INDIAN STATISTICAL INSTITUTE

End-Semester Examination: 2025-26

B. Stat. I Year

Vectors and Matrices I

Date: 21/11/2025 Maximum Marks: 50 Duration: 3 Hours

Note: Give proper justification to all your answers.

State clearly all the results you are using.

Notation:

null A = null space of a matrix or a linear map A,

$\mathcal{R}(A)$ = the row space of a matrix A,

$\mathcal{C}(A)$ = the column space of a matrix A,

$\rho(A)$ = the rank of a matrix A.

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(1) Let V and W be finite-dimensional vector spaces and $T : V \to W$ be a linear map.

(a) If T is one-to-one (injective) and $\{v_1, v_2, \ldots, v_n\}$ is linearly independent in V, then show that $\{Tv_1, Tv_2, \ldots, Tv_n\}$ is linearly independent in W.

(b) If T is onto (surjective) and $\{v_1, v_2, \ldots, v_n\}$ spans V, then show that $\{Tv_1, Tv_2, \ldots, Tv_n\}$ spans W. **[3 + 3]**

(2) Let V_1, V_2 and V_3 be subspaces of a vector space V such that $V_1 \subseteq V_3$. Show that $V_1 + (V_2 \cap V_3) = (V_1 + V_2) \cap V_3$. **[6]**

(3) Suppose V and W are finite-dimensional vector spaces and U is a subspace of V. Prove that the following statements are equivalent.

(a) There exists a linear map $T : V \to W$ such that null $T = U$.

(b) $\dim W \geq \dim V - \dim U$. **[2 + 4]**

(4) Let A and B be matrices with the same number of rows. Let P, Q and R be non-singular matrices such that the products PBQ and PAR are defined. Prove that $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ if and only if $\mathcal{C}(PBQ) \subseteq \mathcal{C}(PAR)$. **[6]**

(5) Let A and B be square matrices of same order. Suppose

$$\rho(A) = \rho(A^2) \quad \text{and} \quad AB = BA = 0.$$

Prove that $\rho(A + B) = \rho(A) + \rho(B)$. **[6]**

[*Hint. Show that there are matrices C and D such that $A = A^2C = DA^2$. Next, show that $\mathcal{C}(A) \cap \mathcal{C}(B) = \{0\}$ and $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$.*]

[P. T. O.]

(6) Reduce the following matrix to a matrix in reduced echelon form by elementary row operations:

$$A = \begin{bmatrix} 2 & 1 & -1 & 0 & 4 \\ 1 & -1 & 0 & 3 & 2 \\ 1 & 2 & -1 & -3 & 2 \end{bmatrix}.$$

Also, find a rank-factorization of A. **[6 + 2]**

(7) Consider the system of linear equations $Ax = b$, where

$$A = \begin{bmatrix} 0 & 2 & 4 & 3 \\ 0 & 1 & 2 & 4 \\ 2 & 3 & 8 & 2 \\ 0 & 4 & 8 & 16 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -3 \\ 1 \\ -5 \\ 4 \end{bmatrix}.$$

(a) Consider the partitioned matrix $\tilde{A} = [A : I_4 : b]$. Apply a sequence of elementary row operations to $\tilde{A}$ so that A is reduced to its Hermite canonical form.

(b) Find the rank of A.

(c) Find a g-inverse of A.

(d) Show that the system $Ax = b$ is consistent.

(e) Find a particular solution of $Ax = b$.

(f) Find a general solution of $Ax = b$. **[8 + 1 + 1 + 2 + 1 + 3]**