

WALRASIAN THEORY WITHOUT AN AUCTIONEER

A Study of Monetary Exchange through Trading Posts and a Diagrammatic View of Disequilibria

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Thesis submitted to the Indian Statistical Institute
in partial fulfilment of the requirements
for the award of the degree of
Doctor of Philosophy
CALCUTTA

1993

ACKNOWLEDGEMENTS

First, I would like to record my indebtedness to Prof. Dipankar Dasgupta for his help and guidance in the preparation of this thesis. He has been highly generous in allowing me free access to his time. But for my association with him I would neither be motivated towards research in Economic Theory nor develop a fascination for the subject.

I have benefited greatly from many useful discussions with Prof. Amitava Bose. I am indebted to him for giving me much of his valuable time and for his patient counseling and criticism.

I would like to thank my husband B. Rajeev for his bemused tolerance of my "research activities" and his non-expert comments at various stages of the thesis. The thesis would not be complete without him.

Many thanks are due to Kavita Rao for her help on numerous occasions, for her ready willingness to "listen" and for being a good friend during the ups and downs of the thesis.

Dr. Subir Bhandari has helped me a lot on the Mathematical side. I am grateful to him for his generous assistance.

I would like to acknowledge the support given by the faculty of the Economic Research Unit, ISI, Calcutta. In particular, I wish to recall with gratitude the valuable advice of Prof. Robin Mukherjee on several occasions. Further thanks in this regard are also due to my fellow research scholars Malabika Ghosh and Arpita Dhar. I would like to thank my friends Patrick Francois and Siwan Anderson for helping me with the proof reading.

During this period, the workers of the Economic Research Unit have extended all possible help and I take this opportunity to express my gratitude towards everybody especially to Sm. Manjusree Benerjee. I would like to

thank Shri V. P. Sharma and Shri S. P. Jakhmola for typing this thesis and Shri Kaushik Gaha for his assistance in handling the word processing package.

Finally, I would like to acknowledge my parents and in-laws for their continuous support during this period.

Meenakshi Rajeev

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INTRODUCTION

1.1 The Problems

The theory of Walrasian equilibrium yields a set of prices at which the aggregate competitive demand for each commodity equals its aggregate competitive supply. Two important issues arise in this context. The first is concerned with discovering laws which guide the behaviour of the many economic variables, especially prices, when the system is *out of equilibrium*. Walras (1900) tackled this problem by providing an algorithm (or the “*tâtonnement* scheme”) which can be viewed as an auctioneer quoting different prices for the various goods and adjusting them according to the signs of the resulting aggregate excess demands. This is the well-known problem of the stability of a Walrasian equilibrium. The other issue revolves around the function of an auctioneer as a clearing house for commodities. Once the equilibrium prices are found, all agents are assumed to deposit their initial endowments with the auctioneer, who in turn reallocates them according to the pattern of excess demands.

The removal of an auctioneer exposes two important weaknesses of the Walrasian system. First, it lacks the institutional scaffolding required to carry out desired exchanges even when prices are at equilibrium. Secondly, there is no guarantee that prices will glide smoothly to their equilibrium values without the guidance of the auctioneer. The thesis is concerned exclusively with these two lacunas in the Walrasian model. In particular, the initial half of our work is geared towards finding institutions for allocating commodities when the system is in equilibrium. The other half is concerned with the issue of finding an appropriate allocation itself when the auctioneer is not present to adjust prices and the latter have somehow got stuck at disequilibrium levels. This justifies the main title of the thesis as well as its subtitles.

The next two sections (1.2 and 1.3) provide a brief overview of the concerned issues.

1.2 Allocation in Equilibrium

Let us begin with the problem of allocation of commodities in equilibrium. In this context we assume that prices have somehow settled at the equilibrium level. This in turn ensures that aggregate excess demand for each commodity equals zero and in this sense everybody's desired allocation is at least feasible. However, even in such a situation, a lack of mutual coincidence of wants at the individual level creates serious obstacles in the way of realizing the equilibrium allocation through a decentralized process of trade.

1.2.1 The role of money as a medium of exchange

When trades take place in a decentralized fashion, (i.e. in the absence of an auctioneer) it is likely that they would be restricted to those between pairs of agents. More importantly such pairwise meetings of a particular agent with different traders need to be *separated in time*. In the absence of a centralized agency, each agent going through such sequential bilateral trades will naturally insist on the value of his incomings to be at least as large as the value of his outgoings. In other words, trades should be *bilaterally balanced* in value terms after each meeting or equivalently maintain *quid pro quo*. However, in the absence of perfect mutual coincidence of wants between the agents, this quid pro quo may have to be maintained by transferring a good to the creditor for which he has no (Walrasian) excess demand. The need for a medium of exchange in a competitive set-up can be best appreciated against this background, for as soon as an agent accepts a good for which he does not have excess demand, it

takes the form of a medium of exchange. In what follows, Chapters 2, 3 and 4 will be concerned with some problems associated with the use and choice of a medium of exchange.

The earliest recognition of this problem came in Menger (1892). There have been many subsequent attempts to look into the role of money as a medium of exchange, but it was Hicks (1969) who posed it in the context of a Walrasian equilibrium. The last couple of decades saw further progress in this branch of the literature. In fact, it has been a shared view (see Starr (1972), Eckalbar (1984), Kiyotaki and Wright (1989) etc.) that money as a medium of exchange is an indispensable tool for attaining one's desired allocation.

An important question that came up in the literature is the following: which good would be a best choice as a medium of exchange? An associated question relates to the basis on which one can justify a medium of exchange as the best. The latter question has been answered in the literature by assuming the presence of costs involved in transactions (see for example Niehans (1967)).

1.2.2 Two ways of introducing transaction costs

Agents trying to attain a desired bundle of commodities would naturally take account of the transaction costs involved in the process of the realization of an allocation and there are at least two ways in which they may do so. The first consists of a two stage procedure. In this case, an agent begins by deciding upon his most preferred consumption bundle quite independent of time or any other costs necessary to achieve it. Having done so, he proceeds next, to solve for a least cost sequence of exchanges, which realizes the desired optimum position. This is the approach mainly of Ostroy-Starr (1974), Ostroy (1973), Starr (1976), Jones (1976), Oh (1989), Kiyotaki and Wright (1989) etc.

The second approach advocated mainly by Niehans (1971) is a single shot procedure which solves for the best possible consumption bundle simultane-

ously with the least cost way of achieving it. A similar approach followed by Karni (1973) also determines through a one shot optimization technique, the optimal trades (involving pairs of goods) for each individual trader.

However, the abstraction employed by Niehans is inadequate to capture the notion of money as a decentralized medium of exchange. For, he has not visualized trades between the pairs of agents to take place sequentially in time. All trades in his model are carried out simultaneously as it were in a timeless vacuum. Thus, the basic problem under consideration viz. how to attain a desired allocation through a decentralized process of trade remains unanswered.

1.2.3 The problem of a sequence of meetings

An attempt to fill this gap led Ostroy and Starr to consider sequential bilateral meetings between the agents. However, the sequence of such pairwise meetings is supposed to be quite arbitrary, given the fact that under decentralized trade no agent has prior information about the excess demand vectors of the other agents in the system. Given this set-up, Ostroy and Starr try to identify the characteristics of a commodity which the agents may use as a medium of exchange to attain equilibrium in a reasonable period of time. We would refer to such a medium of exchange as a *feasible* one.

1.2.4 Specific results on feasibility

In the process, Ostroy and Starr have shown that if the initial endowment of a good, say M , is so large for every agent of the economy that he can cover his desired purchases of all other goods with it, i.e.

$$(Q) \cdots \sum_{c \neq M} p_c [z_{kc}]^+ \leq p_M w_{kM}, \quad \forall \text{ agents } k, p_M > 0$$

where,

c is the index for commodities, p_c is the price of the c th commodity
 z_{kc} is agent k 's excess demand for good c , $[z_{kc}]^+ = \max[0, z_{kc}]$, w_{kM}
 is the initial endowment of good M for agent k

then good M can be used as a medium of exchange to attain equilibrium within one *round* (i.e. after each agent finishes meeting every other agent once and only once). Apart from imposing a strong restriction on the initial endowment of the money good, condition (Q) implies in particular that money received by a trader from his debtors is not required to meet claims on him by his creditors. In other words, money received (through trade) by agent k need not be *passed on*. All his money receipt may just as well be viewed as being acquired to satisfy his final demand for it.

As a simple first step towards removing this undesirable feature of (Q) , we have tried to relax it in a straight forward fashion by asking what happens if the condition is satisfied by two goods (say M and M') taken together (in the RHS) rather than by a single good (M) and which are in turn used simultaneously as media of exchange. It is easily shown in this case that it takes twice the time to attain equilibrium as compared to the time taken by a single medium of exchange. Interestingly, however, the result cannot be generalized to more than two goods (see Chapter 2).

In a subsequent contribution, Starr (1976) dispenses with such strong conditions to be satisfied by money goods and allows every good to be used simultaneously as a medium of exchange. A conclusion of this investigation is

that the time taken to attain equilibrium can mount to undesirably high levels, including perhaps infinity.

1.2.5 Market institution

It should be noted in this context that the trading arrangement considered generally in the literature (by Ostroy-Starr and others) assumes arbitrary meetings between the agents without any reference to the goods they want to exchange. However, casual empiricism suggests that historically speaking monetary trade and specialization of markets go hand in hand. Clearly, the use of money as a medium of exchange is one way of reducing the inefficiencies involved with barter. However, the “blind meeting” between the agents in a marketless trading arrangement may take an unnecessarily long time, especially if the number of agents is large. Keeping this in mind, we have posed the problem of a medium of exchange in a framework characterized by the social institution of markets, i.e. for a system where separate trading posts (or markets) exist to deal with the exchange of each good against money. The advantage of such a system arises from the fact that now an agent visits only those specific markets which are relevant for satisfying his own excess demands.

In this context Chapter 2 has shown that if a good shows up as a positive quantity in the final consumption vector of each agent in an economy characterized by trading posts, then that good can be used as a medium of exchange to attain the equilibrium allocation in *finite* time. (The result holds in the absence of trading posts also.)

However, even this condition on the medium of exchange is too strong. For the condition is weakened further (in Chapter 2) and replaced by one which partly satisfies the Ostroy-Starr condition (Q) also. This weaker condition will later be interpreted in the context of *OLG* models to mean that a commodity can serve as a medium of exchange if it functions as a store of value. This

is true even if the store of value happens to be a fiat money. The question of whether the fiat money would command a positive price in equilibrium is a problem not addressed in this thesis. There is indeed a stream of literature (see e.g. Grandmont (1974), Freeman (1989)) that deals specifically with this issue.

It should be mentioned here that the role of the social institution of markets in the process of trading in fact received its due emphasis elsewhere too. In this context, one can first of all mention the work of Hicks (1969). In another framework Foley (1970) has discussed how there can prevail different kinds of markets (viz. wholesale and retail) in an economy. Similar work done by Hirshleifer (1973) endeavours to explain the presence of middlemen due to differences between the buying and selling prices of a commodity. Wallace (1972) on the other hand tries to provide conditions relating to transactions costs under which it may be profitable to trade in a particular trading post or market.

1.2.6 Efficiency of trading posts

As mentioned above, the feasibility question has been posed in this thesis under the set-up of the social institution of markets. The intuitive justification for this lies in the fact that for large economies a monetary pattern of trade is expected to work better under the trading post set-up. We next (in Chapter 3) try to ask a simple question. Is it possible to formally justify our intuition that the trading post set up functions better (with respect to transaction costs) as compared to the marketless trading arrangement generally considered in the literature? Therefore, as a next step we have taken up this problem in some detail. It is revealed from the exercise that for large economies, the social institution of money functions better when coupled with the social institution of markets. In fact, if the population size increases in the Ostroy-Starr set-up,

the gains from a monetary pattern of trade tend to become negligible when unaccompanied by trading posts. It should be noted in this context that this comparison is made on the basis of the *time* taken to attain equilibrium *given* a feasible medium of exchange for the economy.

1.2.7 Endogenous choice of a medium of exchange

However, the question that might be asked now is who selects the appropriate medium of exchange for the agents? In a set-up where trades are taking place in a decentralized fashion, it is also relevant to examine which good can get endogenously chosen *by the agents themselves* as a medium of exchange. This problem has lately become a well recognized issue (see eg. Kiyotaki and Wright (1989), Aiyagari and Wallace (1991), Jones (1976), Oh (1989) etc.).

Kiyotaki and Wright have dealt with the problem in a more rigorous fashion by incorporating the transactions costs in the objective functions of the agents. As would be clear in the sequel (Chapters 2 and 3), transactions costs remain implicit in the models of Ostroy and Starr (1974), Jones (1976), Oh (1989) etc. In the Kiyotaki and Wright framework, an infinitely long lived agent in a *marketless* trading arrangement is seen to be maximizing his expected life time utility, where the time cost is introduced through a discount factor. An agent has the option of choosing different strategies, viz. to opt for direct barter or indirect trade involving the choice of different good(s) as medium of exchange. On the basis of the concept of a (steady state) Nash equilibrium, different goods are shown to be chosen by the agents as media of exchange in equilibrium.

Suppose now that we introduce trading posts into the above framework. Since transactions costs are already incorporated in the utility functions of the agents, efficiency of a trading post set-up can be examined just by looking at the equilibrium utility levels of the agents. Interestingly, it can be shown

that the (steady state) utility derived by an agent under a trading post set-up dominates the utility levels associated with any possible equilibria under a marketless trading arrangement (see Chapter 3). This again confirms our intuition that the social institution of money works better when coupled with the social institution of markets.

1.2.8 A mix of direct barter and indirect trade

We finally pose the problem of an *endogenous* choice of a medium of exchange in a trading post set-up (in Chapter 4). In this context, a possibility emerges of simultaneous prevalence of direct barter and indirect trade (as that of Oh (1989) as well as Kiyotaki and Wright (1989)).

Under the alternative marketless trading arrangement, an agent wanting to have a mix of direct barter and indirect trade confronts another agent at random to explore the trading opportunities prevailing between them. Naturally, if they find their wants to be mutually coinciding, exchange takes place. Otherwise, on the basis of transaction costs they decide whether or not to go for indirect trade. Thus, the decision regarding *when* to go for direct barter is an obvious one. On the other hand, when we have separate trading posts to deal with different pairs of goods one has to make a specific choice regarding when to visit a trading post relevant for his direct barter, or more precisely, how frequently one should visit his direct barter trading post vis-a-vis the trading posts relevant for his indirect trade. Chapter 4 will mainly emphasize this problem.

It may be noted that Iwai (1988) also addresses the problem of choice of a medium of exchange in a trading post set-up. However, the structure of the trading posts considered by Iwai is somewhat different in nature. Furthermore, a possibility of having a mix of direct barter and indirect trade is not considered.

Thus, to sum up, this exercise analyses the role played by money as a

medium of exchange in the process of trade. It has been revealed from this exercise that the social institution of money works better when coupled with the social institution of markets. Finally, some issues specifically associated with the problem of choice of indirect trade in a trading post set-up have also been analyzed.

1.3 Allocations in Disequilibrium

As already noted, reallocation of goods and services in equilibrium is only one of the functions of an auctioneer. The other important role consists of price adjustments (viz. the Walrasian *tâtonnement*). However, we observe in reality a large number of cases where prices may not necessarily have adjusted to their market clearing levels (for example labour unions may fix the wage rate at a particular level or the government may fix the prices of essential commodities etc.). In this situation, even before we can consider the problem of devising social institutions to realize the agents' desired allocations, it is important to decide what the most desired feasible allocation of an agent is when he is faced with disequilibrium prices. Clearly an agent's notional demands and supplies would generally be unattainable since the prices are not market clearing. In other words, an agent trying to realize his *utility maximizing* allocation would in turn perceive physical constraints preventing him from realizing them (Benassy (1975 a,b)).

Any constraint on the good one desires to supply (e.g., labour) would on the other hand affect his demand for other consumable goods (the so called "spill over effect"). Under these circumstances, it is necessary to see what

allocations would be *acceptable* to all agents in the sense that no individual would have any incentive to depart from it.

The problem is an old one. As is well known, three major concepts of equilibrium in the face of disequilibrium prices have been put forward by Benassy (1975b), Drèze (1975) and Malinvaud and Younes (1977) respectively. Moreover, it has also been shown that the three allocation schemes are actually equivalent (see Silvestre (1982)). Some other major contributions in this literature include Grandmont, Laroque and Younes (1978), Younes (1978) Leijonhufvud (1968), Benassy (1986) etc.

1.3.1 Solution concepts in a pure exchange economy

Chapter 5 of this thesis looks once again into the equilibrium concepts and the equivalence results. More precisely, through a uniform diagrammatic scheme, we have tried to depict the relationships between the three concepts of equilibrium mentioned above. In this respect, the work of this chapter does not present any new results and is mostly expository in nature.

Earlier attempts of a similar nature are to be found in Basu (1992) and Malinvaud and Younes (1977). Basu as well as Malinvaud and Younes were concerned with diagrammatic presentations of the Drèze and the Malinvaud-Younes equilibria respectively. Böhm and Müller (1977) have also provided a diagrammatic representation of a Drèze equilibrium which reveals in particular that a Pareto improvement of an equilibrium requires some form of forced trading.

By contrast the present exercise focuses mainly on the concept of the Benassy equilibrium.

1.3.2. p -efficiency

Having analysed the disequilibrium concepts in terms of a set of diagrams, we proceed next to look into them from a completely different point of view viz. that of *economic welfare*. Prices may be rigid for many different reasons. Price fixation by the government (i.e. administered prices) in a planned economy may not be the least important of these. There are an endless variety of price vectors that an economic planner might choose from.

Given the initial endowments the government can quote a price such that the resulting equilibrium is a competitive one and hence Pareto optimal. However, the government may not be interested in attaining the competitive allocation. While aiming to achieve other possible allocations through price manipulation, is there some alternative notion of efficiency which a planner might take into account ? In this context, Benassy's concept of p -efficiency can be considered to be a natural candidate (though a weaker substitute for Pareto optimality).

In Chapter 6 we have described this concept of efficiency through a set of diagrams. It is then shown that depending just on the notional demands and supplies (at disequilibrium prices), it is possible to infer the p -efficiency of the corresponding Benassy equilibrium. These results are next used to obtain a complete characterization of the price space in terms of p -efficient and p -inefficient Benassy equilibria. The exercise has been performed for a Cobb-Douglas utility specification.

1.3.3 Models with production

The problems described above are restricted to a pure exchange economy. Therefore, as a next step (in Chapter 7), we endeavour to introduce production into the scenario. This leads naturally to a consideration of the phenomenon of

unemployment and allows a detailed representation of the Benassy-Malinvaud classification of the different types of unemployment that may arise in the short run.

In this respect, Chapter 7 provides an alternative to Stoneman's (1979) exercise. However, the present treatment is more complete due to several reasons. First, as opposed to Stoneman, the utility function is not assumed here to be separable. Secondly, the diagrammatic framework developed here gives rise to a more unified presentation of the three well known varieties of macroeconomic equilibria viz., Keynesian equilibrium, classical equilibrium and repressed inflation. The characterization also turns out to be an interesting complement to the corresponding representation of Benassy (1982) and Malinvaud (1977). Thus, while their demonstrations are restricted to the wage price plane, the present work provides a dual characterization of these in quantity space.

In this respect our work bears a close resemblance to the contribution of Muellbauer and Portes (1978), but is distinguished from it with respect to the diagrammatic apparatus itself. In particular, it utilizes the three dimensional space which enables one to visualize the choice problem on the budget plane of the household and read the behaviour of a third but important commodity, viz. money (whose role in the literature would be discussed in the sequel) directly from the diagram. Moreover, this advantage is not negated by any additional complications (arising out of the extra dimension) as the entire analysis is carried out with reference to the budget plane alone which constitutes a two dimensional subspace of the commodity space.

More importantly, our representation gives rise to a characterisation of different varieties of Benassy equilibria, depending on the position of the notional demand point on the budget plane.

This representation in the quantity space brings into sharp focus the problem of nonuniqueness of a Benassy equilibrium (discussed also by Hildenbrand

and Hildenbrand (1978)). In fact, it makes possible a simple demonstration of the fact that the same wage price constellation may be associated with two different types of Benassy equilibria. The diagrams are also utilized to study policy issues, such as the effects of money wage cuts.

Finally, p -efficiency properties of the different varieties of Benassy equilibria (with production) are examined in detail. An interesting conclusion in this context is that a Benassy equilibrium of the Keynesian type need not necessarily be inefficient as claimed by Benassy (1982).

CHAPTER 2

FEASIBILITY

ABSTRACT

This chapter deals with the problem of determining a feasible medium of exchange, i.e., a commodity which can be used by the agents (of any competitive economy) to attain their most desired allocation in a decentralized fashion. Such an attempt on the part of Ostroy and Starr led them to introduce a quantity constraint on the amount of availability of the money good for it to function as a medium of exchange for all possible competitive economies. We examine here how far it is possible to relax this constraint. We also examine an alternative condition under which decentralized exchange is feasible. An important conclusion of the chapter is that a good will act as a decentralized means of exchange provided it is finally demanded by all agents.

All problems dealt with in this chapter are posed for a trading arrangement involving separate markets to deal with different pairs of goods. The efficiency of such an arrangement will be discussed in the following chapter.

2.1 INTRODUCTION

In the introductory chapter it has been mentioned that Niehans developed a single shot procedure which solves for the best possible consumption bundle together with the least cost way of achieving it. The method advocated by Niehans, though rich in various aspects, is inadequate to capture the notion of money as a decentralized medium of exchange. This is mainly due to the fact that Niehans' work does not visualize bilateral trades to be taking place sequentially in time. Indeed, Niehans assumes an individual to be purchasing a commodity against a medium of exchange, simultaneously in time with the

acquisition of the latter through the sale of some other good to another agent. Since an agent cannot simultaneously trade with different traders at the same point of time, however, the trades involving the acquisition of the medium of exchange must *precede* those involving its disposal. Hence the abstraction used by Niehans is inadequate to handle the problem of decentralized trade.

Some pioneer attempts to overcome the problem may be found in Starr (1972), Ostroy (1973) and Ostroy and Starr (1974) who have visualized decentralized trade as an exchange of goods through bilateral meetings. It is natural that each agent engaged in such bilateral exchange would expect the inflow of his commodities through trade to be at least as large as the outflow in value terms. Secondly, the good(s) he receives should be those which he in fact demands, that is, the trades should be *excess demand diminishing* for the involved traders. Starr (1972) however, has shown that the agents are generally *not* able to acquire their most desirable allocations if they insist that each bilateral trade satisfies the above two conditions. One way out is to accept a good as a payment for ones sale, even though an agent may not necessarily have excess demand for it. In other words, an agent should be ready to accept such a good with the belief that he will be able to pass it on to other traders in exchange for goods he desires to have. Once that is complied with, the concept of a *medium of exchange* naturally emerges.

It is then natural to ask which specific commodity is a suitable candidate for a medium of exchange, and it is this problem that captured the attention of Ostroy (1973) and Ostroy and Starr (1974). It was shown by the authors that if for every agent, the initial endowment of a specific commodity, say M , is large enough to cover his purchases of all other goods for which he has excess demands, then that good can be used as a medium of exchange to attain equilibrium within a reasonable period of time. Ostroy and Starr recognize that this is a strong condition to impose. In view of this, we tried first to relax

this condition in a straight-forward fashion. However, the relaxed versions too are not above reproach. Keeping this in mind, we have tried to focus on an entirely different condition. In particular, it has been established that if a good is *finally demanded* by every agent of the economy, then it can be used as a medium of exchange to attain equilibrium in *finite* time. This condition is further relaxed and we have tried to relate the weaker version to the role played by money as a store of value in overlapping generations models (see Rajeev (1990)). It should be mentioned in this context that as a by product of the method of analysis developed here, we are able to provide an alternative proof of the result established by Starr (1976).

As it is observed in reality that the monetary pattern of trade and specialization of market organization go hand in hand, the problem is posed with reference to a system consisting of *monetized trading posts*, i.e. where corresponding to each non-money good, there exists a trading post for exchanging it against the money good. The results proved here remain valid, of course, for the alternative marketless trading arrangement visualized by Ostroy and Starr (1974).

In this context it is shown in the sequel (Section 2.4) that for an appropriately defined market structure, an economy can attain *equilibrium in finite time* even when *all goods* are simultaneously used as media of exchange. This result is to be contrasted with the corresponding result due to Starr (1976) that for a marketless trading arrangement it may take infinitely long to attain equilibrium when all goods are used as media of exchange.

To discuss these problems more rigorously, we first describe (in Section 2.2) the basic framework under consideration. Section 2.3 deals with some extensions of the Ostroy-Starr work for a marketless trading arrangement. We introduce the trading post set-up (in Section 2.4) and study the different feasibility criteria mentioned above. A concluding section sums up the findings.

Proofs of the results presented in this chapter are worked out in the Appendix. At the end of each chapter we include a list of all the symbols used in that chapter.

2.2 THE MODEL AND SOME OBSERVATIONS ON OSTROY AND STARR

The economy is comprised of finitely many individuals, each born with a definite endowment of goods. It is assumed by Ostroy and Starr (1974) that prices have somehow settled at their competitive equilibrium levels and that agents are aware of the fact. The problem is to attain the equilibrium allocation through a decentralized process of trade, i.e. in the absence of any guiding authority. This poses certain complications due to the fact that (i) agents' wants are not mutually synchronized and (ii) agents do not have information regarding the excess demand vectors of others until and unless they directly confront each other. It is natural to expect that traders would desire to attain the competitive allocation as soon as possible. Keeping that in mind, the *time taken to attain the equilibrium allocation* is considered to be the relevant *transaction cost* which every trader wishes to reduce as far as possible.

Let us now describe the notation to be used in the subsequent discussion. The economy consists of K traders (indexed by k) and n commodities (indexed by c), the n -th commodity, denoted by M , being earmarked to serve as the medium of exchange. The set of all traders is also represented by K and the set of commodities other than the n -th good by C . The initial endowments of the K traders are depicted by the $K \times n$ matrix $\tilde{W} = \|w_{kc}\|$, where w_{kc} represents the level of good c in the initial endowment of agent k . The utility function of agent k is $U_k : \mathbb{R}_+^n \rightarrow \mathbb{R}$, $k = 1, 2, \dots, K$. Prices $\tilde{p}' = (p_1, \dots, p_n)$ are exogenously given at the equilibrium levels. Given the prices and the initial endowments, the Walrasian excess demands for the economy are given by the

$K \times n$ matrix $\tilde{Z} = \|z_{kc}\|$, where z_{kc} is the level of agent k 's excess demand for (or supply of) good c . Clearly,

$$\begin{aligned} z_{kc} &> 0, \text{ if } k \text{ is an excess demander of } c, \\ z_{kc} &\leq 0, \text{ otherwise.} \end{aligned}$$

Similarly, the final demands are given by the $K \times n$ matrix $\tilde{X} = \tilde{W} + \tilde{Z} = \|x_{kc}\|$, where x_{kc} is the level of good c finally consumed by agent k .

Thus, the economy under study is characterized by $(\tilde{W}, \tilde{Z}, \tilde{p})$, where $\tilde{p}$ is the transpose of vector p' and it satisfies (as listed in Ostroy and Starr (1974)) the following properties:

- (U.1) $\tilde{Z} \cdot \tilde{p} = 0, p_M > 0$, i.e. the value of total excess demand equals zero, where the price of the n -th good is assumed to be positive.
- (U.2) $\sum_k \tilde{Z}_k = \tilde{0}$, where $\tilde{Z}_k$ represents the k -th row vector of $\tilde{Z}$. This relation ensures that the total quantity of a particular good purchased is exactly equal to the aggregate amount of that good sold.
- (U.3) $\tilde{Z} \geq -\tilde{W}$, the inequality is assumed to hold entrywise. It implies that the final consumption of each good by every agent is non-negative.

As mentioned earlier, an agent belonging to an economy of the above nature is trying to realize his final demands through a process of decentralized bilateral exchange. Here one of our attempts is to visualize these trades against the background of trading-posts. However, we would concentrate initially (i.e. in Section 2.2 and 2.3) on the alternative marketless trading scheme discussed by Ostroy and Starr (1974) and then move on to the trading-post set-up (in Section 2.4).

The trading arrangement taken up by Ostroy and Starr is based on the assumption that in order to explore the trading possibilities, agents meet each other in pairs following an arbitrary sequence of meetings. This ad hoc sequence

in no way depends on the goods an agent desires to demand and supply. Further, when one pair meets, other pairs are allowed to meet simultaneously. After a pair of agents meets, exchange takes place between them following a definite *trading rule*. For example, the trading rule considered by Ostroy and Starr (1974), viz. μ , can be described as follows. Suppose following the arbitrary sequence of meetings (between traders), agent i confronts agent j in period t . In notation, $\Pi^t(i) = j$. When agent i and j meet they scan the excess demands and supplies of each other and make trades that diminish one's excess supply and reduce the others' excess demand to the largest extent possible. Thus let V_{ic}^t represent the value of trader i 's unsatisfied excess demand for good c (at the beginning of time period t). By convention, $V_{ic}^t > 0$ if i has unsatisfied demand and $V_{ic}^t < 0$ if i has unsatisfied supply. Suppose, $WLOC$ $V_{ic}^t > 0$ and $V_{jc}^t < 0$, then i and j exchange good c to the extent that either the unsatisfied excess demand of i or the unsatisfied excess supply of j , whichever is smaller reduces to zero. Similar exchanges take place for all possible goods. It is clear that if both agents are either demanders or suppliers of good $c (\in C)$ no trade involving that good takes place.

These trades will in general keep one agent a net debtor to the other. Thus at the end, quid pro quo is maintained by the debtor transferring the required level of good M (which is assumed to be the medium of exchange) to the creditor. This whole process of exchange between a pair of agents is assumed to take one unit of time.

Thus, if we represent the inflow and outflow of commodities of agent i in his trade with agent j by the vector $b_i^t (\in \mathbb{R}^n)^1$ then it can be separated into two parts viz. $b_i^t = \rho_i^t + y_i^t$. Here $\rho_i^t (\in \mathbb{R}^n)$ captures all possible exchanges that are excess demand diminishing for agent i . $y_i^t (\in \mathbb{R}^n)$ shows the amount

¹Strictly speaking, we should write b_{ij}^t to indicate the fact that i trades with j . However, (given $\Pi^t(i) = j$) we drop subscript j for notational simplicity.

of good M that is transferred from i to j (or vice-versa) in order to maintain quid-pro-quo. A positive component of any vector represents an inflow of the respective commodity whereas a negative component shows an outflow of that good. Similarly, we can consider $b_j^t = \rho_j^t + y_j^t$, for agent j .

Thus, on the basis of the above discussion one can represent ρ_i^t and y_i^t as follows.

$$\begin{aligned}\rho_{\sim i}^t &= (\rho_{i1}^t, \rho_{i2}^t, \dots, \rho_{ic}^t, \dots, \rho_{in}^t) \\ \rho_{\sim j}^t &= (\rho_{j1}^t, \rho_{j2}^t, \dots, \rho_{jc}^t, \dots, \rho_{jn}^t)\end{aligned}$$

s.t.

$$\rho_{ic}^t = -\rho_{jc}^t = \begin{cases} 0, & \text{if } V_{ic}^t \cdot V_{jc}^t \geq 0, \\ \frac{1}{p_c} \min(|V_{ic}^t|, |V_{jc}^t|), & \text{if } V_{ic}^t > 0, \text{ and } V_{jc}^t < 0 \\ -\frac{1}{p_c} \min(|V_{ic}^t|, |V_{jc}^t|), & \text{if } V_{ic}^t < 0, V_{jc}^t > 0. \end{cases}$$

We next come to the second vector viz.

$$y_k^t = (y_{k1}^t, y_{k2}^t, \dots, y_{kc}^t, \dots, y_{kn}^t), k = i, j$$

where,

$$y_{ic}^t = -y_{jc}^t = \begin{cases} 0 & \text{if } c \neq M \\ m^t & \text{when } c = M \end{cases}$$

and m^t is s.t.

$$\sum_{c \in C} p_c \rho_{ic}^t + p_M m^t = 0.$$

Clearly, this trading rule satisfies the following properties (originally proposed by Ostroy and Starr (1974)).

$$(T.1) \quad W_{\sim k}^t + b_{\sim k}^t \geq 0 \quad \forall t \quad k = i, j, \text{ where}$$

$$\begin{aligned}W_{\sim k}^t &= W_{\sim k} + \sum_t b_{\sim k}^{t-1} \\ &= (w_{k1} w_{k2} \dots w_{kc} \dots w_{kn}) + \sum_t b_{\sim k}^{t-1}\end{aligned}$$

$$(T.2) \quad \underset{\sim i}{b}^t = -\underset{\sim j}{b}^t$$

$$(T.3) \quad \underset{\sim i}{b}^t \cdot \underset{\sim}{p} = 0 = \underset{\sim j}{b}^t \cdot \underset{\sim}{p}.$$

The first property implies that at any stage an agent cannot have a negative holding of any good. (T.2) ensures that the goods sold by i are the goods received by j and vice-versa. The third property is called *price consistency* (or, equivalently, the *quid pro quo* property) which implies that the value of goods delivered by i to j equals the value of goods received by i from j .

Furthermore, it is revealed by rule μ that when i and j are trading, they do not require any information regarding excess demands and supplies of other agents in the economy. Thus another property satisfied by the trading rule is that it is *decentralized* in nature. For future reference we will use the notation (D) to represent this feature. That is

- (D) *Decentralization* : trade between i and j does not depend on their knowledge of endowments and final demands of the agents belonging to the set $K \setminus \{i, j\}$.

Given the trading rule, one looks for a condition (to be satisfied by the medium of exchange) under which equilibrium can be attained within a reasonable time span. An attempt to ensure this led Ostroy and Starr (1974) to impose the following quantity constraint on M .

$$(Q) \quad \sum_{c \in C} p_c [z_{kc}]^+ \leq p_M w_{kM}, \quad \forall k$$

where,

$$[z_{kc}]^+ = \max[z_{kc}, 0].$$

Thus, $[z_{kc}]^+$ represents the level of good c that agent k wishes to *purchase*. Therefore, the left hand side of inequality (Q) shows the total value of the agent k 's desired purchases of goods other than M . On the other hand, the

right hand side of (Q) expresses the value of the initial endowment of good M for agent k . Hence, condition (Q) guarantees the fact that an agent possesses enough money in his initial endowment to cover his desired purchases of all the remaining goods. It has been shown that if condition (Q) is satisfied, then using good M as a medium of exchange, the equilibrium state can be attained within *one round*, i.e. with each agent confronting every other agent once and only once.

The logic behind this result is straightforward. Since every agent has enough money to satisfy his excess demand for all other commodities except good M , the excess demand (respectively the excess supply) of all $c (\neq M)$ will be satisfied (respectively disposed of) within one round. Thus by Walras' law (or merely from the budget constraints of the agents), excess demand and supply of M will also be satisfied simultaneously.

Though condition (Q) ensures speedy attainment of equilibrium, it imposes a strong restriction on the initial endowment of the money good. Since (Q) ensures enough money in the initial endowment of an agent to cover all his desired purchases, his money receipts on account of the sale of different goods can, strictly speaking, be viewed as money acquired to satisfy his final demand for M .

2.3 CERTAIN FEASIBILITY CONDITIONS

In the light of the discussion of Section 2.2 it is of interest to examine how far one can relax condition (Q).

One possible way of weakening (Q) is the following. Assume that each agent can cover his desired purchases not by one specific good but by *two goods taken together* (say M and M'). More precisely, let the following condition hold.

$$(Q') \quad \sum_{\substack{c \in C \\ c \neq M'}} p_c [z_{kc}]^+ \leq p_M w_{kM} + p_{M'} w_{kM'}, \quad \forall k.$$

The natural question that arises now is how far can the agents succeed in their attempt to attain equilibrium by using goods M as well as M' simultaneously as media of exchange. In order to examine this issue, it is first necessary to provide a trading scheme.

Suppose in period t agent i meets agent j (i.e. $\Pi^t(i) = j$). Trades that take place between them follow *trading rule* μ' as described below. As in the case of trading rule μ , when agents i and j meet, they examine their excess demands and supplies and get involved in trade that diminish one's excess supply and reduce the other's excess demand as far as possible. As in the case of trading rule μ (Section 2.2), this part of trade is captured by $\rho_{\sim i}^t$ and $\rho_{\sim j}^t$ for agents i and j respectively. Finally, to maintain quid-pro-quo the debtor can settle accounts by sending to the creditor either good M or M' or a combination of both (captured by vector ξ_i^t). Thus,

$$\begin{aligned} b_{\sim k}^t &= (b_{k1}^t, \dots, b_{kc}^t, \dots, b_{kM}^t) \\ &= \rho_{\sim k}^t + \xi_{\sim k}^t, \quad k = i, j \end{aligned}$$

where,

$$\rho_k^t = (\rho_{k1}^t, \dots, \rho_{kc}^t, \dots, \rho_{kM}^t), k = i, j,$$

are such that

$$\rho_{ic}^t = -\rho_{jc}^t = \begin{cases} 0, & \text{if } V_{ic}^t \cdot V_{jc}^t \geq 0, \\ \frac{1}{p_c} \min(|V_{ic}^t|, |V_{jc}^t|), & \text{if } V_{ic}^t > 0, \text{ and } V_{jc}^t < 0 \\ -\frac{1}{p_c} \min(|V_{ic}^t|, |V_{jc}^t|), & \text{if } V_{ic}^t < 0, V_{jc}^t > 0. \end{cases}$$

From the definition of $\xi_k^t = (\xi_{k1}^t, \dots, \xi_{kc}^t, \dots, \xi_{kM'}^t, \dots, \xi_{kM}^t)$ it is clear that it has at most two non-zero components, viz. the M -th and M' -th. Thus,

$$\xi_{kc}^t = 0 \text{ if } c \neq M, \quad c \neq M', \quad k = i, j$$

Clearly, ξ_k^t is constructed to maintain price consistency that is,

$$p \cdot (\rho_i^t + \xi_i^t) = 0 = p \cdot (\rho_j^t + \xi_j^t).$$

Thus it is clear that the trading rule μ' satisfies (T.1) - (T.3) and property (D).

Suppose trade takes place following trading rule μ' . Then each agent can satisfy his excess demands for all the commodities, except perhaps for M and M' , in exactly one round. Therefore, at the beginning of the second round there are at most two commodities for which excess demands remain unsettled. Since there cannot arise any multilateral chain of trades for two goods, one additional round is sufficient to attain equilibrium. Thus we get the following result:

Proposition 2.1 Let $(\tilde{W}, \tilde{Z}, \tilde{p})$ satisfy (U.1) - (U.3) and suppose there exists two commodities M and M' s.t. condition (Q') is satisfied. Then there exists a trading rule (viz. μ') satisfying (T.1) - (T.3) and (D) which allows the attainment of equilibrium in within two rounds.

Proof : Obvious.

Can this result be pushed further? Proceeding one more step in this direction, let us now examine the consequence of condition (Q) being fulfilled by three goods taken together, say M, M' and M'' . More precisely assume that the following condition holds:

$$(Q'') \quad \sum_{\substack{c \neq M \\ \neq M' \\ \neq M''}} p_c [z_{kc}]^+ \leq p_M w_{kM} + p_{M'} w_{kM'} + p_{M''} w_{kM''}, \forall k$$

Now suppose, the agents use all three goods, M, M' and M'' as media of exchange following a trading rule similar to μ' . The problem to look into would be whether the agents are able to attain equilibrium within a finite number of rounds.

Traders can fulfill their excess demands for (as well as excess supplies of) all commodities other than M, M' and M'' in one round if they use all three goods M, M' and M'' simultaneously as media of exchange. Thus, at the beginning of the second round, in general there would exist unsatisfied demands for three goods, viz M, M' and M'' and hence multilateral chains of trade can very

well be present. Obviously therefore, one more round will not be enough to attain equilibrium. On the other hand, if all three goods are used as media of exchange, then equilibrium may not be attained in finite time. To see this, one has merely to appeal to Starr (1976):

Statement (of Theorem, Starr (1976)): Let $(\tilde{W}, \tilde{Z}, \tilde{p})$ satisfy (U.1) - (U.3). If all goods are simultaneously used as media of exchange then $\exists$ a trading rule, viz. γ of Starr (1976), satisfying (T.1) - (T.3) and (D) which ensures that $\forall k$ and $\forall c, V_{kc}^t \rightarrow 0$ as $t \rightarrow \infty$.

Remark 2.1: In this connection we have discussed in the Appendix an alternative (and possibly simpler) method of establishing Starr's result.

Note: Starr was unable to demonstrate convergence in finite time, nor was he able to construct an example where it takes infinitely long to attain equilibrium. However, his work raises the undesirable possibility that trade might continue indefinitely without achieving full execution. Therefore, as already noted, we have the following corollary to Proposition 2.1.

Corollary 2.1 Let $(\tilde{W}, \tilde{Z}, \tilde{p})$ satisfy (U.1) - (U.3) and suppose $\exists$ three commodities M, M' and M'' s.t. condition (Q'') is satisfied. Then an economy using all three goods, viz. M, M', M'' simultaneously as media of exchange may take infinite time to attain equilibrium.

In order to avoid the situation of Corollary 2.1, one may conceive of a trading rule where at the beginning of the second round the number of media of exchange is reduced by one. In other words, instead of all three goods M, M' and M'' continuing to act simultaneously as media of exchange, two of them are arbitrarily chosen to be used as media of exchange during the second round. In particular, suppose without loss of generality, M and M' are used as media of exchange from the second round onwards. The trading procedure mentioned

here can be captured through the rule μ'' .

Let t' be the index of a time period during the *first round* and t'' represent the index for time *for any round succeeding the first*. If

$\Pi'(i) = j$, then

$b_i^{t'} = \rho_i^{t'} + \zeta_i^{t'}$, where $\rho_i^{t'}$ is the same as ρ_i^t of trading rule μ' and $\zeta_i^{t'} = -\zeta_j^{t'}$ are n -dimensional vectors with at most the M -th, M' -th and M'' -th components non-zero. These vectors, viz. $\zeta_i^{t'}$ and $\zeta_j^{t'}$, are constructed so as to maintain the price consistency condition (i.e. (T.3) of Section 2.2) i.e.

$$p.(\rho_i^{t'} + \zeta_i^{t'}) = 0 = p.(\rho_j^{t'} + \zeta_j^{t'})$$

If however, agent i meets agent j at some period t'' during the second or the third round i.e.

$\Pi''(i) = j$, then

$$b_i^{t''} = \rho_i^{t''} + \zeta_i^{t''}, \text{ where}$$

$\rho_i^{t''}$ is exactly similar to $\rho_i^{t'}$ with all t 's replaced by t'' . The n -dimensional vectors $\zeta_i^{t''}$ and $\zeta_j^{t''}$ ($\zeta_i^{t''} = -\zeta_j^{t''}$) have at most the M -th and M' -th components being non-zero and are as before constructed to maintain quid-pro-quo i.e. $p.(\rho_i^{t''} + \zeta_i^{t''}) = 0 = p.(\rho_j^{t''} + \zeta_j^{t''})$.

Clearly, if trade takes place (under condition (Q'')) following trading rule μ'' then at the beginning of the second round we have an economy with unsatisfied demands and supplies for only three goods, viz. M, M', M'' . Therefore, an agent having excess demand for M'' has excess supply of either M or M' or a combination of both, of value at least equal to his excess demand for M'' . Since from the second round onwards, M and M' are used as media of exchange, agents having excess demand for M'' would have enough money to purchase it. Thus, at the end of the second round excess demand for M'' (and hence excess supply of the same) would be completely satisfied. Therefore (at most), one additional round would settle all desired transactions concerning M and M' .

Thus we have the following result:

Proposition 2.2: Let $(W, \tilde{z}, \tilde{p})$ satisfy (U.1) - (U.3) and suppose $\exists$ *three commodities* M, M' and M'' s.t. condition (Q'') is satisfied. Then $\exists$ a trading rule (viz. μ'') satisfying (T.1) - (T.3) and (D) using which the economy can attain equilibrium in *three rounds*.

Proof: Clear from the above discussion.

Remark 2.2: Clearly this result can be generalized to more than three goods acting as media of exchange. Thus for an economy with n goods, we always have

$$\begin{aligned} p_n x_{kn} &\leq p_1 w_{k1} + \dots + p_c w_{kc} + \dots + p_n w_{kn}, \forall k \\ \Rightarrow p_n z_{kn} &\leq p_1 w_{k1} + \dots + p_c w_{kc} + \dots + p_{n-1} w_{kn-1}, \forall k \\ (Q^{(n-1)}) \dots \therefore p_n [z_{kn}]^+ &\leq p_1 w_{k1} + \dots + p_c w_{kc} + \dots + p_{n-1} w_{kn-1}, \forall k \end{aligned}$$

Suppose now during the first round, goods 1 to $(n-1)$ are used as media of exchange. Then from $(Q^{(n-1)})$ it is clear that excess demand (and therefore excess supply as well) for good n will be satisfied during the initial round.

In the second round suppose goods $1, 2, \dots, (n-2)$, are used as media of exchange. Then for similar reasons all desired transactions regarding the $(n-1)$ -th good can be settled during that round and so on.

Proceeding in an exactly parallel fashion, after the end of the $(n-1)$ -th round the economy will attain equilibrium.

Though the process of trade described above does not require any condition to be imposed on the money good (or goods) at all, the trading rule involved here is subject to criticism. This implies that an agent accepting a good as a medium of exchange at a particular period of time might refuse to do so

at a later date. Secondly, and *more importantly*, an agent needs to be aware when a new round is beginning. It will be shown in the next section that an appropriate trading post set-up can remove this drawback.

Though trading rule μ' is free from all such problems, it requires condition (Q') to be satisfied. Clearly, the criticisms for the Ostroy-Starr condition (Q) hold for (Q') also.

These observations naturally raise a need to provide a plausible condition to be satisfied by a good (or a set of goods) which can consistently (all through) be used as a medium (media) of exchange in order to attain equilibrium in finite time.

The next section takes up all these issues for an economy characterized by the social institution of markets.

2.4 CONDITIONS FOR FEASIBILITY UNDER A TRADING POST SET-UP

In reality one observes that the monetary pattern of trade and specialization of markets go hand in hand. As an explanation of this fact it is shown in the next chapter that the social institution of money works better (in terms of transactions costs) when coupled with the social institution of markets. As a consequence, the latter is potentially replaceable by a trading post set-up. This observation provides the motivation to pose the problem under the set-up of monetized trading posts. We begin by describing the trading scheme under consideration.

2.4.1 The Trading Post Set-Up

Under the set-up of monetized trading posts it is assumed that there exists separate markets or trading posts to deal with each good against the money good(s). Suppose good M is the unique medium of exchange. In this case all the traders who wish to trade a particular good $c (\neq M)$ are assumed to

gather in the market for that commodity (where good c is exchanged against M), following a pre-specified sequence of markets. This sequence however, is arbitrarily chosen from the set of all possible sequencing of trading posts. We have assumed, following reality that as soon as a trader enters the post he *can distinguish between buyers and sellers*, which is seen as an informational gain associated with the trading post set-up. Once a trader enters a trading post, he can exchange with as many agents as he wishes. Here buyers and sellers are assumed to be separately lined up in any arbitrary order. The first buyer then confronts the first seller and so on. The traders who are not satisfied in their first meeting and are willing to trade again in the same trading post line up afresh in any random order as before. Once all opportunities for trade in this trading post are over, traders move to the next one following the pre-determined sequence mentioned above.

Time is partitioned into *days* where each day is defined to consist of a number of *periods* to be indexed by t . Each day is long enough to finish all desired trade in a particular trading post. Once a trader finishes his trade in a specific trading post on a particular day, he waits till the beginning of the next day, when he visits the next trading post. It is assumed that an agent is concerned only with the *time spent in trading* (i.e. the number of trading periods) in the relevant markets.

It is assumed throughout the sequel that it takes exactly one period of time for a pair of agents to carry out an exchange in any particular trading post. A *cycle* (indexed r) of trades is said to be completed, when each trader finishes visiting all the trading posts relevant for him once and only once. As already pointed out, however, a trader may be interacting with several traders in a trading post during each such visit. It may in general take more than one cycle to attain the equilibrium allocation.

Consider the meeting between two agents, say $k = i$ and j , at the beginning

of period t in trading post (c, M) , i.e. where good c is exchanged against M . Clearly, when i is a seller j is a buyer and vice-versa. The trade that takes place between them is summarized by the following vectors.

$$\begin{aligned} a_{ic}^t &= (0, \dots, 0, \alpha_{ic}^t, 0, \dots, \beta_{iM}^t) \\ a_{jc}^t &= (0, \dots, 0, \alpha_{jc}^t, 0, \dots, \beta_{jM}^t) \end{aligned}$$

where,

$$\begin{aligned} \alpha_{kc}^t &\in \mathbb{R}, \beta_{kM}^t \in \mathbb{R}, k = i, j \\ \alpha_{ic}^t &= -\alpha_{jc}^t, \quad p_M \beta_{kM}^t = -p_c \alpha_{kc}^t, \quad k = i, j \\ w_{kc} + \sum_t \alpha_{kc}^t &\geq 0 \quad \text{and} \quad w_{kM} + \sum_t \beta_{kM}^t \geq 0, \quad k = i, j \end{aligned}$$

That is, in trading post (c, M) , i and j exchange good c , so as to reduce their respective excess demands and supplies to the largest extent possible. The buyer then settles his account by transferring the required value of good M to the seller.

This provides a general idea regarding the trading arrangement of a monetized trading post set-up.

2.4.2 Feasibility Conditions

Here we look into the conditions of feasibility (as developed in the previous section) for a trading post set-up.

(i) Condition (Q) :

Condition (Q) ensures availability of enough money for every agent of the economy so that each trader can cover all his desired purchases.

Since all the potential demanders and suppliers of good c gather in trading post (c, M) at the same time and each trader has enough money with him, all his excess demands would be satisfied, after he finishes trading in all the

trading posts. As a result, once every agent visits all the relevant trading posts, excess demands for all the $(n - 1)$ goods (and hence excess supplies of them) would be completely satisfied after the end of one cycle. Therefore, Ostroy and Starr's (1974) result would remain valid even in a trading post set-up when a *round* is replaced by a *cycle*

(ii) Condition (Q') :

We next consider Proposition 2.1 for a trading post set-up, where there exists $((n - 1) + (n - 2))$ trading posts to deal with all $(n - 1)$ goods against M and then all $(n - 2)$ goods (excluding M and M') against M' . Thus, there are separate trading posts to use M and M' as media of exchange. Using a similar trading rule where, M and M' are used as media of exchange in the respective trading posts, it is easy to see that after the first cycle (i.e. when the traders finish visiting all the $(2n - 3)$ trading posts once), only the excess demands for M and M' would remain to be satisfied. Thus, during the second cycle, only the (M, M') trading post would be operating and hence equilibrium will be attained after everybody visits the (M, M') trading post for the second time.

(iii) Condition $(Q^{(n-1)})$:

In this context we consider a trading post set-up of a slightly different kind. In an n -good economy, suppose there are n trading posts to deal with each particular commodity, where commodity c is traded against *every other good* in the good c trading post. Following an arbitrary sequence of trading posts, say (WLOG) $\{ \text{good } 1, \text{good } 2, \dots, \text{good } n \}$, all agents wishing to demand or supply good c gather in that market. When all desired trades in that trading post are over, agents move to the $(c + 1)$ -th trading post etc. Trade takes place in the good c trading post following the *trading rule* $\bar{\mu}$ defined below.

Suppose agent i meets agent j in the trading post for good c at period t . Trade that takes place between them can be summarized by the following

vectors:

$$a_{ic}^t = \rho_{\sim i}^t + \lambda_{\sim i}^t = (0, \dots, 0, \rho_{ic}^t, 0, \dots, 0) + \lambda_i^t$$

$$a_{jc}^t = \rho_{\sim j}^t + \lambda_{\sim j}^t = (0, \dots, 0, \rho_{jc}^t, 0, \dots, 0) + \lambda_j^t$$

Where,

$$\rho_{ic}^t = -\rho_{jc}^t = \begin{cases} 0, & \text{if } V_{ic}^t \cdot V_{jc}^t \geq 0 \\ \frac{1}{p_c} \min(|V_{ic}^t|, |V_{jc}^t|), & \text{if } V_{ic}^t > 0, \\ & V_{jc}^t < 0 \\ -\frac{1}{p_c} \min(|V_{ic}^t|, |V_{jc}^t|), & \text{if } V_{ic}^t < 0, \\ & V_{jc}^t > 0 \end{cases}$$

As before, ρ_{ic}^t and ρ_{jc}^t refer to the trades regarding good c that are excess demand diminishing.

Furthermore, vectors λ_i^t and λ_j^t are constructed so as to maintain price consistency, which is assumed to be settled by the debtor transferring a combination of goods of which he has excess supply, to the creditor. This part of the trade can be represented as:

$$\lambda_i^t = -\lambda_j^t \text{ are s.t.}$$

$$\begin{aligned} [V_i^t - \rho_i^t]^- < \lambda_i^t < 0 \quad \text{and} \quad p \cdot (\rho_i^t + \lambda_i^t) &= 0 \\ \text{if } p \cdot \rho_i^t > 0 \\ [V_j^t - \rho_j^t]^- < \lambda_j^t < 0 \quad \text{and} \quad p \cdot (\rho_j^t + \lambda_j^t) &= 0 \\ \text{if } p \cdot \rho_j^t > 0 \end{aligned}$$

Where,

$$\begin{aligned} [V_i^t - \rho_i^t]^- &= [V_{i1}^t - \rho_{i1}^t, \dots, V_{ic}^t - \rho_{ic}^t, \dots]^- \\ &= [\min(0, (V_{i1}^t - \rho_{i1}^t)), \dots, \min(0, (V_{ic}^t - \rho_{ic}^t)), \dots] \end{aligned}$$

etc.

If $p \cdot \rho_i^t < 0$, then $\lambda_i^t = -\lambda_j^t$, where $p \cdot \rho_j^t > 0$ and λ_j^t is constructed as above.

Thus, when the good 1 market is operating agents having excess demand for good 1 purchase the required amount from the suppliers of good 1. The buyers can in return pay through any of the commodities of which he has excess supply. Thus after all desired trades are over, excess demand for good 1 (and hence excess supply as well) is completely satisfied. The agents next move to the good 2 market. Here all the demanders of good 2 have excess supply of (some other good(s)) of at least equal value to cover his demand. We note that no agent has excess supply of good 1. Therefore, the demanders of good 2 transfer any combination of good 3, ..., good n , to the supplier for maintaining price consistency. Therefore when all possible trades are over in this market, excess demand for good 2 (and hence excess supply of the same) is met. Traders then move to good 3 market and so on.

Thus when the agents visit the market for good $(n - 1)$, excess supply of good 1, ..., $(n - 2)$ are completely satisfied. Hence the traders visiting the good $(n - 1)$ market can have excess supply of only good n . Once all possible trades in this market are over, excess demands for and supplies of all n goods are satisfied.

We note that the trading rule $\bar{\mu}$ is similar to the trading rule introduced by Starr (1976) for a market-less trading arrangement with *non-monetary trades* i.e. where each good can be exchanged against every other. The latter rule says that if two agents i and j meet they scan their excess demands and supplies and exchange takes place so as to reduce one's excess demand and the other's excess supply as far as possible. Finally, the debtor settles accounts by transferring to the creditor any combination of goods of which he has excess supply. This rule in a marketless trading arrangement does not ensure that equilibrium would be attained necessarily within finite time. However, for this specially defined trading post arrangement things operate smoothly.

But it is rare to find an economy with all goods acting simultaneously as

media of exchange. We therefore look for a plausible condition to be satisfied by a medium of exchange such that equilibrium can be attained within finite time.

2.4.3 An Alternative Criterion

We now consider a situation where the money good is not so abundantly available, as in the case of condition (Q) or (Q') . Let M_{kc}^t represent the money allocated by trader k out of his total money holding for the purchase of good c in the beginning of some period t . Clearly,

$$M_{kc}^t = 0, \quad \text{if } V_{kc}^t \leq 0.$$

We note that both V_{kc}^t and M_{kc}^t change with the progress of trader k 's trades involving c .

When agent i meets j , the trade that takes place between them can be summarized by the following vectors.

$$\begin{aligned} a_{ic}^t &= (0, \dots, 0, \alpha_{ic}^t, 0, \dots, \beta_{iM}^t) \\ a_{jc}^t &= (0, \dots, 0, \alpha_{jc}^t, 0, \dots, \beta_{jM}^t) \end{aligned}$$

Where,

$$\begin{aligned} \alpha_{kc}^t &\in \mathbb{R}, \beta_{kM}^t \in \mathbb{R}, k = i, j \\ \alpha_{ic}^t &= -\alpha_{jc}^t, p_M \beta_{kM}^t = -p_c \alpha_{kc}^t, k = i, j. \\ w_{kc} + \sum_t \alpha_{ic}^t &\geq 0 \quad \text{and} \quad w_{kM} + \sum_t \beta_{kM}^t \geq 0 \end{aligned}$$

Trades between i and j are carried out according to the following rule to be referred to as Φ :

$$\alpha_{ic}^t = -\alpha_{jc}^t = \begin{cases} 1/p_c \min(V_{ic}^t, |V_{jc}^t|, M_{ic}^t), & \text{if } V_{ic}^t > 0 \\ -1/p_c \min(V_{ic}^t, |V_{jc}^t|, M_{jc}^t), & \text{if } V_{ic}^t < 0 \end{cases}$$

Trading rule Φ says that a seller comes to a market with his unsold supply and a buyer with the money that he allocates for this trading post. His money holding at the beginning of a period t is denoted by M_{kh}^t which breaks up partly into M_{kc}^t , the amount set aside for carrying out transactions involving c . The rest of it, denoted by M_{kf}^t , is held to satisfy the final demand for money. In this context it is assumed that whenever a trader acquires money, his first priority is to satisfy his needs for the non-money goods. This is stated more formally in Assumption 2.1 and 2.2.

Assumption 2.1: At the beginning of a period t

$$M_{kh}^t = \sum_{c \in C} M_{kc}^t + M_{kf}^t$$

where

$$M_{kf}^t = \begin{cases} 0 & \text{if } \sum_{c \in C} [V_{kc}^t]^+ \geq M_{kh}^t \\ M_{kh}^t - \sum_{c \in C} [V_{kc}^t]^+, & \text{if } \sum_{c \in C} [V_{kc}^t]^+ < M_{kh}^t. \end{cases}$$

$$[V_{kc}^t]^+ = \max[0, V_{kc}^t]$$

Regarding M_{kc}^t , it is clear that it would depend on M_{kh}^t and $[V_{kc}^t]^+$. We will assume that an agent is ready to allocate (if necessary) his entire money holding to a good c just before visiting the trading post relevant for that good. Hence we have,

Assumption 2.2: For every $c \in C$

$$M_{kc}^t = \min\{[V_{kc}^t]^+, M_{kh}^t\}$$

It can be easily checked that the trading rule Φ described above satisfies (T.1) - (T.3) and (D). Given the trading rule we now search for a plausible condition under which equilibrium can be attained within finite time. Clearly, as soon as one removes condition (Q) or (Q'), one has to guarantee the availability

of the money good(s) in the hands of the traders till their demands are satisfied. In order to ensure this, the possibility that a group of individuals might leave the system with the entire stock of the money good (to satisfy their final demands for it) prior to the attainment of the full Walrasian equilibrium must be guarded against. Such considerations motivate the following result:

Proposition 2.3: Let $(\tilde{W}, \tilde{Z}, \tilde{p})$ satisfy (U.1) - (U.3) and suppose $\exists$ a commodity M s.t. the final demand for M is strictly positive for all traders, i.e.

$$(E) \quad x_{kM} > 0, \forall k;$$

then $\exists$ a trading rule (viz. Φ) for a monetized trading post set-up that satisfies (T.1) - (T.3) and (D), which leads the economy to equilibrium in finite time.

Note: The proof, given in the Appendix (to Chapter 2) will be established for the trading rule Φ . It has already been mentioned that rule Φ satisfies (T.1) - (T.3) and (D). Thus the proof would be geared towards the finiteness property alone.

Thus, we have two major conditions, viz. (Q) and (E) under which equilibrium can be attained in finite time. It is of obvious interest to examine whether it is possible to relax the extremes represented by each condition in isolation. Such an attempt motivates the following result which represents a combination of the two conditions.

Proposition 2.4: If the economy consists of two groups of individuals, viz. K_1 and K_2 , $K = K_1 \cup K_2$, where all $k \in K_1$, satisfy (Q) and all $k' \in K_2$ satisfy (E), then there exists a trading rule, viz Φ , which satisfies (T.1) - (T.3), (D) and which allows the attainment of equilibrium in finite time.

Proof: See Appendix to Chapter 2.

Remark 2.3 The two sets of agents (K_1 and K_2) defined in Proposition 2.4 can be given an interesting interpretation. Consider an overlapping generations model with two groups of individuals, viz. old and young. An agent is assumed to live for two units of time to be referred to henceforth as years (where each year consists of a finite number of “transaction” days). In his youth, i.e. in the first year of his life, an agent is endowed with consumable goods only. These goods, however, are perishable, i.e. they are not going to last till the second year when he is old. There is, therefore, a strong inducement - indeed a compulsion - for the agents to acquire a store of value during the first period of their lives to be used it when they are old.

Given this perspective, the set K_1 can be identified with the set of old people satisfying condition (Q), K_2 as the set of young people demanding money as a store of value and good M as the unique store of value in the system.

What proposition 2.4 then shows is the following. The ability of money to act as a store of value between this period and the next, enables money to act as an *efficient* medium of exchange in the current period. The word “efficient” essentially implies that both young and old people can attain their desired bundles of commodities in finite number of periods, through the aid of a medium of exchange.

2.5 CONCLUSION

This Chapter develops criteria indicating the circumstances under which it is feasible to use a good as a medium of exchange and attain one’s most desirable allocation through decentralized trade. The problem is represented *against* the background of the social institution of markets. It is revealed from this discussion that if there is a unique store of value in the system which everybody desires, then that commodity can be used as a medium of exchange to attain equilibrium in finite time.

The next chapter tries to justify our use of trading posts by showing their relative efficiency over the marketless trading arrangement.

APPENDIX TO CHAPTER 2:

An Alternative Proof of Starr's Result:

According to trading rule γ (Starr (1976)), when two traders meet they first trade exactly as described by ρ_i^t of trading rule μ' . Quid pro quo is then maintained by the debtor sending *any combination of goods for which he has excess supply* to the creditor (see trading rule $\bar{\mu}$).

As is clear (also shown in Starr (1976)), the trading rule γ satisfies (T.1) - (T.3) and (D).

Suppose equilibrium is not attained even in infinite time. Then there exists a $\bar{k} \in K$ and a $\bar{c} \in C$ such that, for infinitely many θ , $V_{\bar{k}\bar{c}}^\theta < \varepsilon < 0$, where θ stands for the index of a round.

If at the beginning of any round, say θ_0 ,

$$V_{\bar{k}\bar{c}}^{\theta_0} < \varepsilon < 0 \quad (A.2a)$$

then in the θ_0 th round, agent $\bar{k}$'s actual sale of good $\bar{c}$ to be denoted by $s_{\bar{k}\bar{c}}^{\theta_0}$, is such that

$$s_{\bar{k}\bar{c}}^{\theta_0} \geq |V_{\bar{k}\bar{c}}^{\theta_0}| \quad (A.2b)$$

This is so because given (A.2a) there exists (without loss of generality) one trader, k , for whom $V_{k\bar{c}}^{\theta_0} > 0$ and k can use any good for which he has excess supply, as a medium of exchange. A strict inequality might hold in (A.2b) if *within* round θ_0 agent $\bar{k}$ receives a positive amount of good $\bar{c}$ through its being used as a medium of exchange by others.

Trader $\bar{k}$ can pass on $\bar{c}$ in two ways. First, by using it as a medium of exchange, in which case $\bar{k}$ would purchase some other goods for which he has unsatisfied excess demand. Secondly, to satisfy the excess demand of some

other agent. In both these cases excess demand of level $s_{k\varepsilon}^\theta (> 0)$ is satisfied for at least one trader.

If $V_{k\varepsilon}^\theta < \varepsilon < 0$, for infinitely many θ then, from (A.2b)

$$\sum_{\theta} s_{k\varepsilon}^\theta \text{ goes to infinity} \quad (\text{A.2c})$$

But the aggregate value of positive excess demands in the system, i.e. $\sum_{k \in K} \sum_{c=1}^n p_c [z_{kc}]^+$ is finite, which contradicts (A.2c). ■

Proof of Proposition 2.3:

To prove this result, three lemmas are necessary.

Before stating the lemmas, let us first introduce two relevant notations. Let V_{kc}^r represent trader k 's unsatisfied demand for (or supply of) good c at the beginning of cycle r and M_{kR}^r represent agent k 's cumulative gross (of outgoing on account of purchases) money receipt in cycle r .

Lemma 2.1: If there exists a pair $k(\in K)$ and a unique good $c(\in C)$ for which

$$V_{kc}^r > 0 \quad (\text{A.2d})$$

for infinitely many r , then every nonzero subsequence of $\{M_{kR}^r\}$ must converge to zero².

Note: Such a non zero subsequence must exist.

Proof: Suppose the result is false. This implies that there exists a subsequence $\{M_{kR}^{r_\ell}\}$ bounded away from 0. Suppose $M_{kR}^{r_\ell} \geq \alpha > 0$

Let $H = \{r_\ell | \ell = 1, 2, \dots\}$. Thus,

$$\sum_{r_\ell \in H} M_{kR}^{r_\ell} \geq \sum_{r_\ell \in H} \alpha \quad (\text{A.2e})$$

²The lemma extends easily to the case where (A.2d) holds for more than one c .

which goes to infinity.

On the other hand, the rule Φ implies that the money received by k in cycle r , i.e. M_{kR}^r , would be spent completely on c . (This follows from the fact that $V_{kc'}^r \leq 0$ from some r onwards, $\forall c' \neq c$). Denoting this cycle by r_0 , it follows from definition that:

$$V_{kc}^r - M_{kR}^r \geq V_{kc}^{r+2}, \quad r > r_0 \quad (\text{A.2f})$$

In other words, V_{kc}^r diminishes *at least* by M_{kR}^r at the end of the $(r+1)$ th cycle.

But V_{kc}^1 is a finite number, which contradicts (A.2d) and (A.2e). $\blacksquare$

Lemma 2.2: Proposition 2.3 holds if we allow infinite time to attain equilibrium.

Proof: Suppose the equilibrium cannot be reached. Then there exists a pair $k(\in K)$ and $c(\in C)$ such that for infinitely many r

$$V_{kc}^r > 0 \quad (\text{A.2g})$$

Hence Lemma 2.1 implies that all non-zero subsequences $\{M_{kR}^{r_\ell}\}$ of $\{M_{kR}^r\}$ converge to 0. In other words, given any $\varepsilon > 0$,

$$M_{kR}^r \in [0, \varepsilon) \quad (\text{A.2h})$$

for infinitely many r .

Let the set of all r satisfying (A.2g) for the chosen ε be H' . Suppose now that the final demand for money by the k th trader is $x_{kM} = m > 0$. It follows from (A.2g) and (A.2h) that $V_{kM}^r = m$ for infinitely many $r \in H'$. Since the economy is in equilibrium, excess supply of money in the economy is at least m for infinitely many $r \in H'$. This excess supply of money will be spent by the suppliers in the respective markets to fulfil their unsatisfied demands. Thus,

$$\text{the value of aggregate purchase} \geq \sum_{r \in H'} m \quad (\text{A.2i})$$

The right hand side of this inequality goes to infinity. On the other hand, the left hand side of (A.2i) equals $\sum_j \sum_c p_c [z_{jc}]^+$, a finite number.

This yields a contradiction. ■

Lemma 2.3: If there is a trader $\bar{k}$ for whom $V_{\bar{k}c}^r > 0$ for infinitely many r , then there exists a self contained set of traders $\bar{T} (\bar{T} \subseteq K)$ and a set of goods $\bar{C} (\bar{C} \subseteq C)$, such that for each $k \in \bar{T}$, there exists a $c \in \bar{C}$ for which $V_{kc}^r > 0$, for infinitely many r .

Note: By a *self contained set of traders* $\bar{T}$ we mean a set of traders such that after a finite number of cycles, there is no fresh injection (though there may be leakages) of money into (out of) this set from (to) members outside $\bar{T}$.

Proof: Consider the set of goods such that $V_{\bar{k}c}^r > 0$ for infinitely many r . Call that set C_1 . Clearly C_1 is non-null.

Now consider the set of all traders T_1 such that for all $k \in T_1$, there exists $c \in C_1$ for which the following inequality holds for infinitely many r :

$$V_{kc}^r > 0 \tag{A.2j}$$

i.e. T_1 is the set of traders, each of whose demands for some commodities belonging to C_1 are not satisfied in finite time. Then Lemma 2.1 implies, that there exists a subsequence $\{M_{kR}^{r\ell}\}$ which goes to 0.

Thus, for each $k \in T$, there exists at least one commodity, say c' , such that k 's actual sale of c' over infinitely many cycles r satisfies the following inequality:

$$s_{kc'}^r > 0 \tag{A.2k}$$

Let C_2 be the set of all such commodities c' . We note that $C_1 \cap C_2$ need not be null. Now Lemma 2.1 implies that there exists a monotonically decreasing subsequence $\{s_{kc'}^{r\ell}\}$ for all $k \in T_1$ and $c' \in C_2$ for some c' , i.e. there exists a

second set of traders T_2 such that, for all $k \in T_2$ and some $c' \in C_2$, (A.2j) holds.

Therefore, in an analogous manner, we can construct a third set of commodities C_3 such that, for all $k \in T_2$, there exists $c'' \in C_3$ for which (A.2k) holds and so on.

We define a sequence of sets $\{K_h\}$, where $K_1 = T_1$; $K_2 = T_1 \cup T_2$; $K_3 = T_1 \cup T_2 \cup T_3$ etc. As the set of all traders is finite, there must exist a finite L such that:

$$K_L = K_{L+1} = K_{L+2} = \dots$$

We will show that K_L is a self-contained set, i.e. $K_L = \bar{T}$.

Consider an arbitrary trader $k \in K_L$. Suppose, in particular, $k \in T_h$ (in fact k can belong to more than one T_h). Trader k can acquire money only through his supply of some commodity (or commodities). From our construction procedure it is clear that all the commodities that k supplies for infinitely many r are included in C_{h+1} . All the demanders for each good $c \in C_{h+1}$ are included in T_{h+1} and $T_{h+1} \subseteq K_L$. Therefore, K_L is a self-contained set. ■

Proof of Proposition 2.3:

Suppose the result is false.

Lemma 2.3 has established that for each $k \in K_L$, there exists some commodity c such that, for infinitely many r , the following inequality holds.

$$V_{kc}^r > 0 \tag{A.2l}$$

Thus, from Lemma 2.1, for each $k \in K_L$, there exists a monotonically decreasing subsequence $\{M_{kR}^r\}$ such that ,

$$\{M_{kR}^r\} \rightarrow 0 \tag{A.2m}$$

Given (A.2l), trading rule Φ implies that for all $k \in K_L$, M_{kR}^r would be completely spent on the purchase of some c (satisfying (A.2l)) for infinitely many

r .

This fact together with (A.2m) implies that the acquisition of money (to meet the final demand for it) by $k \in K_L$ is not positive for infinitely many r . But for all $k \in K_L$, final demand for M , i.e. $x_{kM} > 0$ and $\{M_{kR}^r\}$ is bounded away from x_{kM} for all but finitely many r .

As K_L is a self contained set of traders, for all $k \in K_L$, the final demand for money is not going to be satisfied even when r goes to infinity.

This contradicts Lemma 2.2. ■

Proof of Proposition 2.4:

All the agents $k \in K_1$ satisfy condition (Q). Hence, they can satisfy their purchases of all commodities $c \in C$ within one cycle. By Proposition 2.3 all $k' \in K_2$ can do so in finite time. Hence, equilibrium can be attained within finite time. ■

List of symbols used in Chapter 2:

- (1) K : the set of all agents as also cardinality of the set.
- (2) k : *index for agent*
- (3) n : total number of commodities
- (4) M : n -th commodity earmarked to play the role of a medium of exchange
- (5) C : the set of all commodities excluding the n -th good.
- (6) c : index for a commodity
- (7) W : matrix of initial endowments
- (8) w_{kc} : level of good c in the initial endowment of agent k .
- (9) U_k : utility function for agent k
- (10) $\mathbb{R}_+^n$: the set of n -dimensional positive real vectors
- (11) $\mathbb{R}$: the set of real numbers.
- (12) p' : row vector of prices for n goods
- (13) p : transpose of p'
- (14) p_c : price of good c
- (15) z_{kc} : agent k 's excess demand for (or excess supply of) good c .
- (16) Z : matrix of excess demands and supplies
- (17) Z_k : k -th row vector of matrix Z
- (18) X : matrix of final demands and supplies of all goods for the agents.

- (19) x_{kc} : level of good c finally consumed by agent k
- (20) μ, μ', μ'' : trading rules in a marketless set-up
- (21) t : index for time
- (22) $\Pi^t(k)$: function indicating trading partner of agent k in period t .
- (23) V_{kc}^t : the value of trader k 's unsatisfied excess demand for (or, supply of) good c , at the beginning of time period t .
- (24) V_k^t : $V_k^t = (V_{k1}^t, \dots, V_{kc}^t, \dots, V_{kn}^t)$
- (25) V_{kc}^θ : agent k 's hitherto unsatisfied demand for or supply of good c at the beginning of round θ (in value terms)
- (26) V_{kc}^r : Same as (25) when "round θ " is replaced by "cycle r ".
- (27) b_k^t : n -dimensional vector showing the (levels of) inflow and outflow of commodities for agent k in the course of his trade during period t
- (28) ρ_k^t : n -dimensional vector showing the (levels of) inflow and outflow of commodities for agent k in the course of his trade during period t that are excess demand diminishing.
- (29) y_k^t : n -dimensional vector showing the (levels of) inflow and outflow of commodities for agent k in the course of his trade during period t , necessary for maintaining quid-pro-quo, M being the unique medium of exchange.
- (30) W_k^t : n -dimensional vector showing the level of each good in agent k 's possession at the beginning of period t .
- (31) W_k : n -dimensional vector showing the level of each good in the initial endowment of agent k .

- (32) $[z_{kc}]^+$: $\max [o, z_{kc}]$
- (33) M', M'' : alternative media of exchange
- (34) ξ_k^t : same as (29) when two goods M and M' are simultaneously acting as media of exchange.
- (35) ζ_k^t : same as (29) when 3 goods M, M' and M'' are simultaneously acting as media of exchange
- (36) r : index for cycle
- (37) (c, M) : notation for a trading post or equivalently a market where good c is exchanged against M
- (38) a_{kc}^t : n -dimensional vector showing the(levels of) inflow and outflow of commodities for agent k in the course of his trade in the (c, M) trading post in period t .
- (39) α_{kc}^t : level of good c purchased or sold by agent k in period t in trading post (c, M) .
- (40) β_{kM}^t : level of good M spent or received by agent k in period t for maintaining price consistency.
- (41) $\bar{\mu}, \bar{\mu}, \Phi$: trading rules for a trading post set-up
- (42) M_{kh}^t : money holding by agent k at the beginning of period t .
- (43) M_{kc}^t : money allocated by agent k , out of M_{kh}^t for the purchase of good c in the beginning of period t .
- (44) M_{kf}^t : money held by k in period t as part of his final demand for M
- (45) K_1, K_2 : subsets of K

- (46) θ : index for a round
- (47) s_{kc}^θ : agent k 's actual *sale* of good c during θ -th round (in value terms)
- (48) M_{kR}^r : agent k 's cumulative money receipt in cycle r gross of outgoing on account of purchases.

TRADING POST SET-UP AND MARKETLESS TRADING ARRANGEMENT : A COMPARISON

ABSTRACT

This chapter endeavours to compare the transactions costs associated with a trading post set-up vis-a-vis a marketless trading arrangement. It establishes that for a large economy, the time cost associated with the attainment of the equilibrium is expected to be higher in a marketless trading arrangement as compared to the trading post set-up. It also shows that for the economy considered by Kiyotaki and Wright (1989), the welfare associated with a steady state Nash equilibrium state would be higher under the social institution of markets.

3.1 INTRODUCTION

The previous chapter addresses the question of feasibility under the set-up of monetized trading posts. The next question that emerges is whether or not such a trading arrangement is more efficient than the marketless set-up. One might expect an affirmative answer here, for in the marketless trading arrangement generally considered in the literature (See Ostroy and Starr (1974), Jones (1976), Kiyotaki and Wright (1989)), an agent arbitrarily encounters another agent (without any specific reference to the goods they want to transact) to explore the trading possibilities prevailing between them. However, in a trading post set-up, agents wishing to exchange a particular pair of goods gather in the trading posts relevant for those goods. Therefore, when trade is performed through trading posts relating to pairs of goods, it appears that many

unnecessary meetings can be dispensed with. As a consequence, from the point of view of time as transactions cost, a trading post set-up seems to be more efficient. However, a pair of agents can exchange as many goods as they want to in one meeting in a marketless trading arrangement whereas in a trading post set-up they are restricted to trade only a pair of goods. Thus, there is a need to formally verify our intuition.

Thus, we first consider the trading post set-up (described in Section 2.4) where an agent visits different trading posts following an ad hoc sequence of markets. We intend to compare this set-up with the marketless trading arrangement of Ostroy and Starr (1974) where again some arbitrary sequence of meetings is assumed to be followed by the traders. It will be established in this connection that the time taken to realize equilibrium allocations to any degree of approximation under a trading post set-up is bounded above by a quantity which is independent of the total number of agents in the economy. On the other hand, the time taken in the alternative set-up will not be so bounded for all possible sequences of meetings. This shows that the gains from a monetary pattern of trade is likely to be minimal when unsupported by the social institution of markets.

The above trading arrangement being restrictive in nature, we have considered as a next step a more general trading post set-up where all markets are functioning simultaneously and an agent chooses a trading post at random from the set of trading posts he desires to visit. Thereby, a trader need not follow any predetermined sequence of markets. As before, the superiority of the trading post set-up is established for large economies under this more general trading arrangement (as compared to the marketless trading scheme).

It was pointed out in the introductory chapter that the framework described by Ostroy and Starr (1974) introduces "time cost" in a rather ad hoc fashion. Therefore, it may be interesting to raise the question of efficiency for a set-up

where the transactions cost is incorporated more explicitly in the agent's objective functions. The framework appropriate for the purpose is the one chosen by Kiyotaki and Wright (1989). Kiyotaki and Wright have established different possible steady state Nash equilibria for a marketless trading arrangement, i.e. where an agent trades randomly with another agent (this model is discussed in Section 3.4). They show, however, that a set of non-equilibrium strategies directing each pair of agents to "*exchange goods whenever they meet*", provide more welfare as compared to any other equilibrium strategies. As opposed to this it will be shown in the sequel that there exists a set of equilibrium (and thus implementable) strategies *under the social institution of markets* which provides even more welfare than the non-implementable strategies described by Kiyotaki and Wright. This fact further establishes the power of the trading post set-up as compared to the marketless system.

The next section compares the transactions cost in the Ostroy and Starr set-up with that in the trading post system. Section 3.3 examines the relative efficiency of trading post set-up by introducing a more general trading arrangement where an agent randomly chooses to visit a market. The penultimate section discusses the problem under the framework of Kiyotaki and Wright described above. A concluding section sums up the findings. Details of proofs of different results established in this chapter are provided in the Appendix to Chapter 3.

3.2 TRANSACTIONS COSTS COMPARISON

In this section we would compare the trading arrangement of Ostroy and Starr (see Section 2.2) with the monetized trading post set-up (discussed in Section 2.4) with respect to the associated transactions costs.

The economy we are going to consider is the same as that of Debreu and Scarf (1963). It consists of different types of agents, where the agents belong-

ing to one specific type have identical characteristics. Each of the types is comprised of a finite number of agents. An economy of this nature will be denoted by $\hat{E}$ in the sequel. Further, its ν -fold replication will be denoted by $\hat{E}_\nu$. For such an economy we are able to show that under the trading post set-up, the time taken to realize the Walrasian demands and supplies to any degree of approximation is bounded above by a quantity which is independent of the total number of individuals in the economy (though dependent on the degree of approximation itself). On the other hand, the time taken in the alternative set-up will not be so bounded for every possible sequence of meetings.

Before stating the main result, further notations must be introduced. Suppose, in the c -th trading post there are respectively ℓ and m types of demanders and suppliers. Let K_c^S and K_c^D represents respectively the sets, of suppliers and demanders of good c . A typical member of K_c^D is denoted by i and that of K_c^S by j . As before, initial excess demands and supplies are represented respectively by $z_{ic}, i \in K_c^D$ and $z_{jc}, j \in K_c^S$. The level of approximation referred to above would then be given by an ε -neighbourhood of these values.

We may choose rational numbers from $[z_{ic} - \varepsilon, z_{ic} + \varepsilon]$ and $[z_{jc} - \varepsilon, z_{jc} + \varepsilon]$ for all c and denote them respectively by χ_i^ε and ω_j^ε respectively. We then consider the problem of achieving these rational values instead of the original z_{ic} 's and z_{jc} 's. The associated time cost would then be denoted by $[T]_\varepsilon$. An exactly analogous problem would be considered for each of the $(n-1)$ trading posts that exists for dealing with each good against M (the medium of exchange). The total time needed to arrive at the ε -neighbourhood of a competitive allocation would be represented by $[\hat{T}_{TP}]_\varepsilon$. Parallel notations, viz. $[\hat{T}_{ML}]_\varepsilon$ would be used to denote the same for the alternative marketless framework. Thus we have,

Proposition 3.1: Given any arbitrary ν , consider an economy $\hat{E}_\nu$ (defined

above) satisfying (U.1) - (U.3) and (Q)¹. Then there exists an integer $N(\varepsilon)$ independent of ν s.t. $[\hat{T}_{TP}]_\varepsilon < N(\varepsilon)$.

Proof: See Appendix to Chapter 3.

However, a conclusion as general as Proposition 3.1 cannot be drawn for the alternative trading arrangement. This is clear from the following result.

Proposition 3.2: There exists an economy $\hat{E}_\nu$ satisfying (U.1) - (U.3) and (Q) such that $[\hat{T}_{ML}]_\varepsilon$ increases unboundedly as ν increases, for at least one given sequence of meetings.

Proof: See Appendix to Chapter 3.

It is important to emphasize the fact that in Proposition 3.1, the rational approximation played a crucial role. It would be impossible to establish the uniform bound if the agents' excess demand vectors were comprised of irrational components and they tried to achieve these demand points exactly. As Example 3.1 below shows clearly, no such uniform bound would exist even if the economy with trading posts consists of two goods.

Example 3.1: Consider an economy $\hat{E}$ having two goods (say, c and M) and 3 demanders and 3 suppliers with excess demands and supplies for good c against M . Let z_i^d and z_j^s ($i, j = 1, 2, 3$) denote respectively the excess demands and excess supplies of the i -th and j -th agents for good c and

$$(z_1^d, z_2^d, z_3^d) = (1, \sqrt{2}, \sqrt{3})$$

$$(z_1^s, z_2^s, z_3^s) = (1, \sqrt{2}, \sqrt{3})$$

Initial endowments of the (excess) demanders are represented by $w_1^d = (0, 1), w_2^d = (0, \sqrt{2}), w_3^d = (0, \sqrt{3})$, where the first components refer to

¹(U.1) - (U.3) and (Q) are described in Section 2.2.

good c and the second components refer to good M . Similarly initial endowments of the (excess) suppliers are given by, $w_1^s = (1, 0)$, $w_2^s = (\sqrt{2}, 0)$ and $w_3^s = (\sqrt{3}, 0)$. It can be easily checked that Ostroy and Starr's condition (Q) is trivially satisfied here (where $p' = (1, 1)$).

From z_i^d 's and z_j^s 's it is clear that $z_2^d, z_3^d, z_2^s, z_3^s \in \mathcal{Q}^c$, the set of irrational numbers. Also $\text{rank}(1, \sqrt{2}, \sqrt{3}) = 3$, in the field of integers.

Given the above competitive economy consider the following sequence of meetings. The first demander meets the first supplier, the second demander meets the second supplier and so on. This is depicted through the following figure.

Demander		Supplier
1	→	1
2	→	2
3	→	3

Let the time taken in attaining equilibrium (starting from the initial endowments) when this economy replicates ν times is denoted by $\hat{T}_{TP}(\nu)$. Clearly, $\hat{T}_{TP}(1) = 1$ for the aforementioned sequence of meetings.

Let us now consider $\nu = 3$ times replication of this economy. That is, there are 9 demanders as well as suppliers. Let

$$(z_1^d, z_2^d, \dots, z_q^d) = (1, \sqrt{2}, \sqrt{3}, 1, \dots, \sqrt{3}).$$

Similarly

$$(z_1^s, z_2^s, \dots, z_1^s) = (1, \sqrt{2}, \sqrt{3}, 1, \dots, \sqrt{3})$$

Consider the following sequence of meetings shown by the arrows in the figure below. Arrangement I is related to the first 3 demanders and suppliers. Similarly, Arrangement II is related to the next 3 demanders and suppliers (viz. 4,5 and 6) and so on.

Arrangement I

Demander		Supplier
1	→	1
2	→	2
3	→	3

Arrangement II

Demander		Supplier
4	↘	4
5	↗	5
6	↘	6

Arrangement III

Demander		Supplier
7	↘	7
8	↗	8
9	↘	9

Let $V_{id}^t(V_{js}^t)$ denote demander (supplier) i 's (j 's) unsatisfied demand (supply) for (of) good c at the beginning of time period t . Thus, from trading arrangement I.

$$V_{1d}^2 = V_{2d}^2 = V_{3d}^2 = V_{1s}^2 = V_{2s}^2 = V_{3s}^2 = 0.$$

From trading arrangement II

$$V_{4d}^2 = V_{5d}^2 = V_{4s}^2 = 0$$

$$V_{6d}^2 = \sqrt{3} - 1, V_{5s}^2 = \sqrt{2} - 1, V_{6s}^2 = \sqrt{3} - \sqrt{2}$$

Thus, 4 agents get satisfied and 4 agents remain to be satisfied. The reason behind this is quite simple. When two agents meet, at least one of them would get satisfied. But not more than 3 traders would get satisfied because $|1 - \sqrt{2}| \neq |\sqrt{2} - \sqrt{3}| \neq |\sqrt{3} - 1| \neq 0$.

In trading arrangement III for the same reason, only 3 agents would be satisfied and

$$V_{8d}^2 = \sqrt{2} - 1, V_{9d}^2 = \sqrt{3} - \sqrt{2}, V_{9s}^2 = \sqrt{3} - 1$$

In period 2, consider the following sequence of meetings.

Demander		Supplier
6	→	9
8	→	5
9	→	6

This would result in all the agents reaching equilibrium in two periods.

Thus,

$$\hat{T}(3) = 2$$

As a next step consider the $\nu = 3^2$ times replication of this economy, where,

$$(z_1^d, z_2^d, \dots, z_{27}^d) = (1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{3})$$

$$(z_1^s, z_2^s, \dots, z_{27}^s) = (1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{3})$$

The trading sequence for the first 9 demanders and suppliers is exactly the same as that depicted through arrangements I, II and III above.

The next 9 demanders and suppliers meet in an exactly similar fashion as the first 9 demanders and suppliers. This is revealed by arrangements IV, V and VI.

Arrangement IV

Demander		Supplier
10	→	10
11	→	11
12	→	12

Arrangement V

Demander		Supplier
13	↘	13
14	↗	14
15	↘	15

Arrangement VI

Demander		Supplier
16	↘	16
17	↗	17
18	↘	18

We arrange the last 9 demanders and suppliers in an exactly parallel manner and call these arrangements VII, VIII and IX.

As is clear from the $\nu = 3$ case, 3 demanders as well as suppliers would remain unsatisfied in arrangements I, II and III taken together. For exactly similar reasons, 3 demanders and 3 suppliers would remain unsatisfied in arrangements IV, V and VI as well as VII, VIII and IX.

Thus, in total we are left with 9 unsatisfied demanders and suppliers with levels of demands and supplies $\sqrt{2} - 1$, $\sqrt{3} - \sqrt{2}$ and $\sqrt{3} - 1$.

Arranging them in a similar fashion as I, II and III we are again left with 3 unsatisfied demanders as well as suppliers after period 2. The levels of unsatisfied demands and supplies are

$$(\sqrt{3} - 1) - (\sqrt{3} - \sqrt{2}) = \sqrt{2} - 1 > 0,$$

$$(\sqrt{3} - 1) - (\sqrt{2} - 1) = \sqrt{3} - \sqrt{2} > 0,$$

$$(\sqrt{2} - 1) - (\sqrt{3} - \sqrt{2}) = 2\sqrt{2} - \sqrt{3} - 1 > 0.$$

In period $t = 3$, let the demander with unsatisfied demand equal to $\sqrt{2} - 1$ meet the supplier with unsatisfied supply of $\sqrt{2} - 1$ and so on, so that after period 3 all agents get satisfied. Hence, $\hat{T}(3^2) = 3$.

We can next consider $\nu = 3^3$ and proceed as before to show that $\hat{T}(3^3) = 4$ etc.

Thus, in general there exists at least one sequence of meetings s.t. $\hat{T}(3^t) = t + 1$ and hence the time taken for attaining equilibrium rises unboundedly as the economy replicates indefinitely.

Note: Such a situation cannot occur if the excess demands and supplies take rational values. The reason is quite simple. Let g be the greatest common divisor of these z_i^d and z_j^s values. Clearly g is positive. Hence, any meeting would reduce the agents' excess demands and supplies at least by g . Since

z_i^d 's and z_j^s 's are finite, after a finite number of stages every agent would get completely satisfied.

Remark 3.2: It can be easily checked that Propositions 3.1 and 3.2 remain valid if condition (Q') is satisfied by the agents (instead of condition (Q)). In this case there would be $(2n - 3)$ trading posts (as described in Section 2.4) to deal with each good against M and M' separately (see proofs of Proposition 3.1 and 3.2 in the Appendix).

3.3. TRANSACTIONS COSTS IN RANDOMIZED TRADING ARRANGEMENTS

The trading post set-up considered hitherto is restrictive in nature due to the following two reasons. First, there exists a predetermined (i.e. ad hoc) sequence in which different trading posts are to be visited. Secondly, each trading post opens for a definite period of time, i.e. for a day, and *all* traders visit that trading post alone during this period. This rules out the possibility of two trading posts operating simultaneously.

As an attempt to relax these restrictions we consider an arrangement where all markets are functioning simultaneously and an agent chooses a trading post at *random* from the set of trading posts he desires to visit (which depends on the goods he wishes to demand and supply). In a parallel fashion we assume that there does not exist any predetermined sequence of meetings for the marketless trading arrangement either. Agents meet each other at random.

Thus, instead of focussing on one specific sequence of meetings (as in the case of Proposition 3.2), we can now compare the average time involved for all possible sequences of meetings, i.e. $E\hat{T}_{ML}$, with the corresponding $E\hat{T}_{TP}$. However, it can be shown that parallel conclusions to those of Propositions 3.1 and 3.2 hold even for such generalized trading arrangements.

To establish these results we consider exactly the same economy $\hat{E}$ and its ν fold enlargement $\hat{E}_\nu$ (see Section 3.2). As mentioned above, unlike the previous trading scheme (of Section 3.2) an agent now chooses to visit a trading post at random. In particular we assume that

Assumption 3.1: An agent chooses each trading post he desires to visit with equal probability.

Suppose in a trading post, δ^t buyers and ψ^t sellers are present in period t and $\psi^t > \delta^t$. Then, the probability of a seller meeting a buyer is $\frac{\delta^t}{\psi^t}$ and a buyer meeting a seller is unity. As soon as a seller confronts a buyer in a particular trading post, exchange of respective goods take place. It is assumed to take one unit of time for an agent to confront a trading partner and trade. If on the other hand a seller visiting a particular trading post could not meet any trading partner in period t (this can happen if $\psi^t > \delta^t$) he *wastes* one unit of time.

In the case of the marketless trading arrangement, an agent meets another agent at random without any reference to the goods they want to exchange. It is assumed to take one unit of time to confront any agent and exchange (if possible).

Since an agent chooses a trading post at random and there is no ad hoc sequence of meetings in a market, $\hat{T}_{TP}$ would be a random variable and hence we would concentrate mainly on $E(\hat{T}_{TP})$. Parallel notation, viz. $E(\hat{T}_{ML})$ would be used to denote the same for the alternative trading scheme.

Even for these general trading arrangements one can expect conclusions similar to Proposition 3.1 and 3.2. In fact, and as mentioned above, we do indeed arrive at parallel results.

Proposition 3.3: For any arbitrary ν , consider an economy $\hat{E}_\nu$ (defined above)

satisfying (U.1) - (U.3) and (Q). Then $\exists$ an integer $\bar{N}(\epsilon)$ independent of ν s.t

$$E(\hat{T}_{TP}(\nu)) < \bar{N}(\epsilon) \quad \forall \nu$$

and hence

$$Prob\{\hat{T}_{TP}(\nu) < \infty, \text{ for any } \nu\} = 1.$$

Proof: See Appendix to Chapter 3.

On the other hand, a conclusion as general as Proposition 3.3 does not hold for the marketless trading arrangement. In fact it is possible to construct a large number of economies (with $z_{kc} \in \mathcal{Q}$ (the set of rational numbers), $\forall k, \forall c$) where expected time taken in a marketless trading scheme diverges as ν increases. More precisely,

Proposition 3.4: There exist competitive economies $\hat{E}_\nu$ satisfying (U.1) - (U.3) and (Q) s.t

$$E(\hat{T}_{ML}(\nu)) \rightarrow \infty \text{ as } \nu \rightarrow \infty$$

Proof: See Appendix to Chapter 3.

So far (in Sections 3.2 and 3.3) we have been considering a framework where the choice of a medium of exchange is exogenously given. We next devote our attention to a situation where a medium of exchange gets endogenously selected through the agents' optimization behaviour. As before our objective is to pose the problem for a trading post set-up as well as for a marketless trading scheme to examine the relative advantages of the two arrangements.

3.4 WELFARE COMPARISONS

It has been mentioned in the introductory chapter that the framework described in the previous two sections introduces transactions costs in a rather ad hoc fashion. In particular, the costs of attainability remain implicit insofar

as an agent is not seen to be carrying out a well defined optimization exercise in connection with the actual realization of his Walrasian demand vector. Therefore, it may be interesting to raise the question of relative efficiency of a trading arrangement for a set-up where transactions cost is incorporated more explicitly. The framework appropriate for the purpose is one similar to Kiyotaki and Wright (1989).

It consists of three types of infinitely lived agents (type 1, 2 and 3), each specialized in consumption and production. Every type consists of an equal number of agents producing one unit of a specific good. There are three indivisible commodities, viz. good 1, 2 and 3. Type k agents derive utility only from the consumption of good k and are able to produce only good $k^* \neq k$ (for notational simplicity we use here the symbols k and k^* to denote commodities also). Each good can be stored at a cost, but an agent's capacity to store is restricted to one unit only. Let b_{kc} denote the cost (in terms of instantaneous disutility) to type k agents of storing good c . It is assumed that

$$0 < b_{k1} < b_{k2} < b_{k3} \quad , \forall k$$

For a type k agent, let U_k denote the instantaneous utility from consumption of good k , L_k instantaneous disutility of producing k^* and $\beta \in (0,1)$ is the common discount factor. Thus, a type k agent's expected discounted life time utility is given as

$$E[\sum_{t=0}^{\infty} \beta^t \{I_k^U(t)U_k - I_{k^*}^L(t)L_k - I_{kc}^b(t)b_{kc}\}]$$

where $I_k^U(t)$ is a random indicator function with

$$\begin{aligned} I_k^U(t) &= 1, \text{ if a type } k \text{ agent consumes good } k \text{ in period } t \\ &= 0, \text{ otherwise} \end{aligned}$$

Similarly,

$$\begin{aligned}
I_{k^*}^L(t) &= 1, \text{ if a type } k \text{ agent produces} \\
&\quad \text{good } k^* \text{ in period } t \\
&= 0, \text{ otherwise}
\end{aligned}$$

and

$$\begin{aligned}
I_{kc}^b(t) &= 1, \text{ if a type } k \text{ agent stores} \\
&\quad \text{good } c \text{ in period } t \\
&= 0, \text{ otherwise}
\end{aligned}$$

In each period, agents are randomly matched with each other in pairs and exchange of goods take place when it is mutually agreeable. It is assumed that the net utility of consuming plus producing, $u_k = U_k - L_k$ is large enough for agents not to want to drop out of the economy. This may be ensured through the following sufficient condition introduced by Kiyotaki and Wright (1989).

$$(3.1) \quad U_k > (b_{kk^*} - b_{kc^*})/(1 - \beta), \forall k \in K, \text{ for all commodities } c$$

If a type k agent can acquire good k in period t , he would immediately consume it and it is assumed that as soon as the agent consumes good k , he produces one unit of k^* .

The economy described so far would be denoted by $\mathcal{E}$ for later reference.

Under this basic framework Kiyotaki and Wright established that there exist two types of pure strategy steady state Nash equilibria². (For the following discussion, however, we would confine ourselves to model I of Kiyotaki and

²A steady-state Nash equilibrium is a set of trading strategies $\{T_k\}$, one for each type k , together with a steady state distribution of inventories $\bar{p}$ (which gives the proportion of type k agents with good c) that satisfies the following conditions:

- (i) each individual k chooses T_k to maximize his expected utility given the best strategies of others and the distribution $\bar{p}$;
- (ii) given T_k , $\bar{p}$ is the resulting steady state distribution

Wright, where $1^* = 2, 2^* = 3$ and $3^* = 1$ i.e. a type 1 agent produces good 2 etc)

(1) Fundamental equilibrium: where good 1 is the unique medium of exchange and type 2 traders who produce good 3 and consume good 1, are the middlemen (i.e. who go through two-stage indirect trade using good 1 as a medium of exchange)

(2) Speculative equilibrium: where both goods 1 and 3 are acting as media of exchange. Type 1 traders (who consume good 1 and produce good 2) use good 3 as a medium of exchange and type 2 traders use good 1 as a medium of exchange.

Further, comparing the individual welfare levels associated with each equilibrium, it can be seen that these two equilibria are Pareto non-comparable.

We now consider the same problem for a trading post set-up, where there exist 3 markets to deal respectively with good 1 against good 2, good 1 against good 3 and good 2 against good 3. An agent is assumed to have two relevant (pure) strategies: either to go for direct barter or to choose indirect trade.

Let $\bar{U}_k^B$ represent the utility derived by a type k agent when he goes through direct barter and $\bar{U}_k^I$ represent the same when he goes through indirect trade.

The set of strategies which would induce good 1 to be the unique medium of exchange (as in the case of fundamental strategies) in a trading post set-up would require type 2 agents to prefer indirect trade to direct barter and type 1 and type 3 agents to stick to direct barter only. More precisely, given that type 1 and type 3 agents opt for direct barter, it should be optimal for type 2 agents to go for indirect trade, which would be ensured if $\bar{U}_2^B < \bar{U}_2^I$ and so on.

It can be shown (see the proof of Proposition 3.5 in the Appendix to Chapter 3) that this strategy profile (to be called fundamental strategies) forms a set of equilibrium strategies for all parameter values³. Having arrived at this

³It has been shown in the Appendix (see Proposition A.3.1) that (depending on the

equilibrium our next attempt is to compare the welfare levels of this equilibrium with the corresponding fundamental equilibrium for a marketless trading arrangement.

We observe that since there exist separate markets to deal with every pair of goods, in the trading post set-up, a type 2 agent who wishes to exchange good 3 against good 2 through indirect trade would first go to trading post (1,3) where only type 3 agents are present. Therefore, he has no chance of (unnecessarily) meeting a type 1 agent. Also, when a type 2 agent acquires good 1, he can visit the trading post that deals with good 1 against 2. Since type 3 agents (who prefer to go for direct barter only) would never visit trading post (1,2), the probability of meeting a complementary trading partner for a type 2 agent with good 1 (i.e. a type 1 agent) would be higher as compared to that in the marketless trading arrangement of Kiyotaki and Wright. By the same logic, trade would be faster for type 1 and type 3 agents. This leads one to expect that the welfare associated with a trading post set-up is higher than that for a marketless trading arrangement. We have in fact shown the following:

Proposition 3.5: For the economy $\mathcal{E}$ defined above, welfare of every agent is higher under the fundamental strategies in a trading post set-up as compared to that of alternative marketless trading arrangement.

Proof: See Appendix to Chapter 3.

Kiyotaki and Wright, however, have shown that none of the strategies defined above (viz. fundamental and speculative) are Pareto optimal. On the other hand, a strategy which directs every pair of agents to exchange the goods parameter values any good acting uniquely as a medium of exchange can give rise to a steady state Nash equilibrium. However, the equilibrium with good 1 acting as a medium of exchange Pareto dominates the other two equilibria.

they possess is Pareto superior to both the equilibrium strategies. But left to themselves, the traders would never opt for this strategy and hence it would not be implementable.

It can be shown, however, (see the proof of Proposition 3.5 in the Appendix), that the fundamental strategies mentioned above for a trading post set-up are Pareto superior to the aforementioned non-implementable strategy. Also, under the social institution of markets our strategies form a set of equilibrium strategies and hence are implementable. This strengthens our conclusion of Proposition 3.5 even further.

3.5 CONCLUSION

This chapter discusses the relative efficiencies of a trading post set-up vis-a-vis a marketless trading arrangement. The trading post set-up being an institution for separating out the buyers and sellers of each pair of goods, complementary trading partners can locate each other faster. Thus, completion of all desired trades under the social institution of markets is expected to take less time as compared to the marketless trading arrangement. This chapter has indeed established this fact formally under different kinds of framework.

APPENDIX TO CHAPTER 3:

Proof of Proposition 3.1:

We will prove this proposition with the help of two lemmas.

Before going to the lemmas we introduce the following notations. Let K_c^S and K_c^D represent respectively the sets of suppliers and demanders of good c . Let V_{kc}^t denote the value of agent k 's unsatisfied demand for good c in the beginning of period t . V_{kc}^t is negative if k has unsatisfied supply. Let g be the greatest common divisor of the elements of the set

$$A = \{\chi_i^c, \Omega_j^c : c \in C, i \in K_c^D, j \in K_c^S\}$$

Lemma 3.1: For all t , $|V_{kc}^t| = g \cdot q_{kc}^t$, $k \in K$ and $c \in C$, where $q_{kc}^t \in \mathbb{N}$, the set of non-negative integers (including zero).

Proof: All the discussions to follow will concern good c only and hence, for notational simplicity, we will omit the subscript c .

Consider an agent $i \in K^D$, i.e. he is an excess demander of good c . The case where the agent is an excess supplier is handled similarly. Clearly, from the definition of g ,

$$(A.3a) \quad V_i^1 = \begin{cases} \chi_i, & \text{if } i \text{ cannot meet any} \\ & \text{supplier in the first period} \\ -\min\{0, \Omega_j - \chi_i\}, & \text{if } i \text{ confronts } j(\in K^S) \end{cases}$$

The first alternative in V_i^2 can occur if

$$\#\{K^S\} < \#\{K^D\}$$

Clearly, for both alternatives $V_i^2 = gq_i^2$, $q_i^2 \in \mathbb{N}$; as both Ω_j and χ_i are integer multiples of g .

In general, if $i \in K^D$

$$V_i^t = \begin{cases} V_i^{t-1}, & \text{if } i \text{ cannot meet any} \\ & \text{supplier in the } (t-1)\text{-th period} \\ -\min\{0, (|V_j^{t-1}| - V_i^{t-1})\}, & \text{if } i \text{ confronts } j(\in K^S) \text{ in the} \\ & \text{time period } (t-1) \end{cases}$$

The first alternative in V_i^t can occur if $\#\{K^{Dt}\} > \#\{K^{St}\}$, where K^{Dt} and K^{St} represent respectively the sets of demanders and suppliers remaining unsatisfied at the beginning of time period t .

Using (A.3a) one can easily check that

$$V_i^t = gq_i^t, \quad q_i^t \in \mathbb{N}$$

Lemma 3.2: Suppose an economy $\hat{E}$ satisfies (U.1) - (U.3) and the condition (Q). Then

$$[T_c]_e < \{ \max_{j \in K_c^S} \frac{\Omega_j^e}{g} \} + \{ \max_{i \in K_c^D} \frac{\chi_i^e}{g} \}$$

Note: As in Lemma 3.1, the subscript c will be omitted in the proof for notational simplicity.

Proof: Without loss of generality, suppose $k = i, j$ meet in the trading post relevant for good c .

From Lemma 3.1 it is clear that if

$$V_i^t > 0 \Rightarrow V_i^t - V_i^{t+1} = g\bar{q}_i^t, \bar{q}_i^t \in \mathbb{N}^+$$

Where, $\mathbb{N}^+$ is the set, of positive integers,

$$\bar{q}_i^t = 0 \quad \text{if} \quad V_i^t = 0.$$

Thus, if $V_i^t > 0$, then agent i 's unsatisfied demand diminishes at least by an amount g when he confronts a supplier.

Therefore, if at every period of time, i confronts a supplier (in the trading post for good c) then the total time taken to satisfy χ_i is at most $\bar{t}_i = \chi_i/g$. An upper bound of these $\bar{t}_i$'s is

$$t' = \max_{i \in K^D} \{ \chi_i/g \} \tag{A.3b}$$

But it may so happen that at some point of time t , i could not meet any supplier. This is possible if $\#\{K^{Dt}\} > \#\{K^{St}\}$. In such a situation, i has to remain idle for one period of time. Therefore, in order to find $[T_c]_e$ one has to take account of the waiting time too.

We define a function,

$$\dot{W}_i(t) = \begin{cases} 1, & \text{if } i \text{ cannot meet any supplier} \\ & \text{at time period } t \\ 0, & \text{otherwise} \end{cases}$$

$\dot{W}_i(t) = 1$ implies that $\#\{K^{Dt}\} > \#\{K^{St}\}$ which implies each $j \in K^{St}$ confronts a demander. Therefore $|V_j^t|$ decreases at least by g . Since $|V_j^t|$ cannot decrease below zero, it takes at most $\underline{t} = \max_{j \in K^S} \frac{\Omega_j}{g}$ units of time to go to zero. (if j confronts a demander in each unit of time). Therefore, $\dot{W}_i(t) = 1$ can hold for less than a total of $\underline{t}$ units of time. Thus, total waiting time

$$\sum_t \dot{W}_i(t) < \max_{j \in K^S} \frac{\Omega_j}{g} \quad (A.3c)$$

Using (A.3b) and (A.3c)

$$[T_c]_\varepsilon < [\{\max_{i \in K^D} \frac{\chi_i}{g}\} + \{\max_{j \in K^S} \frac{\Omega_j}{g}\}]$$

■

Proof of Proposition 3.1: It is clear that,

$$[\hat{T}_{TP}]_\varepsilon = \sum_{c=1}^n [T_c]_\varepsilon$$

From lemma 3.2

$$\begin{aligned} [T_c]_\varepsilon &< N_c(\varepsilon) \\ \Rightarrow [\hat{T}_{TP}]_\varepsilon &< \sum_{c \in C} N_c(\varepsilon) = N(\varepsilon), \text{ say} \end{aligned}$$

■

Remark A.3.1 If we do not consider an economy consisting of fixed types of agents, Proposition 3.1 still holds as long as with the increase in the number of agents, $\max_{j \in K^S} \Omega_j^c$ and $\sum_{i \in K^D} \chi_i^c$ and g remain the same.

Proof of Proposition 3.2:

Consider the competitive economy comprising of four types of agents 1,2,3,4 with an equilibrium price vector $p = (1, 1, 1, 1)$. The excess demand matrix is shown as:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Where a row and a column stand respectively for the corresponding commodity and the agent.

(i) Suppose, in each type there is just one agent and the sequence of meetings is as follows. During the first period, a type 1 agent meets a type 3 and a type 2 agent meets a type 4. In the second period a type 1 meets a type 4 and a type 2 meets a type 3. In the third period a type 1 agent meets a type 2 and a type 3 meets a type 4. Then

$$\hat{T}_{ML} = 3$$

(ii) Suppose, however, that in each type there are two agents and that the sequence of meetings is as follows:

(ii-a) Initially, each agent meets his identical partner. This takes $2c_2 = 1$ unit of time.

(ii-b) then each type 3 agent meets every type 1 agent. That takes $\{(\frac{4}{2})/(1/2)\} - 1 = 2$ units of time.

(ii-c) next each type 3 agent meets every type 2 agent. That takes $\{(\frac{4}{2})/(1/2)\} - 1 = 2$ units of time.

(ii-d) finally type 3 agents meet type 4 agents and the equilibrium is reached. This takes one unit of time.

Thus, $\hat{T}_{ML} = 6$ and $[\hat{T}_{ML}]_\varepsilon > 5$ for any $\varepsilon > 0$.

If the economy replicates ν times, then exactly the same sequence of meetings as that in the case where $\nu = 2$, gives

$$\begin{aligned} \hat{T}_{ML} &= \left\{ \binom{\nu}{2} / (\nu/2) \right\} - 2 \left\{ \binom{2\nu}{2} / \nu \right\} \\ &\quad - \left\{ \binom{\nu}{2} / (\nu/2) \right\} + 1 = 3\nu = N'(\nu), \text{ say.} \end{aligned}$$

Clearly $[\hat{T}_{ML}]_\varepsilon > 3\nu - 1$ for any $\varepsilon > 0$. Hence it diverges with ν . ■

Note: The result is true even when

$$z_{ic} \in N \quad \forall i \in K_c^D \quad \text{and} \quad z_{jc} \in N. \quad \forall j \in K_c^S.$$

Proof of Proposition 3.3:

At each period, an agent chooses a trading post at random from the set of trading posts he wishes of visit. Let η_k^t denote the number of trading posts which k wishes to visit at the beginning of time period t . Let

$$B_k^t = \{c : |V_{kc}^t| > 0, c \neq M\}$$

then

$$\eta_k^t = \#B_k^t$$

Since $|V_{kc}^t|$ is a decreasing function of t , therefore, η_k^t decreases in time.

According to Assumption 3.1, at period t , agent k chooses a trading post with probability $= 1/\eta_k^t$

Let us denote the probability for a particular demander $i (\in K_c^{Dt}$, the set of unsatisfied demanders of good c at the beginning of time period t) being present in trading post (c, M) at period t by ρ_{ic}^t . Hence,

$$\rho_{ic}^t = \frac{1}{\eta_i^t} \geq \frac{1}{n-1} \tag{A.3d}$$

Similarly, for a supplier $j(\in K_c^{St})$

$$\rho_{jc}^t \geq \frac{1}{n-1} \quad (A.3c)$$

We denote by δ_c^t and ψ_c^t respectively the actual number of demanders and suppliers *present* in trading post (c, M) , at time period t . The following, relations can hold between δ_c^t and ψ_c^t

$$(i) \min(\delta_c^t, \psi_c^t) = \delta_c^t$$

This implies each buyer would meet a seller with probability 1 and hence by the same logic as that of Proposition 3.1, unsatisfied demand of each buyer present would reduce at least by g .

$$(ii) \min(\delta_c^t, \psi_c^t) = \psi_c^t$$

This implies unsatisfied supply of each seller present would be reduced at least by g .

$$(iii) \delta_c^t = \psi_c^t$$

This implies excess unsatisfied supply as well as demand of the respective agents would be reduced at least by g .

At each period t , the expected reduction of i 's ($i \in K_c^{Dt}$) unsatisfied demand and/or j 's ($j \in K_c^{St}$) unsatisfied supply is at least

$$E(\rho_{ic}^t g) \geq \frac{g}{n-1}, \quad i \in K_c^{Dt}$$

and

$$E(\rho_{jc}^t g) \geq \frac{g}{n-1}, \quad j \in K_c^{St}$$

The alternatives depend on (i), (ii) and (iii) above.

Let at time period t , $\tilde{R}^t$ be the reduction that occurs to the expression

$$\tilde{V}_c^t = \max_{i \in K_c^{Dt}} V_{ic}^t + \max_{j \in K_c^{St}} |V_{jc}^t|$$

The (c, M) trading post would be completely cleared (i.e. $\tilde{V}_c^t = 0$) after atmost τ periods if

$$\sum_{t=1}^{\tau} \tilde{R}^t \leq \max_t \tilde{V}_c^t = |z_{lc}| + |z_{hc}|$$

where

$$z_{\ell c} \geq z_{ic}, \forall i \in K_c^D, i \neq \ell \in K_c^D$$

and

$$|z_{hc} \geq |z_{jc}|, \forall j \in K_c^S, j \neq h \in K_c^S$$

We have,

$$\begin{aligned} E(\sum_{t=1}^{\tau} \tilde{R}^t) &= EE(\sum_{t=1}^{\tau} \tilde{R}^t | \tau) \\ &= EE(\tilde{R}^1 + \dots + \tilde{R}^{\tau}) \\ &= E\{E\tilde{R}^1 + E\tilde{R}^2 + \dots + E\tilde{R}^{\tau}\} \\ &\geq E\left(\frac{g}{n-1}\tau\right) \\ \therefore E(\tau) &\leq \frac{(n-1)[z_{\ell c} + |z_{hc}|]}{g} \end{aligned}$$

i.e.,

$$E(\hat{T}_C) \leq \frac{(n-1)[z_{\ell c} + |z_{hc}|]}{g},$$

which is independent of ν .

A similar exercise can be performed for other trading posts. Adding over all c we get,

$$E(\hat{T}_{TP}(\nu)) \leq \bar{N}(\varepsilon), \text{ independent of } \nu.$$

Coming now to the second part of Proposition 3.3, consider the sequence of random variables $\hat{T}_{TP}(\nu)$. We have shown that

$$E(\hat{T}_{TP}(\nu)) \leq \bar{N}(\varepsilon) \quad \forall \nu$$

$$\therefore \text{Prob} \{\hat{T}_{TP}(\nu) < \infty \text{ for any } \nu\} = 1. \quad \blacksquare$$

Proof of Proposition 3.4:

Consider an economy with 2 types of agents, viz. type 1 and type 2 and two goods, to be called c and M . The total number of type 1 agents is equal

to the total number of type 2 agents. Each type 1 agent has one unit of good c and demands in return 1 unit of good M . A type 2 trader on the other hand has one unit of good M and demands in return 1 unit of good c .

It is clear that when a type 1 agent meets a type 2 agent, they can trade and attain their desired commodity bundles. Suppose each type consists of X number of agents. Then, the probability that an arbitrary type 1 agent, say k , meets a complementary trading partner (i.e. a type 2 agent) in the first period is

$$p_1 = \frac{X}{X + X - 1} = \frac{1}{2 - \frac{1}{X}} \leq \frac{2}{3} \text{ if } X \geq 2.$$

The denominator of this probability (i.e. $X + X - 1$) refers to the total number of agents excluding k and the numerator (i.e. X) represents the number of complementary trading partners.

The probability of not meeting any type 2 agent by k in the first period is

$$1 - p_1 = 1 - \frac{X}{2X - 1} \geq \frac{1}{3}.$$

Let X_t^U denote the number of type 1 agents (or, equivalently type 2 agents) remaining unsatisfied at the beginning of time period t .

$$\therefore X_1^U = X$$

Clearly, $X_t^U, t > 1$, is a random variable and hence we would be concerned here with EX_t^U . Since the probability of remaining unsatisfied during the first period is $(1 - \frac{X}{2X-1})$ and there are a total of X type 1 traders in the economy

$$E(X_2^U) = X \cdot (1 - \frac{X}{2X-1}) \geq \frac{1}{3}X.$$

Let us define an indicator function

$$\begin{aligned} I_{X_2^U=x} &= 1, \text{ if } X_2^U = x \\ &= 0, \text{ otherwise} \end{aligned}$$

Then we can write

$$\begin{aligned} E(X_2^U) &= E(X_2^U I_{X_2^U=1} + X_2^U I_{X_2^U \geq 2}) \\ &= E(X_2^U I_{X_2^U=1}) + E(X_2^U I_{X_2^U \geq 2}) \end{aligned}$$

Let us now calculate

$$\begin{aligned} E(X_2^U I_{X_2^U \geq 2}) &= \sum_{\gamma=2}^X \gamma Pr\{X_2^U = \gamma\} \\ &= \sum_{\gamma=2}^X \gamma \binom{X}{\gamma} (1-p_1)^\gamma p_1^{X-\gamma} \\ &= X(1-p_1) \sum_{\gamma=1}^{X-1} \binom{X-1}{\gamma-1} (1-p_1)^{\gamma-1} p_1^{X-\gamma} \\ &= X(1-p_1)(1-p_1^{X-1}) \geq \frac{1}{3}X(1-p_1), \text{ if } X \geq 2 \\ E(X_2^U I_{X_2^U \geq 2}) &\geq (\frac{1}{3})^2 X \end{aligned} \tag{A.3f}$$

Let us consider period $t = 3$. Suppose at the beginning of the third period, X_3^U is the number of unsatisfied agents. The probability that a type 1 agent gets satisfied during period 2 is

$$\begin{aligned} p_2 &= \frac{X_2^U}{X_2^U + X_2^U - 1} \\ \therefore 1 - p_2 &= 1 - \frac{X_2^U}{2X_2^U - 1} \geq \frac{1}{3}, \text{ if } X_2^U \geq 2 \end{aligned}$$

At the beginning of period $t = 2$, X_2^U agents have remained unsatisfied and they have a probability $(1-p_2)$ of remaining unsatisfied during the third period

$$\begin{aligned} \therefore E(X_3^U) &= EE(X_3^U | X_2^U) \\ &= E(X_2^U (1 - \frac{X_2^U}{2X_2^U - 1})) \\ &= E[X_2^U (1 - \frac{X_2^U}{2X_2^U - 1}) I_{X_2^U=1}] \end{aligned}$$

$$\begin{aligned}
& +E[X_2^U(1 - \frac{X_2^U}{2X_2^U-1})I_{X_2^U \geq 2}] \\
& = 0 + E[X_2^U(1 - \frac{X_2^U}{2X_2^U-1})I_{X_2^U \geq 2}] \\
& \geq \frac{1}{3}[X_2^U I_{X_2^U \geq 2}] \\
& \geq \frac{1}{3} \cdot (\frac{1}{3})^2 X = \frac{1}{3^3} X \quad (using (A.3f))
\end{aligned}$$

As before in an exactly parallel manner let us calculate

$$\begin{aligned}
& E[X_3^U I_{X_3^U \geq 2}] \\
& = EE[X_3^U I_{X_3^U \geq 2} | X_2^U] \\
& = E[\sum_{\gamma=2}^{X_2^U} \gamma Pr\{X_3^U = \gamma\} \\
& = E[X_2^U(1 - p_2)(1 - p_2^{X_2^U-1})] \\
& = E[X_2^U(1 - \frac{X_2^U}{2X_2^U-1})(1 - p_2^{X_2^U-1})] \\
& \quad [I_{X_2^U=1} + I_{X_2^U \geq 2}] \\
& = 0 + E[X_2^U(1 - \frac{X_2^U}{2X_2^U-1})(1 - p_2^{X_2^U-1})I_{X_2^U \geq 2}] \\
& \geq \frac{1}{3}E[X_2^U(1 - p_2)I_{X_2^U \geq 2}] \\
& \geq (\frac{1}{3})^2 E(X_2^U I_{X_2^U \geq 2}) \geq (\frac{1}{3})^2 (\frac{1}{3})^2 X \\
& \quad \therefore (X_3^U I_{X_3^U \geq 2}) \geq (\frac{1}{3})^4 X \tag{A.3g}
\end{aligned}$$

Using (A.3g), we get as before

$$\begin{aligned}
E(X_4^U) & \geq \frac{1}{3}E(X_3^U I_{X_3^U \geq 2}) \\
& \geq \frac{1}{3} \cdot (\frac{1}{3})^4 X = \frac{1}{3^5} X
\end{aligned}$$

In general,

$$E(X_t^U) = \frac{1}{3^{2(t-2)+1}} = \frac{1}{3^{2t-3}}$$

If we replicate the economy ν times

$$E(X_t^U(\nu)) \geq \frac{X}{3^{2t-3}} \cdot \nu$$

Let us define $\bar{T}_\nu$ as

$$\bar{T}_\nu = \min\{t \mid \frac{\nu X}{3^{2t-3}} \leq 2\}$$

Then,

$$E\hat{T}_{ML}(\nu) \geq \bar{T}_\nu$$

Now given any t , we can find a ν s.t.

$$\frac{\nu X}{3^{2t-1}} \geq 2.$$

$\therefore$ as $\nu \rightarrow \infty$, $\bar{T}_\nu \rightarrow \infty$, and hence $E\hat{T}_{ML}(\nu) \rightarrow \infty$.

Remark A.3.1 From this example it is clear that the result holds even when we consider a similar economy with 4 types of agents (each type having X agents) and 4 commodities. A type 3 agent demands one unit of good 3 against one unit of good 4. A type 4 agent demands one unit of good 4 against one unit of good 3 and so on. Each type (type 3 and 4) has one unit of good M so that condition (Q) is satisfied. Thus, a type 3 agent can trade only when he meets a type 4 agent and vice-versa.

In a parallel fashion we can consider 6 types of agents with 6 goods in the economy and so on. Thereby, one can construct a large member of competitive economies where Proposition 3.4 holds.

Proof of Proposition 3.5:

Step 1:

We first show that a set of steady state Nash equilibrium strategies is constituted by type 1 and type 3 agents opting for direct barter and type 2 agents

going for indirect trade. Thus, we need to establish that given the strategies of type 1 and type 3 agents, it is optimal for type 2 agents to go for indirect trade and so on.

(i) We begin with a type 1 agent's expected utility in opting for direct barter. As soon as a type 1 agent produces good 2 he pays the storage cost b_{12} . Next period he can meet a type 2 agent (with good 1) with probability, say p'_{21} , where p'_{21} is the proportion of types 2 agents present at trading post (1,2). If he could succeed in meeting a type 2 trader, there is perfect mutual coincidence of wants. Thus, through consumption of good 1 and production of good 2, he derives net utility u_1 and there after he has the option of going for direct barter or indirect trade whichever is more profitable.

On the other hand, with probability $(1 - p'_{21})$ he does not meet a complementary trading partner and remains with good 2.

Thus, we get

$$\begin{aligned}\bar{U}_1^B = & -b_{12} + \beta[p'_{21}(u_1 + \max(\bar{U}_1^B, \bar{U}_1^I)) \\ & + (1 - p'_{21})\max(\bar{U}_1^B, \bar{U}_1^I)]\end{aligned}\quad (A.3h)$$

Let us now concentrate on $\bar{U}_1^I$. If a type 1 agent goes for indirect trade, he first visits trading post (2,3) to sell good 2. Given the strategies of type 2 and type 3 agents, a type 1 agent going to trading post (2,3) would not be able to meet any complementary trading partner. Hence,

$$\bar{U}_1^I = -\frac{b_{12}}{1 - \beta} \quad (A.3i)$$

i.e. the life time storage cost of good 2.

To compare $\bar{U}_1^B$ and $\bar{U}_1^I$:

From (A.3h)

$$\bar{U}_1^B \geq -b_{12} + \beta\{p'_{12}(u_1 + \bar{U}_1^B) + (1 - p'_{12})\bar{U}_1^B\}$$

$$\begin{aligned} \Rightarrow (1 - \beta)\bar{U}_1^B &\geq -b_{12} + \beta p'_{12}u_1 \\ \Rightarrow \bar{U}_1^B &\geq -\frac{b_{12}}{1 - \beta} + \frac{\beta}{1 - \beta}p'_{12}u_1 \end{aligned}$$

Comparing (A.3j) with (A.3i) we get

$$\bar{U}_1^B > \bar{U}_1^I, \text{ if } p'_{12} > 0$$

It would be shown in the sequel that

$$p'_{12} > 0.$$

(ii) We concentrate next on type 3 agents. As in the case of a type 1 agent, we can compute $\bar{U}_3^B$ as

$$\begin{aligned} \bar{U}_3^B &= -b_{31} + \beta[p'_{23}\{u_3 + \max(\bar{U}_3^B, \bar{U}_3^I)\} \\ &\quad + (1 - p'_{23})\max(\bar{U}_3^B, \bar{U}_3^I)] \end{aligned} \quad (A.3k)$$

where, p'_{23} is the proportion of type 2 agents in the (1,3) trading post.

We next find expected utility for a type 3 agent opting for indirect trade. A type 3 agent who produces good 1 would first visit the (1,2) trading post in order to sell good 1. Let p'_{31} be the probability of a type 3 agent meeting a type 1 agent in the (1,2) trading post. Thus, with probability p'_{31} he can attain the indirect utility of acquiring a medium of exchange. (i.e. good 2). We denote this indirect utility by $\bar{U}_{32}^I$.

$$\bar{U}_3^I = -b_{31} + \beta[p'_{31}\bar{U}_{32}^I + (1 - p'_{31})\max(\bar{U}_3^I, \bar{U}_3^B)] \quad (A.3l)$$

We concentrate now on $\bar{U}_{32}^I$. As soon as a type 3 agent acquires good 2, the storage cost for holding good 2 has to be incurred. Next period, good 2 would be taken to the (2,3) trading post to be exchanged against good 3. However, given the strategies of type 1 agents as well as of type 2 agents, it is not possible to meet any complementary trading partner in the (2,3) trading post. Thus, he incurs the storage cost indefinitely.

$$\therefore \bar{U}_{32}^I = -b_{32} + \beta \left[-\frac{b_{32}}{1-\beta} \right] \quad (A.3m)$$

To compare $\bar{U}_2^1$ and $\bar{U}_2^B$:

From (A.3k)

$$\begin{aligned} \bar{U}_3^B &\geq -b_{31} + \beta[p'_{23}\{u_3 + \bar{U}_3^B\} + (1 - p'_{23})\bar{U}_3^B] \\ &\Rightarrow (1 - \beta)\bar{U}_3^B \geq -b_{31} + \beta p'_{23}u_3 \end{aligned} \quad (A.3n)$$

From (A.3l) and (A.3m)

$$\begin{aligned} \bar{U}_3^I &= -b_{31} + \beta[p'_{31}(-b_{32})\{1 + \frac{\beta}{1-\beta}\} \\ &\quad + (1 - p'_{31})\max(\bar{U}_3^1, \bar{U}_3^B)] \\ &\Rightarrow \bar{U}_3^1 - \beta(1 - p'_{31})\max(\bar{U}_3^I, \bar{U}_3^B) \\ &= -b_{31} + \beta[p'_{31}(-b_{32})\frac{1}{1-\beta}] \\ &\Rightarrow \{1 - \beta(1 - p'_{31})\}\bar{U}_3^I - \beta(1 - p'_{31})[\max(\bar{U}_3^I, \bar{U}_3^B) - \bar{U}_3^I] \\ &= -b_{31} - \frac{\beta}{1-\beta}p'_{31}b_{32}. \end{aligned} \quad (A.3o)$$

If $\max(\bar{U}_3^I, \bar{U}_3^B) = \bar{U}_3^I$ then comparing (A.3n) and (A.3o)

$$\begin{aligned} \{1 - \beta(1 - p'_{31})\}\bar{U}_3^I &< (1 - \beta)\bar{U}_3^B \\ \therefore \bar{U}_3^I &< \bar{U}_3^B, \quad \text{a contradiction.} \end{aligned}$$

(iii) Finally, we consider a type 2 agent.

If he goes through direct barter, it is not possible to meet any complementary trading partner in the (2,3) trading post. Therefore,

$$\bar{U}_2^B = -\frac{b_{23}}{1-\beta} \quad (\text{A.3p})$$

If he opts for indirect trade he has to first visit trading post (1,3) to sell good 3 and acquire the medium of exchange (i.e. good 1). Given the strategies of type 3 agents, they would be present in the (1,3) trading post and hence the probability of meeting a complementary trading partner is unity for a type 2 agent visiting the (1,3) trading post. Once a type 2 agent acquires good 1, he next visits the (1,2) trading post in order to dispose of the medium of exchange. Here again, for similar reasons, he meets a complementary trading partner with probability one. Thus,

$$\bar{U}_2^I = -b_{23} + \beta[-b_{21} + \beta(u_2 + \max(\bar{U}_2^B, \bar{U}_2^I))] \quad (\text{A.3q})$$

To compare $\bar{U}_2^B$ and $\bar{U}_2^I$:

$$\begin{aligned} (\text{A.3q}) \Rightarrow \bar{U}_2^I &\geq (-b_{23} - \beta b_{21}) + \beta^2 u_2 + \beta^2 \bar{U}_2^I \\ &\Rightarrow (1 - \beta^2) \bar{U}_2^I \geq -(1 + \beta)b_{23} + \beta^2 u_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \bar{U}_2^I &\geq -\frac{1+\beta}{1-\beta^2} b_{23} + \frac{\beta^2}{1-\beta^2} u_2 \\ &= -\frac{1}{1-\beta} b_{23} + \frac{\beta^2}{1-\beta^2} u_2 \\ &> \bar{U}_2^B \quad (\text{see (A.3p)}) \end{aligned}$$

Hence, the initial strategy profile indeed constitutes a set of equilibrium strategies.

Steady State values of p'_{21} and p'_{23} :

From the above discussion (i.e. (iii) above), it is clear that the proportion of type 2 agents with good 1 in a particular period equals proportion of type 2

agents with good 3 in the next period

$$\therefore p'_{21} = p'_{23} = \frac{1}{2}$$

Step 2:

We now intend to compare the welfare derived by a type i agent in a trading post set-up with his welfare in the marketless trading arrangement of Kiyotaki and Wright (under fundamental strategies). It has been shown for the framework of Kiyotaki and Wright (1989, pp.948) that the welfare derived by a type i agent is

$$WF_i = (1 - \beta) \sum_j p_{ij} V_{ij} \quad (A.3r)$$

where,

p_{ij} is the proportion of type i agent with good j ,

V_{ij} is the indirect utility for a type i agent of acquiring good j .

In particular, it has been derived that

$$(A.3s) \quad \begin{cases} WF_1 = \frac{\beta u_1}{6} - b_{12}, WF_2 = \frac{\beta u_2}{6} - \frac{1}{2}(b_{21} + b_{23}) \\ WF_3 = \frac{\beta u_3}{6} - b_{31} \end{cases}$$

Using (A.3r), for a trading post set-up we get corresponding welfare levels

as

$$\begin{aligned} \overline{WF}_1 &= (1 - \beta) \bar{U}_1^B = -b_{12} + \beta(u_1 p_{21}) \\ &= -b_{12} + \frac{u_1 \beta}{2} \\ \overline{WF}_3 &= -b_{31} + \frac{1}{2} \beta u_3 \end{aligned}$$

Using (A.3s)

$$WF_1 < \overline{WF}_1 \quad \text{and} \quad WF_3 < \overline{WF}_3$$

We next consider the type 2 agents. At any point of time half of them are present in trading post (1,3) and half of them are in trading post (1,2).

Let Δ_{21} and Δ_{23} denote respectively the indirect utility (for a type 2 agent) of visiting the (1,2) and (1,3) trading posts. Then

$$\begin{aligned}\Delta_{23} &= -b_{23} + \beta\Delta_{21} \\ \Delta_{21} &= -b_{21} + \beta(u_2 + \Delta_{23})\end{aligned}$$

∴ the steady state utility is

$$\begin{aligned}\frac{1}{2}\Delta_{23} + \frac{1}{2}\Delta_{21} &= \frac{1}{2}[-b_{23} + \beta\Delta_{21}] \\ &\quad + \frac{1}{2}[-b_{21} + \beta(u_2 + \Delta_{23})] \\ \Rightarrow \overline{WF}_2 &= \frac{1}{2}(1 - \beta)[\Delta_{23} + \Delta_{21}] = \frac{\beta u_2}{2} - \frac{b_{23} + b_{21}}{2}\end{aligned}$$

From (A.3s)

$$\overline{WF}_2 > WF_2$$

Step 3:

We finally consider the welfare associated with the non-equilibrium strategy (see Kiyotaki and Wright (1989, pp.948) of “always trading regardless of the match”, as it Pareto dominates the equilibrium strategies (in a marketless setup).

Type 1 agent's welfare

$$WF_1^* = \frac{\beta u_1}{3} - \frac{1}{3}(b_{13} - b_{12}) - b_{12}$$

For the type 2 agent,

$$WF_2^* = \frac{\beta u_2}{3} - \frac{b_{23} + b_{21}}{2} - \frac{1}{6}(b_{23} - b_{21})$$

For the type 3 agent

$$WF_3^* = \frac{\beta u_3}{3} - \frac{1}{3}(b_{32} - b_{31}) - b_{31}$$

Comparing these values with $\overline{WF}_1, \overline{WF}_2$ and $\overline{WF}_3$ of the trading post set-up we get

$$WF_1^* < \overline{WF}_1, WF_2^* < \overline{WF}_2, WF_3^* < \overline{WF}_3$$

Remark A.3.2 It can be shown in a similar manner that either good 2 or good 3, acting uniquely as a medium of exchange, would also give rise to steady state Nash equilibria for certain parameter values.

To see this we first consider the situation where good 2 is uniquely chosen as a medium of exchange. This is possible if in equilibrium the type 3 agents opt for indirect trade and the type 1 and 2 agents go for direct barter. Our purpose is to examine whether these form steady state Nash equilibrium strategies.

(i) We begin with a type 1 agent's expected utility in opting for direct barter. As soon as a type 1 agent produces good 2 he pays the storage cost b_{12} . Next period he can meet a type 3 agent (coming to the (1,2) trading post to sell good 1) and acquire the medium of exchange. Let p'_{31} be the proportion of type 3 agents in the (1,2) trading post. Thus, with probability p'_{31} , a type 1 agent can meet a type 3 agent in trading post (1,2).

Therefore, proceeding exactly as in the case where good 1 was the medium of exchange, we get

$$\begin{aligned} \bar{U}_1^B &= -b_{12} + \beta[p'_{31}(u_1 + \max(\bar{U}_1^B, \bar{U}_1^I)) \\ &\quad + (1 - p'_{31}) \max(\bar{U}_1^B, \bar{U}_1^I)] \end{aligned}$$

Let us now concentrate on $\bar{U}_1^I$. It can be easily checked that

$$\bar{U}_1^I = -b_{12} + \beta[p'_{12}\bar{U}_{13}^I + (1 - p'_{12}) \max(\bar{U}_1^I, \bar{U}_1^B)] \quad (A.3t)$$

where, p'_{12} is the probability of meeting a complementary trading partner (for a type 1 agent) in the (2,3) trading post and $\bar{U}_{13}^I$ is the indirect utility of acquiring good 3 by a type 1 agent.

$$\therefore \bar{U}_{13}^I = -b_{13} + \beta \left[\frac{-b_{13}}{1-\beta} \right]$$

To compare $\bar{U}_1^I$ and $\bar{U}_1^B$

$$\begin{aligned} (A.3t) \Rightarrow \bar{U}_1^I &= -b_{12} + \beta [p'_{12}(-b_{13}) \{1 + \frac{\beta}{1-\beta}\} \\ &\quad + (1 - p'_{12}) \max(\bar{U}_1^I, \bar{U}_1^B)] \\ &< -b_{12} + (1 - p'_{13}) \max(\bar{U}_1^I, \bar{U}_1^B) \\ &\Rightarrow \bar{U}_1^I < \bar{U}_1^B \end{aligned}$$

Otherwise we would get a contradiction

(ii) We now consider a type 2 agent. Let p'_{32} be the proportion of type 3 agent in trading post (1,2). Then, as before,

$$\begin{aligned} \bar{U}_2^B &= -b_{23} + \beta [p'_{32} \{u_2 + \max(\bar{U}_2^B, \bar{U}_2^I)\} \\ &\quad + (1 - p'_{32}) \max(\bar{U}_2^B, \bar{U}_2^I)] \\ \Rightarrow (1 - \beta) \bar{U}_2^B &\geq -b_{23} + \beta p'_{32} u_2 \end{aligned} \quad (A.3u)$$

and

$$\bar{U}_2^I = -\frac{b_{23}}{1-\beta} \Rightarrow (1 - \beta) \bar{U}_2^I = -b_{23}$$

Using (A.3u) we get

$$\bar{U}_2^B > \bar{U}_2^I$$

(iii) Finally, we consider a type 3 agent. Clearly,

$$\bar{U}_3^B = \frac{-b_{31}}{1-\beta}$$

If a type 3 agent opts for indirect trade he has to first visit (1,2) trading post and then the (2,3) trading post. Given the strategies of type 1 and type 2 agents, he would meet a complementary trading partner in both these trading posts with probability 1.

$$\begin{aligned}
\therefore \bar{U}_3^I &= -b_{31} + \beta[-b_{32} + \beta(u_3 + \max(\bar{U}_3^B, \bar{U}_3^I))] \\
\Rightarrow (1 - \beta^2)\bar{U}_3^I &\geq -b_{31} - \beta b_{32} + \beta^2 u_3 \\
\Rightarrow \bar{U}_3^I &\geq -\frac{b_{31}}{1 - \beta^2} + \frac{\beta}{1 - \beta^2}[-b_{32} + \beta u_3] \\
&= -\frac{b_{31}}{1 - \beta} + \frac{\beta}{1 - \beta^2} b_{31} + \frac{\beta}{1 - \beta^2}[-b_{32} + \beta u_3] \\
&= -\frac{b_{31}}{1 - \beta} + \frac{\beta}{1 - \beta^2}[(b_{31} - b_{32}) + \beta u_3]
\end{aligned}$$

$$\therefore \bar{U}_3^B < \bar{U}_3^I \quad \text{if}$$

$$(b_{31} - b_{32}) + \beta u_3 > 0, \quad \text{where} \quad b_{31} - b_{32} < 0$$

Condition (3.3) of Kiyotaki and Wright (1989) implies that

$$u_k > \frac{b_{kk^*} - b_{kc^*}}{1 - \beta}, \forall k \in K, \forall c^* \in C$$

$\therefore$ for a type 3 agent (taking $c^* = 2$) we have,

$$\begin{aligned}
u_3 &> \frac{b_{31} - b_{32}}{1 - \beta}, \\
\Rightarrow (1 - \beta)u_3 &> b_{31} - b_{32} \\
\therefore \text{if } \beta &> 1 - \beta \text{ we have} \\
\beta u_3 &> b_{31} - b_{32} \\
\Rightarrow \bar{U}_3^B &< \bar{U}_3^I
\end{aligned}$$

Thus, $\beta > 1 - \beta$ or $\beta > 1/2$ gives a *sufficient condition* under which there will be a steady state Nash equilibrium with good 2 acting as a unique medium of exchange. As in the case of the previous equilibrium $p'_{31} = p'_{32} = 1/2$.

Let us now consider the welfare derived by a type i agent for a steady state Nash equilibrium with good 2 acting as a unique medium of exchange.

$$\begin{aligned}\overline{WF}_1 &= -b_{12} + \frac{u_1\beta}{2} \\ \overline{WF}_2 &= -b_{23} + \frac{u_2\beta}{2} < -\frac{b_{23} + b_{21}}{2} + \frac{u_2\beta}{2} \\ \overline{WF}_3 &= \frac{-(b_{31} + b_{32})}{2} + \frac{u_3\beta}{2} < -b_{31} + \frac{u_3\beta}{2}\end{aligned}$$

Thus, comparing with the earlier equilibrium where good 1 is the unique medium of exchange, we see that welfare values are lower for type 2 and type 3 agents and the same for type 1, when good 2 is the medium of exchange. Thus, the former equilibrium Pareto dominates the latter.

Proceeding in a exactly parallel fashion, it can be shown that good 3 acting as a unique medium of exchange can be established as a steady state Nash equilibrium (for $\beta > 1/2$). However, the equilibrium where good 1 is acting as a medium of exchange is Pareto superior. Thus, we have

Proposition A.3.1. For the economy $\mathcal{E}$ (described above), there exists, a steady state Nash equilibrium with good 1 acting as the unique medium of exchange under all parameter values. On the other hand, under the condition $\beta > 1 - \beta$, there can exist steady state Nash equilibria with good 2 or good 3 acting as the unique medium of exchange. However, the equilibrium with good 1 acting as a unique medium of exchange dominates the other two equilibria.

List of Symbols Used in Chapter 3:

- (1) $\hat{T}_{TP}$: total time taken to attain equilibrium under trading post set-up.
- (2) $\hat{T}_{ML}$: total time taken to attain equilibrium under marketless trading arrangement.
- (3) K : total number of agents as well as the set of all agents.
- (4) k : index for an agent.
- (5) t_i^j : time taken by agent j to exchange information as well as goods with agent i .
- (6) t^j : total time taken by agent j in one round.
- (7) c : index for commodity
- (8) C : set of commodities except M .
- (9) M : commodity acting as a medium of exchange
- (10) n : total number of commodities
- (11) (c_1, c_2) : trading post or, equivalently market where good c_1 is traded against c_2 .
- (12) T_c^k : time taken to trade by agent k in trading post (c, M)
- (13) $\hat{E}$: an economy with different “types” of agents where the agents belonging to one specific type have identical characteristics.
- (14) $\hat{E}_\nu$: ν fold replication of $\hat{E}$ defined in (13)
- (15) K_c^S : the set of suppliers of good c
- (16) K_c^D : the set of demanders of good c

- (17) z_{kc} : agent k 's excess demand for or excess supply of good c
- (18) χ_i^ε : rational number from the interval
 $[z_{ic} - \varepsilon, z_{ic} + \varepsilon], \varepsilon > 0, i \in K_c^D$
- (19) Ω_j^ε : rational number from the interval
 $[z_{jc} - \varepsilon, z_{jc} + \varepsilon], \varepsilon > 0, j \in K_c^S$
- (20) $[T_c]_\varepsilon$: time taken to attain χ_i^ε 's and Ω_j^ε 's, $\forall i \in K_c^D$ and $\forall j \in K_c^S$
- (21) $[\hat{T}_{TP}]_\varepsilon$: time taken to arrive at the ε neighbourhood of a competitive allocation under trading post set-up
- (22) $[\hat{T}_{ML}]_\varepsilon$: same as (21) when trading post set-up is replaced by marketless trading arrangement.
- (23) $\hat{T}_{TP}(\nu)$: same as (21) when the economy replicates ν times
- (24) $\hat{T}_{ML}(\nu)$: same as (22) when the economy replicates ν times.
- (25) $\mathcal{Q}, \mathcal{Q}^c, \mathcal{N}$: the sets of rational numbers, irrational numbers and positive integers respectively.
- (26) t : index for time period
- (27) δ^t, ψ^t : number of buyers and sellers *present* at a particular trading post in period t .
- (28) b_{kc} : cost incurred by agent k for storing good c for 1 period.
- (29) U_k : utility derived by a type k agent in Kiyotaki and Wright kind of framework.
- (30) $\mathbf{T}_k$: optimal strategy of a type k agent

- (31) $\bar{p}$: steady state inventory distribution
- (32) L_k : agent k 's disutility from production
- (33) β : discount factor
- (34) u_k : net utility, i.e. $U_k - L_k$
- (35) $\bar{U}_k^B, \bar{U}_k^I$: respective utility derived by a type k agent when he goes through direct barter and indirect trade in a trading post set-up.
- (36) g : greatest common divisor of the elements of the set $A = \{\chi_i^C, \Omega_j^C, c \in C, i \in K_c^D, j \in K_c^S\}$.
- (37) K_c^{Dt}, K_c^{St} : the sets of unsatisfied demanders and suppliers (resp.) of good c at the beginning of period t .
- (38) $\tilde{W}_i(t)$: function measuring waiting time.
- (39) η_k^t : number of trading posts that trader k wishes to visit at the beginning of period t .
- (40) ρ_{kc}^t : probability that an agent $k \in K_c^{Dt}$ (or $k \in K_c^{St}$) is present in trading post (c, M) at period t .
- (41) V_{kc}^t : agent k 's unsatisfied demand for good c in the beginning of time period t (in value terms)
- (42) $\hat{V}_i^t$: equals $\max_{i \in K_c^{Dt}} V_{ic}^t + \max_{j \in K_c^{St}} V_{jc}^t$
- (43) $\tilde{R}_t$: reduction that occurs to $\tilde{V}_i^t$ in period t .
- (44) p'_{21} : proportion of type 2 agent present in (1,2) trading post.

- (45) $WF_i, \overline{WF}_i$: welfare derived by a type i agent following fundamental strategies in marketless trading arrangement and trading post set-up respectively.
- (46) WF_i^* : welfare derived by a type i agent under a set of non-implementable strategies given in Kiyotaki and Wright (1989)
- (47) Δ_{21} : indirect utility derived by a type 2 agent trading in the (1,2) trading post.

CHOICE OF INDIRECT TRADE IN A TRADING POST SET-UP

ABSTRACT

This chapter provides a simplified treatment of the problem of an endogenous choice of a medium of exchange in a trading post set-up for a more general endowment demand pattern than that of Jones (1975) and Kiyotaki and Wright (1989). However, the main emphasis of the chapter lies in examining how an optimal mixture of direct barter and indirect trade may be determined for such an economy.

4.1. INTRODUCTION

The discussion of Chapter 3 establishes the fact that the social institution of markets gives rise to a more efficient trading arrangement as compared to the marketless set-up. In this connection we have assumed so far (in Chapters 2 and 3, except in Section 3.4) that there exist only those trading posts which deal with the exchange of commodities against a *given* money good. However, a question that remains unanswered here is how an appropriate medium of exchange gets endogenously selected by the system. This question cannot be ignored since trading occurs in a decentralized fashion. The present chapter is partly concerned with this problem. It then proceeds to look into another related and no less important question, viz. having once chosen the medium of exchange, how do agents apportion their time between direct barter and indirect trade. Indeed, as is evident from the work of Oh (1989), the coexistence of the two forms of trade cannot be ruled out on logical grounds.

These two questions have been answered either singly (by Jones (1975)) or

jointly (by Oh (1989), Kiyotaki and Wright (1989) and others) in the literature for a marketless trading arrangement. It is well-known how these authors have shown the emergence of some particular good (or particular goods) as a medium (the media) of exchange through decentralized trades. As opposed to these exercises, we try to pose the problem for a trading post set-up¹. This is particularly relevant due to the fact that as soon as one introduces trading posts dealing with all the different pairs of goods in Jones' or in Oh's economy (which is characterized by a *complete* mutual coincidence of wants), the problem of decentralized trading as well as the need for a medium of exchange vanishes immediately. This indicates that monetized trade will co-exist with trading posts only if the assumption of a "perfect mutual coincidence of wants" is removed.

The removal of this assumption is also necessary in the interest of reality, since perfect mutual coincidence of wants is rarely observed. In what follows however, the assumption will only be partially relaxed, for it is equally rare to find the polar opposite of an economy characterized by a complete absence of want coincidence (for example, as in Kiyotaki and Wright (1989)).

Accordingly, we consider a three-good economy (in Walrasian equilibrium) with 3 trading posts to deal with the different pairs of goods, where for every pair of goods (c_1, c_2) there exist demanders for as well as suppliers of one good against the other, even though the number of demanders may not exactly match the number of suppliers.

Thus there are two distinctive features of the model under consideration: *the existence of trading posts and imperfect mutual coincidence of wants*. For a system of this nature we consider first the question of an endogenous choice of a medium of exchange². It can be shown, however, that if we approach the

¹In this context one can mention the work of Iwai(1988).

²It should be recalled here that the problem of choice of a medium of exchange for an

problem in the spirit of Kiyotaki and Wright (1989) (Section 3.4), then there will be three *potential* pure strategy Nash equilibria, each one associated with a different good acting as a medium of exchange. But these equilibria would be welfarewise non-comparable with the result that the medium of exchange becomes indeterminate.

On the other hand, a more *definite* conclusion is possible via the approach of Jones and Oh. This line of argument compares the *waiting times* for direct barter and indirect trade and asks if such comparisons allow any inferences to be made regarding the choice of a medium of exchange. Proceeding along these lines, we provide a sufficient condition on the waiting times involved in direct barter (for the different markets) under which a particular good may be chosen uniquely as a medium of exchange.

It is worth noting at the outset however, that this condition, being rather strong, is open to criticisms. Our response to such criticisms would be that the choice of a *particular* medium of exchange is not the major thrust of this chapter. Indeed the emphasis of the chapter is more on the associated problem of finding the optimal mix of direct barter and indirect trade that agents employ in a trading post economy. Under the marketless trading arrangements (of Oh (1989) or for that matter Kiyotaki and Wright (1989)), an agent wishing to have a mix of direct barter and indirect trade confronts another agent at random to explore the trading opportunities prevailing between them. Naturally, if they find their wants to be mutually coinciding, direct barter takes place. Otherwise, on the basis of transaction costs considerations, they decide whether or not to go for indirect trade. Therefore, the decision regarding *when to go for direct barter* is an obvious one. On the other hand, in the presence of separate trading posts dealing with different pairs of goods, there emerges a new problem economy with complete absence of mutual coincidence of wants is already discussed (in the Appendix to Chapter 3) under a trading post set-up.

relating to the choice of the timings of one's visit to trading posts for direct barter vis-a-vis those relevant for indirect trade.

This turns out to be a problem by its own right in our set-up, since the existence of trading posts requires agents to decide on the appropriate probabilities with which the ones relevant for direct barter must be visited vis-a-vis the ones relevant for indirect trade (given of course a chosen medium of exchange). It is worth mentioning here that so far this problem has not received much attention in the literature.

This chapter is organized as follows. The next section describes the basic framework under consideration. Section 4.3 discusses the problem of the choice of indirect trade. In the penultimate section we proceed to develop a *decentralized* method for choosing an optimal mix of direct barter and indirect trade. However, the above exercise has its own *shortcomings* and a concluding section outlines a few of them. Details of the proofs of the result presented in this chapter are worked out in the Appendix.

4.2 THE FRAMEWORK

We consider a simple economy consisting of finitely many infinitely lived agents (index by k) and three indivisible commodities, $c = 1, 2, 3$. As in Kiyotaki and Wright (1989), each individual is initially endowed with a particular good and derives utility only from the consumption of another specific good. An agent has only one unit of a good at a time against which he can attain exactly one unit of another good. The initial endowment of agent k would be denoted by w_k and his final demand by x_k . For the sake of brevity, though at the cost of a misuse of terminology, the expression "demand for" and "supply of" a commodity will also be interpreted all through this chapter to include the acquisition as well as disposal of the medium of exchange. In other words, demand for a good will *not* always refer to demand for one's final consumption

goods. Similarly for the case of supply.

The problem before the agents is to acquire the respective x_k 's through a decentralized process of trade. It is assumed that as soon as an agent obtains a good that he finally demands, he consumes it immediately and gets reendowed with his initial endowment³.

For the 3-good economy under review, there exist three trading posts to deal with the different pairs of goods, viz. (1,2) to exchange good 1 against good 2, (2,3) for trading good 2 against good 3 and (1,3) to deal with good 3 against good 1. Therefore, an agent desiring to exchange w_k against x_k has two choices. First, to exchange the respective goods directly; and secondly to exchange w_k against some other good (say, $\bar{c}$) and then acquire x_k through transactions in the $(\bar{c}, x_k)$ trading post.

Time is divided into discrete intervals of equal length called periods. Agents desiring to exchange a particular pair of goods, say (c_1, c_2) , go to the relevant trading post (where a buyer of c_1 can recognize a seller of the same good). Each individual then tries to meet a complementary trading partner. The term "*complementary trading partner*" of agent k , refers to another agent with whom k can carry out (at least) one desired transaction. It is assumed that it takes one period of time to complete a transaction with a complementary trading partner.

Suppose $D_{c_1c_2}$ denotes the set of *demanders of the (c_1, c_2) trading post*. This term refers to the agents who demand (in the sense defined above) c_1 (i.e. the first component of (c_1, c_2)) against c_2 . Similarly, let $S_{c_1c_2}$ denote the set of suppliers of good c_1 against c_2 . By the term *indigenous demanders* of the (c_1, c_2) trading post, we will refer to those agents who are initially endowed with good c_2 and demand good c_1 for final consumption. Similarly, by the

³Following Kiyotaki and Wright (1989) one can introduce production into the picture. It would not change the basic conclusions of the present model.

term *indigenous suppliers* of the (c_1, c_2) trading post we will refer to the agents who are final demanders of good c_2 against their initial endowment of good c_1 . We use the notations $ID_{c_1c_2}$ and $IS_{c_1c_2}$ to refer respectively to the sets of indigenous demanders and suppliers at the (c_1, c_2) trading post. Clearly, $ID_{c_1c_2} \subseteq D_{c_1c_2}$ and $IS_{c_1c_2} \subseteq S_{c_1c_2}$. Let $\mathcal{I}_{c_1c_2} = IS_{c_1c_2} \cup ID_{c_1c_2}$.

If $\#D_{c_1c_2} \geq \#S_{c_1c_2}$, then the demanders (suppliers) are said to be on the long side (short side) of the market. Let $L_{c_1c_2}$ and $\ell_{c_1c_2}$ denote respectively the set of *indigenous* traders belonging to the long side and short side of the (c_1, c_2) trading post. Clearly, an agent of the set $L_{c_1c_2}$ may not be able to meet a complementary trading partner in some period, in which case he wastes one unit of time. A rational agent, say k , always desires to *reduce the waiting time* associated with the transformation of w_k into x_k . It is on the basis of this time cost that an agent decides whether or not to opt for indirect trade and which good to use as an intermediate commodity. The latter problem is indeed simple in a three good economy. If an agent $k \in ID_{12}$ opts for indirect trade, he will necessarily have to use good 3 as the medium of exchange and so on. The problem therefore is to examine when an agent will in fact find it profitable to go for indirect trade. In order to focus on this issue, one has to know about the state of the excess demand (or, supply) prevailing in the trading posts. Since each agent demands (supplies) one unit of a good, the number of demanders (suppliers) also represents the level of demand (supply) in that market.

Let us denote $\#ID_{c_1c_2}$ by $\delta_{c_1c_2}$ and $\#IS_{c_1c_2}$ by $\psi_{c_1c_2}$. In a three good economy depending on the relationship between $\delta_{c_1c_2}$ and $\psi_{c_1c_2}$ one can arrive at two different situations. Without loss of generality, consider in particular the (1,2) trading post and assume that $\delta_{12} > \psi_{12}$, i.e. the number of indigenous demanders of good 1 against good 2 is larger than the number of indigenous suppliers of good 1 for good 2.

In a competitive economy there has to be a matching (indigenous) excess

supply of good 1 in the (1,3) trading post. This means that $\psi_{13} > \delta_{13}$.

This in turn ensures an excess supply of good 3 in the (2,3) trading post. Thus, $\delta_{23} > \psi_{23}$.

Also, in a competitive equilibrium, the excess demand for good 1 in the (1,2) trading post should exactly match the excess supply of good 1 in the (1,3) trading post. Since each agent demands (and supplies) only one unit of a good, we arrive at the following condition:

$$Z_{12} = |\delta_{12} - \psi_{12}| = Z_{13} = |\delta_{13} - \psi_{13}|$$

For exactly similar reasons,

$$Z_{13} = |\delta_{13} - \psi_{13}| = Z_{23} = |\delta_{23} - \psi_{23}|.$$

Thus given the relation $\delta_{12} > \psi_{12}$ and the level of $Z_{12}(= |\delta_{12} - \psi_{12}|)$, excess demands and supplies of the remaining two trading posts are determined.

Alternatively, if we have $\delta_{12} < \psi_{12}$ then a parallel reasoning ensures that

$$\delta_{13} > \psi_{13} \text{ and } \delta_{23} < \psi_{23}$$

Thus, for a three-good economy, we have either one of the following two kinds of situations.

Model I: Characterized by $\delta_{12} > \psi_{12}$

Model II: Characterized by $\delta_{12} < \psi_{12}$.

Here we are mainly concerned with model I. One can deal with model II in a similar fashion.

In our 3-good economy let us call the indigenous demanders of good 1 against good 2 *type 1* agents and the suppliers of good 1 against good 2 *type 2* agents. Similarly the demanders of good 1 against good 3 are called *type 3* agents and those of good 3 against good 1 are known as *type 4* agents. Demanders of the (2,3) trading post are alternatively called *type 5* agents and the suppliers are known as *type 6* agents.

It should be noted that an agent belonging to the long side of a market may not meet a complementary trading partner every period. This implies

that the waiting time involved here is a random variable. Keeping this in mind, we assume that

Assumption 4.1: The agents choose indirect trade or direct barter according to the average (or expected) waiting time involved with these two operations.

We also assume that

Assumption 4.2: Traders of the same type behave similarly in the sense that they opt for direct and indirect trade with the same probability.

The background motivation for the exercise may be understood by means of the following scenario. Agents are supposed to have been going through direct barter for a long time (say, upto period t_0) and have a fairly clear idea about the expected waiting time involved in direct barter. As of period t_0 , let $T_{c_1c_2}^B(T_{c_1c_2}^I)$ and $\tau_{c_1c_2}^B(\tau_{c_1c_2}^I)$ represent respectively the waiting times involved in direct barter (indirect trade) for agents belonging to the long and short sides of trading post (c_1, c_2) .

Assumption 4.3: Each agent knows the common expected waiting time in the trading post relevant for his direct barter.

Clearly, the waiting time involved in a trading post is derived from the frequency of meeting complementary trading partners in that market. The latter depends on the chance or probability of meeting a complementary trading partner in the trading post. Let q_{12}^B denote the probability of meeting a complementary trading partner in trading post $(1, 2)$ for an agent $k \in L_{12} \cap I_{12}$.

If however k belongs to the short side i.e. $k \in \ell_{12}$, then the probability of meeting a complementary trading partner is always unity. Similarly for other trading posts.

Regarding the probabilities q_{12}^B etc., we assume without loss of generality that

Assumption 4.4: $q_{23}^B < q_{13}^B < q_{12}^B$

This completes the background scenario for the exercise. In the next section

we take up the problem of the choice of a medium of exchange together with an appropriate mix of direct barter and indirect trade.

4.3 CHOICE OF INDIRECT TRADE

Given the basic physical environment, this section is mainly going to address two issues presented in two separate subsections. The first subsection begins with an inquiry into the conditions which would make indirect trade profitable for an agent. It is be revealed in this subsection that an agent desiring to go for indirect trade will in fact opt for a mix of direct barter and indirect trade. Consequently we study here, the problem of choosing an *optimal* mix of the two under a trading post set-up.

On the basis of the discussion of Subsection 4.3.1, conditions under which a medium of exchange is uniquely chosen by the agents are derived in Subsection 4.3.2.

4.3.1 Transition from Direct Barter to Indirect Trade

In an economy with trading posts, each indigenous trader (k) of the (c_1, c_2) trading post (i.e. $k \in \mathcal{I}_{c_1 c_2}$) either belongs to $L_{c_1 c_2}$ or $\ell_{c_1 c_2}$. For an indigenous trader $k \in \ell_{c_1 c_2}$ we have,

Lemma 4.1: An agent $k \in \mathcal{I}_{c_1 c_2} \cap \ell_{c_1 c_2}$ does not have any incentive to go for indirect trade at the end of period t_0 .

Proof: We recall that as of t_0 , $\tau_{c_1 c_2}^I$ and $\tau_{c_1 c_2}^B$ represent the levels of waiting time involved in indirect trade and direct barter respectively, for an agent belonging to the short side of the market. It will also be remembered that upto period t_0 the agents are engaged in direct barter only.

$$\therefore E(\tau_{c_1 c_2}^I) \geq 2$$

whereas as of t_0

$$E(\tau_{c_1 c_2}^B) = 1 < E\tau_{c_1 c_2}^I$$

Thus, direct barter is more profitable. Given Assumption 4.3, any agent $k \in \mathcal{I}_{c_1 c_2} \cap \ell_{c_1 c_2}$ will not go for indirect trade. ■

Lemma 4.2: If in an economy

$$\delta_{c_1 c_2} = \psi_{c_1 c_2} \quad \forall (c_1, c_2)$$

then no indirect trade would prevail.

Proof: Clear from Lemma 4.1

Remark 4.1: Lemma 4.2 shows why a trading post set-up removes the need for a medium of exchange in Jones' and Oh's models.

We next consider an agent $k \in \mathcal{I}_{c_1 c_2} \cap L_{c_1 c_2}$. Clearly, his decision to opt for indirect trade depends on the expected waiting time involved in the direct barter itself. For an indigenous trader (WLOG a demander say) belonging to the long side of trading post (c_1, c_2) , the initial probability (i.e. as of t_0) of meeting a complementary trading partner is

$$q_{c_1 c_2}^B = \frac{\psi_{c_1 c_2}}{\delta_{c_1 c_2}} (< 1)$$

The following lemma gives the corresponding *expected* waiting time $ET_{c_1 c_2}^B$.

Lemma 4.3: $ET_{c_1 c_2}^B = \frac{1}{q_{c_1 c_2}^B} = \frac{\delta_{c_1 c_2}}{\psi_{c_1 c_2}}.$

Proof: Given in the Appendix.

On the other hand, for any arbitrary agent, the expected waiting time involved in indirect trade is at least two units.

Suppose for our 3-good economy any of the following three situations hold.

Case I: $ET_{12}^B < 2$

$$ET_{13}^B < 2$$

$$ET_{23}^B < 2$$

Then no agent will have any incentive to opt for indirect trade.

Case II: $ET_{12}^B < 2$

$$ET_{13}^B < 2$$

$$ET_{23}^B > 2$$

Then any agent $k \in \mathcal{I}_{23} \cap L_{23}$ will have the incentive to explore possibilities of indirect trade involving good 1 as a medium of exchange

Case III: $ET_{12}^B < 2$

$$ET_{13}^B > 2$$

$$ET_{23}^B > 2$$

In this situation, any agent $k \in \mathcal{I}_{23} \cap L_{23}$ or $k' \in \mathcal{I}_{13} \cap L_{13}$ will have the incentive to explore possibilities of indirect trade involving (respectively) goods 1 and 2 as media of exchange.

Note: Under Assumption 4.4 one cannot have $ET_{13}^B > 2$ and $T_{23}^B < 2$ etc.

The following discussion concentrates mainly on Case II. We recall that in the (2,3) trading post, demanders (of good 2) are in the long side of the market (i.e. $\delta_{23} > \psi_{23}$). Thus an agent $k \in ID_{23}$ intends to explore the possibilities of indirect trade.

The process of transition :

Suppose only the *demanders of the (2,3) trading post* decide to opt for indirect trade and all others continue to carry on direct barter. It is important

to ask then how many demanders would decide to get involved in indirect trade. This question is particularly relevant because any transition from direct barter to indirect trade would result in a change in the related waiting time.

In particular, if *only one* demander of the (2,3) trading post opts for indirect trade, he would indeed find indirect trade to take exactly two units of time. This is due to the fact that there is excess demand for good 3 in trading post (1,3) and excess supply of good 2 in the (1,2) trading post. Since direct barter takes more than two units of time it is always profitable for at least one agent to opt for indirect trade. Thus, if δ_{23}^B and δ_{23}^I denote respectively the number of indigenous demanders (of good 2) of the (2,3) trading post opting for direct barter and indirect trade, we have $\delta_{23}^I \geq 1$. On the other hand, it is clearly not optimal to have, $\delta_{23}^B < \psi_{23}$, where ψ_{23} is the number of indigenous suppliers at the (2,3) trading post. For, in that case the demanders opting for direct barter would move to the short side of the market and profit more as compared to those who have gone for indirect trade (Lemma 4.1). Consequently, some agents opting for indirect trade would shift back to direct barter and so on. Thus, the only meaningful equilibrium would be one where (at least eventually) direct and indirect trade become equally profitable. In notation, let T_{23}^{Bh} and T_{23}^{Ih} denote respectively the waiting time involved in direct barter and indirect trade in period t_h . Then for some $h' > 0$

$$ET_{23}^B \simeq ET_{23}^{Ih} \quad \forall h > h' \quad (4.1)$$

The question that naturally emerges now is how would the agents initiate a process of arriving at the optimal mix of direct barter and indirect trade in a decentralized fashion. This problem will be discussed in detail in the next section (i.e. Section 4.4). However, in the present section we would motivate the process of transition under more restrictive assumptions. In particular we

assume for the purpose of illustration that there exists a centralized mechanism, which chooses in an optimally randomized manner a certain number of traders from the set of indigenous demanders ID_{23} , to go for indirect trade every period. More precisely, $\delta_{23}(= \#ID_{23})$ is partitioned optimally into

$$\delta_{23} = \delta_{23}^B + \delta_{23}^I + \delta_{23}^J$$

such that δ_{23}^I agents go for indirect trade to trading post (1,3) at period t_1 (to sell good 3 against the medium of exchange) and δ_{23}^B plus δ_{23}^J agents stay back in the (2,3) trading post for direct barter. In period t_2 another batch of δ_{23}^J agents (out of $\delta_{23}^B + \delta_{23}^J$) go for indirect trade (from trading post (2,3) to the (1,3) market) and δ_{23}^B remain behind for direct barter.

It would be argued in a moment that those who had gone for indirect trade to trading post (1,3) at period t_1 would indeed finish their desired trade in one period. Therefore, in period t_2 they would move to the (1,2) trading post to acquire good 2 which they finally consume. In the (1,2) market, they would again take one unit of time to pass on the medium of exchange against good 2. Thus, at the end of the t_2 -th period, δ_{23}^J agents would finish indirect trade and get reendowed with good 3.

As mentioned above, a new set of δ_{23}^I agents would move to trading post (1,3) in period t_2 . Next period (i.e. in period t_3), out of the $\delta_{23}^I + \delta_{23}^B$ agents (with good 3), δ_{23}^B would be chosen for direct barter and δ_{23}^I for indirect trade. This pattern would then repeat in every period.

Here, δ_{23}^I and δ_{23}^B are optimally chosen such that the following holds after period $t_1 : \forall h \geq 2$

$$ET_{23}^{Bh}(\delta_{23}^B) \simeq ET_{23}^{Ih}(\delta_{23}^I) \quad (4.2)$$

It is revealed from this process that even in the presence of a centralized mechanism, there is an element of voluntariness involved in the above process

in the sense that no agent is forced against his will to become an indirect trader. The equilibrium is so constructed that an agent is indifferent between direct barter and indirect trade.

We now propose to show (through Lemma 4.4) that indirect trade indeed takes two units of time after the transition takes place. More precisely,

Lemma 4.4: Suppose δ_{23} is partitioned into $\delta_{23}^B, \delta_{23}^I, \delta_{23}^I$, i.e. $\delta_{23} = \delta_{23}^B + 2\delta_{23}^I$ s.t. condition (4.2) is satisfied. Then for any agent k opting for indirect trade at period t_k , $ET_{23}^{Ih} = T_{23}^{Ih} = 2$.

Proof: As argued earlier, for optimality we require

$$\delta_{23}^B > \psi_{23}$$

Where, ψ_{23} is the number of indigenous suppliers at trading post (2,3). Therefore,

$$\delta_{23} = \delta_{23}^B + \delta_{23}^I + \delta_{23}^I > \psi_{23} + 2\delta_{23}^I$$

$$\Rightarrow \delta_{23} - \psi_{23} > 2\delta_{23}^I$$

$$\Rightarrow Z_{23} > \delta_{23}^I \quad (4.3)$$

In competitive equilibrium

$$Z_{23} = Z_{13} = Z_{12} \quad (4.4)$$

$\therefore$ (4.3) and (4.4) $\Rightarrow$

$$\delta_{23}^I < Z_{13} = \psi_{13} - \delta_{13} \quad (4.5)$$

and

$$\delta_{23}^I < Z_{12} = \psi_{12} - \delta_{12} \quad (4.6)$$

Since in this context we consider a situation where only the agents belonging to the set ID_{23} opt for indirect trade and all other agents are involved in direct barter therefore, (4.5) and (4.6) ensure that indirect trade takes 2 units of time.

$$\therefore T_{23}^I = 2$$

■

From the above discussion we arrive at the following self-evident lemma.

Lemma 4.5: Indirect trades for a pair of goods cannot take place unless direct barter occurs simultaneously.

Thus this discussion provides us with an idea about the process of transition from direct barter to indirect trade, on the basis of which, we will now (in the next subsection) identify the conditions under which a medium of exchange gets chosen uniquely.

4.3.2 Choice of a Medium of Exchange:

In the previous subsection we broadly discussed situations (through Case I, II and III) which induce an agent to opt for direct barter or indirect trade. Presently we intend to discuss these conditions a little more rigorously. In this respect Case II above emphasizes the following set of inequalities,

$$(4.7) \quad \begin{cases} ET_{12}^B < 2 \\ ET_{13}^B < 2 \\ ET_{23}^B > 2 \end{cases}$$

Clearly, these conditions ensure that as of period t_0 , any agent $k \in L_{23}$ has an incentive to opt for indirect trade. It can however be seen that (from Proposition 4.1 below) even after the process of transition from direct barter to indirect trade takes place in a manner that is discussed earlier (in subsection 4.3.1), no other agent ($k' \in K \setminus L_{23}$) will change his decision. Thus,

Proposition 4.1: Condition (4.7) is sufficient to ensure a unique medium of exchange for the three good economy of Model I. However, this condition is not necessary.

Proof: Sufficiency:

Conditions (4.5) and (4.6) ensures that as of period t_0 , no agent $k \in \mathcal{I}_{12} \cup \mathcal{I}_{13}$ would opt for indirect trade. Since, $ET_{23}^B > 2 = ET_{23}^I$ agents belonging to the set ID_{23} will have incentive to opt for indirect trade at period t_1 .

Our next task is to examine whether, after the transition (of the agents from ID_{23}), indirect trade becomes profitable for indigenous traders of trading posts (1,2) or (1,3).

We first concentrate on the agents belonging to the short side of trading posts (1,2) and (1,3). Regarding the number of agents opting for indirect trade at period t_1 , we have:

$$\Rightarrow \delta_{23}^I < Z_{13} \quad \text{and} \quad (4.8)$$

$$\Rightarrow \delta_{23}^I < Z_{12} \quad (4.9)$$

Thus, even at period t_1 , there will remain an excess supply of good 1 (and hence an excess demand for good 3) in trading post (1,2). From the discussion of the previous subsection, it is clear that at period t_1 no indirect trade prevails in trading post (1,2). Therefore, it is obvious that any agent belonging to the short side of trading post (1,2) at period t_0 will remain so even at period t_1 . Furthermore, (4.6) implies that they will be in the short side at period t_2 too.

Let us now look at the agents belonging to $L_{13} \cap \mathcal{I}_{13}$. For them, this new inflow of δ_{23}^I complementary trading partners in period t_1 can only reduce the

waiting time. Since

$$ET_{13}^{B1} = \frac{\psi_{13}}{\delta_{13} + \delta_{23}^I} < \frac{\psi_{13}}{\delta_{13}} = ET_{13}^B$$

For the indigenous traders belonging to the long side of the (1,2) market, there is no change of waiting time in period t_1 (since according to the discussion of Subsection 4.3.1, no indirect trader will be present in trading post (1,2) at period t_1). Thus, for period t_2 we have

$$ET_{12}^{B2} = \frac{\delta_{12}}{\psi_{12} + \delta_{23}^I} < \frac{\delta_{12}}{\psi_{12}} = ET_{12}^B = ET_{12}^{B1}$$

This shows that there exists no incentive for indirect trade for the agents belonging to $K \setminus L_{23}$, in period t_1 and the same scenario repeats period after period.

Hence, condition (4.7) is sufficient to ensure the emergence of a unique medium of exchange.

To show that condition (4.7) is not necessary. Suppose that the second inequality in (4.7) is reversed (i.e. Case III of subsection 4.3.1 holds). Thus, as of period t_0 both the demanders of trading post (2,3) and the suppliers of trading post (1,3) will decide to opt for indirect trade.

Suppose the equilibrium partition of the indigenous suppliers of trading post (1,3) for choosing direct barter and indirect trade is

$$\psi_{13} = \psi_{13}^B + \psi_{13}^I + \psi_{13}^I$$

and that of agents belonging to ID_{23} is

$$\delta_{23} = \delta_{23}^B + \delta_{23}^I + \delta_{23}^I$$

This implies that there is an additional inflow of δ_{23}^I complementary trading partners (for any agent $k \in IS_{13}$) into trading post (1,3). This in turn will

reduce the waiting time for the indigenous suppliers of the (1,3) market in direct barter and thereby make direct barter less time consuming.

Thus for some values of ψ_{13}, δ_{13} and δ_{23}, δ_{23} , it is possible to have an equilibrium, where

$$\psi_{13}^I = 0 \quad \text{and} \quad \delta_{12}^I > 0$$

Thus there can exist an equilibrium with a unique medium of exchange even when (4.7) is not valid. ■

Remark 4.2: A similar method of analysis can be used to deal with the problem of the choice of a medium of exchange even when it is posed with reference to a more well-defined optimization exercise where time cost is incorporated in the utility calculation of the agents. This is discussed in the Appendix to Chapter 4.

The next section looks into the problem of a decentralized choice of direct and indirect trade.

4.4 CHOICE OF INDIRECT TRADE: A DECENTRALIZED APPROACH

The previous section discussed the conditions under which a good may be uniquely chosen as a medium of exchange. It has also been examined how a mix of direct barter as well as indirect trade can be meaningfully chosen. However, in the entire discussion it is assumed that the *optimal number* of agents required to go for indirect trade is chosen at random from the set of agents. This calls either for some amount of communication between the agents or centralized planning. The former may not be possible in a large economy. Regarding the latter option, it may be mentioned that this entire exercise is oriented towards reducing the role of a central planner.

Under these circumstances, it is relevant to ask whether (for a given choice of a medium of exchange) one can provide a decentralized device to the agents to choose a mix of direct barter and indirect trade. Here we make a *preliminary* attempt to deal with this issue.

Suppose good 1 is uniquely chosen as a medium of exchange. Thus agents belonging to the long side of trading post (2,3), i.e. the demanders of good 2 against 3, will profit if they opt for a mix of direct barter and indirect trade. A meaningful decentralized way of doing so is to choose an appropriate probability p , where p is the probability of going for direct trade. Clearly, in this case the number of agents going for indirect trade (or direct barter) is a random variable.

For this decentralized process of trade we assume the following.

Assumption 4.5: Indirect trade takes two units of time.

This is undoubtably a rather strong assumption to make. However, at this preliminary stage of our investigation, we cannot find a suitable alternative. We note that this assumption was not necessary in the earlier case (of Subsection 4.3.1) as the number of agents going for indirect trade was constant and therefore it was possible to show that indirect trade indeed took two units of time.

Given Assumption 4.5, our approach to the problem of decentralized choice may be broken up as follows.

(1) First, we show that for any *arbitrary* probability p of choosing direct barter, there exists a sequence of partitions of δ_{23} , viz.

$$\{\delta_{23}^{Bh}, \delta_{23}^{Ih}, \delta_{23}^{I(h-1)}\}, h = 1, 2, \dots$$

where δ_{23}^{Bh} is the number of agents from ID_{23} who opt for direct barter in period t_h , δ_{23}^{Ih} is the number of agents from ID_{23} who opt for indirect trade in period t_h (i.e. who visit trading post (1,3) in period t_h). We then demonstrate that this sequence of partitions converges in some sense, provided Assumption 4.5 is valid.

(2) Next, we find the appropriate probability p^* for which we eventually arrive at a partition that satisfies the no arbitrage condition (i.e. Condition (4.1)).

(3) Finally, we show in the Appendix that if such a p^* is chosen, then Assumption 4.5 is satisfied with a probability exceeding .99.

Since we are concerned only with the movements of the demanders of trading post (2,3) we would omit the subscript '23' for notational simplicity. Thus we denote the number of demanders opting for direct barter at period t_0 by δ instead of, $\delta_{23}^B (= \delta_{23})$ and the number of demanders going for direct barter in period t_h by $\delta^{(h)}$, where $\delta^{(h)} = \delta_{23}^{Bh}$. The number of demanders choosing indirect trade in period t_h i.e. δ_{23}^I is replaced by $\eta^{(h)}$. In any period t_h there are $\delta^{(h)}$ demanders in trading post (2,3), $\eta^{(h)}$ indirect traders in trading post (1,3) and $\eta^{(h-1)}$ indirect traders in trading post (1,2) such that

$$\delta^{(h)} + \eta^{(h)} + \eta^{(h-1)} = \delta.$$

We first intend to show that the partition $\{E\delta^{(h)}, E\eta^{(h)}, E\eta^{(h-1)}\}$ converges. To get a clear picture of the problem we proceed in two steps. *Initially*, we will show the convergence of the partition under the assumption that in every period the realized value of $\delta^{(h)}$ coincides the $E\delta^{(h)}$. In the *next step* we remove this assumption and establish the convergence more generally.

Step 1: Let us begin with period t_1 .

In period t_1 , a total of δ agents choose direct barter *with probability* p . Thus $\delta^{(1)}$, i.e. the actual number of demanders coming to trading post (2,3) in period t_1 , can be looked upon as the number of successes out of δ Bernoulli trials with p as the probability of success.

$\therefore$ the distribution of $\delta^{(1)}$ is

$$\binom{\delta}{\delta^{(1)}} p^{\delta^{(1)}} (1-p)^{\delta-\delta^{(1)}}$$

$$\therefore E(\delta^{(1)}) = \delta.p = \tilde{\delta}^{(1)}, \text{ say.}$$

Similarly,

$$E(\eta^{(1)}) = E(\delta - \delta^{(1)}) = \delta(1 - p) = \tilde{\eta}^{(1)}, \text{ (say).}$$

Clearly, nobody has gone to trading post (1,2) for indirect trade in period t_1 since in order to go to trading post (1,2) one has to first acquire good 1 from the (1,3) market.

Therefore, in period t_1 the partition $\{\tilde{\delta}^{(1)}, \tilde{\eta}^{(1)}, \eta^{(0)}\}$ is

$$\{\delta p, \delta(1 - p), 0\}$$

We next move to period t_2 .

Period t_2 :

According to our assumption, the realized value of $\delta^{(1)}$ coincides with $\tilde{\delta}^{(1)}$.

$\therefore$ in period t_2 , a total of $\tilde{\delta}^{(1)}$ traders are opting for direct barter with probability p .

$\therefore$ arguing as before

$$E(\delta^{(2)} | \delta^{(1)} = \tilde{\delta}^{(1)}) = p.\tilde{\delta}^{(1)} = p^2\delta = \tilde{\delta}^{(2)}, \text{ say.}$$

Under Assumption 4.5, those who had gone for indirect trade (i.e. to trading post (1,3)) in period t_1 , will go to trading post (1,2) in period t_2 . Thus the number of agents choosing indirect trade in period t_2 , denoted by $\eta^{(2)}$ is

$$\eta^{(2)} = \delta^{(1)} - \delta^{(2)}$$

$$\begin{aligned} E(\eta^{(2)} | \delta^{(1)} = \tilde{\delta}^{(1)}) &= p.\delta - p.\tilde{\delta}^{(1)} \\ &= p.(1 - p).\delta \end{aligned}$$

$\therefore$ in period t_2 , the partition is

$$\{p^2\delta, p(1 - p)\delta, (1 - p)\delta\}$$

Period t_3 :

The random variable $\delta^{(3)}$, which denotes the number of agents going for direct barter in period t_3 can be separated into two parts.

$$\delta^{(3)} = \bar{\delta}^{(3)} + \tilde{\delta}^{(3)}$$

where $\bar{\delta}^{(3)}$ agents have come from a total of $\delta^{(2)}$ agents who opted for direct barter in period t_2 and $\tilde{\delta}^{(3)}$ agents have come from a total of $\eta^{(1)}$ agents who finished indirect trade at the end of period t_2

$$\begin{aligned} \therefore E(\bar{\delta}^{(3)} | \delta^{(2)} = \tilde{\delta}^{(2)}) &= p \cdot \tilde{\delta}^{(2)} \\ \text{and } E(\tilde{\delta}^{(3)} | \delta^{(1)} = \tilde{\delta}^{(1)}, \delta^{(2)} = \tilde{\delta}^{(2)}) &= p \tilde{\eta}^{(1)} \\ \therefore E\delta^{(3)} &= E(\bar{\delta}^{(3)} + \tilde{\delta}^{(3)}) \\ &= p\tilde{\delta}^{(2)} + p \cdot \tilde{\eta}^{(1)} = \tilde{\delta}^{(3)}, \text{ say} \end{aligned}$$

Similarly,

$$E\eta^{(3)} = (1 - p)(\tilde{\delta}^{(2)} + \tilde{\eta}^{(1)}) = \tilde{\eta}^{(3)}, \text{ say}$$

and so on.

It can be shown (see Appendix to Chapter 4) that the elements of the sequence $\{\tilde{\delta}^{(h)}\}$ satisfy the following useful property :

$$\tilde{\delta}^{(h)} = p \cdot \tilde{\delta}^{(h-1)} + (1 - p) \tilde{\delta}^{(h-2)}.$$

Thus, we arrive at the following result.

Proposition 4.2: Consider the three-good economy described by Model 1 (Section 4.2). Suppose good 1 is the unique medium of exchange. Let $\delta^{(h)}$ and $\eta^{(h)}$ denote respectively the number of agents opting for direct barter and indirect trade in period t_h . Suppose,

$$E\delta^{(1)} = \tilde{\delta}^{(1)}$$

$$E(\delta^{(2)} | \delta^{(1)} = \tilde{\delta}^{(1)}) = \tilde{\delta}^{(2)}$$

$$E(\delta^{(3)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \delta^{(2)} = \tilde{\delta}^{(2)}) = \tilde{\delta}^{(3)} \text{ etc.}$$

then the sequence $\{\tilde{\delta}^{(h)}\}$ converges and hence the partition $\{\tilde{\delta}^{(h)}, \tilde{\eta}^{(h)}, \tilde{\eta}^{(h-1)}\}$ converges too.

Proof: See Appendix to Chapter 4.

Step 2:

Proposition 4.2 is based on a rather restrictive assumption viz. that for all t , $\delta^{(t)}$ coincides with $\tilde{\delta}^{(t)}$. We next try to remove this assumption by showing that even if the number of agents going for direct barter does coincide with the respective mean values in every period, the sequence $\{E\delta^{(t)}\}$ converges .

Proposition 4.3: Consider the set-up described in Proposition 4.2. Then

$$(i) \quad E\delta^{(h)} = \tilde{\delta}^{(h)}, \forall h.$$

Hence,

$$(ii) \quad \{E\delta^{(h)}\} \text{ converges and}$$

$$(iii) \quad \text{the partition } \{E\delta^{(h)}, E\eta^{(h)}, E\eta^{(h-1)}\} \text{ converges too.}$$

Proof: See Appendix to Chapter 4.

$$\text{Corollary 4.3: } \lim_{h \rightarrow \infty} E\delta^{(h)} = \delta p^2 + \delta p(1-p)\frac{1-p}{2-p}$$

Proof: See Appendix to Chapter 4.

Under Assumption 4.5, indirect trade takes two units of time. Therefore, for the no arbitrage condition to be satisfied eventually one needs

$$\lim_{h \rightarrow \infty} ET_{23}^{Bh} \simeq 2$$

Here,

$$ET_{23}^{Bh} = EE(T_{23}^{Bh}|\delta_{23}^{(h)}) = E\left(\frac{\delta^{(h)}}{\psi_{23}}\right) = \frac{E\delta^{(h)}}{\psi_{23}}$$

$\therefore$ we need,

$$\lim_{h \rightarrow \infty} E\delta^{(h)} = 2\psi_{23} \quad (4.10)$$

Using Corollary 4.3 we can now find the value of p such that condition (4.10) is satisfied. That is,

$$\begin{aligned} \delta p^2 + \delta p(1-p) \frac{1-p}{2-p} &= 2\psi, \delta_{23} = \delta, \psi_{23} = \psi. \\ \Rightarrow \delta p \frac{1}{2-p} &= 2\psi \\ \Rightarrow p^* &= \frac{4\psi}{\delta + 2\psi} = \frac{4}{ET_{23}^B + 2} \end{aligned} \quad (4.11)$$

Thus, if the indigenous demanders of trading post (2,3) choose direct barter with probability p^* , they can eventually go to a steady state where the no arbitrage condition is satisfied. As is clear from (4.11), p^* depends only on the initial expected waiting time ET_{23}^B , which is known to the agents (Assumption 4.1). Hence with the knowledge of initial expected waiting time involved in direct barter, one can arrive at the level of p^* , which will in turn result in the necessary steady state.

Finally, as mentioned in the beginning of this subsection, we would justify Assumption 4.5 in the Appendix (to Chapter 4).

4.5 CONCLUSION

We have discussed in this chapter the conditions under which a good may be endogenously chosen as a medium of exchange in a trading post set-up for a Walrasian equilibrium with imperfect want coincidence. This exercise brought into focus another relevant issue, viz. the problem of choosing an appropriate mix of direct barter and indirect trade in a trading post economy. A *preliminary* attempt has been made to develop a decentralized method for dealing with this issue. However, in order to deal with the latter problem we have considered a model where the transaction cost remained implicit insofar

as an agent is not seen to be carrying out a well-defined optimizing exercise with the time cost incorporated in his objective function. Also, as indicated before, the exercise has been carried out under the strong Assumption 4.5.

Therefore, it may be interesting to extend the results to a framework which is free of these shortcomings.

APPENDIX TO CHAPTER 4:

Proof of Lemma 4.3:

$$\begin{aligned}
 ET_{c_1c_2}^B &= 1.Pr\{T_{c_1c_2}^B = 1\} + 2.Pr\{T_{c_1c_2}^B = 2\} + \dots \\
 &= 1.q_{c_1c_2}^B + 2.(1 - q_{c_1c_2}^B).q_{c_1c_2}^B + 3.(1 - q_{c_1c_2}^B)^2.q_{c_1c_2}^B + \dots \\
 \therefore (1 - q_{c_1c_2}^B).ET_{c_1c_2}^B &= (1 - q_{c_1c_2}^B)q_{c_1c_2}^B + 2(1 - q_{c_1c_2}^B)^2.q_{c_1c_2}^B + \dots
 \end{aligned}$$

$$\begin{aligned}
 \therefore ET_{c_1c_2}^B &= (1 - q_{c_1c_2}^B)ET_{c_1c_2}^B \\
 &= q_{c_1c_2}^B + (1 - q_{c_1c_2}^B)q_{c_1c_2}^B + (1 - q_{c_1c_2}^B)^2.q_{c_1c_2}^B + \dots \\
 &= q_{c_1c_2}^B[1 + (1 - q_{c_1c_2}^B) + (1 - q_{c_1c_2}^B)^2 + \dots]
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow q_{c_1c_2}^B ET_{c_1c_2}^B &= \frac{q_{c_1c_2}^B}{1 - (1 - q_{c_1c_2}^B)} \\
 \Rightarrow ET_{c_1c_2}^B &= \frac{1}{q_{c_1c_2}^B}
 \end{aligned}$$

■

Remark 4.2 (of Subsection 4.3.1)

In Section 4.3.1 we have discussed the choice of a medium of exchange in a set-up where time cost enters implicitly in the model, without any reference to the underlying utility function of the agents. This is done with a view to provide a clear picture about the choice of an optimal mix of direct barter and indirect trade.

Here we intend to discuss a model where time cost is introduced in the objective function of the agents through a discount factor where an agent is assumed to maximize his expected discounted life time utility. We proceed in

the following manner. We will first compute an agent's expected utility when he goes through direct and indirect trade. It will then be discussed how the comparison of the two provides a basis for deciding if one is in favour of direct barter and/or indirect trade.

Let us first consider a particular agent $k \in ID_{23}$. Given all $K \setminus \{k\}$ agents opting for direct barter, let q^B be the probability of meeting a complementary trading partner in trading post (2,3). On the other hand, k fails to meet a complementary trading partner with probability $(1 - q^B)$. Let V^B denote his expected utility of going through direct barter.

Thus agent $k(\in ID_{23})$ gets endowed with good 3 at a particular period. Next period he visits the (2,3) trading post in order to meet a complementary trading partner. With probability q^B he meets one and derives utility, say u , from consumption of good 2 and gets reendowed with good 3. Thus at the end of this period he has the option of going for direct barter (and thereby attaining utility V^B) or indirect trade and getting V^I (i.e. the expected utility of going through indirect trade).

With probability $(1 - q^B)$ he fails to meet a trading partner and remains stuck with good 3 and faces the subsequent option of attaining either V^B or V^I . Therefore, if an agent going through direct barter follows a maximizing strategy then

$$V^B = \rho[q^B(u + \max(V^B, V^I)) + (1 - q^B)\max(V^B, V^I)]$$

where $0 \leq \rho \leq 1$, is the discount factor.

We now concentrate on the alternative strategy of indirect trade. In this case, k first visits trading post (1,3) to acquire good 1 as a medium of exchange. Let q'_{13} be the probability of meeting a complementary trading partner in the (1,3) trading post. Thus, with probability q'_{13} he can attain the indirect utility of acquiring good 1 to be denoted by $v(1)$. Hence with probability $(1 - q'_{13})$ he

will not meet any complementary trading partner and will be back to square one. Clearly, under our Model I $q_{13}^I = 1$. In general we can write

$$V^I = \rho[q_{13}^I v(1) + (1 - q_{13}^I) \max(V^B, V^I)]$$

When an agent succeeds in acquiring good 1, in the next period he will visit trading post (1,2). Let q_{12}^I be the probability of meeting a complementary trading partner in trading post (1,2). Once again, we note in particular that under our Model I, for $k \in ID_{23}$, $q_{12}^I = 1$. In general we can write

$$\begin{aligned} v(1) = & \rho[q_{12}^I (u + \max(V^B, V^I))] \\ & + (1 - q_{12}^I) v(1)] \end{aligned}$$

Clearly, if $V^B < V^I$ then agent k will prefer to go for indirect trade.

In our 3-good economy let us call the indigenous demanders of good 1 against good 2 as type 1 agents and the suppliers of good 1 against good 2 as type 2 agents. Similarly the demanders of good 1 against good 3 are called type 3 agents and those of good 3 against good 1 are known as type 4 agents. Demanders of the (2,3) trading post are alternatively called type 5 agents and the suppliers are known as type 6 agents. Let V_i^B and V_i^I denote respectively the expected discounted utilities derived by a type i agent when he goes through direct barter and indirect trade.

As before (Section 3.3), if for the agents belonging to the long side of the respective trading posts satisfy the following set of inequalities (as period t_0)

$$V_1^B > V_1^I$$

$$V_4^B > V_4^I$$

$$V_5^B < V_5^I$$

then it will ensure a unique medium of exchange in the economy (where the demanders of (2,3) trading post go for a mix of direct barter and indirect

trade). Similarly all other observations made through Lemma 4.1 to 4.5 remain valid for this kind of treatment of the problem.

Proof of Proposition 4.2:

Let us concentrate on period t_3 :

$$\begin{aligned} E(\delta^{(3)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \delta^{(2)} = \tilde{\delta}^{(2)}) \\ &= E(\bar{\delta}^{(3)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \delta^{(2)} = \tilde{\delta}^{(2)}) \\ &+ E(\bar{\bar{\delta}}^{(3)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \delta^{(2)} = \tilde{\delta}^{(2)}) \\ E(\bar{\delta}^{(3)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \delta^{(2)} = \tilde{\delta}^{(2)}) &= p.\tilde{\delta}^{(2)} = p^3.\delta \\ E(\bar{\bar{\delta}}^{(3)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \delta^{(2)} = \tilde{\delta}^{(2)}) &= p.\tilde{\eta}^{(1)} \\ &= p(1-p)\delta = (1-p)p\delta = (1-p)\tilde{\delta}^{(1)} \\ \therefore \tilde{\delta}^{(3)} &= p\tilde{\delta}^{(2)} + (1-p)\tilde{\delta}^{(1)}. \end{aligned}$$

We have,

$$\tilde{\delta}^{(3)} + \tilde{\eta}^{(3)} + \tilde{\eta}^{(2)} = \delta$$

$$\begin{aligned} \therefore \tilde{\eta}^{(3)} &= \delta - \tilde{\delta}^{(3)} - \tilde{\eta}^{(2)} \\ &= \delta - p\tilde{\delta}^{(2)} - (1-p)\tilde{\delta}^{(1)} - p(1-p)\delta \\ &= \delta - p.p^2\delta - p(1-p)\delta - p(1-p)\delta \\ &= \delta(1-p)(1-p+p^2) \end{aligned}$$

Period t_4 :

$$\begin{aligned} \tilde{\delta}^{(4)} &= E(\delta^{(4)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \delta^{(2)} = \tilde{\delta}^{(2)}, \delta^{(3)} = \tilde{\delta}^{(3)}) \\ &= E(\bar{\delta}^{(4)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \delta^{(2)} = \tilde{\delta}^{(2)}, \delta^{(3)} = \tilde{\delta}^{(3)}) + E(\bar{\bar{\delta}}^{(4)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \dots) \end{aligned}$$

$$\begin{aligned} \therefore E(\bar{\delta}^{(4)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \delta^{(2)} = \tilde{\delta}^{(2)}, \delta^{(3)} = \tilde{\delta}^{(3)}) &= p.\tilde{\delta}^{(3)} \\ E(\bar{\bar{\delta}}^{(4)}|\delta^{(1)} = \tilde{\delta}^{(1)}, \dots) &= p\tilde{\eta}^{(2)} = p^2(1-p)\delta. \\ \therefore \tilde{\delta}^{(4)} &= p.\tilde{\delta}^{(3)} + (1-p)(p^2\delta) = p\tilde{\delta}^{(3)} + (1-p)\tilde{\delta}^{(2)}. \end{aligned}$$

Period t_5 :

$$\begin{aligned}\tilde{\delta}^{(5)} &= p\tilde{\delta}^{(4)} + p\tilde{\eta}^{(3)} \\ p\tilde{\eta}^{(3)} &= p(1-p)(1-p+p^2)\delta \\ &= (1-p)\{p(1-p+p^2)\delta\} \\ &= (1-p)\tilde{\delta}^{(3)}\end{aligned}$$

$\therefore \tilde{\delta}^{(5)} = p.\tilde{\delta}^{(4)} + (1-p)\tilde{\delta}^{(3)}$ and so on.

In general, we get

$$\begin{aligned}\tilde{\delta}^{(h)} &= p\tilde{\delta}^{(h-1)} + (1-p)\tilde{\delta}^{(h-2)} \\ \Rightarrow \tilde{\delta}^{(h)} - \tilde{\delta}^{(h-1)} &= (-1)(p-1)[\tilde{\delta}^{(h-1)} - \tilde{\delta}^{(h-2)}] \\ \because p < 1, \{ \tilde{\delta}^{(h)} \} &\text{ converges}\end{aligned}$$

We have,

$$\tilde{\delta}^{(h)} = p\tilde{\delta}^{(h-1)} + p\tilde{\eta}^{(h-2)}$$

$\because \tilde{\delta}^{(h)}$ converges

$\tilde{\eta}^{(h)}$ cannot but converge

$\therefore$ the sequence of partitions $\{\tilde{\delta}^{(h)}, \tilde{\eta}^{(h)}, \tilde{\eta}^{(h-1)}\}$, converges. ■

Proof of Proposition 4.3:

$$\begin{aligned}(1) \ E(\delta^{(1)}) &= p.\delta = \tilde{\delta}^{(1)} \\ (2) \ E(\delta^{(2)}) &= EE(\delta^{(2)}|\delta^{(1)}) \\ &= Eg(\delta^{(1)}) = \sum_j g(\delta^{(1)} = j)p_j, \text{ where } g \text{ is a function of } \delta^{(1)} \\ &\text{and} \\ p_j &= prob\{\delta^{(1)} = j\}.\end{aligned}$$

Here,

$$\begin{aligned}g(\delta^{(1)} = j) &= E(\delta^{(2)}|\delta^{(1)} = j) = jp \\ \therefore E(\delta^{(2)}) &= \sum_j jp.p_j = p \sum_j jp_j \\ &= pE(\delta^{(1)}) = \tilde{\delta}^{(2)} \quad (A.4a)\end{aligned}$$

$$\begin{aligned}
E(\delta^{(3)}) &= E(\bar{\delta}^{(3)}) + E(\bar{\bar{\delta}}^{(3)}) \\
E(\bar{\delta}^{(3)}) &= EE(\bar{\delta}^{(3)}|\delta^{(2)}) \\
&= Eg_1(\delta^{(2)}), \text{ where } g_1 \text{ is a function of } \delta^{(2)} \\
&= \sum_j g_1(\delta^{(2)} = j)p'_j
\end{aligned}$$

Where,

$$p'_j = Pr\{\delta^{(2)} = j\}$$

Now,

$$g_1(\delta^{(2)} = j) = E(\bar{\delta}^{(3)}|\delta^{(2)} = j) = jp$$

Using (A.4a)

$$\therefore E(\bar{\delta}^{(3)}) = p \sum_j jp'_j = pE(\delta^{(2)}) = p\bar{\delta}^{(2)} \quad (A.4b)$$

Similarly,

$$\begin{aligned}
E(\bar{\bar{\delta}}^{(3)}) &= EE(\bar{\bar{\delta}}^{(3)}|\eta^{(1)}) \\
&= EE(\bar{\bar{\delta}}^{(3)}|(\delta - \delta^{(1)})) \\
&= Eg_2(\delta - \delta^{(1)}) \text{ where, } g_2 \text{ is a function of } \delta^{(1)} \\
&= \sum_j p''_j g_2(\delta - \delta^{(1)} = j)
\end{aligned}$$

Where,

$$\begin{aligned}
p''_j &= Pr\{\delta - \delta^{(1)} = j\} \\
g_2(\delta - \delta^{(1)} = j) &= E(\bar{\bar{\delta}}^{(3)}|\delta - \delta^{(1)} = j) \\
&= jp
\end{aligned}$$

$$\begin{aligned}
\therefore E(\tilde{\delta}^{(3)}) &= p \sum_j j p_j'' = pE(\delta - \delta^{(1)}) \\
&= p[\delta - \delta p] = p\delta(1 - p) \\
&= (1 - p)E(\delta^{(1)}) = (1 - p)\tilde{\delta}^{(1)} \quad (A.4c)
\end{aligned}$$

$$\begin{aligned}
\therefore E\delta^{(3)} &= p\tilde{\delta}^{(2)} + (1 - p)\tilde{\delta}^{(1)} \quad (\text{From (A.4b) and (A.4c)}) \\
&= \tilde{\delta}^{(3)}
\end{aligned}$$

In general,

$$E\delta^{(h)} = \tilde{\delta}^{(h)}$$

$\therefore$ using Proposition 4.2, the sequence

$$\{E\delta^{(h)}\}$$

converges.

Hence, the sequence of partitions

$$\{E\delta^{(h)}, E\eta^{(h)}, E\eta^{(h-1)}\}$$

converges. ■

Proof of Corollary 4.3:

We have from Propositions 4.2 and 4.3 above

$$\begin{aligned}
&E\delta^{(h)} - E\delta^{(h-1)} \\
&= (-1)(1 - p)[E\delta^{(h-1)} - E\delta^{(h-2)}] \\
&= (-1)^2(1 - p)^2[E\delta^{(h-2)} - E\delta^{(h-3)}] \\
&\quad \vdots \\
&= (-1)^{h-2}(1 - p)^{h-2}[E\delta^{(h-(h-2))} - E\delta^{(h-(h-1))}] \\
&= (-1)^{h-2}(1 - p)^{h-2}[E\delta^{(2)} - E\delta^{(1)}]
\end{aligned}$$

$$\therefore E\delta^{(h)} = E\delta^{(h-1)} + (-1)^{h-2}(1-p)^{h-2}[E\delta^{(2)} - E\delta^{(1)}] \quad (A.4d)$$

Using (A.4d)

$$E\delta^{(h-1)} = E\delta^{(h-2)} + (-1)^{h-3}(1-p)^{h-3}[E\delta^{(2)} - E\delta^{(1)}]$$

Thus, replacing $E\delta^{(h-1)}$ by the R.H.S., we can write (A.4d) as

$$E\delta^{(h)} = E\delta^{(h-2)} + (-1)^{h-3}(1-p)^{h-3}[E\delta^{(2)} - E\delta^{(1)}] + (-1)^{h-2}(1-p)^{h-2}[E\delta^{(2)} - E\delta^{(1)}]$$

Proceeding similarly,

$$\begin{aligned} E\delta^{(h)} &= E\delta^{(2)} + (-1)(1-p)[E\delta^{(2)} - E\delta^{(1)}] \\ &\quad + \dots + (-1)^{h-2}(1-p)^{h-2}[E\delta^{(2)} - E\delta^{(1)}] \end{aligned}$$

$$E\delta^{(h)} = E\delta^{(2)} + \{E\delta^{(2)} - E\delta^{(1)}\} \{(-1)(1-p)\} [1 + (-1)(1-p) + \dots + (-1)^{h-3}(1-p)^{h-3}] \quad (A.4e)$$

Let

$$\Delta = 1 + (-1)(1-p) + \dots + (-1)^{h-3}(1-p)^{h-3}$$

$$(1-p)\Delta = (1-p) + (-1)(1-p)^2 + \dots + (-1)^{h-2}(1-p)^{h-3} + (-1)^{h-3}(1-p)^{h-2}$$

Adding

$$\begin{aligned} \Delta + (1-p)\Delta &= 1 + (-1)^{h-3}(1-p)^{h-2} \\ \Rightarrow \Delta &= \frac{1 + (-1)^{h-3}(1-p)^{h-2}}{2-p} \end{aligned}$$

Putting this value in (A.4e)

$$\begin{aligned} E\delta^{(h)} &= E\delta^{(2)} + (-1)(1-p)[E(\delta^{(2)} - E\delta^{(1)})] \\ &\quad \left(\frac{1 + (-1)^{h-3}(1-p)^{h-2}}{2-p} \right) \end{aligned}$$

$$\begin{aligned}
\therefore \lim_{h \rightarrow \infty} E\delta^{(h)} &= E\delta^{(2)} + (-1)(1-p)[E\delta^{(2)} - E\delta^{(1)}] \cdot \frac{1}{2-p} \\
&= \delta[p^2 + (\delta p - \delta p^2)] \frac{1-p}{2-p} \\
&= \delta p^2 + \delta p(1-p) \left(\frac{1-p}{2-p} \right)
\end{aligned}$$

■

Justification of Assumption 4.5:

We have noted repeatedly our awareness of the fact that Assumption 4.5 is somewhat strong. The main conclusion the assumption has led us to is of course the steady state value of the probability p^* of choosing direct barter. In what follows, we prove in some sense the converse result, viz. that if agents were indeed to behave according to the probability distribution $(p^*, 1 - p^*)$, then Assumption 4.5 would be satisfied with high probability. We have shown in Subsection 4.3.2, the optimal probability of choosing direct barter is

$$p^* = \frac{4\psi_{23}}{\delta_{23} + 2\psi_{23}}$$

$\therefore$ probability of choosing indirect trade is

$$1 - p^* = \frac{\delta_{23} - 2\psi_{23}}{\delta_{23} + 2\psi_{23}}$$

Let $\eta^{(h)}$ be the number of agents present in trading post (1,3) in period t_h .

Given the values of $\delta^{(h-1)}$ and $\eta^{(h-2)}$ in period t_h , $\eta^{(h)}$ can be thought of as the number of success out of $\delta^{(h-1)} + \eta^{(h-2)}$ Bernoulli trials with probability of success $(1 - p^*)$.

We have considered the situation where, δ_{23} and ψ_{23} are large enough, so that no communication between the agents is possible. At the beginning of period t_1 , $\eta^{(1)}$ is the number of success out of a total (constant) of $\delta (= \delta_{23})$ Bernoulli trials. Clearly, $\eta^{(1)}$ follows binomial $(\delta, 1 - p^*)$. We have assumed δ

to be large enough so that it is permissible to use the normal approximation of the binomial distribution.

We intend to show now that

$$Prob\{-\infty \leq \eta^{(1)} \leq Z_{13}\} \geq .99$$

$$\begin{aligned} \text{LHS} &= Prob\{-\infty \leq \eta^{(1)} \leq Z_{13}\} \\ &= Prob\{-\infty \leq \frac{\eta^{(1)} - E\eta^{(1)}}{\sqrt{\text{Var}\eta^{(1)}}} \leq \frac{Z_{13} - E\eta^{(1)}}{\sqrt{\text{Var}\eta^{(1)}}}\} \end{aligned}$$

Our problem would be solved if we can show

$$\frac{Z_{13} - E\eta^{(1)}}{\sqrt{\text{Var}\eta^{(1)}}} \geq 3$$

$$E\eta^{(1)} = \delta \cdot \frac{\delta - 2\psi}{\delta + 2\psi}, \quad \psi = \psi_{23}$$

$$E\eta^{(1)} = (\delta - 2\psi) \frac{\delta}{\delta + 2\psi}$$

$$\Rightarrow E\eta^{(1)} \geq \delta - 2\psi \quad (\text{A.4f})$$

and

$$\sqrt{\text{Var}(\eta^{(1)})} = \sqrt{\delta \left(\frac{\delta - 2\psi}{\delta + 2\psi} \right) \left(\frac{4\psi}{\delta + 2\psi} \right)} \quad (\text{A.4g})$$

Using (A.4f) and (A.4g)

$$\begin{aligned} &\frac{Z_{13} - E\eta^{(1)}}{\sqrt{\text{Var}\eta^{(1)}}} \geq \frac{Z_{13} - (\delta - 2\psi)}{\sqrt{\delta \left(\frac{\delta - 2\psi}{\delta + 2\psi} \right) \left(\frac{4\psi}{\delta + 2\psi} \right)}} \\ &= \frac{\psi}{\sqrt{(\delta - 2\psi) \left(\frac{\delta}{\delta + 2\psi} \right) \cdot p^*}} \quad \text{if } Z_{13} = Z_{23} = \delta - \psi \\ &\geq \frac{\psi}{\sqrt{(\delta - 2\psi)p^*}} \end{aligned}$$

$$\text{Claim: } \frac{\psi}{\sqrt{(\delta - 2\psi)p^*}} \geq 3 \text{ if } \psi \geq 36$$

Proof: $\frac{\psi}{\sqrt{(\delta-2\psi)p^*}} \geq 3$

$$\text{if } \frac{\psi}{3} \geq \sqrt{(\delta-2\psi)\left(\frac{4\psi}{\delta+2\psi}\right)} = \sqrt{(\psi)\frac{\delta-2\psi}{\delta+2\psi}} \leq 2\sqrt{\psi}$$

$\therefore$ our claim would hold if

$$\begin{aligned} \frac{\psi}{3} &\geq 2\sqrt{\psi} \\ \Rightarrow \sqrt{\psi} &\geq 6 \\ \Rightarrow \psi &\geq 36 \end{aligned}$$

$\therefore \psi \geq 36$ gives a sufficient condition under which $\frac{Z_{13}-E\eta^{(1)}}{\sqrt{\text{Var}_{\eta^{(1)}}}} \geq 3$ and hence from the normal probability table

$$\text{Prob}\{-\infty \leq \eta^{(1)} \leq Z_{13}\} \geq .99 \quad \text{if } \psi \geq 36$$

Let us now consider $\eta^{(h)}, h \geq 2$. Given $\delta^{(h-1)}$ and $\eta^{(h-2)}$, $\eta^{(h)}$ follows Binomial $(\delta^{(h-1)} + \eta^{(h-2)}, 1 - p^*)$ and $\delta^{(h-1)} + \eta^{(h-2)} \leq \delta \quad \forall h \geq 2$

$\therefore$ for $h \geq 2$

$$\text{Pr}\{-\infty \leq \eta^{(h)} \leq Z_{13}\} \geq \text{Pr}\{-\infty \leq \eta^{(1)} \leq Z_{13}\} \geq .99$$

Hence, if δ_{23} and ψ_{23} are large, our Assumption 4.5 is valid with probability (exceeding) .99. ■

- (1) k : index for agent
- (2) c : index for commodity
- (3) w_k : initial endowment of agent k
- (4) X_k : final demand of agent k
- (5) (c_1, c_2) : the trading post where good c_1 is traded against c_2
- (6) $D_{c_1 c_2}, S_{c_1 c_2}$: the sets of demanders and suppliers (respectively) of good c_1 against c_2 .
- (7) $ID_{c_1 c_2}, IS_{c_1 c_2}$: the sets of indigenous demanders and suppliers respectively of the (c_1, c_2) trading post.
- (8) $\mathcal{I}_{c_1 c_2}$: union of $ID_{c_1 c_2}$ and $IS_{c_1 c_2}$
- (9) $L_{c_1 c_2}, \ell_{c_1 c_2}$: agents belonging respectively to the long side and short side of trading post (c_1, c_2)
- (10) $\delta_{c_1 c_2}, \psi_{c_1 c_2}$: cardinality of the sets $ID_{c_1 c_2}$ and $IS_{c_1 c_2}$ respectively.
- (11) $Z_{c_1 c_2}$: the level of excess demand in the $(c_1 c_2)$ trading post $(= |\psi_{c_1 c_2} - \delta_{c_1 c_2}|)$
- (12) $T_{c_1 c_2}^{Bh}, \tau_{c_1 c_2}^{Bh}$: the waiting time involved in direct barter for the agents belonging to the long and short sides (respectively) of trading post (c_1, c_2) at period t_h .
- (13) $T_{c_1 c_2}^B, \tau_{c_1 c_2}^B$: same as (12) when $h = 0$
- (14) $T_{c_1 c_2}^{Ih}, \tau_{c_1 c_2}^{Ih}$: same as (12) when direct barter is replaced by indirect trade

- (15) $T_{c_1 c_2}^I, \tau_{c_1 c_2}^I$: same as (14) when $h = 0$
- (16) $q_{c_1 c_2}^B$: the probability of meeting a complementary trading partner in the (c_1, c_2) trading post by an agent $k \in L_{c_1 c_2} \cap \mathcal{I}_{c_1 c_2}$.
- (17) $\delta_{c_1 c_2}^{Bh}, \delta_{c_1 c_2}^{lh}$: number of agents belonging to $ID_{c_1 c_2}$ going for direct barter and indirect trade respectively at period t_h
- (18) $\delta^{(h)}, \eta^{(h)}$: $\delta^{(h)} = \delta_{c_1 c_2}^{Bh}, \eta^{(h)} = \delta_{c_1 c_2}^{lh}$ when $c_1 = 2, c_2 = 3$
- (19) p : any arbitrary probability of choosing direct barter
- (20) p^* : optimal probability of choosing direct barter
- (21) V_i^B, V_i^I : expected discounted utility of going through direct barter and indirect trade by a type i agent.
- (22) $v_i(c)$: indirect utility of acquiring good c by a type i agent
- (23) ρ : discount factor.
- (24) q_i^B, q_i^I : probabilities of meeting a complementary trading partner by a type i agent when he goes through direct barter and indirect trade respectively.

DISEQUILIBRIUM THEORY : A UNIFIED PRESENTATION

ABSTRACT

This chapter develops a diagrammatic scheme to provide a unified presentation of three major solution concepts used by disequilibrium theory. These concepts were developed respectively by Benassy, Drèze and Malinvaud and Younes. In particular, the diagrams would clarify their similarities and differences.

5.1 INTRODUCTION

This chapter is pedagogic in nature insofar as there are no new results to be presented. On the contrary, its thrust lies in a unified presentation of old results in an area which is concerned with problems complementary to the ones dealt with so far. In particular, it develops a diagrammatic apparatus whose purpose is to bring out the essence of existing concepts and results of disequilibrium theory as well as to provide a framework suitable for easy comparisons of the diverse models in the area.

Disequilibrium analysis started mainly as an attempt to provide the micro-foundations of macro economic concepts like Keynesian involuntary unemployment. Clearly, in an economy where wages and prices are completely flexible, there is no room for sustained unemployment. However, we observe situations in reality where money wages do not adjust freely enough to ensure full employment. Such rigidities in turn would effect an agent's demand for consumption goods and so on.

Disequilibrium theory studies the micro behaviour of each agent in the face of such sluggish prices at non-market clearing levels. Clearly, the allocation an agent attains in such circumstances would be entirely different from its competitive counterpart. From the very beginning, economists dealing with this area have tried to choose appropriate allocations that are acceptable to the agents facing a set of non-flexible prices. The attempt to deal with the problem gave rise to three allocative concepts under disequilibrium. They were developed respectively by Benassy (1975), Drèze (1975) and Malinvaud and Younes (1977).

The disequilibrium concepts can be viewed from the following angle. As opposed to the problem of Chapter 2, where the auctioneer had failed in his role of commodity reallocation at equilibrium prices, we now have a competitive set-up in which the auctioneer is unable to adjust prices when the markets are in disequilibrium. Put differently, disequilibrium theory studies the situation where a fundamental assumption of Walrasian theory, viz. that market forces adjust prices, does not hold good. In this context, it is undoubtedly interesting to ask how an agent would attain his desired allocations in the face of disequilibrium prices. But before looking into that issue, a less ambitious question may be posed. Clearly, under disequilibrium prices, some *restrictions* or *rations* impede an agent's free realization of notional demands or supplies (i.e. his utility maximizing choices). In these circumstances, it is natural to ask what allocation would be the most desirable one for an agent amongst the ones which are also realizable. Answers to this question are provided in the works of Benassy, Malinvaud and Younes and Drèze.

Each of these solution concepts have been developed independently and there is a substantial literature surrounding each of them. As already noted, our purpose here is to develop a diagrammatic scheme to compare and contrast these concepts in some detail. The literature being vast, it is considered worth-

while to collect the salient features alone and construct a simple unified frame of analysis in terms of which one can present a major part of the literature.

It may be mentioned in this context that similar attempts by Basu (1992) as well as Malinvaud and Younes (1977) relate respectively to the diagrammatic presentation of the Drèze and the Malinvaud-Younes equilibrium. On the other hand, our purpose here is to develop the corresponding diagrams for the Benassy equilibrium in the context of a pure exchange economy.

Though the three equilibrium concepts have been the result of different approaches to the problem, it is established by Silvestre (1982) that they indeed lead to identical solutions (under reasonable assumptions regarding the underlying preference pattern). We next proceed to provide a clear visual representation of the situations under which the three solution concepts coincide and when they do not.

The next section develops the basic diagrammatic framework. Section 5.3 discusses the three equilibrium concepts and a diagrammatic representation of the same. The interconnections between the three equilibria (for different situations) is presented in Section 5.4. A diagrammatic exposition of Silvestre's results is taken up in the penultimate section. A concluding section sums up the chapter.

5.2 THE FRAMEWORK

To understand a disequilibrium framework more clearly, let us first describe the model under consideration. We take up a simple 3-good ($c = 1, 2, 3$), 2-agent ($k = a, b$) economy. An agent k 's preference pattern is captured by a quasi-concave, differentiable utility function

$$U_k = U_k(x_{k1}, x_{k2}, x_{k3}), \quad k = a, b$$

which depends on x_{k1}, x_{k2} and x_{k3} , i.e. his consumption of goods 1, 2 and 3.

The initial endowment matrix is given by

$$\tilde{W} = \|w_{kc}\|_{3 \times 3}$$

where, w_{kc} denotes the initial endowment of agent k of good c . Let $X_k = (x_{k1}, x_{k2}, x_{k3})$ and $W_k = (w_{k1}, w_{k2}, w_{k3})$ represent respectively the vectors of final consumption and initial endowments of agent k . Therefore,

$$Z_k = W_k - X_k = (z_{k1}, z_{k2}, z_{k3})$$

represents the vector of excess demands of agent k . The prices are rigidly fixed at a disequilibrium level, $p = (p_1, p_2, p_3)$. Thus, the budget plane for agent k can be represented as

$$p_1 x_{k1} + p_2 x_{k2} + p_3 x_{k3} \leq p_1 w_{k1} + p_2 w_{k2} + p_3 w_{k3}, k = a, b$$

In particular, if we try to represent agent a 's budget plane in three dimensional Euclidian space, it would look like the triangle IJJ (Fig. 5.1).

In the case of two traders and three goods, the Edgeworth box will be a three dimensional rectangular parallelepiped, where we assume agent a 's origin to be at a *bottom corner* and b 's at the diagonally opposite *top corner*. Fig. 5.1 depicts both these budget planes IJJ (for agent a) and $I'J'L'$ (for agent b). The common region between these two planes can have different shapes. Here we would deal with a triangular region as shown in Fig. 5.2 (i.e. IJJ). Other types of figures can be dealt with in an analogous fashion.

The point of initial endowment of agent a is represented by $W_a = (w_{a1}, w_{a2}, w_{a3})$. Suppose OA represents w_{a1} level of good 1. We denote by $N'AN$ (Fig. 5.2) the plane $x_{a1} = w_{a1}$, which intersects the budget plane along the line NN' . Clearly, NN' would be parallel to JL and it represents w_{a1} level of good 1 on the budget plane. Therefore, the terminologies " NN' level of good 1" and " w_{a1} level of good 1" would be used interchangeably. Similarly, RR' represents w_{a3}

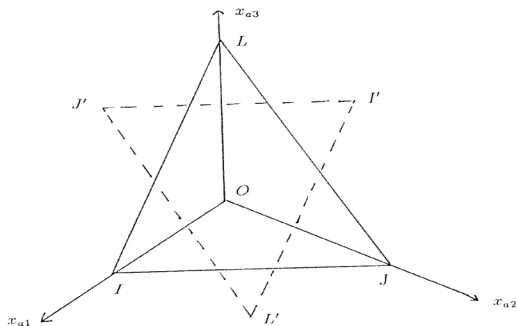


Fig. 5.1

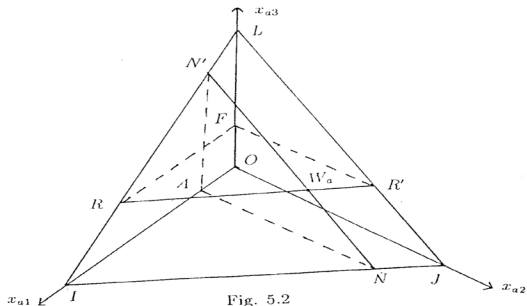


Fig. 5.2

level of good 3 on the budget plane. One can draw lines parallel to IL through W_a to find the level of w_{a2} . As in the case of the two dimensional Edgeworth box diagram, the point W_a also represents the initial endowment of agent b measured from his origin.

Indifference surfaces would intersect with the budget plane in closed contours as depicted in Fig. 5.3 below, where inner contours are associated with higher levels of satisfaction. Finally, we get the best point for agent a (say) on the budget plane (denoted by B_a), where an indifference contour is tangential to the plane IJL . Indifference contours for agent b can be drawn in a similar fashion where B_b shows his best point.

Thus, points B_a and B_b represent respectively the utility maximizing choice of agents a and b for goods 1,2 and 3, given the budget constraint. In Clower's terminology B_a and B_b refer respectively to *notional demands (and supplies)* of agents a and b . As is clear from Fig. 5.3, under non-market clearing prices the notional demands of the two agents would not match, i.e. the best points B_a and B_b do not coincide. Therefore, an agent trying to realize his most desirable allocation under the given prices would in turn face constraints impeding his trade. We next visualize (diagrammatically) the nature of these constraints or equivalently the *rations* faced by an agent, say a . In Fig. 5.4, W represents the point of initial endowment. At W , agent a has FF' level of good 1 and EE' level of good 2. At any point to the right of FF' (say at A or at B_a) agent a is an excess supplier of good 1. Similarly, at A (or at B_a) he is an excess supplier of good 2 also. Without loss of generality suppose agent a 's ration on his demand for and supply of good 1 is given by FF' and RR' (Fig. 5.4). That is he can supply upto a level RR' whereas he is not allowed to be an excess demander of good 1 at all. Assume now that together with the ration on good 1, agent a is also faced with QQ' and $A'A'$ levels of rations respectively on his supply of and demand for good 2. Thereby he is restricted to choose an

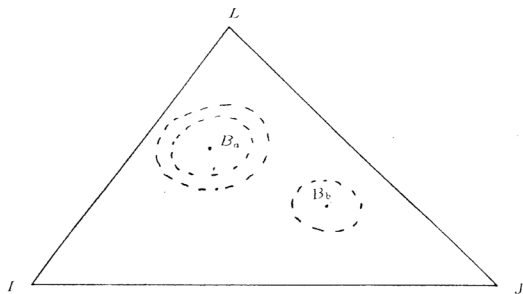


Fig. 5.3

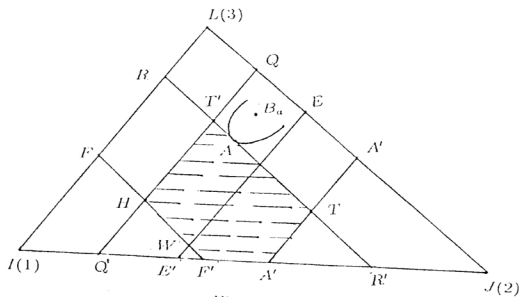


Fig. 5.4

allocation within the shaded region $HF'A'TT'$ (Figure 5.4). It is clear from the diagram that in the face of these rations point A turns out to be the best point for agent a .

Thus, the given rations prevent an agent from choosing his notional demand point. However, given a set of rations, an agent is free to opt for any allocation within the constrained region (viz. $HF'A'TT'$). That is, he can decide to supply good 2 at any level *less* than QQ' and demand at any level *less* than $A'A'$. Similarly for good 1.

Note that at the optimal point A , agent a 's constraint on good 1 viz RR' is binding, while that on good 2 viz. QQ' is non-binding. That is, he optimally chooses to supply less of good 2 than he is allowed to. This indicates the existence of an element of *voluntariness* in the agent's choice, even in the presence of rationing.

This completes a description of the basic framework within which we will discuss the solution concepts referred to earlier. In the next section, we would introduce the equilibrium concepts developed by Drèze, Benassy and Malinvaud and Younes.

5.3 THE EQUILIBRIUM CONCEPTS

It is clear from the above that in the absence of any restrictions save the budget constraint, an agent would always opt for his best point on the budget plane. But as the prices are rigid at a disequilibrium level, an agent trying to realize his notional demands would face restrictions impeding his trade. Furthermore, a constraint on an agent's demand for a good would in turn affect his desired sale of other goods and vice versa. However, these changed levels of desired purchases and sales may again be non-matching (in the aggregate) and so on.

In these circumstances, one may ask whether it is possible to conceive of a

set of *consistent rations* for the agents, so that their ration-constrained choices at least are realizable. The search for such rations motivates the concept of a *Drèze equilibrium*.

Suppose the rationing scheme is given by

$$\begin{pmatrix} r \\ R \end{pmatrix} = \begin{pmatrix} r_1 & r_2 \\ R_1 & R_2 \end{pmatrix}$$

where r_c and R_c ($c = 1, 2$ and $r_c \leq 0 \leq R_c$) denote respectively the bounds on the excess supply of and excess demand for good c . Given a price vector p and a rationing scheme the *ration-constrained budget set* (RCB) of agent k is defined as

$$Bd_k(p, R, r) = \{x \in X_k | p(x - W_k) \leq 0, R \geq x - W_k \geq r\}$$

where, X_k is the consumption set of agent k , $R = (R_1, R_2)$ and $r = (r_1, r_2)$.

For agent k , the utility maximizing choice within the RCB is

$$(D.1) \quad X_k^D = \max_{x \in Bd_k} U_k(x)$$

Clearly, the choices would be meaningless if not realizable. Hence, Drèze appends the following condition. For $k = a, b$

$$(D.2) \quad (X_a^D - W_a) + (X_b^D - W_b) = 0.$$

Suppose however, that the rations for a particular good, say good 1, are such that they impose simultaneously binding constraints on agent a 's supply as well as agent b 's demand. In other words, the rations on good 1 forcefully forbid some amount of mutually advantageous and realizable trades. Drèze guards against this undesirable possibility by imposing the following condition on the allocation scheme:

$$(D.3) \text{ For all } c, \text{ if } R_c = x_{ic}^D - w_{ic}, \text{ for some agent } i \text{ then } x_{jc}^D - w_{jc} > r_c, \text{ for all agents } j \neq i. \text{ Similarly, } \forall c, \text{ if } r_c = x_{ic}^D - w_{ic}, \text{ for some } i \text{ then } x_{jc}^D - w_{jc} < R_c, \forall j \neq i.$$

That is, if an agent is demand (supply) constrained, no other agent is supply (demand) constrained so that there does not exist binding constraints on the demand for as well as the supply of a good. This condition in a sense ensures that the agents would have no incentive to violate these rations; they would rather accept them voluntarily.

Thus, $X_k^D, k = a, b$, satisfying (D.1), (D.2) and (D.3) gives the equilibrium allocations for agent k , brought through an appropriate set of rations $(r, R)^1$.

We now proceed to represent diagrammatically, the solution concepts developed by Drèze. In Fig. 5.5, agent a 's constraint on his supply of good 1 is given by $E_a E'_a$. (i.e. $E_a E'_a$ represents the level corresponding to r_1). To show the constraint on agent b 's supply of good 1, we draw the line $E_b E_b$ (note that agent b is an excess supplier at any point to the left of W). We depict agent a 's demand ration on good 1 by the line $F_a F_a$. To show the constraint on agent b 's demand for good 1, we draw the line $F_b F_b$ (to the right of W). Both these lines represent the levels corresponding to R_1 .

Similarly, supply and demand rations on good 2 can be shown. In Fig. 5.6 we depict the optimal choices of agents a and b in the face of the rations. To keep the diagram simple, we have not shown agent a 's demand rations and b 's supply rations.

Let $E_a E'_a$ and $S_a S_a$ represent respectively the rations faced by agent a on his supply of goods 1 and 2. Similarly, suppose $F_b F_b$ and $T_b T'_b$ show the demand rations for agent b on goods 1 and 2 respectively. Given the rations ($E_a E'_a$ and $S_a S_a$), agent a 's utility maximizing choice is at point E where his indifference contour is tangential to $E_a E'_a$. Similarly, given the restrictions on b 's demand ($F_b F_b$ and $T_b T'_b$), his most desirable choice is also at point E .

We note that at E , agent a feels a *binding* constraint on his supply of good

¹This is a simplified version of Drèze equilibrium. Drèze in fact allows the prices to vary within a given interval.

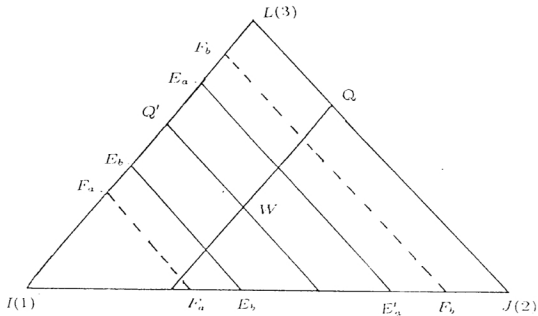


Fig. 5.5

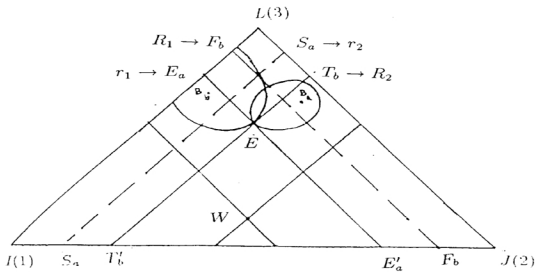


Fig. 5.6

1 whereas agent b 's demand constraint on good 1 (i.e. $F_b F_b$) is non-binding. On the other hand, agent b faces a binding constraint on his demand for good 2, whereas agent a 's ration on the supply of good 2 (i.e. $S_a S_a$) is non-binding. Therefore, if an agent is demand constrained for a good no agent is supply constrained for it and vice versa. Thus, conditions (D.1), (D.2) as well as (D.3) of Drèze equilibrium are satisfied. Hence, E represents a Drèze equilibrium.

Suppose however, that instead of imposing rations and asking the agents to choose their most preferred allocations, one were to assign the allocations to the agents straightaway. Under what circumstances would the agents be satisfied with these allocations? The solution concept proposed by Malinvaud and Younes (1977) is best viewed as an attempt to answer this question.

Any allocation, say $X_k^y = (x_{k1}^y, x_{k2}^y, x_{k3}^y)^2$, $k = a, b$, could generally involve certain amount of divergence from the agents' initial endowment point. An agent is likely to consider the allocation an imposition if, left to himself, he would prefer to demand or supply a lesser quantity of a good than the amount assigned to him in the allocation.

To understand this point clearly, we consider an extreme example. Suppose agent k 's initial endowment W_k gives him more utility than the assigned allocation X_k^y . Then it is natural for him not to shift from his initial endowment and accept X_k^y . Similar reasoning ensures that he would never supply a good a against another good c' , if c is more preferred to c' .

Keeping this in mind Malinvaud and Younes (1977) require that any acceptable allocation X_k^y ($k = a, b$) satisfy the following condition:

$$\begin{aligned} \text{(M.1)} \quad & \max_{x_k} U_k(X_k) = U_k(X_k^y) \\ & \text{s.t. } pX_k \leq pW_k \end{aligned}$$

²Superscript y stands for Younes.

1 whereas agent b 's demand constraint on good 1 (i.e. $F_b F_b$) is non-binding. On the other hand, agent b faces a binding constraint on his demand for good 2, whereas agent a 's ration on the supply of good 2 (i.e. $S_a S_a$) is non-binding. Therefore, if an agent is demand constrained for a good no agent is supply constrained for it and vice versa. Thus, conditions (D.1), (D.2) as well as (D.3) of Drèze equilibrium are satisfied. Hence, E represents a Drèze equilibrium.

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Keeping this in mind Malinvaud and Younes (1977) require that any acceptable allocation X_k^y ($k = a, b$) satisfy the following condition:

$$(M.1) \max_{x_k} U_k(X_k) = U_k(X_k^y)$$

$$\text{s.t. } pX_k \leq pW_k$$

²Superscript y stands for Younes.

$$\begin{aligned}x_{kc}^y &\leq x_{kc} \leq w_{kc} \text{ if } x_{kc}^y \leq w_{kc} \\x_{kc}^y &\geq x_{kc} \geq w_{kc} \text{ if } x_{kc}^y \geq w_{kc}\end{aligned}$$

for $k = a, b, \forall c$

and

$$(X_a^y - w_a) + (X_b^y - w_b) = 0.$$

This condition guarantees that by demanding or supplying less of a commodity as compared to X_k^y , agent k would not be better off.

Moreover, as in the case of Drèze, there would be an incentive to move away from the allocation if mutually advantageous trades were to exist between a pair of agents. Suppose in fact that X_a^y and X_b^y (satisfying (M.1)) are allocated respectively to agents a and b . Assume further that good 3 is conventionally accepted as a *medium of exchange* in the economy. Clearly, if agents a and b (with X_a^y and X_b^y) can engage in mutually advantageous additional trade (against the medium of exchange, viz. good 3), then they would always prefer to do so. To guard against the emergence of such potential trades the following condition is imposed:

(M.2) For each agent i , $\nexists$ another agent j , a good c and an $\varepsilon > 0$, s.t.

$$U_i(X_i^y + a_c \varepsilon) > U_i(X_i^y)$$

$$U_j(X_j^y - a_c \varepsilon) > U_j(X_j^y)$$

hold simultaneously, where

$$a_c = \begin{pmatrix} 0 \\ \frac{1}{p_c} \\ -1 \end{pmatrix}.$$

Thus, allocations X_a^y and X_b^y satisfying conditions (M.1) and (M.2) are expected to be sustainable if they are assigned respectively to a and b . Such

an allocation defines a *Malinvaud-Younes equilibrium*. It is interesting now to compare, in terms of a diagram, the Malinvaud-Younes allocation with the Drèze allocation. To do so, we refer again to Fig. 5.6. At E , agent a is supplying $E_a E'_a$ level of good 1 and $T_b T'_b$ level of good 2. By supplying less of good 1 and/or good 2, agent a can choose any point in the region $WAEA'$. However, since E represents a Drèze equilibrium, any other allocation in the above mentioned region would be less preferred as compared to E . Similarly, by demanding less of good 1 and/or good 2 agent b cannot go to a better position.

This observation reveals the fact that at E , condition (M.1) of a Malinvaud-Younes equilibrium is automatically satisfied. Also, it is clear from Fig. 5.6 that though agent a is allowed to supply $S_a S_a$ level of good 2, he optimally chooses to supply at a lower level, viz. $T_b T'_b$. This is due to the reason that given the ration on good 1, any further trade relating good 2 is not utility improving for agent a . We observe from Fig. 5.6 that agent a 's ration on good 1 is binding at point E and therefore any additional trade (at E) involving good 2 need to be performed against good 3, which implies a movement along $\overline{EE_a}$. From the diagram (Fig. 5.6) it is clear that no such movement is utility improving for agent a . Thus, even though at E additional trade of good 2 against good 3 (i.e. a movement along $\overline{EE_a}$) is desirable to agent b , such trades cannot materialize since agent a will always be unwilling to do so.

Analogous set of arguments can be given to ensure that it is not utility improving for agent b to trade good 1 against 3 (i.e. a movement along $\overline{ET_b}$).

Thus, condition (M.2) of a Malinvaud-Younes equilibrium is satisfied. Hence, E also represents a Malinvaud-Younes equilibrium.

It is revealed from the above discussion that in order to arrive at both these equilibria it is necessary to provide quite a bit of information to the central planner, including the knowledge of the agents' utility functions. One might,

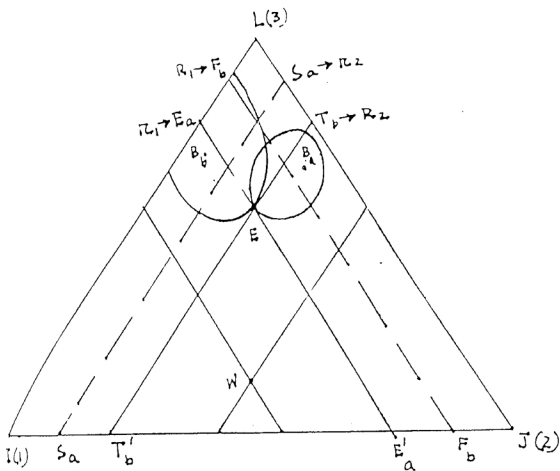


Fig. 5.6

however, wish to economize on such information requirements.

In particular, it might be interesting to know whether full knowledge on the part of the planner of the agents' utility functions can be dispensed with, as in the case of the Walrasian tâtonnement scheme. Benassy's work can be interpreted as an attempt to answer this question. However, major point to be noted in this connection is that unlike a Walrasian auctioneer, Benassy's planner cannot adjust the prices. Therefore, Benassy is concerned with *quantity* rather than price tâtonnement. In this connection, two major concepts are introduced into the literature, viz. *effective demand* and *perceived constraint*.

Given the prices, an agent's best choice would be his notional demands and supplies. An attempt to realize these demands makes the traders aware of some constraints, the levels of which they "perceive" in the process of trade. Precisely how one arrives at these "perceived constraints" will be discussed in the sequel.

Given the perceptions, the next issue is to decide what the desired purchase and sale of any good c is. This desire should represent an agent's unrestricted choice in some sense. For, if we restrict an agent's demand for a particular good completely, then his resulting choice would not capture the desired levels of purchases of that good. But in a rigid price situation, it cannot be totally unconstrained either, because in that case the desired purchase (or sale) will coincide with the notional demand (supply). Considering these features, Benassy defined an agent's desired purchase (or sale) of good c , i.e. his *effective demand* for c as his utility maximizing choice subject to the budget constraint and constraints on all goods other than c and a good reserved to play the role of money. Thus, the effective demand reveals the amount an agent *would have liked to demand* had his transaction in this good been unrestricted.

It would be mentioned in this context that if one considers rations on all goods other than c , then these rations together with the budget constraint

might automatically impose a restriction on his consumption of good c . Therefore, in order to facilitate free choice of good c while evaluating the effective demand for it, one additional good had to be kept unrationed. It is this latter good which is also interpreted as the money good.

More concretely, the level of effective demand $\tilde{x}_{a1}$ of agent a for good 1 is determined as follows:

$$\begin{aligned} \max \quad & U_a(x_{a1}, x_{a2}, x_{a3}) \\ \text{s.t.} \quad & p \cdot X_a \leq p \cdot W_a \\ & \underline{x}_{a2} \leq x_{a2} \leq \bar{x}_{a2} \end{aligned}$$

where $\underline{x}_{a2}$ and $\bar{x}_{a2}$ represent the perceived constraints on the supply of and demand for good 2. The solution to this problem can be presented as (x'_{a1}, x'_{a2}) . Thus, agent a 's effective demand for good 1 is $\tilde{x}_{a1} = x'_{a1}$. Similarly one can derive the effective demand for good 2.

Since effective demand for a good does not take into account the agents' perceptions of rations on that good, the aggregate effective demand for a good will not in general match with the aggregate effective supply. As a result, the agents will realize during the process of trade that their feasible transactions do not coincide with their effective demands. (This feature too will become more clear when we represent the concept through diagrams.) The passage from effective demands to feasible transactions would then necessitate a rationing scheme.

Denoting a feasible transaction for agent a in good c by $\tilde{z}_{ac}$, it follows that $\tilde{z}_{ac}$ is functionally related to $\tilde{z}_{ac}$ and $\tilde{z}_{bc}$, where $\tilde{z}_{kc} = \tilde{x}_{kc} - w_{kc}$, $k = a, b$. When an agent becomes aware of the levels of feasible transactions, it gives him an idea about the constraints prevailing in the market. Thus, starting from a set of perceived constraints, agents arrive at a set of effective demands which in turn leads to a *new set of perceived constraints*.

The general rule to be followed here is that the agents on the long side of a market perceive binding constraints preventing them from realizing their desired trades while those operating on the short side perceive themselves to be unconstrained.

Thus, beginning from an initial set of perceived constraints, each agent determines his effective demand for each commodity. On the basis of these signals, a new set of perceived constraints are determined, usually different from the initial ones. This leads to a fresh computation of effective demands. Equilibrium is attained when two consecutive sets of effective demands coincide.

In equilibrium, the levels of realized transactions $\bar{Z}_k = (\bar{z}_{k1}, \bar{z}_{k2}, \bar{z}_{k3})$ for a three good economy is the solution vector for the problem

$$\max U_k(X_k), \quad X_k = W_k + Z_k$$

$$s.t. \quad p \cdot Z_k \leq 0$$

$$\underline{z}_{kc} \leq z_{kc} \leq \bar{z}_{kc}, \quad c \neq 3$$

As already noted, quantity constraints are clamped on the trades of both good 1 and good 2, though no such constraint is imposed on the third. A plausible reason for this is mentioned earlier. In search of a further explanation of this fact, it may be taken into consideration that Benassy as well as Malinvaud and Younes try to ascribe an additional role to the third good, viz. that of a medium of exchange. It is clear from Chapter 2 above that a good would act as a medium of exchange if people believe they can pass it on to some other agent in exchange for goods they desire to possess finally. Hence, in order for every agent to accept a particular good as a medium of exchange (in the absence of complete information) they should be confident that it is possible to acquire the good or pass it on whenever necessary. This can be ensured by making the agents feel unconstrained in this good. Keeping that in mind, Benassy

has constructed an equilibrium where one good can always be visualized as an unrationed commodity. We note, however, that this condition is not sufficient for good 3 to act as a medium of exchange, as is clear from the discussion of Chapter 2.

Finally, we proceed to visualize the concept of a Benassy equilibrium diagrammatically. Once again referring back to Fig. 5.6 we notice that at the (Drèze) equilibrium E , agent a 's ration on good 2 (i.e. $S_a S_a$) is non-binding, whereas the ration on good 1 (i.e. $E_a E'_a$) restricts him from choosing a more desirable allocation (as compared to E). Therefore, in this situation, it is natural for agent a to *perceive* a binding constraint for good 1 given by $E_a E'_a$ and feel *unconstrained* in the case of good 2. For parallel reasons agent b can *perceive* a binding constraint for good 2 given by $T_b T'_b$, whereas, he feels unconstrained in his demand for good 1.

As agent a perceives that his supply of good 1 is constrained by $E_a E'_a$, his *effective supply* of good 2 (given by his utility maximizing choice on the budget plane subject to $E_a E'_a$) is $T_b T'_b$ (Fig. 5.7). On the other hand, as agent b feels unconstrained in his demand for good 1, his utility maximizing choice would be at B_b . This would imply that his effective demand for good 2 is $Q_b Q_b$. Thus, there is an excess effective demand to the tune of $T_b Q_b$. Naturally, agent b would feel constrained (at a level $T_b T'_b$) concerning his purchase of good 2, whereas, agent a feels unconstrained.

Given the perceived constraint $T_b T'_b$ for agent b (regarding good 2), his effective demand for good 1 is given by $E_a E'_a$ (Fig. 5.7). Agent a on the other hand, feels unconstrained in his supply of good 2 and hence chooses his bliss point and correspondingly determines $N_a N_a$ to be his effective supply of good 1. Thus, there is excess effective supply of good 1 to the tune of $E_a N_a$. Hence, agent a feels constrained in his supply of good 1 given by $E_a E'_a$, whereas b feels unconstrained.

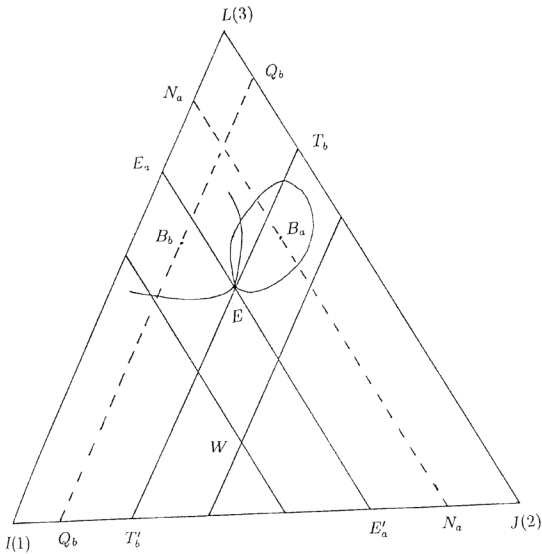


Fig. 5.7

Since we get back the same set of perceived constraints with which we started, they are indeed the *equilibrium perceived constraints*. In equilibrium, agent a faces only one binding constraint viz. $E_a E'_a$ and hence his utility maximizing choice on the budget plane subject to $E_a E'_a$, i.e. the point E , refers to the *realized transactions*. Similarly, for agent b facing the sole constraint $T_b T'_b$, the realized transaction would once again be represented by the point E .

Thus, point E can be sustained as the point of actual transactions for a Benassy equilibrium. From now on, the *point of actual transactions* corresponding to a Benassy equilibrium would also be referred to as the *Benassy equilibrium allocation*, although this involves a misuse of terminology.

From this discussion it is revealed that the same point E can be simultaneously sustained as a Drèze equilibrium, a Benassy equilibrium and also as a Malinvaud-Younes equilibrium. It is natural to wonder whether this coincidence is accidental. Interestingly enough, it is not so, as is shown by Silvestre (1982).

The next section depicts other possible types of allocations for the three solution concepts.

5.4 A USEFUL DEVICE

With the help of the existing literature (see Basu (1992)), we will develop here an interesting device to locate a Benassy equilibrium.

Let us begin with agent a . Suppose he faces the ration $A_1 A_1$ on his supply of good 1 (Fig. 5.8). In other words, he is not allowed to supply good 1 at all. Then his most preferred allocation is given by point o . Now if we relax this ration to $A_2 A_2$ agent a would choose o' at which his indifference contour is tangential to $A_2 A_2$. Thus, varying the restriction on good 1 and tracing the optimal choices of agent a , we get a path like oB_a , which we would refer to as agent a 's *effective demand curve* for good 2. Similarly, varying rations on his

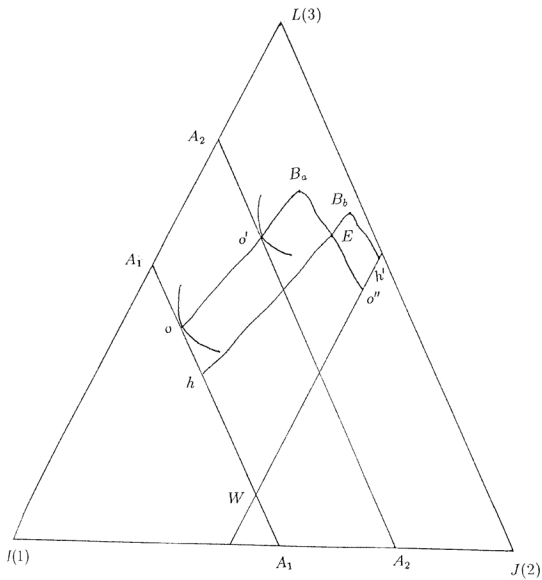


Fig. 5.8

supply of good 2 and connecting his respective optimal choices, the effective demand curve for good 1 can be drawn as $o''B_a$ (see Fig. 5.8). The curve $oB_a o''$ will be referred to simply as a 's *effective demand curve*. In a parallel fashion, agent b 's effective demand curve can be drawn as $hB_b h'$ (Fig. 5.8).

From Fig. 5.8, it can be easily checked that the point of intersection between $oB_a o''$ and $hB_b h'$ (i.e. point E) represents a Benassy equilibrium allocation (as Basu (1992) shows, the intersection point also represents a Drèze equilibrium).

Some other typical examples of a Benassy equilibrium are represented through the figures below (for simplicity we have omitted the budget plane).

In Fig. 5.9, agent a is a notional excess supplier of both goods and b is a notional excess demander of good 1 and notional excess supplier of good 2. Here point E represents the level of actual transactions associated with a Benassy equilibrium. It can be easily checked that at E , condition (M.1) and (M.2) of the Malinvaud-Younes equilibrium are fulfilled and E can also be sustained as a Drèze equilibrium.

We next consider Fig. 5.10. In this situation, point B_a is the Benassy equilibrium. It can also be sustained both as a Malinvaud-Younes equilibrium as well as a Drèze equilibrium.

In Fig. 5.11 both a and b are notional excess suppliers of goods 1 and 2 and hence no trade is possible between them. Therefore, the point of initial endowment W represents the Benassy equilibrium allocation. As before W also represents a Drèze equilibrium and a Malinvaud-Younes equilibrium.

A Benassy equilibrium may also be non-unique. This is depicted in Fig. 5.12. In Fig. 5.12 the effective demand curves ($oB_a o''$ and $hB_b h'$) coincide partly (shown by the segment $B_b B_a$). and any point on the segment $B_b B_a$ can be sustained as any of the three equilibria.

As an extreme case, when B_a and B_b coincide, we in fact get a Walras equilibrium (Fig. 5.13). This equilibrium can be sustained as a Benassy equilibrium

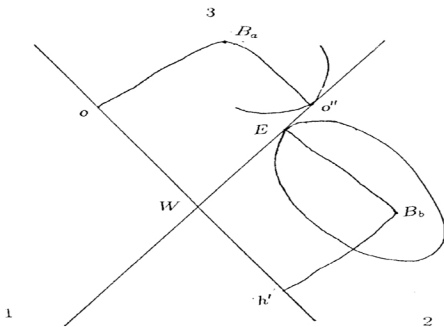


Fig. 5.9

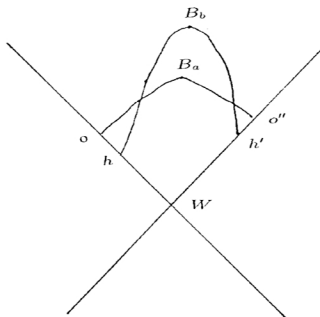


Fig. 5.10

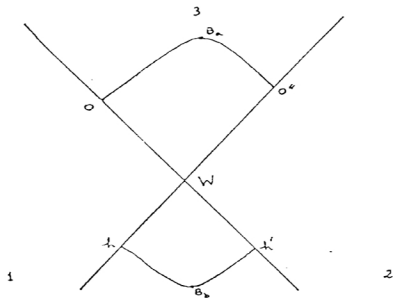


Fig. 5.11

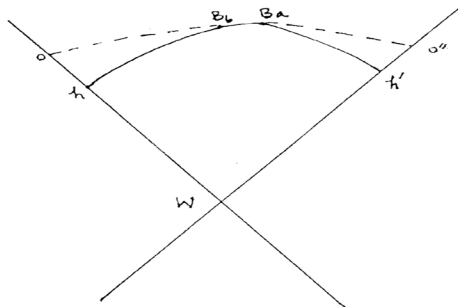


Fig. 5.12

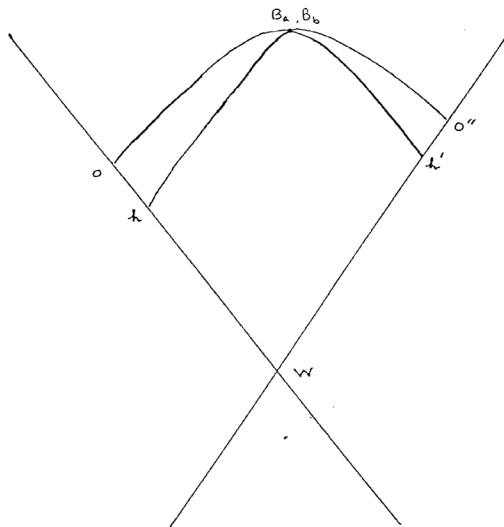


Fig. 5.13

where both agents feel unconstrained in both goods 1 and 2.

5.5 WILL THE THREE EQUILIBRIA ALWAYS COINCIDE ?

From the discussion of Sections 5.3 and 5.4, one gets the impression that the three equilibria would generally coincide. In fact it has been shown formally by Silvestre (1982) that if

- (i) the prices are positive,
- (ii) the consumption set is bounded,
- (iii) the preferences are strictly convex,
- (iv) no satiation point exists,
- (v) the utility functions are differentiable, and
- (vi) there exist consumption vectors x_1 and x_0 with $U_i(x_1) > U_i(x_0)$ and $\lim_{\theta \rightarrow 0} \frac{1}{\theta} [U_i(x_0 + \theta(x_1 - x_0)) - U_i(x_0)] > 0, 0 < \theta < 1$, then the three solution concepts give the same equilibrium allocation.

This result, however, need not necessarily hold if the utility functions do not satisfy the mentioned properties. Here we intend to demonstrate a few diagrammatic examples of this fact.

Example 5.1 : If one assumes away convexity of indifference contours, there can exist a Drèze equilibrium which is not a Malinvaud-Younes equilibrium. This is depicted by Fig. 5.14. Here point E can be sustained as a Drèze equilibrium as follows. Agent a is rationed in his supply of good 1, where the rationing is shown by AA' . Similarly, he is restricted in his supply of good 2 and the rationing is given by WA . Given these two constraints trading possibilities

of agent a are restricted to the line segment $\overline{WA}$. In this region his utility maximizing choice would be the point E . Thus, a 's ration on good 1 is non binding and that on good 2 is binding.

On the other hand, agent b faces rationings given by QQ' for good 1 and WA for good 2. Given these two constraints, his optimal choice would be at point E , where both rationings are binding. Though agent a as well as b face binding constraints for good 2, it does not violate Drèze's criterion as both of them are supply rationed for that good. Thus, the allocation represented by E can be sustained as a Drèze equilibrium.

The fact that point E is not a Malinvaud-Younes equilibrium can be checked very easily. As is clear from Fig. 5.14, by trading good 2 against 3 and choosing an allocation on the line segment FF' (where the indifference contour of agent b cuts QQ'), both agents can arrive at a mutually advantageous position.

Example 5.2 : If we assume away differentiability property of the indifference contours, there can arise a Malinvaud-Younes equilibrium which does not qualify as a Drèze equilibrium. This is shown in Figure 5.15. Here agent a 's indifference contour is non-differentiable at E . It can be easily checked that at E both the criteria of a Malinvaud-Younes equilibrium are satisfied.

To sustain E as a Drèze equilibrium, agent b has to be given binding constraints AA' and QQ' for goods 1 and 2 respectively. On the other hand, for agent a to choose E , he has to face at least one binding constraint for one of the goods. *WLOG* let agent a face binding constraint for good 1 given by QQ' . Thus, for good 1 both the demander (i.e. b) and supplier (i.e. a) are constrained and hence E is not a Drèze equilibrium.

To sustain E as "realized transactions" for a Benassy equilibrium, agent b has once again to perceive (binding) constraints AA' (on good 1) and QQ' (on good 2). Agent a on the other hand has to perceive at least one binding

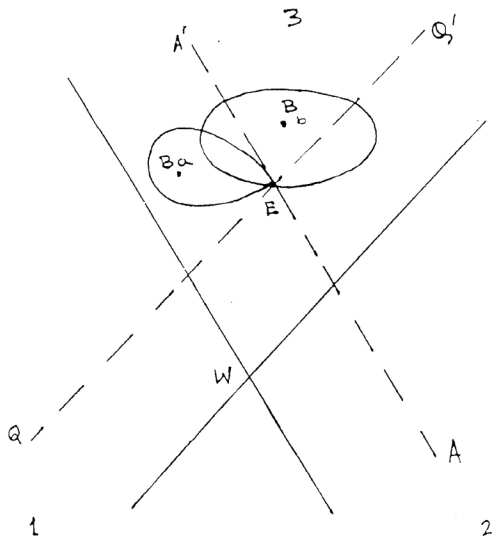


Fig. 5.15

constraint in equilibrium (otherwise he would choose B_a rather than E as equilibrium transactions). *WLOG* suppose agent a faces constraint AA' on its supply of good 1 and in good 2 market he feels unconstrained.

However, it can be checked that AA' cannot be sustained as a (binding) perceived constraint for both a and b . Therefore, E cannot be sustained as a Benassy equilibrium.

Example 5.3 : The following diagram (Fig. 5.16) shows a situation where point E can be sustained as a Malinvaud-Younes and a Benassy equilibrium but not as a Drèze equilibrium.

We note here that the indifference contour of agent a is neither strictly convex along the stretch EE' nor differentiable at E . In particular, in Figure 5.16 a part of the indifference contour of agent a viz EE' , merges with the line AA' (parallel to RR').

5.6 CONCLUSION

This chapter has presented a clear visual analysis of how in a price rigid situation, a particular allocation can be sustained through a set of appropriate rationing schemes which under a monetary pattern of trade would be voluntarily accepted by the agents. This fact may turn out to be an aid to a planner trying to bring about a particular distribution of commodities through appropriate choice of policy prescriptions (viz. quantity rationings and price fixation).

In the next chapter we would take up this issue in some detail. In particular we would try to provide an easily implementable device for a planner while choosing an appropriate allocation for the system.

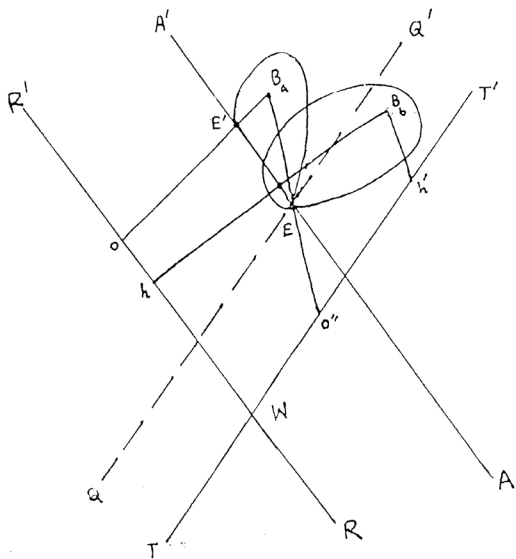


Fig. 5.16

List of Symbols used in Chapter 5:

- (1) c : index for a commodity
- (2) k : index for an agent
- (3) U_k : utility function of agent k
- (4) x_{kc} : level of good c finally consumed by agent k
- (5) W : initial endowment matrix
- (6) w_{kc} : initial endowment of good c for agent k
- (7) X_k : (x_{k1}, x_{k2}, x_{k3})
- (8) W_k : (w_{k1}, w_{k2}, w_{kc})
- (9) z_{kc} : excess demand for good c for k
- (10) Z_k : (z_{k1}, z_{k2}, z_{k3})
- (11) IJL : budget plane
- (12) B_a, B_b : points of notional demands for agents a and b respectively
- (13) r_c, R_c : rations on the excess supply of and demand for good c (respectively)
- (14) p : price vector
- (14) p_c : price of good c
- (15) Bd_k : budget set of agent k
- (16) X_k^D : allocation attained by agent k in Dréze equilibrium
- (17) X_k^y : same as (16) in Malinvaud-Younes equilibrium

(18) $\underline{x}_{kc}, \bar{x}_{kc}$: perceived constraints of agent k regarding good c

CHAPTER 6

THE P -EFFICIENCY PROPERTY

ABSTRACT

This chapter is concerned with a diagrammatic representation of the concept of p -efficiency associated with a Benassy equilibrium. It also provides a complete characterization of the price space into p -efficient and p -inefficient Benassy equilibria. The latter exercise is done for a Cobb-Douglas utility specification. One interesting conclusion of the chapter is that in a 3-good, 2-agent economy, p -efficiency properties of a large variety of Benassy equilibria can be established with reference to the national demands alone.

6.1 INTRODUCTION

The fundamental theorem of welfare economics states the equivalence of Walras equilibria and Pareto optimal allocations. The important part of the theorem consists of showing that a Pareto optimal allocation is usually attainable as a competitive equilibrium. However, to the extent that agents' endowments may not coincide with the Pareto optimal allocation in question, the theorem amounts to an empty exercise, unless the planner may carry out arbitrary transfers of endowments between the agents. In practice, such physical reallocations are difficult if not impossible. On the other hand, price manipulation is generally known to be a more implementable instrument than direct reallocation of agent's endowments. If the planner opts for price manipulations alone, an obvious choice for him is the market clearing competitive price vector for the given set of initial endowments. But the competitive equilibrium allocation though Pareto optimal need not maximize a planner's social welfare

function. Consequently, even if prices are manipulable the planner may not be interested in the competitive allocation.

The interesting question to ask then is whether there may be a (weaker) concept of efficiency which can aid the planner in his choice of (administered) prices. In this context Benassy's concept of p -efficiency offers some help.

As opposed to a competitive equilibrium, a Benassy equilibrium need not always be efficient or even p -efficient. As a first step, therefore, we attempt to describe the concept of p -efficiency through a set of diagrams and identify situations where a Benassy equilibrium need not be p -efficient. This is done in Section 6.2 for a 2-agent and 3-good economy. In the next section, we show that in our framework, it is not necessary to know the Benassy equilibrium allocation in order to examine its efficiency properties. In fact depending only on the notional demands of the agents, one can infer whether the resulting Benassy equilibrium (or equilibria) are p -efficient or not. These results are then used (in Section 6.4) to obtain a complete characterization of the price space in terms of p -efficient and p -inefficient Benassy equilibria. Given such a characterization, all that the planner has to do is to announce the prices which lead to the socially most desirable p -efficient allocation.

It should be mentioned in this context that these characterizations will change depending on the preference patterns of the agents. We perform the exercise here for a Cobb-Douglas type of utility specification.

6.2 THE CONCEPT OF P -EFFICIENCY

We consider an economy exactly similar to the one described in the previous Chapter (Section 5.2). As is observed in Figures 5.10, 5.11, 5.12 and 5.13 (of Section 5.4), depending on the nature of the parameter values, one can arrive at different types of Benassy equilibria. We intend to take up each such situation and examine its efficiency properties.

Case I : If the given prices correspond to a competitive equilibrium, we arrive at a Benassy equilibrium that coincides with the competitive equilibrium. This Benassy equilibrium can be sustained through the agents' perception of being unconstrained in all the markets. Clearly this allocation is Pareto-optimal.

However, Case I does not represent a typical scenario. For in a disequilibrium set-up, prices are usually taken to be rigid at a non-market clearing level. Keeping this in mind we next look for other types of Benassy equilibrium (Figures 6.1 and 6.2). These equilibria, though not Pareto optimal, are efficient in a *restricted sense*. More precisely they have the property that, there does not exist any alternative allocation *on the fixed budget plane* which makes at least one agent better off without making the other worse off. Case II below illustrates some such situations through Figures 5.10, 6.1 and 6.2.

Case II : In Figure 5.10, B_a represents the level of realized transactions corresponding to a Benassy equilibrium. We will equivalently refer to B_a as the Benassy equilibrium. Here agent a achieves his highest possible utility and any other allocation on the budget plane is going to make him worse off. Thus, it is not possible to arrive at an alternative allocation that is superior in this restricted sense.

We now refer to Fig. 6.1, where again B_a can be shown to be a Benassy equilibrium.

Point E of Fig. 6.2 also depicts a Benassy equilibrium. Here the indifference contours of the agents are tangential to each other at E . Hence, it is not possible to move to a new allocation on the budget plane without making at least one agent worse off.

Case III : Indeed, there are a large number of cases (see for example Fig. 6.3) where a Benassy equilibrium does not qualify as efficient even in this restricted sense.

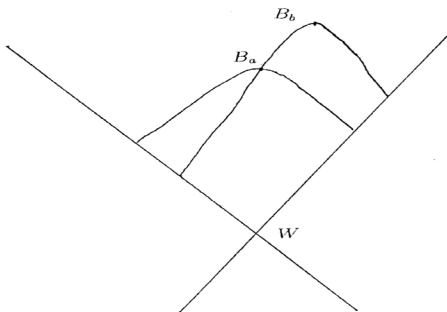


Fig. 6.1

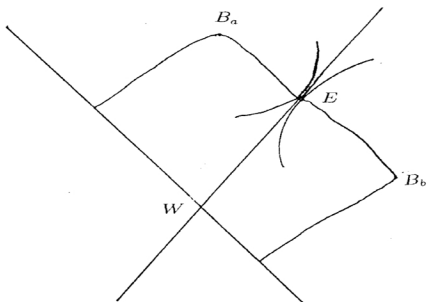


Fig. 6.2

Suppose an allocation, say E , of Figure 6.3 (which is a Benassy equilibrium) is assigned to the respective agents. Clearly, any allocation in the shaded region (say, point E') makes both agents better off. However, the question that arises is whether the agents with allocation E will be able to move out from it through *free trade* involving pairs of goods, when the price vector is fixed at p . Benassy points out that this is not possible.

In a 3-good economy there can be at most three trading posts viz. (1,2), (1,3) and (2,3). Since E is a Benassy equilibrium, no further trade involving good 3 is mutually agreeable. Thus, we are left only with the (1,2) trading post. It is clear from Figure 6.3 that the two agents with allocation E would like to move in opposite directions (viz. $\overrightarrow{EF}$ and $\overrightarrow{EF'}$). This shows that at E there does not exist any mutually agreeable trades (which makes both a and b better off) involving pairs of goods alone. From this point of view, Benassy recognizes E to be an efficient allocation of sorts, viz. as a *p-efficient* allocation.

It is natural to ask when there can exist a *p*-inefficient Benassy equilibrium. From the very definition of equilibrium, it is clear that no mutually advantageous trades involving good 3 can ever exist. This is especially clear from the definition of a Malinvaud-Younes equilibrium (see (M.2) of Section 5.3). Therefore, all that remains to be checked is whether by trading in the (1,2) trading post, the agents can improve upon the existing allocation. The representation of Fig. 6.4 makes the point clear. As discussed earlier (Section 5.4, Fig. 5.11), the initial endowment vector W represents a Benassy equilibrium in the case of Figure 6.4. However, as is clear from Fig. 6.4, at W trade in the (1,2) trading post i.e. a movement along $\overrightarrow{WA}$ makes both traders better off.

This example reveals that even after the attainment of a Benassy equilibrium, it may be possible to have certain potentially profitable trades unrealized. The problem arises from the fact that the Benassy equilibrium rules out by definition direct exchange of good 1 against 2.

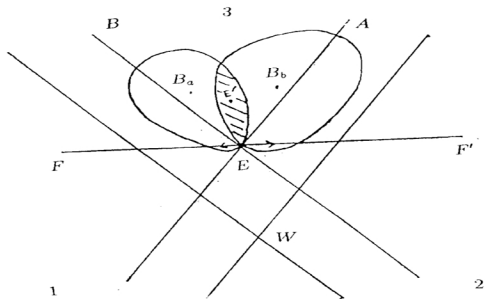


Fig. 6.3

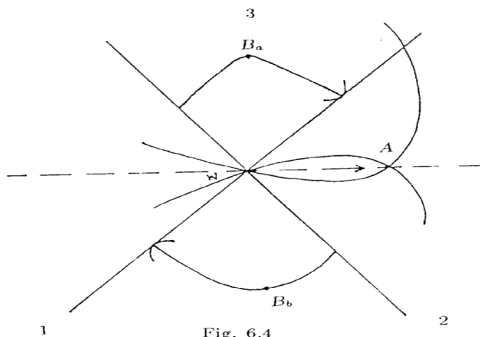


Fig. 6.4

Keeping this in mind a p -efficient allocation is defined as follows:

WLOG consider agent a . Let x_{a1} and x_{a2} denote respectively the amounts of goods 1 and 2 with agent a . We define relation $\mathcal{R}_a$ as

$$1(\mathcal{R}_a)2 \iff \frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}} - \frac{1}{p_2} \frac{\partial U_a}{\partial x_{a2}} > 0, x_{a2} > 0$$

which implies a has a positive quantity of good 2 and would like to supply good 2 against good 1 at the going set of prices. One can now define a *Pareto improving chain of traders as follows*.

$$\begin{aligned} c_1(\mathcal{R}_a)c_2, c_2(\mathcal{R}_b)c_1, \quad c_1 &= 1, 2, 3, \\ c_2 &= 1, 2, 3, \\ c_1 &\neq c_2. \end{aligned}$$

An allocation is p -efficient if no such Pareto improving chain exists. (Note that Fig.6.4 provides an example of a Pareto improving chain of trades starting from W .)

Thus, it is clear that a Pareto improving chain can exist even after reaching a Benassy equilibrium, because a Benassy equilibrium recognizes only potential trades against a specific good (viz. good 3 here, that is viewed as a money good). If, however, an agent is allowed to quote his effective demand for *every good against every other good*, then benefits from all potential bilateral trades can be realized.

More precisely suppose λ_{a12} denotes the money value of the volume of agent a 's transactions of good 1 against good 2:

$\lambda_{a12} > 0$, if a buys 1 against 2

$\lambda_{a12} < 0$, if a sells 1 against 2.

Thus, the final holding by agent k of good 1 will be

$$x_{a1} = w_{a1} + \frac{\lambda_{a12}}{p_2} + \frac{\lambda_{a13}}{p_3}$$

Here, the effective demand in the (1,2) trading post, denoted by $\bar{\lambda}_{a12}$ is found out by solving

$$\max U_a(X_a)$$

$$(6.1) \quad \text{s.t. } p.X_a \leq p.W_a$$

$$(6.2) \quad x_{ac_1} = w_{ac_1} + \sum_{c_h \neq c_1} \frac{\lambda_{ac_1c_h}}{p_h}, h = 2, 3$$

$$(6.3) \quad X_a \geq 0$$

$$(6.4) \quad \underline{\lambda}_{a13} \leq \lambda_{a13} \leq \bar{\lambda}_{a13}$$

$$(6.5) \quad \underline{\lambda}_{a23} \leq \lambda_{a23} \leq \bar{\lambda}_{a23}$$

where $\underline{\lambda}_{ac_1c_2}$ and $\bar{\lambda}_{ac_1c_2}$ represent agent a 's perceived constraints in the (c_1, c_2) trading post etc. Note that for the calculation of effective demand in a *market* we now take into account *two* perceived constraints instead of only one as in the regular case. We may similarly define effective demands in the (1,3) and (2,3) trading posts. Revision of these effective demands are performed in a manner parallel to that of the monetized trading post set-up described in the previous chapter.

As before, equilibrium is attained when two consecutive sets of effective demands coincide. The equilibrium levels of realized transactions are found by taking into consideration all the perceived constraints simultaneously. That is, agent a 's realized transaction is given by the solution of the following problem:

$$\max U_a(X_a)$$

$$\text{s.t. } (6.1) - (6.5)$$

and

$$\underline{\lambda}_{a12} \leq \lambda_{12} \leq \bar{\lambda}_{a12}.$$

An equilibrium of this nature is called a *barter equilibrium*. It is shown by Benassy that a barter equilibrium is always p -efficient.

We now visualize the concept of a barter equilibrium by means of Figure 6.5. Let us begin with the following set of perceived constraints.

Perceived constraints for agent a (Fig. 6.5):

(1,3) trading post : he perceives that no trade is possible

(2,3) trading post : he perceives that no trade is possible.

(1,2) trading post : he feels unconstrained in his *demand* for good 2 against good 1. However he perceives that he cannot *supply* good 2 for good 1. More precisely, he is free to choose any point along WN' .

Perceived constraints for agent b (Fig. 6.5):

For the (1,3) and (2,3) trading posts : exactly the same as those for agent a .

(1,2) trading post : he perceives that he can demand RR' (level of good 1) against TT' (level of good 2) but cannot *demand* good 2 for good 1 at all. That is, he is restricted to move along the line segment $\overline{We}$.

Beginning with a set of perceived constraints, our next task is to examine the effective demand vectors for the two agents.

Effective demand in the (1,2) trading post:

In this situation one has to take into account the perceived constraints of the (1,3) and (2,3) trading posts. Given the "no trade possible" constraints in both these markets the 'feasible set' of allocations from which agent a can choose his optimal point is given by the line NN' . In this region, the utility maximizing volume of trade for agent a in the (1,2) trading post is given by the point 'e' where his indifference contour is tangential to the horizontal line NN' (Fig. 6.6). More specifically, he wants to supply RR' level of good 1 against TT' level of good 2. Since agent b faces the same set of constraints in trading posts (1,3) and (2,3), the feasible set of allocations from which he has to choose his effective demand in the (1,2) market is given by the points on the line NN' . Utility maximizing choice of agent b in this region would be

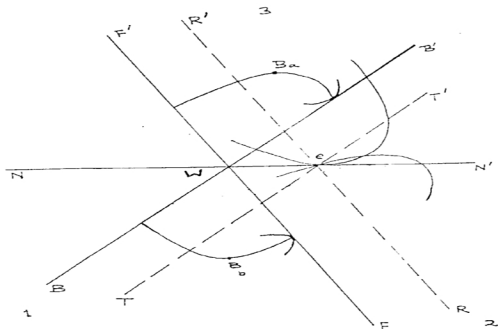


Fig. 6.5

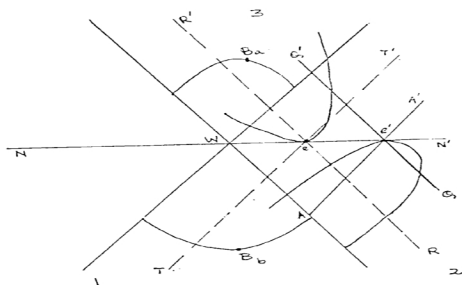


Fig. 6.6

at point e' (Fig. 6.6) where his indifference contour is tangential to NN' . In particular, b demands QQ' level of good 1 against AA' level of good 2.

From the position of e and e' it is clear that agent a belongs to the short side of the (1,2) market and hence feels unconstrained in his demand for good 2 against good 1. However, since agent b is an excess effective supplier of good 2 against good 1, agent a obviously perceives that he cannot supply good 2 against good 1 in the (1,2) trading post.

On the other hand agent b feels a binding constraint in trading post (1,2) restricting him to demand RR' level of good 1 against TT' level of good 2. Furthermore, since agent a is an excess effective demander of good 2 against good 1, b naturally perceives that it is not possible for him to demand good 2 against good 1. This restricts him further from choosing any point to the left of W , i.e. along $\overline{WN}$. Thus, he is constrained to move along the line segment $\overline{We}$. This shows that we have arrived at the same set of perceived constraints in the (1,2) market, with which we started out.

Effective demand in the (1,3) trading post (Fig. 6.7) :

In this case one has to consider the respective perceived constraints in the (1,2) and (2,3) trading posts for agents a and b and evaluate their unconstrained (desired) trade in the (1,3) market. Let us begin with agent a . Agent a feels unconstrained in his demand for good 2 against good 1 in trading post (1,2) and perceives that no trade is possible in the (2,3) market. Furthermore, any desired trade in the (1,3) trading post diagrammatically implies a movement along a line parallel to AA' . Thus, the feasible set of allocations from which a is free to choose his effective demand in the (1,3) trading post is given by the points lying to the right of AA' . Therefore, agent a 's utility maximizing volume of trade in the (1,3) market is attained at f which represents his effective supply of nn' level of good 1 against rr' level of good 3.

Let us now find agent b 's effective demand. Agent b perceives that he can

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choose any point in the line segment $\overline{We}$ through trade in trading post (1,2) and no trade is possible in the (2,3) market. Thus, the feasible set of allocations from which b would choose his desired level of trade in the (1,3) market is given by the region lying between the line $T'eT$ and AWA' . Clearly, B_b represents his best point in this region. Thus, he would declare the corresponding levels of good 1 and good 3 as his effective supply of good 1 against good 3.

From the effective demand configurations for a and b , it is clear that both agents are excess effective demanders of good 3 (against good 1) and hence they arrive at the perception that no trade is possible in that trading post.

Effective demand in the (2,3) trading post (Fig. 6.8) :

It can similarly be shown by using Fig.6.8 that given the rations on trading posts (1,2) and (1,3), a 's utility maximizing choice is at B_a and b 's choice is at b' . Since both the agents are suppliers of good 2, they again get back to the perception that no trade is possible in the (1,3) market.

From the above discussion it is clear that the set of perceived constraints we began with are indeed equilibrium perceived constraints.

To find the level of realized transactions in equilibrium we consider simultaneously all three perceived constraints relevant for each trader. This reveals the fact that each agent can choose any point in the line segment $\overline{We}$, where e is the best point for both a and b (Fig. 6.5).

As is clear from the diagram, e refers to a p -efficient allocation. This representation shows how the agents can improve upon a Benassy-equilibrium by the opening up of the (1,2) trading post and also the way effective demands in the (1,3) and (2,3) markets change as one moves to a barter equilibrium.

6.3 P -EFFICIENCY AND NOTIONAL DEMANDS

Assuming without loss of generality that the Benassy equilibrium for a given set of parameters (viz. preferences, endowments and prices) is unique,

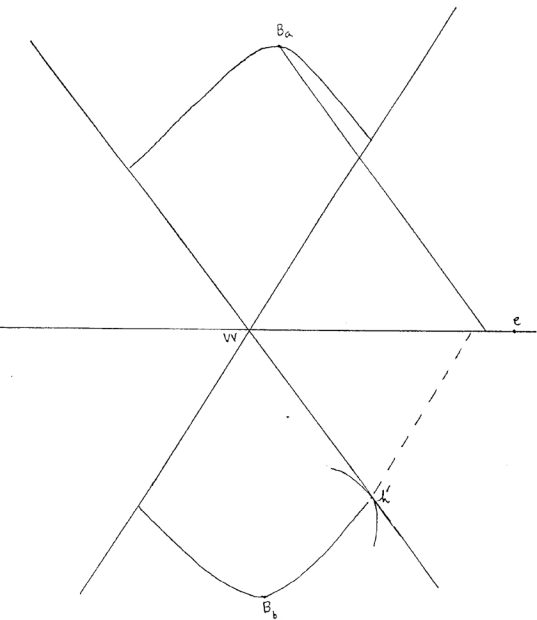


fig. 6.8

one can appeal to the definition of efficiency given by Benassy (1975) in order to determine if it is p -efficient or not. However, this requires in the first place the computation of the equilibrium allocation itself.

The thrust of the present exercise on the other hand is to examine the efficiency properties of a Benassy equilibrium without actually computing the equilibrium. In order to present the problem through diagrams, we concentrate as before on a simple 2-person, 3-good economy. The results derived in this situation are yet to be generalized for a large economy.

In particular, we ask whether the notional demands alone can be used as indicators for determining the efficiency properties of the resulting Benassy equilibrium. Fortunately, for a large class of utility specifications we can answer this question affirmatively. The advantage of this exercise lies in the fact that notional demands are far easier to calculate than the equilibrium effective demands.

We specify below the assumptions on the preference patterns under consideration :

Assumption 6.1 : Suppose $B_a \neq W, B_b \neq W$. We assume that the sign of an agent's excess effective demand for a good is the same as the sign of his notional excess demand for that good. More specifically, if an agent, say a is a notional excess supplier of good 1 in the sense that at B_a he wants to sell some amount of good 1, Assumption 6.1 implies that he can never be made an excess effective demander of good 1 by imposing rations on good 2.

If we represent the corresponding effective demand curve diagrammatically, it lies entirely inside one of the four regions where the no trade restrictions in the (2,3) and (1,3) markets, divide the budget plane with (e.g. refer to Figures 6.1, 6.2, 6.3, 6.4 etc.). Figure 6.9 refers to a situation which does not satisfy Assumption 6.1 and hence is not captured by our exercise.

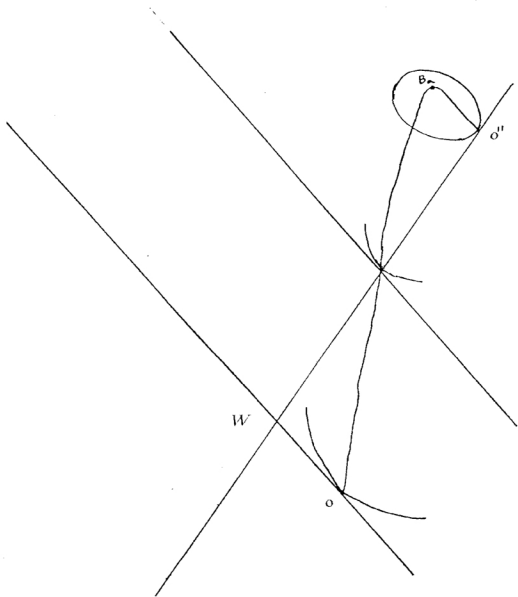


Fig.6.9

Let us now examine various situations characterized by the signs of the notional excess demands of agents a and b for the different goods. Our findings in this respect are presented in the following propositions.

Proposition 6.1 : Let $B_a \neq B_b \neq W$. Further, suppose (without loss of generality) that a is a notional excess supplier of both goods 1 and 2. Then, the Benassy equilibrium will be p -efficient if agent b has complimentary desires at B_a . That is

$$(6.6) \quad \left[\frac{1}{p_1} \frac{\partial U_b}{\partial x_{b1}} - \frac{1}{p_3} \frac{\partial U_b}{\partial x_{b3}} \right]_{B_a} \geq 0$$

$$(6.7) \quad \left[\frac{1}{p_2} \frac{\partial U_b}{\partial x_{b2}} - \frac{1}{p_3} \frac{\partial U_b}{\partial x_{b3}} \right]_{B_a} \geq 0$$

Note : (i) We assume that U_b is differentiable at B_a .

(ii) This situation is represented in Figure 6.1.

Proof : See Appendix to Chapter 6.

Remark 6.1 : If (6.6) and (6.7) holds with equality we get a Walras equilibrium.

Corollary 6.2 : If either $B_a = W$ or $B_b = W$, we get a p -efficient Benassy equilibrium at W .

After this discussion concerning the equilibrium occurring either at B_a or at B_b , we now move to the cases where the notional demand points cannot be sustained as equilibria. Thus, in what follows we would assume the parameter values to be such that Benassy equilibrium does not coincide with the notional demands of any agent.

Proposition 6.2 : (i) If agent a is a notional excess supplier and agent b is a notional excess demander of both goods 1 and 2 or vice versa then $\exists$ a Benassy equilibrium that is p -efficient.

(ii) The same conclusion holds if both agents are notional excess demanders of good 1 and notional excess suppliers of good 2.

Proof : See Appendix to Chapter 6.

Note : It can be easily checked that the result holds if one agent's excess notional demand or supply equals zero (for good 1 and/or good 2).

Remark 6.2 : The excess demand specifications given in Proposition 6.2 are represented by Figures 6.3 and 6.10 respectively.

Proposition 6.3 : If agent a is notional excess supplier of good 1 and demander of good 2 and b is a notional excess demander of good 1 and supplier of good 2 or vice-versa, then the Benassy equilibrium is never p -efficient.

Note : As mentioned earlier, here we are assuming that the equilibrium does not coincide with either B_a or B_b .

Proof : See Appendix to Chapter 6.

Remark 6.3 : The excess demand specifications mentioned above are illustrated in Figure 6.11.

Proposition 6.4 : If agent a is a notional excess supplier (demander) of both goods 1 and 2 and b is a notional excess demander (supplier) of good 1 and supplier (demander) of good 2, then the Benassy equilibrium may or may not be p -efficient.

Proof : See Appendix to Chapter 6.

Note : The diagrams depicting different possibilities are presented together with the discussion of the proof of this proposition.

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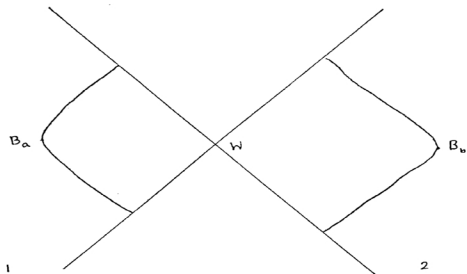


Fig. 6.10

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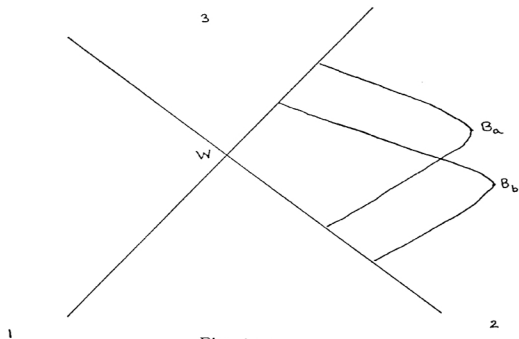


Fig. 6.11

6.4 A CHARACTERIZATION OF THE PRICE SPACE

In the light of the above propositions, it is possible to examine whether an inefficient Benassy equilibrium can be sustained as a p -efficient equilibrium by readjusting the parameter values, in particular by adjusting the prices. This in turn results in a characterization of the price space in terms of p -efficient and p -inefficient Benassy equilibria. Clearly, such characterizations would differ for different utility specifications. We would perform the exercise for a simple 2-person, 3-good economy where the agents are endowed with *identical Cobb-Douglas preferences*.

The utility function of agent i is given by

$$U_i(x_{i1}, x_{i2}, x_{i3}) = x_{i1} \cdot x_{i2} \cdot x_{i3}, i = a, b.$$

We assume the parametric specifications to be such that Assumption 6.1 remains valid. We have a further assumption regarding the parameter values. It has already been shown (Proposition 6.3) that if one agent is a notional excess demander of one good and a notional excess supplier of the other, while the other has opposite characteristics, the Benassy equilibrium is p -inefficient. Keeping this in mind we assume that to begin with the parametric specifications satisfy the above conditions regarding the agents' notional demands. This automatically ensures the initial equilibrium to be p -inefficient. More specifically, we have the following assumption.

Assumption 6.2 : the initial parameter values are such that

$$(6.8) \quad \begin{cases} x_{a1}^* < w_{a1}, & x_{a2}^* > w_{a2} \\ x_{b1}^* > w_{b1}, & x_{b2}^* < w_{b2} \end{cases}$$

Also, the equilibrium does not occur at B_a or at B_b .

Under the given Cobb-Douglas utility specifications

$$B_a(x_{a1}^*, x_{a2}^*, x_{a3}^*) = \left(\frac{p.W_a}{3p_1}, \frac{p.W_a}{3p_2}, \frac{p.W_a}{3p_3} \right)$$

and

$$B_b(x_{b1}^*, x_{b2}^*, x_{b3}^*) = \left(\frac{p \cdot W_b}{3p_1}, \frac{p \cdot W_b}{3p_2}, \frac{p \cdot W_b}{3p_3} \right)$$

where $p \cdot W_a$ = total value of agent a 's initial endowment = $p_1 w_{a1} + p_2 w_{a2} + p_3 w_{a3}$ etc.

We now examine whether any adjustment of the price vectors can restore efficiency in the system. If so, our next task is to identify these values on the price space.

According to Proposition 6.2, the Benassy equilibrium would turn efficient if agent a becomes a notional excess demander of good 1 and notional excess supplier of good 2, with b remaining a notional excess demander of good 1 and a supplier of good 2. This situation is depicted in Figure 6.10.

Initially we have,

$$\begin{aligned} x_{a1}^* &= \frac{(p_1/p_3)w_{a1} + (p_2/p_3)w_{a2} + w_{a3}}{3(p_1/p_3)} < w_{a1} \\ x_{a2}^* &= \frac{w_{a2}}{3} + \frac{(p_1/p_3)w_{a1} + w_{a3}}{3(p_2/p_3)} > w_{a2} \end{aligned}$$

Thus, we begin with a pair of values $\frac{p_1}{p_3}$ and $\frac{p_2}{p_3}$ such that (6.8) holds. As a next step, we increase $\frac{p_2}{p_3}$ keeping $\frac{p_1}{p_3}$ fixed. That raises x_{a1}^* and reduces x_{a2}^* . For the fixed p_1/p_3 , consider the set p_2/p_3 which satisfies

$$(6.9) \quad x_{a1}^* - w_{a1} \geq 0$$

and let $(\frac{\bar{p}_2}{p_3})$ be the minimum value of $\frac{p_2}{p_3}$ such that (6.9) holds. Since x_{a1}^* is a continuous function of $\frac{p_2}{p_3}$ it follows that for $\frac{p_2}{p_3} = (\frac{\bar{p}_2}{p_3})$

$$x_{a1}^* = w_{a1}$$

Similarly, given the same $\frac{p_1}{p_3}$, select the minimum value of $\frac{p_2}{p_3}$ such that

$$(6.10) \quad x_{a2}^* - w_{a2} \leq 0$$

holds and denote it by $(\frac{\bar{p}_2}{p_3})$. Clearly, for $(\frac{\bar{p}_2}{p_3})$, $x_{a2}^* = w_{a2}$.

Thus, $\max((\frac{\bar{p}_2}{p_3}), (\frac{\bar{\bar{p}}_2}{p_3})) = (\frac{\bar{p}_2}{p_3})$, (say) gives a value of $\frac{p_2}{p_3}$ for which agent a is a notional excess demander of good 1 and a notional excess supplier of good 2 (corresponding to the initial value of $\frac{p_1}{p_3}$). Clearly, at $(\frac{\bar{p}_2}{p_3})$ at least one good has 0 excess demand or supply.

We now concentrate on agent b . As $(\frac{p_2}{p_3})$ increases, x_{b1}^* rises and x_{b2}^* falls. Hence, b remains a notional excess demander of good 1 and a notional excess supplier of good 2. This shows that the $(\frac{\bar{p}_2}{p_3})$ (defined above) gives a level of $\frac{p_2}{p_3}$ that is sufficient to ensure p -efficiency.

As is clear from the above discussion $(\frac{\bar{p}_2}{p_3})$ and $(\frac{\bar{\bar{p}}_2}{p_3})$ are functions of $\frac{p_1}{p_3}$. We now proceed to find the exact relationship between them. Let us call $\frac{p_1}{p_3} = P_1$, $\frac{p_2}{p_3} = P_2$, $(\frac{\bar{p}_2}{p_3}) = \bar{P}_2$ and $(\frac{\bar{\bar{p}}_2}{p_3}) = \bar{\bar{P}}_2$.

$\bar{P}_2(Q)$ is such that

$$\begin{aligned} & \frac{p_1 w_{a1} + p_2 w_{a2} + p_3 w_{a3}}{3p_1} = w_{a1} \\ (6.11) \quad & \Rightarrow P_1 = \frac{w_{a2}}{2w_{a1}} \bar{P}_2 + \frac{w_{a3}}{2w_{a1}} \end{aligned}$$

Similarly, $\bar{\bar{P}}_2(Q)$ is that value of $\frac{p_2}{p_3}$ which satisfies

$$\begin{aligned} & \frac{p_1 w_{a1} + p_2 w_{a2} + p_3 w_{a3}}{3p_2} = w_{a2} \\ (6.12) \quad & \Rightarrow P_1 = \frac{2w_{a2}}{w_{a1}} \bar{\bar{P}}_2 - \frac{w_{a3}}{w_{a1}} \end{aligned}$$

The relations (6.11) and (6.12) are represented in Figure 6.12. According to Assumption 6.2 the initial values of P_1 and P_2 are located somewhere inside the dotted region (WLOG, say at N of Figure 6.12). We have shown so far that we can increase p_2/p_3 by NN' and arrive at an efficient allocation. In fact, for any $\frac{p_2}{p_3} \geq \max(\bar{P}_2, \bar{\bar{P}}_2)$, efficiency can be attained. Thus, the region shaded by the '+' signs consists of p -efficient allocations. However, in the region $(\frac{\bar{p}_2}{p_3}) > \frac{p_2}{p_3} > (\frac{\bar{\bar{p}}_2}{p_3})$ (shown by FOAR), or in $(\frac{\bar{p}_2}{p_3}) > \frac{p_2}{p_3} > (\frac{\bar{\bar{p}}_2}{p_3})$ (shown

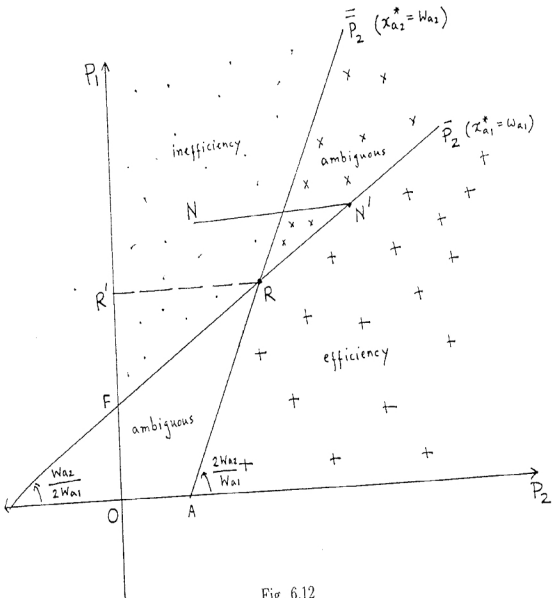


Fig. 6.12

by the crossed (X) region), it is not yet possible to infer about efficiency. Therefore, we have marked these two regions as “ambiguous”.

Our next task is to characterize the ambiguous region to see if with an even smaller increment in P_2 we might attain efficiency. In order to do so, we begin with the situation where the initial P_1 value is larger than R' . Thus, our interest is to characterize at first the ambiguous region above R (i.e. the cross-shaded region). For convenience, the corresponding truncated diagram is shown in Figure 6.13. Clearly, in this region we have $\bar{P}_2 \geq \bar{\bar{P}}_2$. For any P_2 such that $\bar{P}_2 > P_2 > \bar{\bar{P}}_2$, agent a is a notional excess supplier of both goods 1 and 2, while b is a notional excess demander of good 1 and supplier of good 2. In such a situation, the Benassy equilibrium may or may not be efficient (Proposition 6.4). Depending on the parameter values, two distinct possibilities can arise (they are discussed in the Appendix by means of Figures A.6.1 - A.6.3). We would now examine for what values of P_1 and P_2 these possibilities can arise. However, it should be noted that these situations also depend on other parameter values, in particular on the initial endowments of the agents. Thus, we begin by considering two alternative relations between the initial endowments of the two agents which are going to play a major role in our characterization exercise.

Case 1: $w_{a1} > w_{b1}$

Case 2: $w_{a1} < w_{b1}$

Needless to say, depending on the relation between w_{ac} and w_{bc} , $c = 2, 3$, there arise various subcases which we will take up in the sequel.

Let us begin with Case 1. Two different possibilities (as mentioned above) can be considered under this parametric specification. They are described by the following subcases.

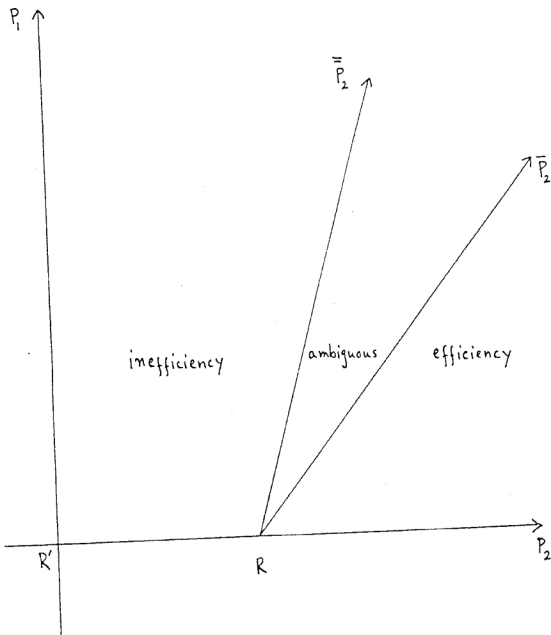


Fig. 6.13

Subcase 1: Given a ration on good 2 at a level GG' (Figure A.6.1), agent a 's effective supply of good 1 $\leq$ agent b 's effective demand for good 1. In this case, the resulting Benassy equilibrium is p -efficient (see the proof of Proposition 6.4 presented in the Appendix).

Subcase 2: Given a ration on good 2 at a level GG' (Figures A.6.2, A.6.3), agent a 's effective supply of good 1 is greater than agent b 's effective demand for the same. In this situation the Benassy equilibrium may or may not be p -efficient (Proposition 6.4).

Let us first consider **Subcase 1** for **Case 1**. Given a ration on agent i 's consumption of good 2 at a level $\bar{x}_{i2}$, his effective demand for good 1 can be computed as

$$\bar{x}_{i1} = \frac{p_i W_i - p_2 \bar{x}_{i2}}{2p_1}, \quad i = a, b$$

The corresponding level of x_{i3} is

$$\bar{x}_{i3} = \frac{p_i W_i - p_2 \bar{x}_{i2}}{2p_3} \quad \text{etc.}$$

Therefore, when good 2 is rationed at w_{i2} , effective demand for good 1 is given by

$$x_{i1} = \frac{p_1 w_{i1} + p_3 w_{i3}}{2p_1}, \quad i = a, b,$$

which is independent of p_2

Thus, from Subcase 1 it is clear that

$$(6.13) \quad \frac{p_1 w_{a1} + p_2 w_{a3}}{2p_1} \leq \frac{p_1 w_{b1} + p_3 w_{b3}}{2p_1}$$

implies p -efficiency. That is, p -efficiency requires

$$(6.14) \quad p_1(w_{a1} - w_{b1}) \leq p_3(w_{b3} - w_{a3})$$

$$(6.15) \quad \text{or,} \quad \frac{p_1}{p_3} \leq \frac{w_{b3} - w_{a3}}{w_{a1} - w_{b1}} \quad \text{if} \quad w_{a1} - w_{b1} > 0$$

Thus, the part of the ambiguous region in Figure 6.14 lying under the horizontal line,

$$P_1 = \frac{w_{b3} - w_{a3}}{w_{a1} - w_{b1}}$$

is revealed to be p -efficient. Note that if $w_{b3} < w_{a3}$, the entire ambiguous region is p -efficient.

In order to characterize the remaining ambiguous regions we next consider Subcase 2 (under Case 1), where agent a 's effective supply of good 1 is greater than b 's effective demand for the same good. In this context, Figure A.6.2. shows that if at A , agent a wants to demand more of good 2 against good 1, (keeping the level of good 3 constant), then the original Benassy equilibrium is p -inefficient. On the other hand, if at A agent a wants to supply good 2 against good 1 (Figure A.6.3) the initial equilibrium (at A) is p -efficient.

Let us now find conditions under which these two alternatives hold. Clearly, the level of good 3 for agent b at A is equal to the level of good 3 he would have had, had he been rationed in good 2 at the level w_{b2} (see figures A.6.2 and A.6.3). Thus, the corresponding x_{b3} is

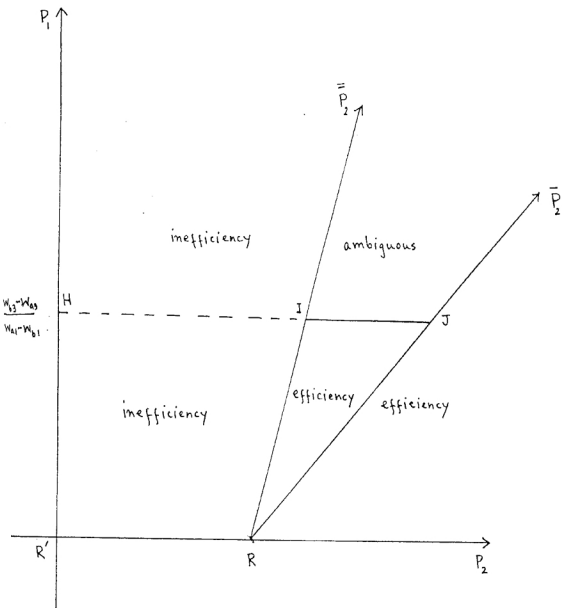
$$\tilde{x}_{b3} = \frac{p_1 W_a - p_2 w_{b2}}{2p_3} = \frac{p_1 w_{b1} + p_3 w_{b3}}{2p_3}$$

Therefore, at point A , agent a 's consumption of good 3 is

$$x_{a3} = w_{a3} + w_{b3} - \tilde{x}_{b3} = \frac{2p_3 w_{a3} + p_3 w_{b3} - p_1 w_{b1}}{2p_3}$$

Now let us examine the level of agent a 's potential demand for good 2 against good 1 at A . This is given by

$$\begin{aligned} x_{a2} &= \frac{p_1 w_{a1} + p_2 w_{a2} + p_3 w_{a3} - \frac{1}{2}[2p_3 w_{a3} + p_3 w_{b3} - p_1 w_{b1}]}{2p_2} \\ &= \frac{p_1 w_{a1} + p_2 w_{a2} - \frac{1}{2}[p_3 w_{b3} - p_1 w_{b1}]}{2p_2} \\ \Rightarrow x_{a2} - w_{a2} &= \frac{p_1 w_{a1} - p_2 w_{a2} - \frac{1}{2}[p_3 w_{b3} - p_1 w_{b1}]}{2p_2} \end{aligned}$$



$$W_{a1} > W_{b1}, W_{b3} > W_{a3}$$

Fig. 6.14

For an equilibrium to be p -efficient, it is sufficient that $x_{a2} - w_{a2} \leq 0$. Now,

$$\begin{aligned}
 & x_{a2} - w_{a2} \leq 0 \\
 \Rightarrow & p_1 w_{a1} \leq p_2 w_{a2} + \frac{1}{2}[p_3 w_{b3} - p_1 w_{b1}] \\
 \Rightarrow & \frac{p_1}{p_3}(w_{a1} + \frac{1}{2}w_{b1}) - \frac{1}{2}w_{b3} \leq \frac{p_2}{p_3}w_{a2} \\
 (6.16) \quad \Rightarrow & P_1 \leq (\frac{w_{a2}}{w_{a1} + \frac{1}{2}w_{b1}})P_2 + \frac{(1/2)w_{b3}}{w_{a1} + w_{b1}}
 \end{aligned}$$

This relationship (i.e. (6.16)) is shown in the untruncated diagram by the line $\overleftrightarrow{AB}$ (Figure 6.15). Here we have assumed $\text{WLOG } \frac{w_{a2}}{2w_{a1}} < \frac{w_{a2}}{w_{a1} + \frac{1}{2}w_{b1}}$. The truncated version of the diagram is shown in Figure 6.16, where A' represents the point at which RR' intersects AB (see Figure 6.12).

From Figure 6.14 it is clear that any configuration of P_1 and P_2 lying below the line segment HJ in the ambiguous region (of Figure 6.13) can be characterized as efficient. However for those values of P_1 and P_2 lying above the line segment HJ , the ambiguous region is still not characterized. Clearly in this region the Benassy equilibrium is of the nature described in Subcase 2. Therefore, the position of the line $A'B$ (Figure 6.16) would, help us to characterize the above mentioned ambiguous region. From relation (6.16), it is clear that to the right of line segment FB (Figure 6.16, shaded region) we get an efficient Benassy equilibrium. On the other hand to the left of line segment FB (dotted region, where F is the point of intersection between HJ and $A'B$) we get inefficiency (Figure 6.16). Thus, the entire region is characterized. Let us now move on to Case 2, where $w_{a1} < w_{b1}$. As before, we again consider Subcase 1 for Case 2. In this situation, the inequality (6.15) is reversed that is, p -efficiency would be ensured if

$$(6.17) \quad \frac{p_1}{p_3} \geq \frac{w_{b3} - w_{a3}}{w_{a1} - w_{b1}}$$

Depending on the sign of $w_{b3} - w_{a3}$ we get two types of possibilities. They are represented through Figures 6.17 and 6.18.

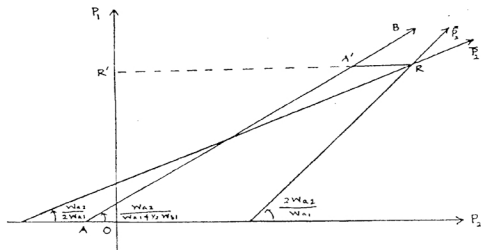


Fig. 6.15

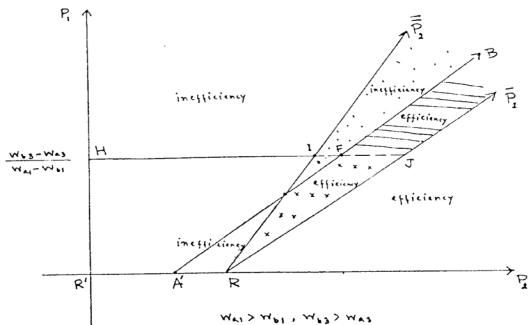


Fig. 6.16

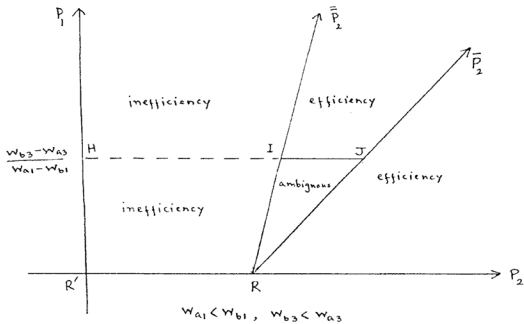


Fig. 6.17

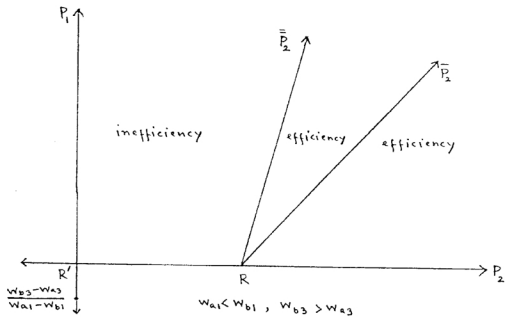


Fig. 6.18

The second possibility includes the interesting situation where $w_{b3} > w_{a3}$. In this case, the entire ambiguous region can be characterized as efficient (Figures 6.18) just on the basis of Subcase 1.

Note that if $w_{a1} - w_{b1} = 0$ and $w_{b3} - w_{a3} > 0$ we can characterize the entire ambiguous region as efficient as in the case of Figure 6.18.

In order to characterize the remaining ambiguous region of Fig. 6.17, we need to consider as before Subcase 2. Therefore, as in the case of Figure 6.16, we show the relation (6.10) by the line $A'B$ (Figure 6.19). Thus, the ambiguous region to the right of $A'B$ can be characterized as p -efficient and to the left of $A'B$ is inefficient.

Thus, in both Case 1 and Case 2 the entire region above RR' can be characterized.

Remark 6.1 Apart from the parametric specifications discussed so far there remains some finer possibilities.

(i) If $w_{a1} - w_{b1} = 0 = w_{b3} - w_{a3}$

(6.14) holds as an equality and we get a situation described in Figure 6.2, which ensures p -efficiency.

(ii) If $w_{b3} - w_{a3} = 0$ and $w_{a1} - w_{b1} > 0$, (6.14) never holds.

We next come to the region where $\bar{P}_2 < \bar{\bar{P}}_2$ (i.e. the ambiguous region lying physically below the line $R'R$). At a value of P_2 such that $P_2 < \bar{P}_2 < \bar{\bar{P}}_2$, agent a is a notional excess demander of good 1 and good 2. This situation is represented in Figure 6.20.

It can be easily checked that in this situation the characterization depends on the relation between w_{a2} and w_{b2} . Thus, we consider the following possibilities:

Case 3: $w_{a2} > w_{b2}$

Case 4: $w_{a2} < w_{b2}$

As before under each case, two subcases need to be studied.

Subcase 3: Given a ration on good 1 at a level SS' (Figure 6.20), agent a 's effective demand for good 2 $\leq$ agent b 's effective supply of good 2. In this case, the resulting Benassy equilibrium would be p -efficient.

Subcase 4: The inequality in Subcase 3 gets reversed. In this situation the Benassy equilibrium may or may not be p -efficient.

Depending on these possibilities, in an exactly parallel fashion as above one can characterize the region below RR' . The following diagrams (Figures 6.21, 6.22) show the complete characterization for different parametric specifications. In the following Figures 6.21 and 6.22, the line CD is drawn to depict the situation

$$P_1 \geq \frac{w_{a2} + (1/2)w_{b2}}{w_{a1}} P_2 - (1/2) \frac{w_{b3}}{w_{a1}}$$

which gives the condition for p -efficiency under Subcase 4.

The case when $w_{a1} < w_{b1}$ and $w_{a2} > w_{b2}$, the characterization of the upper part of the line RR' of Figure 6.21 will change and exactly resemble Figure 6.19. Similarly, in the situation where $w_{a1} < w_{b1}$ and $w_{a2} > w_{b2}$, the characterization of the region lying above RR' (Figure 6.22) will change and will be the same as in Figure 6.19.

Other finer possibilities can be dealt with in a similar fashion. This completes the characterization of the price space.

6.5 CONCLUSION

The above exercise reveals how the price space can be characterized in terms of p -efficient and p -inefficient Benassy equilibria. The characterization as is shown depends on the level of initial endowments of the agents. The exercise

is performed under the assumption of a Cobb-Douglas preference pattern. It is yet to be examined how this characterization changes for other types of utility specifications.

APPENDIX TO CHAPTER 6:

Proof of Proposition 6.1:

Here we intend to show that B_a can be sustained as a Benassy equilibrium. The p -efficiency would then follow immediately since no possible trades can improve a 's utility any further. In order to show that B_a is a Benassy equilibrium, it is enough to prove that B_a can be sustained as a Malinvaud-Younes equilibrium.

(i) At B_a , agent a would not desire to have any trade against good 3. Thus, no trade between a and b can occur. Therefore, the condition (M.2) of a Malinvaud-Younes equilibrium (Section 5.3) is satisfied.

(ii) At B_a , agent a is an excess supplier of both the goods. Therefore, at B_a agent b is an excess demander of good 1 and good 2. Now (6.1) and (6.2) imply that at B_a , agent b wants to demand more of both goods. Hence, by demanding less of good 1 and 2 agent b cannot improve his utility. Thus, at B_a condition (M.1) of a Malinvaud-Younes equilibrium is also satisfied.

Therefore, B_a is a Malinvaud-Younes equilibrium. Hence for all convex and differentiable indifference contours B_a can be sustained as a Benassy equilibrium (see Section 5.5). ■

Proof of Proposition 6.2:

(i) Let the realized transactions in equilibrium, be $x_a = (\bar{x}_{a1}, \bar{x}_{a2}, \bar{x}_{a3})$ and $\bar{x}_b = (\bar{x}_{b1}, \bar{x}_{b2}, \bar{x}_{b3})$

According to our assumption (stated immediately before Proposition 6.2)

$$\bar{x}_a \neq (x_{a1}^*, x_{a2}^*, x_{a3}^*)$$

$$\bar{x}_b \neq (x_{b1}^*, x_{b2}^*, x_{b3}^*)$$

Also we have $x_{a1}^* - w_{a1} < 0, x_{a2}^* - w_{a2} < 0, x_{b1}^* - w_{b1} > 0, x_{b2}^* - w_{b2} > 0$. This implies each agent faces a binding constraint in *exactly one* market. This is true due to the following reasons.

(i) if an agent faces no binding constraint then he would be able to attain his notional demands, which is ruled out here.

(ii) if an agent say a , faces *two* binding constraints and b faces at least one binding constraint (say, without loss of generality in good 1 market), then good 1 will be both demand rationed as well as supply rationed. This is not possible by the definition of a Drèze equilibrium.

WLOG suppose agent a (agent b) faces binding constraint for good 1 (good 2) and feels unconstrained in the good 2 (good 1) market. This gives us the following four relations:

$$(A.6a) \quad \frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}} - \frac{1}{p_3} \frac{\partial U_a}{\partial x_{a3}} \Big|_{\bar{x}_a} < 0$$

$$(A.6b) \quad \frac{1}{p_2} \frac{\partial U_a}{\partial x_{a2}} - \frac{1}{p_3} \frac{\partial U_a}{\partial x_{a3}} \Big|_{\bar{x}_a} = 0$$

$$(A.6c) \quad \frac{1}{p_1} \frac{\partial U_b}{\partial x_{b1}} - \frac{1}{p_3} \frac{\partial U_b}{\partial x_{b3}} \Big|_{\bar{x}_b} = 0$$

$$(A.6d) \quad \frac{1}{p_2} \frac{\partial U_b}{\partial x_{b2}} - \frac{1}{p_3} \frac{\partial U_b}{\partial x_{b3}} \Big|_{\bar{x}_b} > 0$$

Thus,

$$(A.6a) \text{ and } (A.6b) \Rightarrow \frac{1}{p_2} \frac{\partial U_a}{\partial x_{a2}} - \frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}} \Big|_{\bar{x}_a} > 0$$

$$(A.6c) \text{ and } (A.6d) \Rightarrow \frac{1}{p_2} \frac{\partial U_b}{\partial x_{b2}} - \frac{1}{p_1} \frac{\partial U_b}{\partial x_{b1}} \Big|_{\bar{x}_b} > 0$$

Therefore, in equilibrium no further trade between goods 1 and 2 is possible. Hence, the result follows.

(ii) We now come to the second part of the proposition, where both a and b are notional excess demanders of good 1 and notional excess suppliers of good 2. Clearly, in this situation the no trade allocation i.e. the initial endowment W is the Benassy equilibrium. Indeed, at W , agent a satisfies

$$\begin{aligned}\frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}} - \frac{1}{p_3} \frac{\partial U_a}{\partial x_{a3}} \Big|_{W_a} &\geq 0 \\ \frac{1}{p_2} \frac{\partial U_a}{\partial x_{a2}} - \frac{1}{p_3} \frac{\partial U_a}{\partial x_{a3}} \Big|_{W_a} &\leq 0\end{aligned}$$

Therefore,

$$\frac{1}{p_2} \frac{\partial U_a}{\partial x_{a2}} - \frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}} \leq 0$$

Exactly the same relations hold for b . Here no further trade between goods 1 and 2 is possible. Thus, the point W (Figure 6.10) represents a p -efficient Benassy equilibrium.

Proof of Proposition 6.3:

Arguing as in the case of Proposition 6.2 (i), one can easily show that each agent faces exactly one binding constraint (in equilibrium). WLOG assume that agent a is rationed in the good 2 market and b is rationed in the good 1 market. This gives us the following four relations.

$$\begin{aligned}(A.6e) \quad & \frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}} - \frac{1}{p_3} \frac{\partial U_a}{\partial x_{a3}} \Big|_{\bar{x}_a} = 0 \\ (A.6f) \quad & \frac{1}{p_2} \frac{\partial U_a}{\partial x_{a2}} - \frac{1}{p_3} \frac{\partial U_a}{\partial x_{a3}} \Big|_{\bar{x}_a} < 0 \\ (A.6g) \quad & \frac{1}{p_1} \frac{\partial U_b}{\partial x_{b1}} - \frac{1}{p_3} \frac{\partial U_b}{\partial x_{b3}} \Big|_{\bar{x}_b} < 0 \\ (A.6h) \quad & \frac{1}{p_2} \frac{\partial U_b}{\partial x_{b2}} - \frac{1}{p_3} \frac{\partial U_b}{\partial x_{b3}} \Big|_{\bar{x}_b} = 0\end{aligned}$$

Where, as before, $\bar{x}_a$ and $\bar{x}_b$ represent the realized transactions in equilibrium for agents a and b respectively. Now

$$(A.6e) \text{ and } (A.6f) \Rightarrow \frac{1}{p_2} \frac{\partial U_a}{\partial x_{a2}} - \frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}} \Big|_{\bar{x}_a} < 0.$$

$$(A.6g) \text{ and } (A.6h) \Rightarrow \frac{1}{p_1} \frac{\partial U_b}{\partial x_{b2}} - \frac{1}{p_2} \frac{\partial U_b}{\partial x_{b2}} \Big|_{\bar{x}_b} < 0.$$

Hence a Pareto improving chain of trades exists.

Proof of Proposition 6.4:

Let $\bar{x}_a$ and $\bar{x}_b$ represent respectively the realized transactions in equilibrium for agents a and b . In equilibrium agent b satisfies

$$(A.6i) \quad \frac{1}{p_1} \frac{\partial U_b}{\partial x_{b1}} - \frac{1}{p_3} \frac{\partial U_b}{\partial x_{b3}} \Big|_{\bar{x}_b} \geq 0$$

$$(A.6j) \quad \frac{1}{p_2} \frac{\partial U_b}{\partial x_{b2}} - \frac{1}{p_3} \frac{\partial U_b}{\partial x_{b3}} \Big|_{\bar{x}_b} \leq 0$$

$$(A.6i) \text{ and } (A.6j) \Rightarrow$$

$$(A.6k) \quad \frac{1}{p_2} \frac{\partial U_b}{\partial x_{b2}} - \frac{1}{p_1} \frac{\partial U_b}{\partial x_{b1}} \Big|_{\bar{x}_b} \leq 0$$

On the other hand a satisfies

$$(A.6l) \quad \frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}} - \frac{1}{p_3} \frac{\partial U_a}{\partial x_{a3}} \Big|_{\bar{x}_a} \leq 0$$

$$(A.6m) \quad \frac{1}{p_2} \frac{\partial U_a}{\partial x_{a2}} - \frac{1}{p_3} \frac{\partial U_a}{\partial x_{a3}} \Big|_{\bar{x}_a} \leq 0$$

Hence, the sign of $\frac{1}{p_2} \frac{\partial U_a}{\partial x_{a2}} - \frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}}$ is not definite.

Under the circumstances there may or may not arise an opportunity for further barter exchange (of good 1 against good 2). Different alternative possibilities that emerge are described below.

Note that if all four inequalities ((A.6i),(A.6j),(A.6l),(A.6m)) equal 0, B_a and B_b coincide. On the other hand if both (A.6i) and (A.6j) equal 0, equilibrium is attained at B_b and so on. In all these situations Benassy equilibrium is obviously p -efficient. As mentioned above here we are not concerned with the situations where either B_a or B_b can be sustained as an equilibrium.

Since a is an excess supplier of good 1 and b is an excess demander for the same good, in equilibrium *at most* one agent is subject to a binding constraint.

Case 1: Suppose a does not feel a binding constraint for good 1. Then

$$(A.6n) \quad \frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}} - \frac{1}{p_3} \frac{\partial U_a}{\partial x_{a3}} \Big|_{\bar{x}_a} = 0$$

and

$$\frac{1}{p_2} \frac{\partial U}{\partial x_{a2}} - \frac{1}{p_1} \frac{\partial U_a}{\partial x_{a1}} \Big|_{\bar{x}_a} < 0$$

Therefore, no barter trade is possible. This situation is shown in Figure A.6.1.

Case 2: If however agent b is unconstrained in the good 1 market and a feels the binding constraint, we have an ambiguous situation. Clearly in this situation, if at the original Benassy equilibrium, (i.e. at point A of Figure A.6.2) agent a wants to demand good 2 against good 1 in the (1,2) trading-post, then A is *not* a p -efficient Benassy equilibrium (since mutually agreeable trade exists).

However, if at A , agent a wants to supply good 2 against good 1, b is not ready to trade (Figures A.6.3). Thus, A represents a p -efficient Benassy

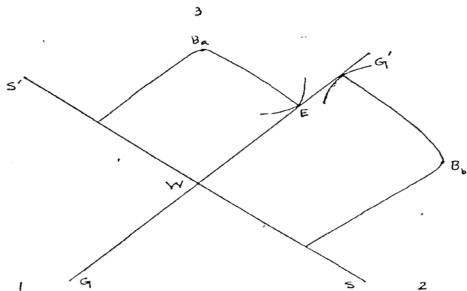


Fig. A.6.1

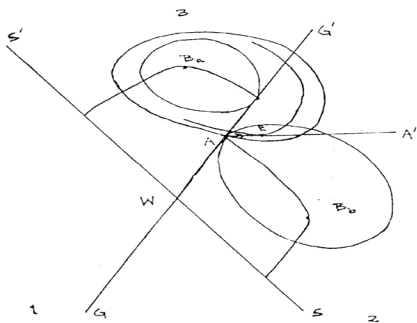


Fig. A.6.2

3

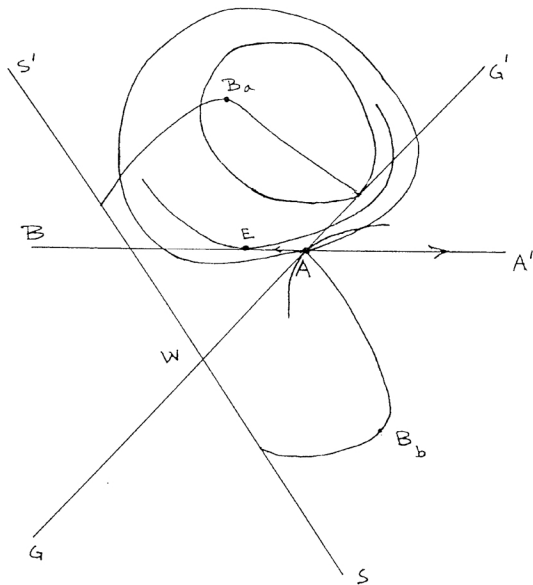


Fig. A.6.3

2

equilibrium.

Case 3: If both a and b feel unconstrained in the good 1 market, we get an efficient Benassy equilibrium. This situation is already represented in Figure 6.2, where E is the Benassy equilibrium.

List of Symbols Used in Chapter 6:

- (1) B_a, B_b : notional demand points of agents a and b respectively.
- (2) W : point of initial endowment on the budget plane.
- (3) w_{kc} : initial endowment of good c for agent k .
- (4) W_k : (w_{k1}, w_{k2}, w_{k3})
- (5) U_k : utility function of agent k
- (6) x_{kc} : level of good c finally consumed by agent k
- (7) X_k : (x_{k1}, x_{k2}, x_{k3})
- (8) $\lambda_{kc_1c_2}$: money value of the volume of agent k 's transactions of good c_1 against c_2 .
- (9) p_c : price of good c
- (10) x_{kc}^* : notional demand of agent k for good c .
- (11) $\bar{x}_{kc}$: realized demand for good c for agent k

DISEQUILIBRIUM MODELS WITH PRODUCTION

ABSTRACT

This chapter introduces production into the non-Walrasian framework. In particular, it provides a detailed and unified diagrammatic presentation of the Benassy-Malinvaud classification of different types of equilibria (viz. Keynesian, classical and repressed inflation) that may arise in the short run.

7.1 INTRODUCTION

In the previous two chapters we have discussed some of the major concepts of disequilibrium theory for a pure exchange economy. As a next step we bring production into the scenario. This naturally leads to the consideration of the phenomenon of unemployment and allows us a detailed representation of different types of unemployment (viz. Keynesian, classical and repressed inflation) that may arise in a price rigid set-up.

The diagrammatic representation to be depicted here bears a close resemblance to the contribution of Muellbauer and Portes (1978). However, the technique of representation differs in several respects. In particular, ours utilizes the three dimensional commodity space which enables one to visualize the choice problem on the budget plane of the household and read the coordinates pertaining to the money good simultaneously with those for other goods. However, since we confine ourselves to the budget plane alone, which constitutes a two dimensional subspace of the commodity space, an extra dimension does not create any special problem.

More interestingly, our representation provides a complete characterization

of the different types of Benassy equilibria on the household's budget plane. This is done on the basis of the position of the notional demand point on the household's budget plane.

The characterization turns out to be an interesting complement of the corresponding representation due to Benassy and Malinvaud. Thus, while their demonstrations are restricted to the wage price plane, the present work provides a dual characterization of different equilibria in the quantity space. The quantity space representation also brings into sharp focus the problem of non-uniqueness of Benassy equilibria (discussed also by Hildenbrand and Hildenbrand (1978)).

The next section describes the basic framework under consideration together with the preliminary features of the diagrams. In Section 7.3 different types of Benassy equilibria (viz. Keynesian, classical and repressed inflation) are depicted separately. A complete characterization of the three types of equilibria is then presented. This representation reveals the problem of non-uniqueness of Benassy equilibria. In Section 7.4, the diagrams are used to discuss the effect of a money wage cut policy. Efficiency properties of the Benassy equilibrium are taken up in Section 7.5. A concluding section sums up the findings.

7.2 THE MODEL AND PRELIMINARY FEATURES OF THE DIAGRAMS

7.2.1 The Model

The economy in question is comprised of three types of agents, viz., households, firms and the government. By assumption, the households have common characteristics while the firms share an identical technology. As a result, aggregate household (respectively firm) behaviour is a simple n -fold replica of

that of a single household (respectively firm) and this permits the system to be represented by a single household and a single firm.

The household supplies labour ℓ which the firm uses as an input to produce an output called y . The household can supply a maximum of $\bar{\ell}$ units of labour (say, 24 hours per day). Equivalently $\bar{\ell}$ represents the (least) upper bound on its leisure time. Similarly, it has an endowment of $\bar{m}$ units of money. In what follows, the household's consumption of leisure will be denoted by x and its supply of labour by ℓ . Thus, $x = \bar{\ell} - \ell$. The household's utility function depends on its consumption of y , of x and of the quantum of real balances $\frac{m}{p}$ it ends up with. This is captured in the utility function $U = U(x, y, \frac{m}{p})$ which satisfies the following properties:

- (H.1.a) U is monotonically increasing in all arguments;
- (H.1.b) U is strictly quasi-concave and differentiable; and
- (H.1.c) all goods are normal goods.

The firm on the other hand is characterized by a production function $y = F(\ell)$, where

$$(F.2.a) \quad \frac{\partial F}{\partial \ell} > 0 \text{ and}$$

$$(F.2.b) \quad \frac{\partial^2 F}{\partial \ell^2} \leq 0$$

Its objective is to maximize profits, given the price of the good (p) and the money wage rate (w).

The government has a fixed real demand G of y which is assumed to be financed by money creation. Consistent with standard practice in disequilibrium theory, it is assumed that ' w ' and ' p ' are fixed.

The basic problem arises from the fact that at least one of the two price variables is set at a (Walrasian) disequilibrium level. This implies that the household and the firm will not be able to realize their 'notional' demands and supplies. Under these circumstances the household's problem is to maximize U , subject to its budget constraint and other restrictions on its realizable trades.

Thus the household may be assumed to

$$(7.1) \quad \text{maximize } U(y, x, \frac{m}{p})$$

$$(7.2) \quad \text{s.t. } y + \frac{w}{p}x + \frac{m}{p} = \frac{w}{p}\bar{\ell} + \frac{\bar{m}}{p}$$

and

$$(7.3) \quad \underline{\ell}_h \leq \ell \leq \bar{\ell}_h$$

$$(7.4) \quad \underline{y}_h \leq y \leq \bar{y}_h$$

where (7.3) and (7.4) represent the constraints perceived by the household on its supply of ℓ and demand for y .

As should be obvious from (7.2), current profits are assumed to remain undistributed. The profits may of course be distributed with a lag and these make their appearance through $\frac{\bar{m}}{p}$.

Similarly the firm's problem is to

$$(7.5) \quad \text{maximize } y - \frac{w}{p}\ell$$

$$(7.6) \quad \text{s.t. } y = F(\ell)$$

and

$$(7.7) \quad \underline{\ell}_f \leq \ell \leq \bar{\ell}_f$$

$$(7.8) \quad \underline{y}_f \leq y \leq \bar{y}_f$$

where as in the case of the household, (7.7) and (7.8) represent the respective perceived constraints of the firm.

7.2.2 Diagrams : Preliminary Features

To begin with, we visualize the household's unconstrained maximization problem (i.e. (7.1) and (7.2)) in three dimensions, represented in Figure 7.1 by axes x, y and m/p . The budget plane (i.e. (7.2)) is given by the triangle IJL.

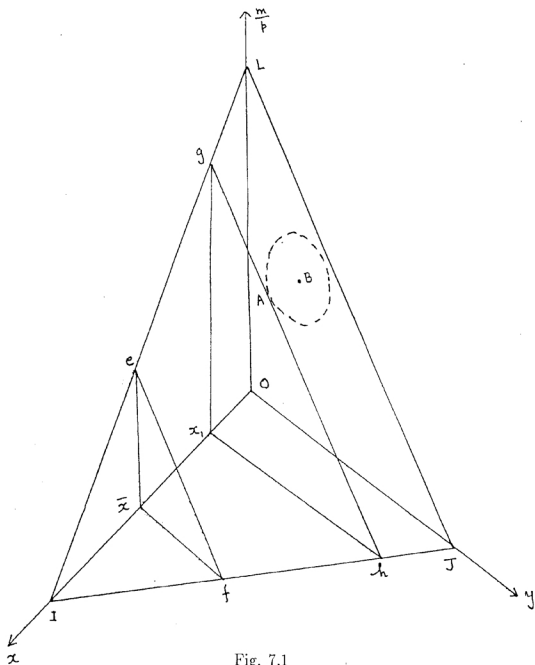


Fig. 7.1

An indifference surface in the three dimensional plane cuts the budget plane along the closed contour shown by the broken line in Fig. 7.1. The solution to (7.1) and (7.2), viz the notional demands, are shown by the point of tangency (B).

The household's initial endowments of x, y and $\frac{m}{p}$ are represented by the point e (Figure 7.1) with coordinates $(\bar{x}, 0, \frac{m}{p})$. The point $\bar{x}$ marks off the leisure coordinate on the Ox axis. Labour is measured in a direction opposite to that of leisure. Thus, while O represents the origin for measuring leisure, $\bar{x}$ is the one relevant for labour. The vertical plane $x = \bar{x}$ drawn through the point $\bar{x}$ ($e\bar{x}f$) will intersect the budget plane in a line which represents the alternative combinations of y and $\frac{m}{p}$ the household can afford when restricted to consume $\bar{x}$ units of leisure. As before, the constant levels of leisure (or, labour) associated with parallel lines such as ef, gh etc. will be referred to as ef level of leisure (labour), gh level of leisure (labour) etc.

Given the restriction gh , the household's trades are restricted to the zone delineated by the trapezium $efhg$, which is an example of the type of restriction that might be imposed by (7.3). This construction clarifies another point, viz. that lines (like gh) lying progressively away from the vertex I represent successively lower levels of leisure (or higher levels of labour). The restriction imposed by $efhg$ implies that the household cannot go to the right of gh and therefore the solution to (7.1) - (7.3) is given by the point A of tangency between a (restricted) indifference contour and gh .

Similarly, available choices of x and $\frac{m}{p}$ for given levels of y (respectively, x and y for given levels of $\frac{m}{p}$) can be shown by lines parallel to IL (respectively IJ) and it is easily checked that lines away from the vertex J (and parallel to IL) represent lower levels of y and those away from the vertex L (and parallel to IJ) represent lower levels of $\frac{m}{p}$. For each such quantity constraint the household's choices can be depicted as a tangency between the corresponding

line on the budget plane and the restricted indifference contour.

We may now introduce the firm. In Figure 7.2, the firm's production function is shown in the two dimensional $x - y$ plane by the curve $\bar{x}bn$ and the household's budget plane is shown as before by the triangle $IJJL$. From (7.2) it is clear that (the absolute value of) the slope of the line IJ in the two dimensional $x - y$ plane is $\frac{w}{p}$. For profit maximization, the firm equates its marginal product F' to $\frac{w}{p}$ and this is shown in Figure 7.2 by the point 'n' on the production function.

Suppose the household supplies $\bar{x}s$ level of labour. This will be represented on the budget plane by the line HR . With $\bar{x}s$ level of labour the firm can at most produce Ot level of output, which is represented on the budget plane by the line ij . Therefore, by supplying HR level of labour the household could hope to purchase a maximum of ij level of output, provided that it is prepared at the same time to reduce its initial endowment of real balance by an amount rq . This allocation is represented by point u on $IJJL$. Using the same procedure and varying the choice of the labour supply (and simultaneously the choice of the real balance) by the household, the entire schedule of the household's trading possibilities are traced out. This is none other than the locus of points like u corresponding to different levels of labour. Alternatively, this curve may also be viewed as the mirror image of the production function on the budget plane. In Figure 7.3 the curve in question is represented by the line euv and will be subsequently referred to as the '(gross) consumption possibility frontier'.

Now we introduce a government that purchases a fixed amount G of y from the firm. For each level of labour, this will obviously reduce the amount of y that is available to the household by a fixed amount G and will shift the production function on the horizontal plane. The mirror image of the curve on the budget plane may be looked upon as the consumption possibility frontier

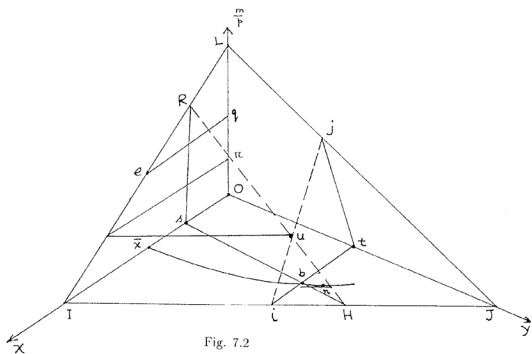


Fig. 7.2

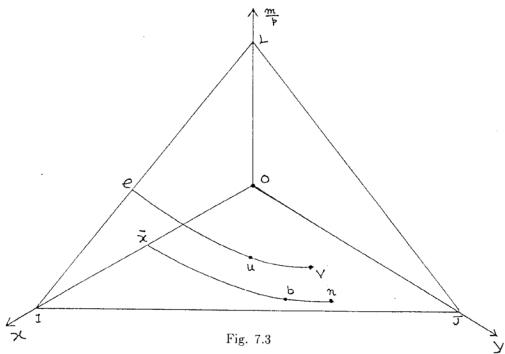


Fig. 7.3

net of the government's purchase of G . This curve will be referred to as the 'net consumption possibility frontier' (or, NCPF in short). The relevant curves are shown in Figure 7.4 which depicts the budget plane without reference to the coordinate axes. Thus, the NCPF is shown in Figure 7.4 by $e'u'u'$ while the (gross) consumption possibility frontier is euv . Thus the firm, by absorbing the supply of labour equal to the level ij , produces an output ab . But after deduction of the public consumption of G , the maximum amount that remains for private consumption is cd .

We may now obtain the locus of the household's optimal choices ($y, \frac{m}{p}$) associated with alternative restrictions on its supply of labour. This is done in Figure 7.5. Thus, when the household is restricted to cf , its utility maximizing choice is given by o . Similarly, by relaxing the labour supply constraints we get the optimal choices of the household, shown by the points on the curve oB . The curve oB designates the household's effective demand curve (EDC) for y .

In an analogous fashion one can find the household's optimal choices for different levels of rations on y . The locus of such points (viz. $o'B$) yields the household's effective supply curve (ESC) of labour. The union of these two paths, represented by the curve oBo' will be called the household's *effective demand curve* (EDC).

The assumptions made so far are not sufficient to establish any specific properties of the two curves oB and $o'B$. The classification of equilibria will be difficult, however, unless more precise information is available. We proceed, therefore, to strengthen the assumptions on household's preferences further.

Given any level of ration imposed on labour supply, one may define the budget for consuming y and $\frac{m}{p}$ as the *labour rationed budget*. Thus, when the household is restricted to consume at least $\bar{x}$ units of leisure, its *labour rationed budget* is given by $(\frac{w}{p}(\bar{\ell} - \bar{x}) + \frac{r_0}{p})$. Similarly, with demand for y rationed to lie below $\bar{y}$, the *goods rationed budget* is $(\frac{w}{p}\bar{\ell} + \frac{r_0}{p} - \bar{y})$.

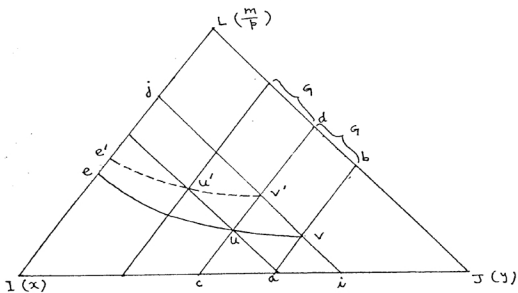


Fig. 7.4

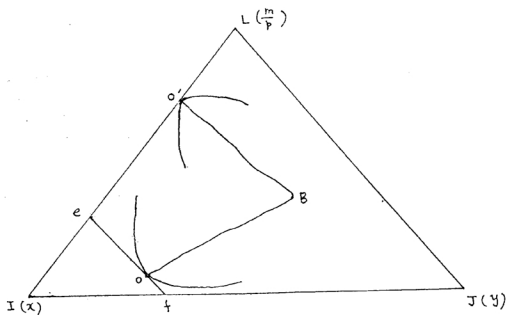


Fig. 7.5

It has already been assumed in (H.1.C) that all goods are normal i.e. the notional demands for all goods rise with a rise in the consumer's budget. We now assume further that :

(H.1.d) y^d and $(\frac{m}{p})^d$ rise with a rise in the labour rationed budget.

Under this assumption, the effective demand curve for y must satisfy the property that the tangent to it at each point is flatter than IL and steeper than IJ (Figure 7.6).

As opposed to the case of the oB curve no analogous assumption will be currently made regarding the household's effective supply curve for labour. It is worth pointing out, however, that if following Benassy (1982,1986) and certain parts of Malinvaud (1977), labour supply is assumed to be perfectly inelastic (with respect to w and p), then a definite characteristic of the curve $o'B$ also emerges. Suppose for example that $\hat{\ell}$ (which could equal $\bar{\ell}$, but not necessarily so) is the level of labour the household supplies inelastically to the firm. This level is shown in figure 7.6 by the line $o'Z$. We assume that o' lies to the right of e' , otherwise it would not be possible to meet the government demand G . It is easily checked in this case that the NCPF will either lie inside the region $oBo'e$ (Figure 7.8) or intersect the EDC at a *unique* point, say E (Figures 7.6, 7.9). The next section will show that depending on the relative positions of the two curves (i.e., whether they intersect or not), one may classify Benassy equilibria into the Keynesian, classical and repressed inflation types.

7.3 TYPES OF BENASSY EQUILIBRIA

7.3.1 Nomenclature

Benassy equilibria may be classified according to the patterns of excess effective demands and supplies characterizing the markets for x and y . The classifications, which are due to Malinvaud (1977) as well as Benassy (1982, 1986), fall into three parts:

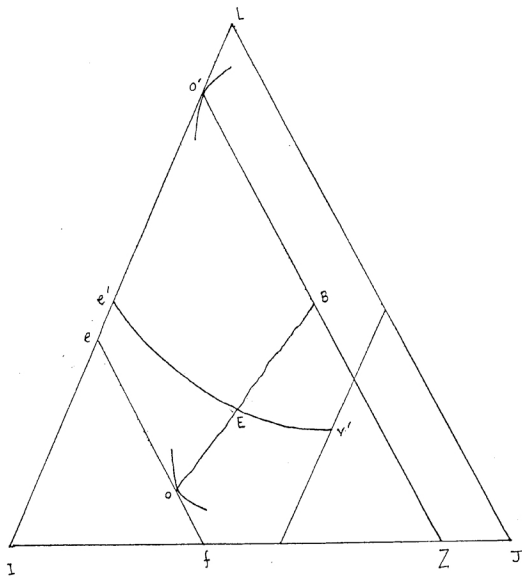


Fig. 7.6

(i) equilibrium of the Keynesian type, characterized by excess effective supply in both the markets;

(ii) classical equilibrium, characterized by excess effective supply in the labour market and demand in the goods market; and finally

(iii) a situation of repressed inflation, characterized by excess effective demand in both the markets.

Unless otherwise stated, it will be assumed for the purpose of this section that either the NCPF intersects the household's EDC at a unique point or is entirely enclosed by it.

7.3.2 Keynesian Equilibrium

We proceed now to demonstrate the first type of equilibrium i.e., the Keynesian type (Figure 7.7). As before $e'v'$ represents the NCPF and oBo' the household's EDC. As demonstrated below, the point of intersection of the two curves, viz. E , represents the optimal transactions corresponding to a position of Keynesian equilibrium.

The equilibrium is characterized by the household's perception that it cannot supply more than ab level of labour. Hence, its effective demand for y is given by ij which is its utility maximizing choice subject to ab . The firm on the other hand is assumed to feel unconstrained in the labour market and chooses fg as its effective supply of y which represents the unconstrained profit maximizing output net of G .

Under the assumed perceived constraints in the labour market, there is aggregate excess effective supply in the goods market to the tune of ig which represents the difference between the output levels fg and ij . The household therefore feels unconstrained in the y market whereas the firm feels that its demand is constrained at the level ij .

Since the household feels unconstrained in the goods market, its effective

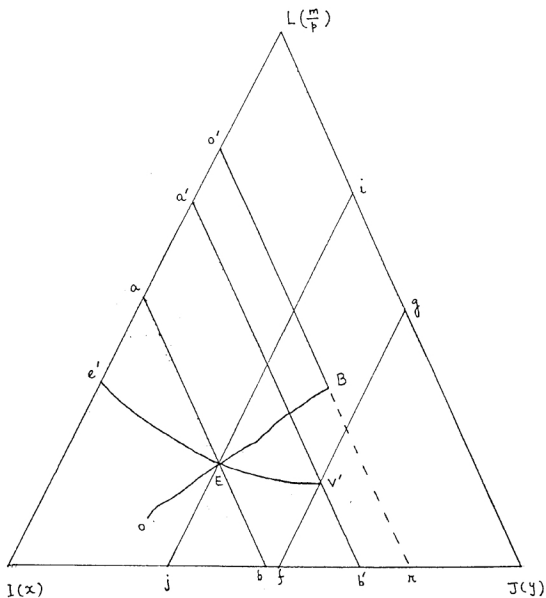


Fig. 7.7

supply of labour is given by the notional supply $o'r$. Simultaneously, the firm perceives that it is constrained to sell no more than ij in the goods market and hence chooses its effective demand for labour at the level ab . This means that the effective supply of labour by the household is in excess of the effective demand by the firm (the difference being ao'). The latter therefore feels unconstrained in the labour market while the household feels constrained to supply up to a maximum level of ab . Thus, the perceived constraints for the two agents in the labour market lead them to behaviour patterns which vindicate their perceptions. The same arguments establish further that the firm's perception of a demand constraint at the level ij along with the household's feeling of being unconstrained in the goods market also need no revision. The configuration of effective demands and supplies thus qualifies as a Benassy equilibrium. It is of the Keynesian type given that effective supply exceeds demand in both the y and ℓ markets.

The actual transactions in the two markets are now found by solving (7.1), (7.2), (7.3) and (7.4) for the household and (7.5), (7.6), (7.7) and (7.8) for the firm. The household is constrained in the labour market by ab and feels unconstrained in the goods market. Thus, (7.3) amounts to the restriction that the household lies on or to the left of the line ab in Figure 7.7. The solution to (7.1), (7.2), (7.3) and (7.4) is thus given by E .

Similarly, the firm's perception of constraints in the two markets reduces (7.7) to the requirement that it does not move to the right of ij . Consequently the solution to (7.5), (7.6), (7.7) and (7.8) is also the point E .

Hence, as claimed earlier, the optimal transactions corresponding to the position of Keynesian equilibrium are described by the point E .

It may be noted in passing that the extent of involuntary unemployment associated with this equilibrium is captured by ao' . The explanation of the persistence of such unemployment, moreover, is to be found in the shortfall of

effective demand y (below effective supply) by an amount ig .

7.3.3 Classical Equilibrium

The case of classical unemployment is taken up next. This is typically associated with an EDC of the type oBo' in Figure 7.8. In this situation the firm feels unconstrained in the labour market so that its effective supply of output is the profit maximizing (notional) one, viz., fg . The household, however, perceives the constraint ab in the labour market which leads it to an effective demand for y at the level ij . Thus, aggregate excess effective demand for y equals ig .

The firm obviously feels unconstrained in the goods market whereas the household discovers itself to be constrained by fg . Therefore, the firm's effective demand for labour coincides with the notional demand ab . The household's perceived constraint at fg results in the effective supply of labour $o'r$ (which is its utility maximizing choice subject to fg , since an indifference contour is tangential to fg at q). There is, therefore, excess effective supply of ao' in the labour market. As a consequence the firm perceives no constraint in this market while the household perceives the constraint ab , which are identical to the perceptions they began with. The same arguments demonstrate that the perceived constraints in the y market (viz., fg for the household and the firm visualizing a seller's market) are also in a state of equilibrium.

To determine the actual transactions for each agent, we consider the constraints perceived by it in the two markets simultaneously. Since the firm feels constrained in neither market, the solution to (7.5), (7.6), (7.7) and (7.8) occurs at v' . The household being constrained in both markets (by ab and fg respectively), v' is also a solution to (7.1), (7.2), (7.3) and (7.4). Hence v' represents the realized transactions pertaining to a Benassy equilibrium. Moreover, since y constitutes a seller's market and ℓ a buyer's one, the equilibrium is of the

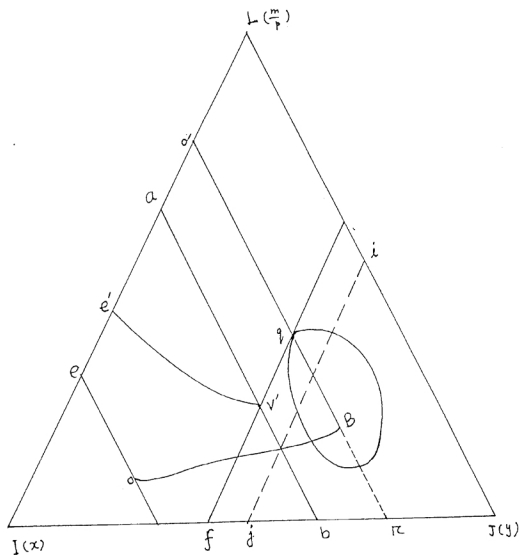


Fig. 7.8

classical type¹.

7.3.4 Repressed Inflation

The point of intersection viz. E between the NCPF and oBo' of Figure 7.9 represents the optimal transactions associated with a situation of repressed inflation.

The household feels unconstrained in the labour market and its effective demand for y is given by ji . The firm's demand for labour is constrained by ro' , so that its effective supply of y to the household is fg . There is, therefore, excess effective demand equal to ig in the goods market.

On the other hand, the firm perceives no constraint in the goods market and its effective demand for labour is given by its notional demand ab . The household perceives the constraint fg and comes out with $o'r$ as its effective supply of labour (we note that an indifference contour is tangential to fg at E). Therefore, there is excess effective demand (equal to ao') in the labour market also.

Thus, the perceived constraints satisfy the conditions of a Benassy equilibrium. That the realized transactions occur at E is easily checked. Thus, as an instance of Benassy equilibrium with excess effective demands in both the markets, this is a situation of repressed inflation.

7.3.5 The Complete Characterization

Depending on the position of the household's notional demand point B , different types of Benassy equilibria are obtained. This permits a complete characterization of the three types of equilibria described above.

¹The concept of classical unemployment needs to be distinguished from that of voluntary unemployment due to Keynes (1936). Keynes, it may be recalled, found voluntary unemployment to be consistent with full employment in the usual sense, i.e., the situation where B merges with v' . It is not possible to make the corresponding statement with respect to classical unemployment.

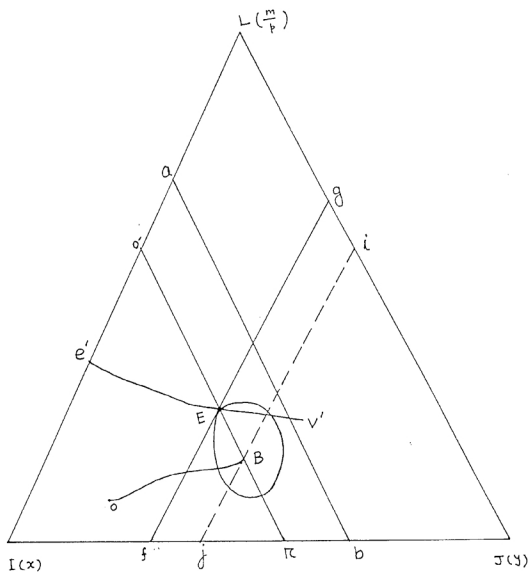


Fig. 7.9

In Figure 7.10, e is the household's initial endowment point and $e'v'$ represents as before the NCPF. The labour needed to produce the profit maximizing output $v'g$ is shown by ab . The lines $v'g$ and ab are parallel to IL and JL respectively and $v'j$ and ef are drawn parallel to IJ and LJ respectively.

If point B lies in the interior of the region $e'v'gL$ (Figure 7.10), then the effective demand curve for y intersects $e'v'$ in its interior². It is clear from subsection 7.3.2 that this gives rise to a Benassy equilibrium of the Keynesian unemployment type.

If on the other hand point B falls in the interior of the region $v'bJj$, then the NCPF lies inside the region $oBo'e$ as in the case of Figure 7.8. In this case, one gets a Benassy equilibrium of the classical unemployment type, with actual transactions at the equilibrium given by v' .

Suppose now that point B falls in the interior of the region $efbv'e'$. In such a case it is the ESC of labour that intersects $e'v'$. As is clear from Section 7.3.4, one now obtains a Benassy equilibrium of the repressed inflation type, where the actual transactions are given by the point of intersection between the ESC and $e'v'$.

In the case where point B falls inside the triangle $v'jg$, the equilibrium could be of either the Keynesian unemployment or the classical unemployment variety depending respectively on whether the EDC intersects $e'v'$ or not.

Finally, when point B coincides with v' we get a full Walras equilibrium.

It should be noted, however, that such a precise characterization is possible only under the assumption made at the end of Section 3.1. In other words, the characterization fails in more general situations, e.g., when the household's EDC intersects $e'v'$ at two different points say E and E' (shown in Figure 7.11). This case is very likely as when the household is permitted to consume

²The interior of $e'v'$ implies the curve excluding the end points e' and v' .

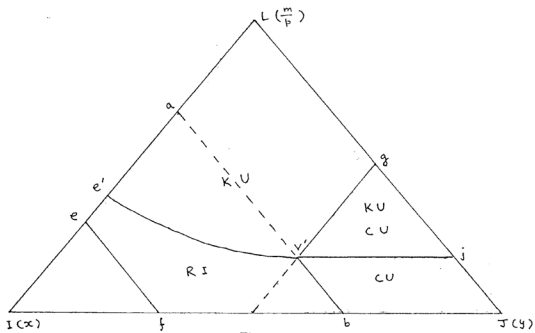


Fig. 7.10

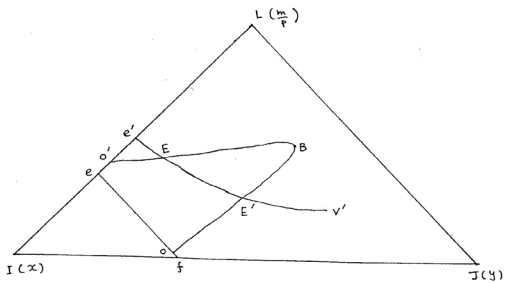


Fig. 7.11

a negligible amount of good y , it is natural on the part of the household to supply a negligible amount of labour too. This gives rise simultaneously to the situations of repressed inflation (at E) and Keynesian unemployment (at E') for the same $w - p$ constellation. Similarly, it is possible to show the simultaneous occurrence of both repressed inflation and classical unemployment. Worse still, no equilibrium will exist at all, if the EDC lies entirely below the NCPF.

First, we point out that if the point o' lies to the left of e' , then equilibrium, if it exists, must be nonunique (barring the borderline case where $o'B$ is tangential to $e'v'$) which represents simultaneous occurrence of either repressed inflation and Keynesian unemployment (Figure 7.11) or repressed inflation and classical unemployment. When o' lies to the right of e' , the Benassy equilibrium may or may not be unique. In fact, the question of uniqueness is closely linked to whether the demands for x and $\frac{m}{p}$ are positively related to the size of the goods rationed budget or not.

To see this we begin with the assumption that the consumption of x decreases when the goods rationed budget increases. In this case, the FSC for labour must always be steeper than LJ (alternatively ef , shown in Figure 7.12). This is true, since, with a rise in the rationed supply $\bar{y}$ of y , the budget available for x and $\frac{m}{p}$ (viz., $\frac{w}{p}\bar{e} + \frac{m}{p} - \bar{y}$) falls and this, along with the corresponding property of oB referred to earlier, ensures that either oBo' intersects $e'v'$ at exactly one point, or $e'v'$ lies in the interior of the region.

On the other hand, suppose consumption of x is positively related to the size of the budget. This situation may give rise to a nonunique equilibrium. Suppose that x is positively related to the size of the goods rationed budget, whereas the opposite is the case for $\frac{m}{p}$. In such a situation the EDC would intersect with NCPF at most once with the result that the Benassy equilibrium is unique.

If, however, both x and $\frac{m}{p}$ are positively related to the size of the goods

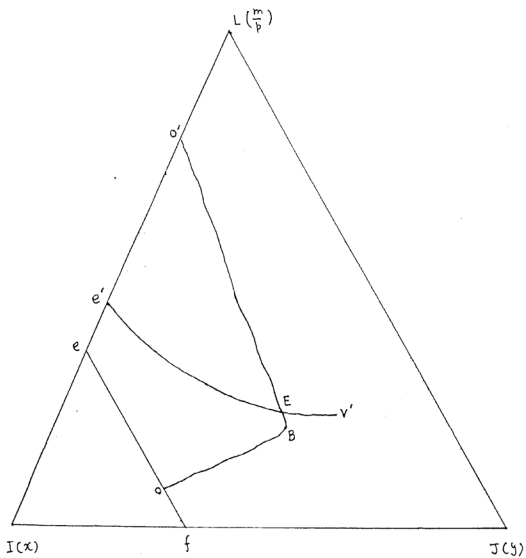


Fig. 7.12

rationed budget, then one gets the possibility of a nonunique Benassy equilibrium. This possibility is illustrated in Figure 7.13.

In between these two extremes, so to speak, lies the case of perfectly inelastic labour supply. This case, as already argued, produces a unique equilibrium.

The above discussion points out that except for particular specifications of the production and utility functions, it may not be possible to classify Benassy equilibria according to wage-price constellations. Indeed, as shown above, a particular wage price specification may be associated with more than one type of equilibrium.

7.4 POLICY ISSUES

For the sake of completeness, the diagrams may now be used to discuss the effect of a money wage cut on effective demand. (Note that it is elementary to study the effect of a change in autonomous government expenditure by means of these diagrams.)

A change in money wage rate (from w to w') causes a change in the position of the budget plane itself from IJJ to $I'J'L'$ as shown in Figure 7.14 by the broken lines.

From (7.2) it can be seen that the slope of the line JL in the two dimensional $(y - m/p)$ plane is -1 , which is independent of w . Therefore, JL and $J'L'$ will have the same slopes. Also, it can be easily checked that the two planes:

$$y + \frac{w}{p}x + \frac{m}{p} = \frac{w}{p}\bar{x} + \frac{\bar{m}}{p}$$

and

$$Y + \frac{w'}{p}x + \frac{m}{p} = \frac{w'}{p}\bar{x} + \frac{\bar{m}}{p}.$$

have a common intersection ef with the vertical plane $x = \bar{x}$. Since the household cannot supply less than zero labour, we are interested in the region lying to the right of the line ef and in this region the new budget plane

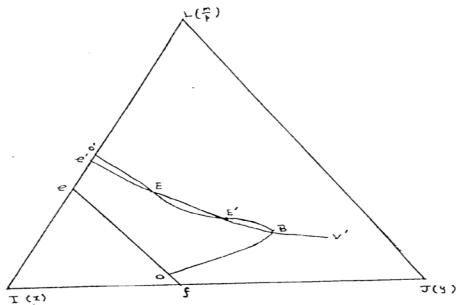


Fig. 7.13

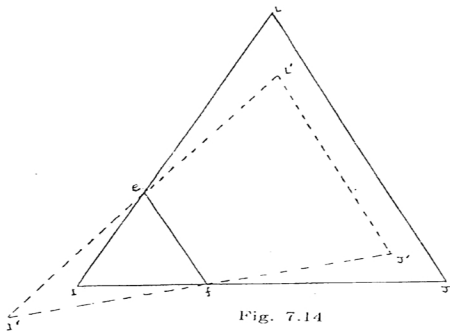


Fig. 7.14

(corresponding to w') lies entirely below the original plane (see Figure 7.15).

Hence, the household's net consumption possibility frontier on $I'J'L'$ (i.e., $e''v''$) also lies physically below that on the original budget plane (i.e., $e'v'$).

Suppose the household's EDC (Figure 7.15) on IJL intersects $e'v'$ at E which represents the transactions corresponding to a Keynesian equilibrium. Let E' be the projection of E on $I'J'L'$. Therefore, E' represents the same levels of ℓ and y as at E . Now the effective demand (for y) curve OB' on $I'J'L'$ cannot pass through E' or to its right. The levels of labour corresponding to E and E' are equal, but since the wage rate has fallen, the same supply of labour will generate a lower wage income at E' than at E . Under assumption (H.1.d), the optimum consumption of y and $\frac{m}{p}$ at this same level of labour supply will thus be lower. In other words, for the same supply of labour as at E , the household will demand a smaller quantity of y on the new budget plane. Therefore, the effective demand (for y) curve in the new plane must pass to the left of E' . The effective demand (for y) curve is thus shown by oB' and the new equilibrium occurs at E'' (say). At this new equilibrium, transactions in output markets are smaller to those at E ³⁴.

³⁴The money wage cut in the face of a fixed p thus leads to a *fall* in employment rather than a rise. The result is to be found in Malinvaud (1977), though proved there for a particular utility function. It is, however of vintage which is much older, for Kelecki (1971) had arrived at precisely the same conclusion. The intuition underlying the result is obvious too. A fall in the money wage rate at a fixed p would reduce incomes of wage earners at each level of employment and this would not be compensated either by a rise in *profits*, which remain undistributed, or a rise in $\frac{m}{p}$. Consequently expenditures fall and, via the multiplier reduce employment.

⁴The conclusion would change drastically in flex-price model. Malinvaud (1977) obtains exactly the opposite result while Keynes (1936) and Mukherji and Sanyal (1986) claim ambiguity. This case is too complicated to be amenable to diagrammatic treatment. See also Bose (1990) and Dasgupta (1991) in this connection

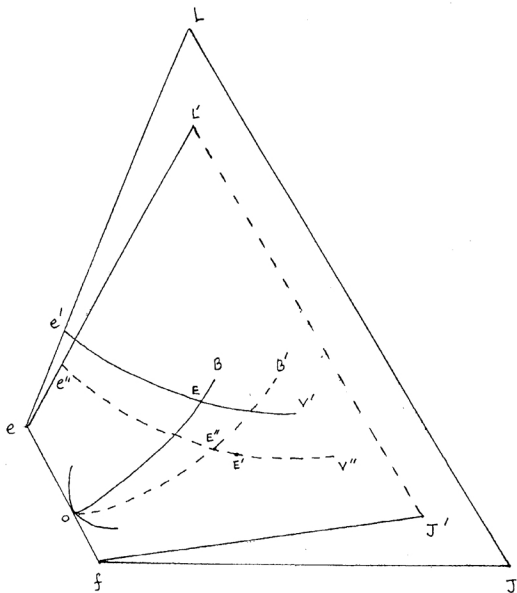


Fig. 7.15

Remark 7.1: The diagrammatic scheme developed so far can be easily extended to incorporate the Barro-Grossman (1971) equilibrium (see Dasgupta and Rajeev (forthcoming)). More interestingly, such a representation would reveal that the Keynesian unemployment equilibrium discussed by Barro and Grossman would remain completely unaffected by a money wage cut.

7.5 *P*-EFFICIENCY

We next focus on the *p*-efficiency properties of different varieties of Benassy equilibria associated with production. Out of the three types of equilibria discussed so far, only the Keynesian variety may be inefficient, though it need not necessarily be so. Thus we proceed to identify situations where a Benassy equilibrium with production is *p*-efficient.

In what follows, we will consider a Benassy equilibrium to be *p*-efficient if, by opening up the market for good y against ℓ , one cannot move to a better equilibrium (see Benassy (1982), pp.219)).

We first consider a classical case (Fig. 7.9). Since in equilibrium the firm is attaining its best point, no further improvement is feasible. Similarly, the case of repressed inflation can be shown to be *p*-efficient.

Before going to the Keynesian case we prove a lemma.

Lemma 7.1 : Under voluntary trading the firm would always choose to stay on the *NCPF* curve.

Proof: It is technically not feasible for the firms to choose a point that lies physically below the $e'v'$ curve.

On the other hand, it is not profitable for the firm to choose a point that lies physically above the $e'v'$ curve. This is shown in Fig. 7.16. Suppose for example the firm chooses point E. Indeed, the firm would be earning more profit by producing the same quantity for lesser labour at E'' .

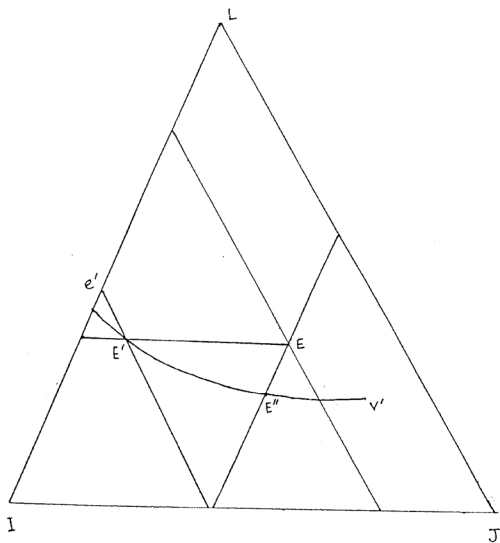


Fig. 7.16

Benassy has mentioned that (1982, pp.131) an equilibrium of the Keynesian type is always p -inefficient. However, we have constructed a situation below where this need not necessarily be true (under Lemma 7.1).

Proposition 7.1: There exists a p -efficient Benassy equilibrium of Keynesian type.

Proof: Consider Keynesian equilibrium as shown by E (Fig.7.17). Note that at E , the household's money holding is the same as its initial endowment.

Suppose E is not a p -efficient Benassy equilibrium. This implies that by opening up (y, L) market, both the household and the firm can go to a better position say, to point H . We will show that this leads to a contradiction. Clearly, H lies on $e'v'$ (Lemma 7.1) and to the right of E .

(i) **Claim:** At H , the household's money holding is less than that at E (Fig. 7.17).

Proof of the Claim : Suppose not. That implies H lies physically above the line ej . In that case ej (parallel to IJ) cuts $e'v'$ at two points say, E and E' (not shown in the diagram) and the point H lies above E' . Therefore, the profit maximizing point cannot lie to the right of E' . Thus, v' lies to the left of E' . Hence, we get a contradiction.

Therefore, at H , the household's money holding is less than that at E . Thus, at H the household is an excess supplier of money (Fig.7.18). The household can supply money only against y . Therefore, whatever amount of money the household is supplying (i.e. eQ) must be against y . This makes the household perceive that it can supply at least eQ amount of money and get the corresponding amount of y .

At H , the household is supplying AA' level of labour. Let AA' cut oB at

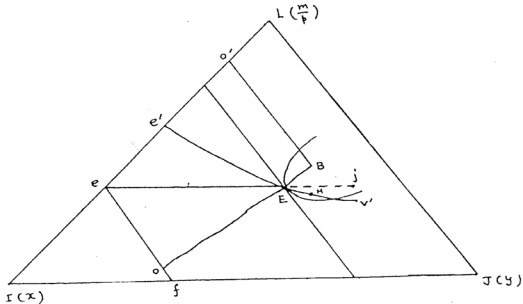


Fig. 7.17

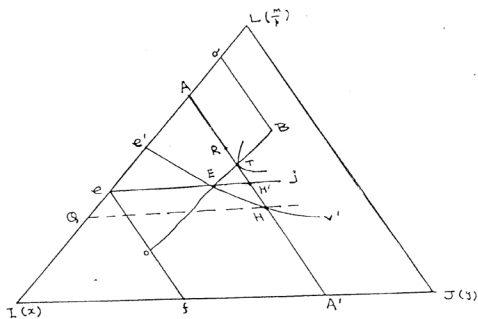


Fig. 7.18

T (Fig.7.18)

(ii) **Claim:** T lies physically above H .

Proof of the Claim: If not, then while increasing the labour rationed budget, the household has less real balances. This violates Assumption (H.1.d).

Thus, the indifference contour that goes through point H would cut AA' at another point that lies physically above T . Call this point R . Thus, H and R are on the same indifference contour. We can express H' (i.e. where AA' cuts ej) as a strict convex combination of H and R . Clearly, at H' the household's utility is higher than that at H or R .

Under our supposition that H is the barter equilibrium, the household perceives that it can at least supply eQ , level of m/p against y . But by *supplying less of m/p* (and demanding correspondingly less of y), that is, by trading less in the $(y, m/p)$ market, the household can acquire H' which gives more utility. Thus, H cannot be sustained as a barter equilibrium. Hence, a contradiction.

■

7.6 CONCLUSION

This chapter studies through a diagrammatic device different varieties of Benassy equilibria that may arise in the short run. The exercise reveals the problem of nonuniqueness of a Benassy equilibrium. As a by product, some of the time-worn policy questions surrounding the Keynesian equilibrium are also discussed.

List of Symbols used in Chapter 7:

- (1) ℓ : labour
- (2) $\bar{\ell}$: maximum possible supply of labour
- (3) $\bar{m}$: initial endowment of money
- (4) x : leisure
- (5) y : consumption good
- (6) p : price of y
- (7) U : utility function of the household
- (8) F : production function
- (9) w : money wage rate
- (10) G : government's demand for y
- (11) IJL : budget plane
- (12) e : the point of initial endowment for the household
- (13) B : best point for the household (on the budget plane)

REFERENCES

- Aiyagari, S.R. and N. Wallace (1991): Existence of steady states with positive consumption in Kiyotaki-Wright model, *Review of Economic Studies* 58, 901-916.
- Barro, R.J. and H.I. Grossman (1971): A general disequilibrium model of income and employment, *American Economic Review* 61, 82-93.
- Basu, K. (1992): A geometry for non Walrasian general equilibrium theory, *Journal of Macroeconomics* 14, 87-103.
- Benassy, J.P. (1975a): Disequilibrium exchange in barter and monetary economies *Economic Inquiry* 13, 131-156.
- Benassy, J.P. (1975b): Neo-Keynesian disequilibrium theory in a barter and monetary economy, *Review of Economic Studies* 42, 503-523.
- Benassy, J.P. (1982): *The economics of market disequilibrium*, Academic Press Inc., Orlando, Florida.
- Benassy, J.P. (1986): *Macroeconomics: An introduction to the non-Walrasian approach*, Accademic Press Inc., Orlando, Florida.
- Böhm, V. and H. Müller (1977): Two examples of equilibria under price rigidities and quantity rationing, *Zeitschrift für Nationalökonomie Journal of Economics*, 37, 165-173.
- Bose, A. (1990): Wage cuts and self-fulfilling beliefs, in *Economic theory and policy: Essays in honour of Dipak Banerjee* (B.Dutta, S. Gangopadhyay, D. Mukherjee and D.Ray ed.), Oxford University Press.

- Clower, R.W. (1965): The Keynesian counter-revolution: a theoretical appraisal, in *The theory of interest rates* (F.H. Hahn and F.P.R. Brechling, eds), Macmillan, London.
- Clower, R.W. (1967): A reconsideration of the microfoundations of monetary theory, *Western Economic Journal* 6, 1-8.
- Dasgupta, D. (1991): On the influence of high wages on the rate of private investment: the Keynesian story, in *Essays in Economic analysis and policy* (D. Benerjee ed.), Oxford University Press.
- Dasgupta, D. and M. Rajeev : Choice under disequilibrium: A diagrammatic exposition, *Journal of Quantitative Economics*, forthcoming.
- Debreu, G. and H. Scarf (1963): A limit theorem on the core of an economy, *International Economic Review* 4, 235-246.
- Dréze, J. (1975): Existence of an exchange equilibrium under price rigidities, *International Economic Review* 16, 301-320.
- Eckalbar, J.C. (1984): Money, barter, and convergence to the competitive allocation: Menger's problem, *Journal of Economic Theory* 32, 201-211.
- Foley, D.K. (1970): Equilibrium with costly marketing, *Journal of Economic Theory* 2, 276-291.
- Freeman, S (1989): Fiat money as a medium of exchange, *International Economic Review* 30, 137-151.
- Grandmont, J.M. (1974): On the Short run equilibrium in a monetary economy, in *Allocations under uncertainty, equilibrium and optimality* (J.H. Dréze ed.), Macmillan, London.

- Grandmont, J.M., Laroque, G., and Y.Younes (1978): Equilibrium with quantity rationing and recontracting, *Journal of Economic Theory* 19, 84-102.
- Hahn, F. and F.P.R. Brechling (eds) (1965): *The theory of interest rates*, Macmillan, London.
- Harcourt, G (ed.) (1977): *The microeconomic foundations of macroeconomics*, Macmillan, London.
- Hicks, J. (1967): *Critical essays in monetary theory*, ELBS and Oxford University Press.
- Hicks, J. (1969): *Theory of economic history*, Oxford University Press.
- Hildenbrand K. and W. Hildenbrand (1978): On Keynesian equilibria with unemployment and quantity rationing, *Journal of Economic Theory* 18, 255-277.
- Hirshleifer, J. (1973): Exchange theory: the missing chapter, *Western Economic Journal* 2, 129-146.
- Iwai, K. (1988): The evolution of money: a search-theoretic foundation of monetary economics, CARESS Working paper # 88-03, University of Pennsylvania.
- Jones, R.A. (1976): The origin and development of media of exchange, *Journal of Political Economy* 84, 757-775.
- Kalecki, M. (1971): *Selected essays on the dynamics of the capitalist economy*, Cambridge University Press, Cambridge.
- Kalecki, M. (1943): The political aspects of full employment, in Kalecki (1971).

- Karni, E. (1973): Transactions costs and the demand for media of exchange, *Western Economic Journal* 2, 71-80.
- Keynes, J.M. (1936): *The general theory of employment, interest and money*, Harcourt, Brace and World Inc., New York.
- Kiyotaki, N. and R. Wright (1989): On money as a medium of exchange, *Journal of Political Economy* 97, 927-954.
- Leijonhufvud, A. (1968): *On Keynesian economics and the economics of Keynes*, Oxford University Press, London and New York.
- Malinvaud, E. and Y. Younes (1977): Some new concepts for the microeconomic foundations of macroeconomics in Harcourt (1977).
- Malinvaud, E. (1977): *The theory of unemployment reconsidered*, Basil Blackwell, Oxford, England.
- Menger, K. (1892): On the origin of money, *Economic Journal* 2, 239-255.
- Muellbauer, J. and R. Portes (1978): Macroeconomic models with quantity rationing, *Economic Journal* 88, 783-821.
- Mukherjee, A. and A. Sanyal (1986): Price flexibility and unemployment: microeconomics of some old fashioned question, *Kio Economic Studies*, 19-35.
- Niehans, J. (1969): Money in a static theory of optimal payment arrangements, *Journal of Money, Credit and Banking* 1, 706-726.
- Niehans, J. (1971): Money and barter in general equilibrium with transactions costs, *American Economic Review* 61, 773-783.

- Oh, Seonghwan (1989): A theory of generally acceptable medium of exchange and barter, *Journal of Monetary Economics* 23, 101-119.
- Ostroy, J.M. (1973): The informational efficiency of monetary exchange, *American Economic Review* 63, 597-610.
- Ostroy, J.M. and R.M. Starr (1974): Money and the decentralization of exchange, *Econometrica* 42, 1093-1113.
- Rajeev, Meenakshi (1990) : The medium of exchange role of money and the competitive economy with monetized trading posts, Tech. Report No. ERU/7/90, Indian Statistical Institute.
- Silvestre, J. (1982): Fix price analysis in exchange economies, *Journal of Economic Theory* 26, 28-58.
- Starr, R.M. (1972): The structure of exchange in barter and monetary economies, *Quarterly Journal of Economics* LXXXVI, 290-302.
- Starr, R.M. (1974): The price of money in a pure exchange monetary economy with taxation, *Econometrica* 42, 43-54.
- Starr, R.M. (1976): Decentralized non-monetary trade, *Econometrica* 44, 1087-1089.
- Stoneman, D. (1979): A simple diagrammatic apparatus for the investigation of a macroeconomic model of temporary equilibria, *Economica* 46, 61-63.
- Wallace, N. (1972): An approach to the study of money and non-money exchange situations, *Journal of Money, Credit and Banking*, 838-847.
- Walras, L. (1900): *Elements of pure economics*, Translated and edited by W. Jaffe, Homewood, Illinois, U.S.A.

Younes, Y. (1975): On the role of money in the process of exchange and the existence of a non Walrasian equilibrium, *Review of Economic Studies* 42, 489-501.