F_l^{\chi'} \tag{5.1.74}$$
$$= \tau(\chi')\sum_{k,l} \omega_{jk}a_{kl}^{\chi'}F_l^{\chi'}. \tag{5.1.75}$$

From (5.1.70) and (5.1.75), we will get

$$\tau(\chi')\omega a^{\chi'} = d_{\chi'} a^{\chi'} \omega', \tag{5.1.76}$$

where $\omega' = (\omega'_{ij}), a^{\chi'} = (a'_{ij})$ are 2× 2 matrices.

□

$$a^{\chi_{010}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a^{\chi_{111}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, a^{\chi_{011}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, a^{\chi_{110}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.1.77}$$

If $\psi' \equiv 1$ then from equation (5.1.76) one can observe that

$$\begin{aligned} w' &= \tau(\chi')e_{\chi'}(a^{\chi'})^{-1} w a^{\chi'} \\ &= \tau(\chi')e_{\chi'} w \\ &= \tau(\chi')e_{\chi'} \lambda Id. \end{aligned}$$

So there exists a constant k_0 such that $\tau(\chi')e_{\chi'} = k_0$ for all $\chi' \in \{\chi_{ijk}\}$ and also $k_0^2 = \frac{a'_0}{a_0}$.

If $\psi'(\chi_{100}) = 1, \psi'(\chi_{001}) = -1$ then

$$w' = \tau(\chi')e_{\chi'}(a^{\chi'})^{-1}wa^{\chi'}.$$

From the matrices (5.1.77), it is a routine check that

1) For $\chi' = \chi_{010}$, $w' = \tau(\chi_{010})e_{\chi_{010}}\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}$,

2) For $\chi' = \chi_{110}, w' = e_{\chi_{110}}\tau(\chi_{110})\begin{pmatrix}\lambda & 0\\ 0 & -\lambda.\end{pmatrix}$,

3) For $\chi' = \chi_{011}, w' = -e_{\chi_{011}}\tau(\chi_{011})\begin{pmatrix}\lambda & 0\\ 0 & -\lambda.\end{pmatrix}$,

4) For $\chi' = \chi_{111}, w' = -e_{\chi_{111}}\tau(\chi_{111})\begin{pmatrix}\lambda & 0\\ 0 & -\lambda.\end{pmatrix}$.

So, $\tau(\chi_{010})e_{\chi_{010}} = \tau(\chi_{110})e_{\chi_{110}} = -\tau(\chi_{011})e_{\chi_{011}} = -\tau(\chi_{111})e_{\chi_{111}}$.

Lemma 5.1.19. $\lambda^2 = a_0$.

Proof. The space $Mor(\pi_4,\ \pi_2{\otimes}\pi_2{\otimes}\pi_4)$ is 4-dimensional, observe that $C_{\chi_{i0k}} = (\Upsilon^{\pi_2\otimes\pi_2}_{\chi_{i0k}}\otimes Id_{\pi_4})\Upsilon^{\chi_{i0k}\otimes\pi_4}_{\pi_4}$. $\{C_{\chi_{i0k}}\}$ is a basis of $Mor(\pi_4,\ \pi_2\otimes\pi_2\otimes\pi_4)$, where

$$C_{\chi_{000}} = (Id_{\pi_2}\otimes T_1)T_1 + (Id_{\pi_2}\otimes T_2)T_2, \tag{5.1.78}$$
$$C_{\chi_{100}} = (Id_{\pi_2}\otimes T_1)T_1 - (Id_{\pi_2}\otimes T_2)T_2, \tag{5.1.79}$$
$$C_{\chi_{001}} = (Id_{\pi_2}\otimes T_1)T_2 + (Id_{\pi_2}\otimes T_2)T_1, \tag{5.1.80}$$
$$C_{\chi_{101}} = (Id_{\pi_2}\otimes T_1)T_2 - (Id_{\pi_2}\otimes T_2)T_1. \tag{5.1.81}$$

One can observe that

$$\begin{aligned}\widetilde{C_{\chi_{i0k}}} = \lambda_{\chi_{i0k}}\tau(\chi_{i0k})C_{\chi_{i0k}} &= a_o c_{\chi_{i0k}}\tau(\chi_{i0k})C_{\chi_{i0k}}\\ &= a_0\tau(\chi_{i0k})\psi(\chi_{iok})\tau(\chi_{i0k})C_{\chi_{i0k}}\\ &= a_0\psi(\chi_{i0k})C_{\chi_{i0k}}.\end{aligned}$$

From equations (5.1.78), (5.1.79),(5.1.80) and (5.1.81), it follows that

i) For $\psi \equiv 1$, We will get $\lambda^2 = a_0$.

ii) For $\psi(\chi_{100}) = 1,\ \psi(\chi_{001}) = -1$, we know $\widetilde{T_1} = \lambda T_1,\ \widetilde{T_2} = -\lambda T_2$. Now, it is a straightforward computation to prove $\lambda^2 = a_0$.

□

Let S_1, S_2 be two linear maps defined by,

$$S_1(e_1) = x_1 \otimes x_1 + x_3 \otimes x_3, \quad S_1(e_2) = x_2 \otimes x_2 + x_4 \otimes x_4,$$
$$S_2(e_1) = x_2 \otimes x_4 + x_4 \otimes x_2, \quad S_2(e_2) = x_1 \otimes x_3 + x_3 \otimes x_1.$$

$\{S_1, S_2\}$ is a basis of the vector space $Mor(\pi_2, \pi_4 \otimes \pi_4)$.

Let us define

$$G_i^{\chi} := (\Upsilon_{\pi_4}^{\chi \otimes \pi_4} \otimes Id_{\pi_4}) S_i, \tag{5.1.82}$$
$$H_i^{\chi} := (Id_{\chi} \otimes S_i) \Upsilon_{\pi_2}^{\chi \otimes \pi_2}, \tag{5.1.83}$$

where $G_i^{\chi}, H_i^{\chi} \in Mor(\pi_2, \chi \otimes \pi_4 \otimes \pi_4)$.

It is straightforward to verify that

$$G_1^{\chi 000} = H_1^{\chi 000}, \quad G_2^{\chi 000} = H_2^{\chi 000}, \tag{5.1.84}$$
$$G_1^{\chi 001} = H_2^{\chi 001}, \quad G_2^{\chi 001} = H_1^{\chi 001}, \tag{5.1.85}$$
$$G_1^{\chi 100} = H_1^{\chi 100}, \quad G_2^{\chi 100} = -H_2^{\chi 100}, \tag{5.1.86}$$
$$G_1^{\chi 101} = -H_2^{\chi 101}, \quad G_2^{\chi 101} = H^{\chi 101}. \tag{5.1.87}$$

Let $\phi(S_l) = \widetilde{S}_l = \sum_{m=1}^{2} \theta_{lm} S_m$, where $\theta_{lm} \in \mathbb{C}, l = 1$ *to* 2 and also assume that $G_l^{\chi i0k} = \sum_{y=1}^{2} n_{ly}^{\chi i0k} H_y^{\chi i0k}$, where $n_{ly}^{\chi i0k} \in \mathbb{C}$.

Lemma 5.1.20.

$$\tau(\chi_{i0k}) \theta n^{\chi_{i0k}} = c_{\chi_{i0k}} n^{\chi_{i0k}} \theta,$$

where $\theta = (\theta_{lm}), n^{\chi_{i0k}} = (n_{ly}^{\chi_{i0k}})$ *are* 2×2 *matrices.*

Proof. We have,

$$\widetilde{G_l^{\chi i0k}} = \sum_{y=1}^{2} n_{ly}^{\chi i0k} \widetilde{H_y^{\chi i0k}} \tag{5.1.88}$$

$$= c_{\chi i0k} \sum_{y=1}^{2} n_{ly}^{\chi i0k} (Id_{\chi i0k} \otimes \widetilde{S_y}) \Upsilon_{\pi_2}^{\chi i0k \otimes \pi_2} \tag{5.1.89}$$

$$= c_{\chi i0k} \sum_{y,z=1}^{2} n_{ly}^{\chi i0k} \theta_{yz} (Id_{\chi i0k} \otimes S_z) \Upsilon_{\pi_2}^{\chi i0k \otimes \pi_2} \tag{5.1.90}$$

$$= c_{\chi i0k} \sum_{y,z} n_{ly}^{\chi i0k} \theta_{yz} H_z^{\chi i0k}, \tag{5.1.91}$$

and also

$$\widetilde{G_l^{\chi i0k}} = (\widetilde{\Upsilon_{\pi_4}^{\chi i0k \otimes \pi_4}} \otimes Id_{\pi_4}) \widetilde{S_l} \tag{5.1.92}$$

$$= \tau(\chi i0k) (\Upsilon_{\pi_4}^{\chi i0k \otimes \pi_4} \otimes Id_{\pi_4}) \sum_{m=1}^{2} \theta_{lm} S_m \tag{5.1.93}$$

$$= \tau(\chi i0k) \sum_{m} \theta_{lm} (\Upsilon_{\pi_4}^{\chi i0k \otimes \pi_4} \otimes Id_{\pi_4}) S_m \tag{5.1.94}$$

$$= \tau(\chi i0k) \sum_{m} \theta_{lm} G_m^{\chi i0k} \tag{5.1.95}$$

$$= \tau(\chi i0k) \sum_{m,z} \theta_{lm} n_{mz}^{\chi i0k} H_z^{\chi i0k}. \tag{5.1.96}$$

After comparing equations (5.1.98) and (5.1.103), one can conclude that

$$\tau(\chi i0k) \theta n^{\chi i0k} = c_{\chi_{i0k}} n^{\chi i0k} \theta, \tag{5.1.97}$$

where $\theta = (\theta_{lm}), n^{\chi i0k} = (n_{ly}^{\chi i0k})$ are 2×2 matrices. □

Remark 5.1.21. *From the equation (5.1.102), it is easy to conclude that* $\psi(\chi_{i0k})\theta n^{\chi i0k} = n^{\chi i0k}\theta$.

We already know that

$$n^{\chi 000} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, n^{\chi 001} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, n^{\chi 101} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, a^{\chi 100} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.1.98}$$

1) If $\psi \equiv 1$ then θ commutes with $n^{\chi 100}, n^{\chi 101}$. So, $\theta = diag(a, b)$ for two complex numbers a, b. θ also commutes with $n^{\chi 001}$, from that one can easily reduce that $\theta = diag(\lambda_1, \lambda_1)$ for a non-zero complex number λ_1.

2) If $\psi(\chi_{100}) = 1, \psi(\chi_{001}) = -1$ then $\theta = diag(\lambda_1, -\lambda_1)$ where $\lambda_1 \in \mathbb{C} - \{0\}$.

Similarly, there is a basis $\{S'_1, S'_2\}$ of the vector space $Mor(\pi'_2, \pi_4 \otimes \pi_4)$. Here we introduce another set of notations:

$$G^{\chi_{i1k}}_{l'} := (\Upsilon^{\chi_{i1k}\otimes\pi_4}_{\pi_4} \otimes Id_{\pi_4})S'_l \tag{5.1.99}$$

$$H^{\chi_{i1k}}_{l} := (Id_{\chi_{i1k}} \otimes S_l)\Upsilon^{\chi_{i1k}\otimes\pi_2}_{\pi'_2}, \tag{5.1.100}$$

where $G^{\chi_{i1k}}_{l'}, H^{\chi_{i1k}}_{l} \in Mor(\pi'_2, \chi_{i1k} \otimes \pi_4 \otimes \pi_4)$.

Assume that

$$\phi(S'_l) = \widetilde{S}'_l = \sum_{m'} \theta'_{l'm'} S'_m, \tag{5.1.101}$$

$$G^{\chi_{i1k}}_{m'} = \sum_{y} n'^{\chi_{i1k}}_{m'y} H^{\chi_{i1k}}_{y}, \tag{5.1.102}$$

where $\theta'_{l'm'}, n'^{\chi_{i1k}}_{m'y} \in \mathbb{C}$.

$$\phi(G^{\chi_{i1k}}_{l'}) = \widetilde{G^{\chi_{i1k}}_{l'}} = \tau(\chi_{i1k})((\Upsilon^{\chi_{i1k}\otimes\pi_4}_{\pi_4} \otimes \pi_4)\widetilde{S}'_l).$$

Similarly, we can observe that

$$\tau(\chi_{i1k})\theta' n'^{\chi_{i1k}} = d_{\chi_{i1k}} n'^{\chi_{i1k}}\theta. \tag{5.1.103}$$

1) If $\psi \equiv 1$ then $\theta' = \tau(\chi_{i1k})d_{\chi_{i1k}}\lambda_1 Id$, where $\tau(\chi_{i1k})d_{\chi_{i1k}} = \frac{1}{k_0}$.

2) If $\psi(\chi_{100}) = 1, \psi(\chi_{001}) = -1$, then

i) For $\chi' = \chi_{010}$, $\theta' = \tau(\chi_{010})d_{\chi_{010}} \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}$,

ii) For $\chi' = \chi_{110}, \theta' = d_{\chi_{110}}\tau(\chi_{110}) \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1. \end{pmatrix}$,

iii) For $\chi' = \chi_{011}, \theta' = -d_{\chi_{011}}\tau(\chi_{011}) \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1. \end{pmatrix}$,

iv) For $\chi' = \chi_{111}, \theta' = -d_{\chi_{111}}\tau(\chi_{111}) \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1. \end{pmatrix}$.

Hence $\tau(\chi_{010})e_{\chi_{010}} = \tau(\chi_{110})e_{\chi_{110}} = -\tau(\chi_{011})e_{\chi_{011}} = -\tau(\chi_{111})e_{\chi_{111}}$.

Let us assume that

$$K_{i0k} := (\Upsilon^{\pi_4\otimes\pi_4}_{\chi_{iok}} \otimes Id_{\pi_4})\Upsilon^{\pi_2\otimes\chi_{i0k}}_{\pi_2} \in Mor(\pi_2, \pi_2 \otimes \pi_4 \otimes \pi_4).$$

We already know that $\{S_i : i = 1, 2\}$ is a basis of $Mor(\pi_2, \pi_4 \otimes \pi_4)$ and $\{T_i : i = 1, 2\}$ is a bais of $Mor(\pi_4, \pi_2 \otimes \pi_4)$. Let

$$D_{ij} = (T_i \otimes Id_{\pi_4})S_j \in Mor(\pi_2, \pi_2 \otimes \pi_4 \otimes \pi_4).$$

Then the following relations hold,

$$D_{11} = (K_{000} - K_{100}), \tag{5.1.104}$$
$$D_{12} = (K_{001} - K_{101}), \tag{5.1.105}$$
$$D_{21} = (K_{001} + K_{101}), \tag{5.1.106}$$
$$D_{22} = (K_{000} + K_{100}). \tag{5.1.107}$$

Lemma 5.1.22. $\lambda_1 \lambda = \mu$.

Proof.

$$\phi(D_{11}) = \widetilde{D_{11}} = (\widetilde{T_1} \otimes Id_{\pi_4})\widetilde{S_1} \tag{5.1.108}$$
$$= (\mu\tau(\chi_{000})c_{\chi_{000}} K_{000} - \mu\tau(\chi_{100})c_{\chi_{100}} K_{100}), \tag{5.1.109}$$

from which we can conclude that

$$\widetilde{S_1} = (\widetilde{T_1^*} \otimes Id_{\pi_4})\mu(\psi(\chi_{000})K_{000} - \psi(\chi_{100})K_{001}). \tag{5.1.110}$$

Similarly we can observe that

$$\widetilde{S_1} = (\widetilde{T_2^*} \otimes Id_{\pi_4})\mu(\psi(\chi_{001})K_{001} + \psi(\chi_{101})K_{101}), \tag{5.1.111}$$
$$\widetilde{S_2} = (\widetilde{T_1^*} \otimes Id_{\pi_4})\mu(\psi(\chi_{001})K_{001} - \psi(\chi_{101})K_{101}), \tag{5.1.112}$$
$$\widetilde{S_2} = (\widetilde{T_2^*} \otimes Id_{\pi_4})\mu(\psi(\chi_{000})K_{000} + \psi(\chi_{100})K_{100}). \tag{5.1.113}$$

1) If $\psi \equiv 1$ then

$$\widetilde{S_1} = (\widetilde{T_1^*} \otimes Id_{\pi_4})\mu(\tau(\chi_{000})K_{000} - \tau(\chi_{100})K_{001})$$
$$= \frac{1}{\lambda}(T_1^* \otimes Id_{\pi_4})\mu(K_{000} - K_{100}) = \frac{\mu}{\lambda}S_1.$$

Similarly, we can prove that $\widetilde{S_2} = \frac{\mu}{\lambda}S_2$.

2) If $\psi(\chi_{100}) = 1, \psi(\chi_{001}) = -1$ then $\widetilde{S_1} = \frac{\mu}{\lambda}S_1$ and $\widetilde{S_2} = -\frac{\mu}{\lambda}S_2$.

From this, we can easily conclude that $\lambda_1 \lambda = \mu$. □

Lemma 5.1.23. *For any choice of* $(\tau, \lambda, k_0, \mu)$ *and* $\psi \equiv 1$ *the corresponding fiber functor* ϕ *is monoidally isomorphic to the identity fiber functor.*

Proof. Let $v_\chi : \mathbb{C}_\chi \to \mathbb{C}_\chi, v_{\pi_2} : \mathcal{H}_{\pi_2} \to \mathcal{H}_{\pi_2}, v_{\pi_2'} : \mathcal{H}_{\pi_2'} \to \mathcal{H}_{\pi_2'}, v_{\pi_4} : \mathcal{H}_{\pi_4} \to \mathcal{H}_{\pi_4}$ be the unitary linear maps given by

1) $v_\chi(1_{\mathbb{C}_\chi} =)\tau(\chi)1_{\mathbb{C}_\chi}$,

2) $v_{\pi_2} = \lambda Id_{\mathcal{H}_{\pi_2}}$,

3) $v_{\pi_2'} = \lambda k_0 Id_{\mathcal{H}_{\pi_2'}}$,

4) $v_{\pi_4} = \mu^{1/2} Id_{\pi_4}$.

One can check that $(v_a \otimes v_b)(\Upsilon_c^{a\otimes b})v_c^* = \phi(\Upsilon_c^{a\otimes b})$ for any $a, b, c \in \{\chi_{ijk}, \pi_2, \pi_2', \pi_4\}$. Hence ϕ corresponds to the identity tensor functor on $Corep(C(G))$. □

Lemma 5.1.24. *When* $\psi(\chi_{100}) = 1$ *and* $\psi_(\chi_{001}) = -1$, *for any two choices of* $(\tau_1, \lambda_1, k_01, \mu_1)$ *and* $(\tau_2, \lambda_2, k_{02}, \mu_2)$ *the corresponding fiber functor* ϕ_1, ϕ_2 *are monoidally isomorphic.*

Proof. It is easy to observe that $\phi_1^{-1}\phi_2 \cong Id$. Hence ϕ_1 is isomorphic to ϕ_2. □

Lemma 5.1.25. *Any two fiber functors as in lemma (5.1.23) and as in lemma (5.1.24), are not monoidally isomorphic.*

Proof. Without loss of generality, we can take first functor to be identity tensor functor and other to be ϕ corresponding $\lambda = 1, k_0 = 1, \mu = 1, \tau = 1$. Suppose $\phi \cong Id$. So there exists a unitary morphism $V_{\pi_2\otimes\pi_4} \in Mor(\pi_2 \otimes \pi_4, \pi_2 \otimes \pi_4)$ such that $V_{\pi_2\otimes\pi_4}(T_1)V_{\pi_4}^* = T_1, V_{\pi_2\otimes\pi_4}(T_2)V_{\pi_4}^* = -T_2$ where $T_1, T_2 \in Mor(\pi_4, \pi_2 \otimes \pi_4)$. As π_4 is irreducible, V_{π_4} must be of the form cI_{π_4} for some constant $c \in \mathbb{C} - \{0\}$. Let $V_{\pi_2\otimes\pi_4} = A$. We already know $T_1(x_1) = e_1 \otimes x_4$ and $T_2(x_1) = e_2 \otimes x_2$. Hence $c^{-1}A(e_1 \otimes x_4) = e_1 \otimes x_4$ and $c^{-1}A(e_2 \otimes x_2) = -e_2 \otimes x_2$. This implies $c^{-1}A \neq Id$ and therefore ϕ is not isomorphic to the identity tensor functor. □

As a corrollary, we get our final result

Theorem 5.1.26. $H^2_{uinv}(C^*(G), S^1) = Z_2$.

Proof. The proof follows from lemmas (5.1.23),(5.1.24),(5.1.25). The only non-trivial class is given by any functor in lemma (5.1.25). □

Remark 5.1.27. *Similarly we can prove that* $H^2(C^*(G), \mathbb{C} - \{0\})$ *is* Z_2.

We use the above result to give an example where the group Γ as in the theorem (4.2.18) for the CQG $C^*(G)$ is different from $H^2_{uinv}(C^*(G), S^1)$. It has been proven that $H^2_{uinv}(C^*(Z_8 \rtimes Aut(Z_8), S^1) = Z_2$. It is also proven in [Kas98] that Nontrivial invariant 2-cocycle Ω is given by $\Omega = (v \otimes v)\Delta(v^*)$, where

$$v = \frac{1}{2}(1 + u^4) + \frac{\sqrt{2}}{4}u(1 - u^2 - u^4 + u^6).$$

One can check that v is a self adjoint unitary element and $v^2 = 1$.

Theorem 5.1.28. *For the group ring $Z_8 \rtimes Aut(Z_8)$, Γ is trivial.*

Proof. It follows from lemma 4.2.17 that the equivalence class $K_{Id_{\mathbb{C}} \otimes 1} = K_{Id_{\mathbb{C}} \otimes v}$. By our construction, the group Γ is in one-one correspondence with the equivalence classes K_U, where U is any irreducible strongly projective corepresentation. Hence Γ is the trivial group. □

5.2 2-cocycles of dual of Kac-Paljutkin algebra

Let us recall the Tambara-Yamagami tensor category [TY98] .

Tambara–Yamagami tensor categories [TY98] is equivalent to the the category of representations of the Kac–Paljutkin Hopf algebra [TY98], which is arising from the Klein 4-group $K_4 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Elements of $K_4 = \{e, s, t, st\}$ satisfies the relations $s^2 = t^2 = (st)^2 = e$. $\chi = \chi_c$ is a nondegenerate symmetric bicharacter of K_4 which is given by

$$\chi_c(a, a) = \chi_c(b, b) = -1, \ \chi_c(a, b) = 1,$$

and considering the parameter $\tau = \frac{1}{2}$. Now, we define the category $\mathcal{C}(\chi, \tau)$ and Its objects are finite direct sums of elements in $S = K_4 \cup \{\rho\}$. Sets of morphisms between elements in S are given by

$$Mor(s, s') = \begin{cases} \mathbb{C} & s = s', \\ 0 & s \neq s', \end{cases}$$

so S is the set of irreducible classes of $\mathcal{C}(\chi, \tau)$. Tensor products of elements in S are given by

$$s \otimes \rho = \rho = \rho \otimes s, \ \rho \otimes \rho = \bigoplus_{s \in K_4} s, s \otimes t = st, \ (s, t \in K_4)$$

and the unit object is e. Associativities φ are given by

$$\varphi_{s,t,u} = \mathrm{id}_{stu}, \qquad \varphi_{s,t,\rho} = \varphi_{\rho,s,t} = \mathrm{id}_\rho,$$
$$\varphi_{s,\rho,t} = \chi_c(s,t)\mathrm{id}_t, \qquad \varphi_{s,\rho,\rho} = \varphi_{\rho,\rho,s} = \bigoplus_{k\in K_4} \mathrm{id}_k,$$
$$\varphi_{\rho,s,\rho} = \bigoplus_{k\in K_4} \chi_c(s,t)\mathrm{id}_k, \qquad \varphi_{\rho,\rho,\rho} = \left(\frac{1}{2}\chi_c(k,l)^{-1}\mathrm{id}_\rho\right)_{k,l} : \bigoplus_{k\in K_4} \rho \to \bigoplus_{l\in K_4} \rho,$$

for $s,t,u \in K_4$. Now, if we choose the natural fiber functor of this category then this category is identified with the corepresentation category of Kac–Paljutkin quantum group $\mathcal{Q}_{kp}$, that is

$$\mathcal{C}\left(\chi_c, \frac{1}{2}\right) \simeq Rep(\mathcal{Q}_{kp}) \simeq Corep(\hat{\mathcal{Q}_{kp}})$$

as tensor categories.

Moreover, using the discussion and calculation in [TY98], we observe that there is a fiber functor ϕ_0 from which $\mathcal{Q}_{kp}$ is obtained by the Tannaka-Krein reconstruction. It can be seen from [TY98] that $\phi_0(s) \cong \phi_0(t) \cong \phi_o(st) \cong \mathbb{C}$ and $\phi_0(\rho) = \mathbb{C}^2$. Moreover we can choose the basis element U_s, U_t, U_{st} and V_s, V_t, V_{st}:

$$U_s = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, U_t = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}, U_{st} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $U_s \in Mor(\rho, s\otimes\rho), U_t \in Mor(\rho, t\otimes\rho), U_{st} \in Mor(\rho, st\otimes\rho)$ and

$$V_s = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}, V_t = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, V_{st} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $V_s \in Mor(\rho, \rho\otimes s), V_t \in Mor(\rho, \rho\otimes t)$ *and* $V_{st} \in Mor(\rho, \rho\otimes st)$. Now, let ϕ be a dimension preserving fiber functor on this category.

Let $\phi(\Upsilon_\rho^{\rho\otimes t}) := \widetilde{\Upsilon}_\rho^{\rho\otimes t}$, $\phi(\Upsilon_\rho^{t\otimes\rho}) := \widetilde{\Upsilon}_\rho^{t\otimes\rho}$ and $\phi(\Upsilon_s^{\rho\otimes\rho}) := \widetilde{\Upsilon}_s^{\rho\otimes\rho}$.

Here we introduce some notations which are,

$$\widetilde{\Upsilon}_{st}^{s\otimes t} = \theta(s,t)\Upsilon_{st}^{s\otimes t},$$
$$\widetilde{\Upsilon}_\rho^{\rho\otimes t} = d_t\Upsilon_\rho^{\rho\otimes t} = d_t V_t,$$
$$\widetilde{\Upsilon}_\rho^{t\otimes\rho} = c_t\Upsilon_\rho^{\rho\otimes t} = c_t U_t,$$
$$\widetilde{\Upsilon}_t^{\rho\otimes\rho} = k_t\Upsilon_t^{\rho\otimes\rho}.$$

Now if we choose the unitary linear maps $v_1 : \mathbb{C}_1 \to \mathbb{C}_1, v_s : \mathbb{C}_s \to \mathbb{C}_s, v_t : \mathbb{C}_t \to \mathbb{C}_t, v_{st} : \mathbb{C}_{st} \to \mathbb{C}_{st}$ *and* $v_\rho : H_\rho \to H_\rho$ such that v_i are identity maps from $\mathbb{C}_i$ *to* $\mathbb{C}_i$

and $v_\rho = k_1^{-1/2} Id_{\mathcal{H}_\rho}$ then it follows from the proof of Proposition (3.5) [BRV06] that ϕ is isomorphic to a fiber functor ϕ' where $\phi'(\Upsilon_c^{a\otimes b}) = (v_a \otimes v_b)\phi(\Upsilon_c^{a\otimes b})v_c^*$ for which $\phi'(\Upsilon_e^{\rho\otimes\rho}) = \Upsilon_e^{\rho\otimes\rho}$. Without loss of generality, Let us assume that $k_1 = 1$.

Lemma 5.2.1. *θ is a 2-cycle on K_4.*

Proof. Proof of this lemma similar to the proof of lemma (5.1.2),hence omitted. □

Without loss of generality, we assume that θ is a normalized 2-cycle.

Lemma 5.2.2. *$c_x c_y = \theta(x,y)c_{xy}$, where $x, y \in K_4$.*

Proof. From the diagram

$$\begin{array}{ccc} x \otimes y \otimes \rho & \xrightarrow{(Id_{\mathbb{C}_x} \otimes c_y^* U_y^*)} & x \otimes \rho \\ {\scriptstyle \theta^*(x,y)\otimes Id_{H_\rho}} \downarrow & & \downarrow {\scriptstyle c_x^* U_x^*} \\ xy \otimes \rho & \xrightarrow{c_{xy}^* U_{xy}^*} & \rho, \end{array}$$

we can conclude that $c_x c_y = \theta(x,y)c_{xy}$. □

Lemma 5.2.3. *$d_x d_y = \theta(x,y)d_{xy}$, where $x, y \in K_4$.*

Proof. Proof of this lemma similar to the previous lemma, hence omitted. □

Lemma 5.2.4. *$c_x = \tau(x)d_x$, where τ is a character on K_4.*

Proof. From lemma (5.2.2) and (5.2.3), one can conclude that

$$\theta(x,y) = \frac{c_x c_y}{c_{xy}} = \frac{d_x d_y}{d_{xy}}. \tag{5.2.1}$$

Hence, $(c_x d_x^{-1})(c_y d_y^{-1}) = c_{xy} d_{xy}^{-1}$. This implies that $c_x d_x^{-1} = \tau(x)$ for a 2 cycle of K_4. □

From the associativity relation,

$$\varphi_{s,\rho,t} = \chi(s,t)\mathrm{id}_t,$$

we observe that

$$U_x V_y = \chi(x,y)V_y U_x. \tag{5.2.2}$$

Let P_x be range of $\Upsilon_x^{m\otimes m}$ and assume that ϵ_x is the image of P_x.
We already know that $\varphi_{s,\rho,t} = \chi_c(s,t)id_t$.

Now, one can easily observe from the associativity relations that

$$U_x V_y = \chi(x, y) V_y U_x. \tag{5.2.3}$$

It is a straight forward computation to verify the following

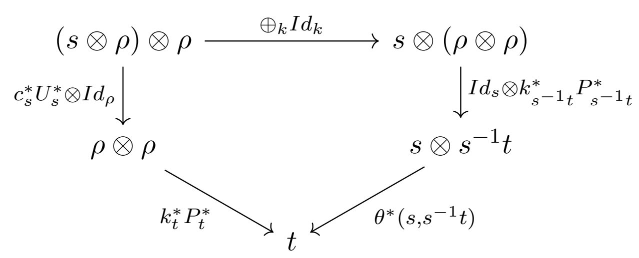
$$(U_s \otimes Id_\rho)\epsilon_s = i\epsilon_1, \tag{5.2.4}$$
$$(U_t \otimes Id_\rho)\epsilon_t = i\epsilon_1, \tag{5.2.5}$$
$$(U_{st} \otimes Id_\rho)\epsilon_{st} = (-1)\epsilon_1. \tag{5.2.6}$$

Lemma 5.2.5. *The following identities hold:*

1) $c_s k_t = \theta(s, s^{-1}t) k_{s^{-1}t}$,

2) $d_s k_t = \theta(s, s^{-1}t) k_{s^{-1}t}$.

Proof. 1) From this diagram



one can easily conclude that $c_s k_t = \theta(s, s^{-1}t) k_{s^{-1}t}$.

2) Similarly, we can prove that $d_s k_t = \theta(s, s^{-1}t) k_{s^{-1}t}$.

□

Lemma 5.2.6. $c_x k_x = 1$ *and* $c_x = d_x$, *where* $x \in K_4$.

Proof. If we choose $s = t$ then it follows $c_s k_s = \theta(s, 1) k_1 = k_1 = 1$ from lemma (5.2.5). Similarly, we can deduce that $d_s k_s = k_1$. Hence $c_s = d_s$. □

Theorem 5.2.7. $H^2_{uinv}(\hat{\mathcal{Q}}_{kp}, S^1) \cong 1$, $H^2_{inv}(\hat{\mathcal{Q}}_{kp}, \mathbb{C} - \{0\}) \cong 1$.

Proof. We define unitary linear maps $v_1 : \mathbb{C}_1 \to \mathbb{C}_1, v_s : \mathbb{C}_s \to \mathbb{C}_s, v_t : \mathbb{C}_t \to \mathbb{C}_t, v_{st}$:

$\mathbb{C}_{st} \to \mathbb{C}_{st}$ *and* $v_\rho : H_\rho \to H_\rho$, which are given by

$$v_1(1_{\mathbb{C}_1}) = c_1 1_{\mathbb{C}_1},$$
$$v_s(1_{\mathbb{C}_s}) = c_s 1_{\mathbb{C}_s},$$
$$v_t(1_{\mathbb{C}_t}) = c_t 1_{\mathbb{C}_t},$$
$$v_{st}(1_{\mathbb{C}_{st}}) = c_{st} 1_{\mathbb{C}_{st}},$$
$$v_\rho = Id_{H_\rho}.$$

Now, one can check that $\phi(\Upsilon_c^{a\otimes b}) = (v_a \otimes v_b)(\Upsilon_c^{a\otimes b})v_c^*$.

Hence $H^2_{uinv}(\hat{\mathcal{Q}}_{kp}, S^1) = 1$.

Similarly, $H^2_{inv}(\hat{\mathcal{Q}}_{kp}, \mathbb{C} - \{0\}) \cong 1$ as $v_c^* = v_c^{-1}$ for $c \in \{1, s, t, \rho\}$. □

Remark 5.2.8. *In fact, it is well known that* $\mathcal{Q}_{kp} \cong \hat{\mathcal{Q}}_{kp}$ *as Hopf *-algebras, hence Theorem (5.2.7) is also valid if* $\hat{\mathcal{Q}}_{kp}$ *is replaced by* $\mathcal{Q}_{kp}$.

Bibliography

[Ban05] Teodor Banica, *Quantum automorphism groups of small metric spaces*, Pacific J. Math. **219** (2005), no. 1, 27–51. MR 2174219

[Bar54] V. Bargmann, *On unitary ray representations of continuous groups*, Ann. of Math. (2) **59** (1954), 1–46. MR 58601

[BB07] Teodor Banica and Julien Bichon, *Quantum automorphism groups of vertex-transitive graphs of order 11*, Journal of Algebraic Combinatorics **26** (2007), no. 1, 83–105.

[BBC07] Teodor Banica, Julien Bichon, and Benoît Collins, *Quantum permutation groups: a survey*, Noncommutative harmonic analysis with applications to probability, Banach Center Publ., vol. 78, Polish Acad. Sci. Inst. Math., Warsaw, 2007, pp. 13–34. MR 2402345

[BCV20] Michael Brannan, Alexandru Chirvasitu, and Ami Viselter, *Actions, quotients and lattices of locally compact quantum groups*, Documenta Mathematica **25** (2020), 2553–2582.

[Bro94] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original. MR 1324339

[BRV06] Julien Bichon, An De Rijdt, and Stefaan Vaes, *Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups*, Communications in mathematical physics **262** (2006), 703–728.

[CGLW13] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen, *Symmetry protected topological orders and the group cohomology of their symmetry group*, Phys. Rev. B **87** (2013), 155114.

[Che15] Chuangxun Cheng, *A character theory for projective representations of finite groups*, Linear Algebra Appl. **469** (2015), 230–242. MR 3299063

[Che16] Xie Chen, *Symmetry protected topological phases*, https://xiechen.caltech.edu/documents/27791/lecture.pdf (2016).

[CP95] Vyjayanthi Chari and Andrew N Pressley, *A guide to quantum groups*, Cambridge university press, 1995.

[DC09] Kenny De Commer, *Galois objects for algebraic quantum groups*, Journal of Algebra **321** (2009), no. 6, 1746–1785.

[DC11a] ______, *Galois objects and cocycle twisting for locally compact quantum groups*, Journal of Operator Theory (2011), 59–106.

[DC11b] ______, *On projective representations for compact quantum groups*, J. Funct. Anal. **260** (2011), no. 12, 3596–3644. MR 2781971

[DCKSSt18] Kenny De Commer, Paweł Kasprzak, Adam Skalski, and Piotr M. Soł tan, *Quantum actions on discrete quantum spaces and a generalization of Clifford's theory of representations*, Israel J. Math. **226** (2018), no. 1, 475–503. MR 3819700

[DCMN24] Kenny De Commer, Rubén Martos, and Ryszard Nest, *Projective representation theory for compact quantum groups and the quantum baum–connes assembly map*, Journal of Noncommutative Geometry (2024).

[DCY12] Kenny De Commer and Makoto Yamashita, *Tannaka-krein duality for compact quantum homogeneous spaces. i. general theory*, arXiv preprint arXiv:1211.6552 (2012).

[DCY15] ______, *Tannaka–krein duality for compact quantum homogeneous spaces ii. classification of quantum homogeneous spaces for quantum su (2)*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2015** (2015), no. 708, 143–171.

[Dri85] V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR **283** (1985), no. 5, 1060–1064. MR 802128

[Dri89] ______, *Quasi-Hopf algebras*, Algebra i Analiz **1** (1989), no. 6, 114–148. MR 1047964

[EG20] Pavel Etingof and Shlomo Gelaki, *Invariant Hopf 2-cocycles for affine algebraic groups*, Int. Math. Res. Not. IMRN (2020), no. 2, 344–366. MR 4073189

[GK10] Pierre Guillot and Christian Kassel, *Cohomology of invariant Drinfeld twists on group algebras*, Int. Math. Res. Not. IMRN (2010), no. 10, 1894–1939. MR 2646345

[GS25] Debashish Goswami and Suchetana Samadder, *Quantum groups of discrete outer symmetry of von neumann algebra*, in preparation (2025).

[Jim85] Michio Jimbo, *A q-difference analogue of $U(g)$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), no. 1, 63–69. MR 797001

[Kas98] Ch. Kassel, *Quantum groups*, Algebra and operator theory (Tashkent, 1997), Kluwer Acad. Publ., Dordrecht, 1998, pp. 213–236. MR 1643398

[KS97] Anatoli Klimyk and Konrad Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997. MR 1492989

[KV99] Johan Kustermans and Stefaan Vaes, *A simple definition for locally compact quantum groups*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 10, 871–876. MR 1689849

[KV00] ———, *Locally compact quantum groups*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 6, 837–934. MR 1832993

[KV03] ———, *Locally compact quantum groups in the von Neumann algebraic setting*, Math. Scand. **92** (2003), no. 1, 68–92. MR 1951446

[Maj95] Shahn Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995. MR 1381692

[MVD98] Ann Maes and Alfons Van Daele, *Notes on compact quantum groups*, Nieuw Arch. Wisk. (4) **16** (1998), no. 1-2, 73–112. MR 1645264

[NT] Sergey Neshbeyev and Lars Tuset, *Compact quantum groups and their representation categories.*

[NT11a] Sergey Neshveyev and Lars Tuset, *On second cohomology of duals of compact groups*, International Journal of Mathematics **22** (2011), no. 09, 1231–1260.

[NT11b] ———, *Symmetric invariant cocycles on the duals of q-deformations*, Advances in Mathematics **227** (2011), no. 1, 146–169.

[NT12] ———, *Autoequivalences of the tensor category of u q -modules*, International mathematics research notices **2012** (2012), no. 15, 3498–3508.

[NY16] Sergey Neshveyev and Makoto Yamashita, *Classification of non-kac compact quantum groups of su (n) type*, International Mathematics Research Notices **2016** (2016), no. 11, 3356–3391.

[Pod95] Piotr Podleś, *Symmetries of quantum spaces. Subgroups and quotient spaces of quantum* SU(2) *and* SO(3) *groups*, Comm. Math. Phys. **170** (1995), no. 1, 1–20. MR 1331688

[Rag94] M. S. Raghunathan, *Universal central extensions. Appendix to: "Symmetries and quantization: structure of the state space" [Rev. Math. Phys.* **6** *(1994), no. 2, 167–205; MR1269297 (95g:22036)] by P. P. Divakaran*, Rev. Math. Phys. **6** (1994), no. 2, 207–225. MR 1269298

[TY98] Daisuke Tambara and Shigeru Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, Journal of Algebra **209** (1998), no. 2, 692–707.

[VD94] A. Van Daele, *Multiplier Hopf algebras*, Trans. Amer. Math. Soc. **342** (1994), no. 2, 917–932. MR 1220906

[VD96] ———, *Discrete quantum groups*, J. Algebra **180** (1996), no. 2, 431–444. MR 1378538

[VDVK94] A. Van Daele and S. Van Keer, *The Yang-Baxter and pentagon equation*, Compositio Math. **91** (1994), no. 2, 201–221. MR 1273649

[VDW96] Alfons Van Daele and Shuzhou Wang, *Universal quantum groups*, Internat. J. Math. **7** (1996), no. 2, 255–263. MR 1382726

[VV02] Stefaan Vaes and Leonid I. Vainerman, *On low-dimensional locally compact quantum groups*, arXiv: Quantum Algebra (2002).

[Wal47] G. E. Wall, *Finite groups with class-preserving outer automorphisms*, J. London Math. Soc. **22** (1947), 315–320 (1948). MR 25461

[Wan98] Shuzhou Wang, *Quantum symmetry groups of finite spaces*, Comm. Math. Phys. **195** (1998), no. 1, 195–211. MR 1637425

[Wor87] S. L. Woronowicz, *Compact matrix pseudogroups*, Communications in Mathematical Physics **111** (1987), 613–665.

[Wor88] ———, *Tannaka-Kreĭn duality for compact matrix pseudogroups. Twisted* SU(N) *groups*, Invent. Math. **93** (1988), no. 1, 35–76. MR 943923

[Wor98] ———, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 845–884.