INDIAN STATISTICAL INSTITUTE
Back Paper Examination
M. Tech. CS I year, 1st Sem, AY 2025–2026
Probability and Stochastic Processes

Date: 18. 12. 2025 | Time: 3:00 Hours | Total Marks: 100

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1. Answer **all** the questions.

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1. Let $[n] = \{1, 2, \ldots, n\}$, and Perm be the set of all permutations $\pi : [n] \to [n]$. For a permutation $\pi \in \mathsf{Perm}$, if there exists an $i \in [n]$ such that $\pi(i) = i$, then we say that i is a fixed point of π. Choose a permutation π from Perm uniformly at random. Using arguments of conditional probability, show that the probability of the event that the chosen permutation π has no fixed point is e^{-1}. [10]

2. Two gamblers, A and B, bet on the outcomes of successive flips of a coin. On each flip, if the coin comes up heads, A collects \$1 from B, whereas if it comes up tails, A pays \$1 unit to B. They continue to do this until one of them runs out of money. Assumed that the successive flips of the coin are independent and each flip results in a head with probability p. A starts with \$$i$ and B starts with \$$(N - i)$. What is the probability that A ends up with all the money? [10]

3. Let X be a Geometric random variable with parameter p. Compute mean and variance. [5+5=10]

4. Let $A_1, A_2, \ldots, A_n$ be n disjoint events that creates a partition of the sample space Ω. Let B be an event such that $A_i \cap B > 0$ for all i. Let X be a discrete random variable defined on Ω. Prove that

$$\Pr(X = x|B) = \sum_{i=1}^{n} \Pr(X = x|A_i \cap B) \Pr(A_i \mid B).$$

Prove all the relevant identities you need. [10]

5. Let $X_1, X_2, \ldots, X_n$ be any finite collection of discrete random variables with finite expectations and $a_1, a_2, \ldots, a_n$ be any any scalars. Show that for any random variable Y

$$\mathbf{E}\left[\sum_{i=1}^{n} a_i X_i \mid Y = y\right] = \sum_{i=1}^{n} a_i \mathbf{E}\left[X_i \mid Y = y\right].$$

Prove all the relevant identities you need. [10]

6. Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i = 1) = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbf{E}[X]$. Show that the following Chernoff bounds hold:

   (a) for any $\delta > 0$,

$$\Pr(X \geq (1 + \delta)\mu) \leq \left(\frac{e^{\delta}}{(1 + \delta)^{(1+\delta)}}\right)^{\mu};$$

   (b) for $0 < \delta \leq 1$,

$$\Pr(X \geq (1 + \delta)\mu) \leq e^{-\mu\delta^2/3};$$

(c) for $R \geq 6\mu$,

$$\Pr(X \geq R) \leq 2^{-R}.$$

[5+5+5= 15]

7. Suppose we have a probability space Ω and a random variable X defined on Ω such that $\mathbf{E}[X] = \mu$. Show that

$$\Pr(X \geq \mu) > 0 \text{ and } \Pr(X \leq \mu) > 0.$$

[5]

8. If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph. Note that we denote a complete graph of n vertices by K_n. [10]

9. Compute the Moment Generating Function (MGF) associated with continuous random variable $Y \sim N(\mu, \sigma^2)$. Prove all results that you need. [10]

10. State the Weak Law of Large Numbers (WLLN) and Strong Law of Large Numbers (SLLN). Give a comparative discussion on how they are different. [10]