Measure Theory

M. Math. 1st year
Final exam
3 hours
Full marks: 60

21 November 2025

Answer **any four** questions. Each question carries 15 marks. The best four scores will be taken if more than four questions are answered. Results from the class may be quoted and used.

1. Suppose that μ is a σ-finite measure on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ such that $\mu \ll \lambda$, where λ is the Lebesgue measure on $\mathbb{R}$, and let
$$f = \frac{d\mu}{d\lambda}\,.$$
Let $A = \big\{(x,y) \in \mathbb{R}^2 : 0 < x < y\big\}$.

   (a) [**5 marks**] Show that
$$A \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})\,.$$

   (b) [**1 mark**] Show that
$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_A(x,y)\, dx\, \mu(dy) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_A(x,y)\, \mu(dy)\, dx\,.$$

   (c) [**9 marks**] Hence or otherwise, prove that
$$\int_0^\infty y\, f(y)\, dy = \int_0^\infty \mu\big((x,\infty)\big)\, dx\,.$$

2. Suppose $(\Omega, \mathcal{A}, \mu)$ is a measure space and $f, f_1, f_2, \ldots : \Omega \to \mathbb{R}$ are measurable functions such that for all $\varepsilon > 0$,
$$\lim_{n\to\infty} \mu\left([|f_n - f| > \varepsilon]\right) = 0\,.$$

   (a) [**7 marks**] Show that there exist $1 \le n_1 < n_2 < \ldots$ such that
$$\mu\left(\left[|f_{n_k} - f| > \frac{1}{k}\right]\right) \le 2^{-k} \text{ for all } k \ge 1\,.$$

5. Suppose $(\Omega, \mathcal{A}, \mu)$ is a finite measure space and $f : \Omega \to [0, \infty)$ is an $\mathcal{A}$-measurable function with

$$\int_\Omega f \, d\mu = \infty \, .$$

Suppose $\mathcal{F} \subset \mathcal{A}$ is a σ-field.

(a) **[4 marks]** For all $n \geq 1$, show that there exists $g_n \in L^1(\Omega, \mathcal{F}, \mu)$ such that

$$\int_A g_n \, d\mu = \int_A (f \wedge n) \, d\mu \text{ for all } A \in \mathcal{F} \, .$$

(b) **[2 marks]** Show that

$$\int_A (g_{n+1} - g_n) \, d\mu \geq 0 \text{ for all } n = 1, 2, \ldots, \; A \in \mathcal{F} \, .$$

(c) **[4 marks]** Denote $g = \sup_{n \geq 1} g_n$ and show that

$$0 \leq g_n \uparrow g \text{ a.e.}$$

(d) **[2 marks]** Prove that

$$\int_A f \, d\mu = \int_A g \, d\mu \text{ for all } A \in \mathcal{F} \, .$$

(e) **[1 mark]** Show that

$$\int_\Omega g \, d\mu = \infty \, .$$

(f) **[2 mark]** Prove or disprove that $g < \infty$ a.e.