Indian Statistical Institute
Second Semester Examinations: 2025-26 (Back paper)

Course Name: M. Math, 2^{nd} year
Subject Name : Probability Theory
Maximum Marks: 100, Duration: Three hours
2:30 PM – 5:30 PM
Date : 23.12.2025

- You may use any results proved in class unless explicitly asked to prove it. Any other results require proof.
- You will be awarded the full marks for a question if you justify using correct arguments that there is a mistake in the question.

1. (a) [**10 marks**] Construct a process $(X_i)_{i=1}^{\infty}$ of independent real-valued random variables such that X_i has law
$$\lambda_i = \frac{1}{2}\left(\delta_{1/2-1/2^i} + \delta_{1/2+1/2^i}\right),$$
where δ_x denotes the dirac mass supported at x. Then write
$$S_n = \sum_{i=1}^{n} X_i.$$
Prove or disprove that
$$\frac{S_n - n/2}{s_n} \xrightarrow{n\to\infty} N(0,1) \quad \text{in distribution},$$
for some sequence $(s_n)_n$ with $s_n \to \infty$.

2. (a) [**10 marks**] For $f \geq 0$ measurable in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, show that
$$\sum_{k=1}^{\infty} \mathbb{P}[\{f > k+1\}] \leq \mathbb{E}_{\mathbb{P}}[f] \leq \sum_{k=0}^{\infty} \mathbb{P}[\{f > k\}].$$
(b) [**5 marks**] State the second Borel-Cantelli lemma, that is, the partial converse of the first Borel Cantelli lemma.
(c) [**10 marks**] Let $(X_i)_{i=1}^{\infty}$ be iid RVs. Write $S_n = \sum_{i=1}^{n} X_i$. Prove that if $\frac{S_n}{n}$ converges almost surely to $a \in \mathbb{R}$, then
$$\mathbb{E}[|X_1|] < \infty.$$

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X, Y \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub sigma algebra.
(a) [**12 marks**] Suppose X, Y have joint probability density φ, which is a continuous function of the real line, so that for any intervals $(a,b) \subset \mathbb{R}$, $(c,d) \subset \mathbb{R}$,
$$\mathbb{P}[X \in (a,b), Y \in (c,d)] = \int_a^b \int_c^d \varphi(x,y)\, dx dy.$$
Suppose g is a measurable real valued function defined in $\mathbb{R}$ such that $\mathbb{E}[|g \circ X|] < \infty$. Compute
$$\mathbb{E}[g \circ X | \sigma(Y)].$$

4. Let $S = \{a, b, c, d\}$. Define P by $P_{ad} = 1 = P_{dc}$, $P_{ba} = P_{bc} = P_{cb} = P_{cd} = 1/2$.

(a) [**2 marks**] Is (S, P) irreducible? Explain.

(a) [**2 marks**] Is (S, P) aperiodic? Explain.

(b) [**8 marks**] Describe the stationary distribution explicitly.

(c) [**10 marks**] Find the distribution of Z_4 starting from a, that is, describe the probability $(Z_4)_*\mathbb{P}_a$.

5. Consider the Markov chain in $\mathbb{Z}$ with stochastic matrix

$$P_{xy} = \begin{cases} 3/4, & y \in \{x-1, x+1\},\ 0 < |y| < |x| \\ 1/4, & y \in \{x-1, x+1\},\ 0 < |x| < |y| \\ 1/2, & y \in \{-1, 1\},\ x = 0. \end{cases}$$

(a) [**7 marks**] Describe the stationary measure explicitly.

(b) [**2 mark**] Let $\tau_a^{(k)}$ be the k-th time of hitting $a \in \mathbb{Z}$. Argue that $\lim_{k\to\infty} \tau_a^{(k)}/k$ exists $\mathbb{P}_y$-a.s.

(c) [**2 mark**] Determine the $\mathbb{P}_y$-a.s limit $\lim_{k\to\infty} \tau_a^{(k)}/k$ in terms of a.

(d) [**3 marks**] Prove or disprove that 0 is positive recurrent.

6. Let (S, P) be a finite state Markov chain with stationary probability λ. Suppose

$$(P^n)_{xy} \xrightarrow{n\to\infty} \lambda(y), \tag{1}$$

for all $x, y \in S$.

(a) [**7 marks**] Let T be the left shift. Use (1) to show that for any

$$f = \mathbb{1}_{\{a_0\}} \circ Z_0 \cdot \mathbb{1}_{\{a\}} \circ Z_1, \quad \text{and} \quad g = \mathbb{1}_{\{b_0\}} \circ Z_0 \cdot \mathbb{1}_{\{b_1\}} \circ Z_1,$$

for $a_0, a_1, b_0, b_1 \in S$,

$$\lim_n \mathbb{E}_{\mathbb{P}_\lambda}[f \cdot g \circ T^n]$$

exists. What is the limit?

(b) [**2 marks**] Show that (S, P) is irreducible if (1) holds.

(c) [**5 marks**] Prove that the period function is constant. Recall $\mathrm{per}(x) = gcd(\{k \in \mathbb{N} \mid (P^k)_{xx} > 0\})$.

(d) [**3 marks**] Show that (S, P) is aperiodic when (1) holds.