INDIAN STATISTICAL INSTITUTE, KOLKATA
Midterm , First Semester 2025-26
Analysis 3 , B. Stat 2nd year
Full Marks : 30
Time : 2 hours
Date : 9/9/2025

1. Construct a homeomorphism between the subset $S := \{(x,0)|x \geq 0\} \cup \{(0,y)|y \geq 0\}$ (together with the subspace topology coming from $\mathbb{R}^2$) and $\mathbb{R}$. [4]

2. Is the union of $x$-axis and $y$-axis homeomorphic to $\mathbb{R}$ ? Justify your answer. [4]

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $a \in \mathbb{R}^n$. Show that $f$ is differentiable at $a$ if and only if $f$ satisfies the following two properties.[4+3]

   (a) For any $v \in \mathbb{R}^n$, the directional derivative along $v$ at $a$, $\nabla_v f(a)$, exists. Also
   $$\nabla_{\alpha_1 v_1 + \alpha_2 v_2}(f)(a) = \alpha_1 \nabla_{v_1}(f)(a) + \alpha_2 \nabla_{v_2}(f)(a),$$
   for all $\alpha_1, \alpha_2 \in \mathbb{R}$ and $v_1, v_2 \in \mathbb{R}^n$.

   (b) For any $\epsilon > 0$ there exists $\delta > 0$ such that
   $$\frac{|f(a+tv) - f(a) - t.\nabla_v(f)(a)|}{|t|} < \epsilon,$$
   whenever $|v| = 1$ and $|t| < \delta$.

4. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (xy, x^2 + y^2 + e^{(x-2)(y-1)})$. Show that there exists $r > 0$ such that for any $(a,b) \in N_r((2,6))$ there exists $(p,q) \in \mathbb{R}^2$ such that $f(p,q) = (a,b)$. [5]

5. Let $f(x), g(x) \in \mathbb{R}[x]$, $\alpha, \beta \in \mathbb{R}$ with $f'(\alpha) = g'(\beta) = 0$ and $f''(\alpha) > 0, g''(\beta) > 0$. Show that the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(x,y) = f(x) + g(y)$ has a local minima at $(\alpha, \beta)$.[6]