On the Jordan-Chevalley-Dunford Decomposition of Certain Classes of Operators and Convergence of Their Normalized Power Sequences

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On the Jordan-Chevalley-Dunford Decomposition of Certain Classes of Operators and Convergence of Their Normalized Power Sequences

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Dedicated to my Mumma and Papa

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Notations & Abbreviations

$[n]$	Set of first n natural numbers, $\{1, 2, \ldots, n\}$.
$\mathbb{N}$	Set of all natural numbers.
$\mathbb{N}_0$	Set of all whole numbers, $\mathbb{N} \cup \{0\}$.
$\mathbb{Z}, \mathbb{R}, \mathbb{C}$	Set of all integers, real numbers, and complex numbers, respectively.
$\mathbb{R}_{\geq 0},\ \mathbb{R}_{>0}$	Set of all non-negative, positive real numbers, respectively.
r^+	$\max\{0, r\}\ ;\ r \in \mathbb{R}$.
$\Re z$	Real part of $z \in \mathbb{C}$.
$\mathbb{D}_r$	Closed disc of radius r centered at the origin in $\mathbb{C}$.
$\mathbb{H}_r$	Closed half-plane $\{z \in \mathbb{C} : \Re z \leq r\}$.
$\mathcal{R}$	Unital commutative ring.
$\mathbb{K}, \mathbb{L}$	Fields.
$\mathbf{e}_i\ ;\ i \in [n]$	Standard basis vectors of the n-dimensional vector space $\mathbb{K}^n$ over $\mathbb{K}$ with $\mathbb{K}$ understood from the context.
$M_{m,n}(\mathcal{R})$	Set of all $m \times n$ matrices with entries coming from $\mathcal{R}$.
$\mathbf{0}_{m,n}$	Zero matrix in $M_{m,n}(\mathcal{R})$; with $\mathcal{R}$ understood from the context.
$\mathbf{e}_{m,n}$	Matrix in $M_{m,n}(\mathcal{R})$ whose $(1,1)$–entry is equal to the identity of $\mathcal{R}$, and rest of the entries are equal to the zero of $\mathcal{R}$.

$M_n(\mathcal{R})$	Set of all $n \times n$ matrices over $\mathcal{R}$.
$\mathbf{0}_n, I_n$	Zero and identity matrices, respectively, in $M_n(\mathcal{R})$.
E_{ij} ; $i, j \in [n]$	Elementary matrix in $M_n(\mathcal{R})$, whose $(i,j)^{\text{th}}$ entry is equal to the identity of $\mathcal{R}$ and rest of the entries are equal to the zero of $\mathcal{R}$.
$GL_n(\mathcal{R})$	Set of all invertible matrices in $M_n(\mathcal{R})$.
$UT_n(\mathcal{R})$	Set of all upper-triangular matrices in $M_n(\mathcal{R})$.
$\mathcal{R}[x]$	Set of all polynomials, in the variable x, over $\mathcal{R}$.
$\chi_A(x)$	Characteristic polynomial of the matrix $A \in M_n(\mathcal{R})$, in $\mathcal{R}[x]$.
$\det A$	Determinant of $A \in M_n(\mathbb{K})$; with $\mathbb{K}$ understood from the context.
$\mathrm{dvec}(A)$	Principal diagonal of $A \in M_n(\mathbb{K})$, viewed as a vector in $\mathbb{K}^n$.
$\mathrm{diag}(\vec{v})$	Diagonal matrix in $M_n(\mathbb{K})$, with principal diagonal $\vec{v} \in \mathbb{K}^n$.
$\mathfrak{B}$	Set of all Borel subsets of $\mathbb{C}$.
Ω, Ω'	Topological spaces.
$C(\Omega, \Omega')$	Set of all continuous functions from Ω to Ω'.
$C(\Omega)$	Set of all complex continuous functions on Ω.
$\mathbf{0}, \mathbf{1}$	Constant functions in $C(\Omega)$ with value 0 and 1, respectively; with Ω understood from the context.
$\mathfrak{X}, \mathfrak{Y}$	Complex Banach spaces.
$\mathscr{B}(\mathfrak{X}, \mathfrak{Y})$	Set of all bounded linear maps from $\mathfrak{X}$ to $\mathfrak{Y}$.
$\mathscr{B}(\mathfrak{X})$	Set of all bounded linear operators on $\mathfrak{X}$.
$\mathrm{sp}\,(A)$	Spectrum of the operator $A \in \mathscr{B}(\mathfrak{X})$.
$\mathrm{r}(A)$	Spectral radius of $A \in \mathscr{B}(\mathfrak{X})$.
$R(\xi; A)$	$(\xi I - A)^{-1}$; resolvent of $A \in \mathscr{B}(\mathfrak{X})$ with respect to $\xi \in \mathbb{C} \backslash \mathrm{sp}\,(A)$.

$\|A\|$	Operator norm of $A \in \mathscr{B}(\mathfrak{X})$.
$\mathrm{dom}(f)$	Domain of a function f.
$\mathrm{ran}(f)$	Range of a function f.
$\ker(f)$	Kernel of a linear transformation f of vector spaces.
$\mathscr{H}$	Complex Hilbert space.
$\mathcal{K}(\mathscr{H})$	Set of all compact operators in $\mathscr{B}(\mathscr{H})$.
WOT	Weak operator topology on $\mathscr{B}(\mathscr{H})$.
SOT	Strong operator topology on $\mathscr{B}(\mathscr{H})$.
$\mathbf{R}(A)$	Projection onto the range of $A \in \mathscr{B}(\mathscr{H})$.
$\mathscr{R}$	von Neumann algebra acting on Hilbert space $\mathscr{H}$.
$0_{\mathscr{R}}, I_{\mathscr{R}}$	Zero and the identity operators, respectively, in $\mathscr{R}$; the subscript is dropped when $\mathscr{R}$ is understood from the context.
$\mathrm{Aff}(\mathscr{R})$	Set of all densely-defined closed linear operators acting on $\mathscr{H}$ that are affiliated with $\mathscr{R}$.
$\mathscr{M}, \mathscr{N}$	Finite von Neumann algebras.
$\|A\|$	$(A^*A)^{\frac{1}{2}}$; for $A \in \mathscr{B}(\mathscr{H})$, or a densely-defined closed linear operator A affiliated with a finite von Neumann algebra.
NPS	Normalized power sequence, $\{\|A^k\|^{\frac{1}{k}}\}_{k\in\mathbb{N}}$, of operator A described above.

Introduction

A central theme in mathematics is the reduction of complex structures into simpler, more tractable parts. Among the most classical and elegant results, exemplifying this principle, is the *Jordan–Chevalley decomposition*. Originating in the 19th century from the work of Camille Jordan, and later algebraically formalized by Claude Chevalley in the context of algebraic groups and Lie algebras (see [Che51a]), this decomposition expresses a linear transformation T of a finite-dimensional vector space $\mathscr{V}$ over a field $\mathbb{K}$ as a unique sum,

$$T = D + N$$

where D is semisimple (diagonalizable over the algebraic closure of $\mathbb{K}$), and N is nilpotent, with the additional condition that D and N commute.

For a linear transformation given in a Jordan canonical form, this decomposition is readily described : the semisimple part corresponds to the diagonal component, while the nilpotent part is the strictly upper-triangular (off-diagonal) component. Furthermore, the Jordan-Chevalley decomposition exists under significantly weaker assumptions than those required for the existence of a Jordan canonical form. For instance, the Jordan canonical form of $T \in M_n(\mathbb{K})$ exists (in $M_n(\mathbb{K})$) only when the characteristic polynomial of T splits over $\mathbb{K}$, whereas the Jordan-Chevalley decomposition exists (in $M_n(\mathbb{K})$) under the weaker assumption that the minimal polynomial of T is a product of separable polynomials over $\mathbb{K}$ (see [CEZ11], [Hum80]). In this sense, the Jordan-Chevalley decomposition refines the Jordan canonical form.

Jordan-Chevalley decomposition underpins several important results in the theory of associative algebras, representation theory, and the structure theory of Lie algebras, most notably in connection with the Wedderburn principal theorem (see [Hum80], [Beh72]).

Although traditionally rooted in algebra, the Jordan–Chevalley decomposition has found significant applications in the analysis of asymptotic behaviour of matrix powers. In [Nay23], Nayak proved that for any matrix $A \in M_n(\mathbb{C})$, the sequence $\{|A^k|^{\frac{1}{k}}\}_{k\in\mathbb{N}}$, with the notation $|T| := (T^*T)^{\frac{1}{2}}$, is convergent, and provided a concrete description of the limit in terms of certain orthogonal projections associated with the diagonalizable part of A in its Jordan-Chevalley decomposition. We refer to the sequence, $\{|A^k|^{\frac{1}{k}}\}_{k\in\mathbb{N}}$, as the *normalized power sequence* (NPS) of A. More generally, the notion of normalized power sequence can be defined in the same way for bounded linear operators acting on a Hilbert space, and for densely-defined closed linear operators affiliated with a finite von Neumann algebra.

It is natural to ask whether the structural insights offered by the Jordan-Chevalley decomposition in finite dimensions, carry over to infinite-dimensional Hilbert space operators. N. Dunford, in the foundational work [Dun54], developed an analogous theory for bounded linear operators on Banach spaces by introducing the notion of *spectral operators*, a broad class of (generally non-normal) operators that still admit a spectral resolution. Algebraically, spectral operators are characterized as those operators that admit a unique decomposition into a sum of a scalar-type operator and a quasinilpotent operator, which commute with each other, closely mirroring the Jordan–Chevalley decomposition (see [Dun58], [DS88]).

However, the landscape in infinite dimensions is more nuanced. Not every bounded linear operator on a Hilbert space is spectral – an example due to Kakutani (see [DS88, §XV.2]) highlights this limitation. The issue becomes more intricate when considering operators in type II_1 von Neumann algebras, particularly the hyperfinite II_1 factor as these operators are often informally regarded as "continuous matrices", making questions of spectrality – whether they admit Dunford-type decompositions – both natural and compelling. In recent years, this direction has attracted attention in the work of Dykema and Krishnaswamy-Usha (see[DKU21], [DKU20], [DKU24]), yet a comprehensive and definitive framework remains elusive.

In [HS09], using tools from free probability theory and ultrapower techniques, Haagerup and Schultz proved that the normalized power sequence of operators in a type II_1 von Neumann factor converges in the strong operator topology (and being a norm-bounded sequence, in the ultra-strong topology), and the spectral resolution of the limiting positive operator is described in terms of the so-called Haagerup-Schultz projections. Furthermore, an elementary example – due to Voiculescu (see [HS09, Example 8.4]) – of a weighted shift operator on an infinite-dimensional Hilbert space shows that the normalized power sequence of (infinite-dimensional) Hilbert space operators need not converge in SOT.

Furthering the inquiry into the convergence of normalized power sequences for infinite-dimensional Hilbert space operators, Bhat and Bala have shown in [BB24], that the NPS of a compact operator acting on a complex separable Hilbert space, $\mathscr{H}$, is norm convergent, making essential use of Nayak's theorem for matrices. In [BB24, Example 3.12], they consider yet another example of a weighted shift operator whose normalized power sequence does not converge in WOT. In the context of real semisimple Lie groups, Huang and Tam proves a generalization of Nayak's theorem in [HT24]. In §3.4 we show that the normalized power sequence of a spectral operator acting on a complex Hilbert space is norm-convergent.

Interestingly, in [DKU21], one starts with the Haagerup-Schultz theorem about SOT-convergence of the normalized power sequence of operators in a type II_1 factor, say $\mathscr{L}$, and uses its consequences to establish 'spectrality' of a large class of operators in $\mathscr{L}$, in terms of the UNZA (uniformly non-zero angles) property (see [DKU21, Theorem 4.7-(ii)]). An operator in $\mathscr{L}$ is said to have the UNZA property if the angles between its Haagerup–Schultz projections are uniformly bounded away from zero. This suggests an intimate connection between the spectrality of operators and the convergence of their normalized power sequences.

In view of Bhat and Bala's results, one is thus prompted to investigate whether compact operators acting on a complex separable Hilbert space, $\mathscr{H}$, are spectral. We show in §4.3, that a compact operator in $\mathscr{B}(\mathscr{H})$ need not be spectral in the Dunford's sense. However, if we relax the boundedness conditions on the components of the Dunford-decomposition, and allow densely-defined closed linear operators, compact operators in $\mathscr{B}(\mathscr{H})$ do admit a Dunford-type decomposition (§4.4).

It is perhaps not surprising that the setting of (unbounded) densely-defined closed linear operators emerges as the appropriate framework to formulate a Dunford-type decomposition for operators acting on an infinite-dimensional Hilbert space. The germs of this phenomenon already arise in finite dimensions : In §2.6, we show that the mapping on $M_n(\mathbb{C})$, which sends a matrix to the diagonalizable part in its Jordan-Chevalley decomposition is unbounded for $n \geq 3$. Another instance of such phenomenon will be illustrated in Chapter 5, where we establish *Jordan-Chevalley-Dunford decomposition* for operators in a finite type I von Neumann algebra $\mathscr{M}$, which necessarily involves unbounded affiliated operators in $\mathrm{Aff}(\mathscr{M})$.

In this thesis we study Jordan–Chevalley type decompositions in several contexts along these developments, and discuss the convergence of the normalized power sequence of the operators that admit such decompositions. In particular, our main contributions are as follows:

1. We present an elementary computational proof of the existence and uniqueness of the Jordan-Chevalley decomposition of a matrix $A \in M_n(\mathbb{K})$, viewed as a matrix in $M_n(\mathbb{L})$, where $\mathbb{L}$ is the splitting field of the characteristic polynomial of A. Further, using standard techniques, we show that the *potentially-diagonalizable* and nilpotent parts of A are matrices over the fixed field of the set of all automorphisms of $\mathbb{L}$ which fixes every element of $\mathbb{K}$. While at it, we also present an elementary computational proof of the Jordan canonical form of A in $M_n(\mathbb{L})$.

   We reiterate that, despite its seemingly algebraic nature, the Jordan–Chevalley decomposition encodes subtle analytic properties; we show that the map on $M_n(\mathbb{C})$, which sends A to its diagonalizable part (or nilpotent part), in its Jordan-Chevalley decomposition, is not norm-bounded in general.

2. We show that for a bounded spectral operator A acting on a complex Hilbert space, the normalized power sequence, $\{|A^k|^{\frac{1}{k}}\}_{k\in\mathbb{N}}$, converges in norm. Moreover, we provide an explicit description of the limit in terms of the *idempotent-valued spectral resolution* of A. When restricted to the finite-dimensional case, this gives a different, succinct proof of Nayak's theorem [Nay23, Theorem 3.8].

Furthermore, we characterize all unilateral weighted shift operators for which the normalized power sequence converges in norm.

3. We prove a spatial version of Nayak's theorem [Nay23, Theorem 3.8] for compact operators acting on an infinite-dimensional complex Hilbert space. When restricted to complex *separable* Hilbert spaces, this gives a concise proof of Bhat and Bala's result [BB24, Theorem 3.10]. We also provide an explicit description of the limit in terms of the Riesz idempotents associated with the compact operator.

   Furthermore, we show that a compact operator on a complex Hilbert space is not necessarily spectral, however it admits a Dunford-type decomposition.

4. We extend the classical notion of the Jordan–Chevalley decomposition to matrices in $M_n(\mathcal{N}(\mathcal{X}))$, where $\mathcal{N}(\mathcal{X})$ is the $*$-algebra of normal (unbounded) functions on a Stonean space $\mathcal{X}$, containing $C(\mathcal{X})$ as a $*$-subalgebra (see Definition. Leveraging the unboundedness of the Jordan–Chevalley decomposition in the case of $M_n(\mathbb{C})$, we show that when $n \geq 3$ and $\mathcal{X}$ has infinitely many points, the diagonalizable and nilpotent components of an operator in $M_n(C(\mathcal{X}))$ need not lie in $M_n(C(\mathcal{X}))$. Building on our results for the type I_n Murray-von Neumann algebra, which are precisely of the form $M_n(\mathcal{N}(\mathcal{X}))$ for some Stonean space $\mathcal{X}$, we prove the existence and uniqueness of the Jordan–Chevalley–Dunford decomposition in type I Murray von Neumann algebras, which involves what we call $\mathfrak{u}$-*scalar-type* and $\mathfrak{m}$-*quasinilpotent operators* (see Definition 5.4.5).

   Furthermore, we show that the normalized power sequence of an operator in a type I Murray-von Neumann algebra converges in the $\mathfrak{m}$-topology, the intrinsic measure topology on Murray von Neumann algebras (see [Nay21, Definition 3.3]).

We now outline the structure of the thesis. The work is divided into a preliminary chapter and four chapters containing the main results.

Chapter 1: Preliminaries. This chapter lays the foundational material necessary for the rest of the thesis. It begins in §1.1 reviewing basic concepts from matrix analysis, with particular focus on the notion of similarity. In §1.2 we discuss elementary definitions from field theory, relevant in Chapter 2. In §1.3, we list some pertinent results from operator theory, including operator inequalities, spectral resolution of the identity, spectral and compact operators. These results are crucially used in Chapter 3 and 4. Section 1.3.3 is devoted to the theory of unbounded linear operators acting on a Hilbert space. In §1.4.1, we briefly discuss Stonean spaces, highlighting few properties that are useful to us, and §1.4.2 provides an overview of partially ordered sets and the Scott topology, which is used in establishing one of the key results (Proposition 5.3.10) of Chapter 5.

We conclude the chapter with a concise treatment of the structure theory of type I_n $AW*$-algebras and type I_n von Neumann algebras in §1.5, and basic definitions concerning affiliated operators and Murray von Neumann algebras in §1.6.

Chapter 2: Jordan–Chevalley decomposition of matrices over arbitrary fields. The primary goal of this chapter is to present an elementary computational proof of the existence and uniqueness of the Jordan-Chevalley decomposition and the Jordan canonical form, for square matrices over an arbitrary field $\mathbb{K}$. The main strategy is to arrive at an appropriate block-diagonal form of a given matrix via similarity transformations obtained from solutions of certain Sylvester equations associated with the matrix, which is discussed in §2.2.

Let A be a matrix in $M_n(\mathbb{K})$, and $\mathbb{L}$ be the splitting field of the characteristic polynomial of A. In Theorem 2.3.2, we show that A, viewed as a matrix in $M_n(\mathbb{L})$, is similar to a block-diagonal matrix in $M_n(\mathbb{L})$, where each diagonal block is an upper-triangular matrix having exactly one eigenvalue, and distinct diagonal blocks have distinct eigenvalues. Using this, we prove two fundamental (basis-independent) results about $\mathbb{K}$-linear transformations on a $\mathbb{K}$-space. Firstly, the Cayley-Hamilton theorem (see Theorem 2.4.2) follows as an immediate corollary. Secondly, we prove the existence and uniqueness of the Jordan-Chevalley decomposition of A in $M_n(\mathbb{L})$ (see Theorem 2.4.5). Then using standard techniques, in Corollary 2.4.8, we show that the potentially-diagonalizable (see Definition 2.4.3) and nilpotent parts of A are in fact matrices over the fixed field of $\mathrm{Aut}(\mathbb{L}/\mathbb{K})$ (see definition 1.2.7). In particular, when $\mathbb{K}$ is a perfect field (field of characteristic zero, finite field, etc.), they are both matrices in $M_n(\mathbb{K})$, as noted in Theorem 2.4.9. In Theorem 2.5.5, we algorithmically obtain Jordan canonical form of $A \in M_n(\mathbb{L})$.

We conclude this chapter in §2.6, by uncovering a somewhat surprising phenomenon that the map $A \mapsto D(A) : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, where $D(A)$ is diagonalizable part in the Jordan-Chevalley decomposition of A, is norm-unbounded on any neighbourhood of the zero matrix in $M_n(\mathbb{C})$ for $n \geq 3$.

Chapter 3: Dunford decomposition and norm convergence of normalized power sequence. The main goal of this chapter is to show that the normalized power sequence of a spectral operator in $\mathscr{B}(\mathscr{H})$, for $\mathscr{H}$ a complex Hilbert space, converges in norm. We set up the context by reviewing Yamamoto's classical theorem on singular values alongside Nayak's generalization, which is the main motivation of our quest.

In §3.2 we recall the algebraic characterization, the *Dunford-decomposition* of spectral operators, thereby providing a generalization of Jordan–Chevalley decomposition in $\mathscr{B}(\mathscr{H})$. In Remark 3.2.4, we note that an operator in $\mathscr{B}(\mathscr{H})$ is spectral if and only if it is similar to a commuting sum of a normal operator and a quasinilpotent operator in $\mathscr{B}(\mathscr{H})$. Lemma 3.2.6 provides an approximation argument for the normalized powers of spectral operators of the form $M + N$, where M and N are commuting normal and quasinilpotent operators, respectively, in $\mathscr{B}(\mathscr{H})$.

The main result of §3.3 is Theorem 3.3.5, which shows that for a positive operator H and an invertible operator S, in $\mathscr{B}(\mathscr{H})$, the sequence $\{(S^*H^kS)^{\frac{1}{k}}\}_{k\in\mathbb{N}}$ converges in norm, and provides an explicit description of the limiting positive operator in terms of the spectral projections of H. This result, along with the technical tools, Lemma 3.4.1 and Lemma 3.2.6, serves as a crucial step towards the main theorem of the chapter, Theorem 3.4.2. As an application, in Theorem 3.4.4, we describe the long-term behaviour of one-parameter groups in $\mathscr{B}(\mathscr{H})$ whose infinitesimal generator is a spectral operator.

In the final section §3.5, we characterize all unilateral weighted shift operators for which the normalized power sequence converges in norm.

Chapter 4: Dunford-type decomposition for compact operators Drawing on the techniques developed in Chapter 3, in §4.2, we prove that the normalized power sequence of a compact operator acting on a complex Hilbert space $\mathscr{H}$, converges in norm to a positive operator in $\mathscr{B}(\mathscr{H})$. This supplements the result of Bhat and Bala [BB24, Theorem 3.10], which establishes the norm convergence of the normalized power sequence of compact operators acting on a complex separable Hilbert space. We begin with Proposition 4.2.3, which proves the result in the setting of finite-rank operators in $\mathscr{B}(\mathscr{H})$, and subsequently extend the argument to general compact operators in Theorem 4.2.4. Our approach also yields an explicit description of the limiting positive operator in terms of the Riesz idempotents associated with the compact operator under consideration.

In §4.3, making essential use of Theorem 2.6.4, we construct two examples of compact operators acting on the separable Hilbert space $\bigoplus_{k\in\mathbb{N}}\mathbb{C}^3$, that are not spectral.

In §4.4, we explore the resemblance between compact and spectral operators and highlight several spectral-theoretic properties exhibited by compact operators. Motivated by this analogy, Theorem 4.4.10 establishes that every compact operator in $\mathscr{B}(\mathscr{H})$ admits a Dunford-type decomposition, where the scalar-type and quasinilpotent components are realized as densely-defined closed linear operators acting on $\mathscr{H}$, that are simultaneously quasi-similar to bounded commuting normal and nilpotent operators, respectively.

Chapter 5: The Jordan–Chevalley–Dunford decomposition for operators in type *I* Murray–von Neumann algebras. The primary goal of this chapter is to examine the Jordan-Chevalley decomposition in the setting of type I_n von Neumann algebras (for $n \in \mathbb{N}$).

As noted in §1.5, every type I_n ($n \in \mathbb{N}$) von Neumann algebra is algebraically of the form $M_n\big(C(\mathcal{X})\big)$ for a (hyper-)Stonean space $\mathcal{X}$; more generally, every type I_n AW^*-algebra is algebraically of the form $M_n\big(C(\mathcal{X})\big)$, for a Stonean space $\mathcal{X}$. We are thus interested in the study of Jordan-Chevalley decomposition for matrices over $C(\mathcal{X})$, where $\mathcal{X}$ is a Stonean space.

In §5.2.1, exploiting the extremally-disconnected nature of the Stonean spaces, we establish a key lemma (Lemma 5.2.11), which facilitates a transference of results from $M_n(\mathbb{C})$ to $M_n\big(C(\mathcal{X})\big)$. The 'unboundedness' of the Jordan-Chevalley decomposition in $M_n(\mathbb{C})$ discussed in §2.6, suggests that the candidates for the diagonalizable and the nilpotent parts in the (infinite-dimensional) analogue of the Jordan-Chevalley decomposition in $M_n\big(C(\mathcal{X})\big)$ are not necessarily bounded. However, we observe that they are affiliated operators, when

$M_n\big(C(\mathcal{X})\big)$ is a von Neumann algebra.

For a finite von Neumann algebra $\mathscr{N}$, we denote the Murray-von Neumann algebra (see Definition 1.6.2) of densely-defined closed linear operators affiliated with $\mathscr{N}$ by $\text{Aff}(\mathscr{N})$. In [Kad86], Kadison introduced the notion of normal functions on a Stonean space, providing a function representation for unbounded operators affiliated with abelian von Neumann algebras. More precisely, the set of all normal functions on $\mathcal{X}$, denoted by $\mathcal{N}(\mathcal{X})$, is the $*$-algebra $\text{Aff}(C(\mathcal{X}))$ for a hyper-Stonean space $\mathcal{X}$. Using [Kad86, §3, §4] and [Nay21, Theorem 4.15], one identifies $\text{Aff}\big(M_n(C(\mathcal{X}))\big)$ with $M_n\big(\text{Aff}(C(\mathcal{X}))\big)$. In §5.2 we define the notion of $\mathcal{Y}$-valued normal functions on a Stonean space, for a locally compact Hausdorff space $\mathcal{Y}$, and in Remark 5.2.9, we identify $M_n(\mathbb{C})$-valued normal functions on a Stonean space $\mathcal{X}$ with matrices in $M_n\big(\mathcal{N}(\mathcal{X})\big)$ in a natural manner.

In Theorem 5.3.11, we establish the existence and uniqueness of the Jordan-Chevalley-Dunford decomposition for matrices in $M_n\big(\mathcal{N}(\mathcal{X})\big)$, which is a key step towards the main result of this chapter, Theorem 5.5.3. This immediately yields a version of the Jordan-Chevalley decomposition in the context of type I_n Murray-von Neumann algebras, which we record in Proposition 5.5.1-(i). This decomposition involves a '$\mathfrak{u}$-scalar-type operator' (see Definition 5.4.5), and a nilpotent operator, which commute with each other. In Proposition 5.5.1-(ii) we note the convergence of the normalized power sequence of such operators in the $\mathfrak{m}$-topology (see Definition 1.6.3).

Using restriction and direct sum techniques afforded by §5.4.1, the results from type I_n Murray-von Neumann algebra case are pieced together to establish the main theorem, Theorem 5.5.3, of this chapter which asserts the existence and uniqueness of the Jordan-Chevalley-Dunford decomposition for operators in type I Murray-von Neumann algebras, and the $\mathfrak{m}$-convergence of the normalized power sequence of such operators. In Remark 5.5.2, we note that while the property of being a $\mathfrak{u}$-scalar-type is preserved under infinite direct sums, nilpotency is not. However, the property of being an $\mathfrak{m}$-quasinilpotent (see Definition 5.4.5) is preserved under infinite direct sums (see Lemma 5.4.12), making it an appropriate choice for the 'nilpotent part' in the Jordan-Chevalley-Dunford decomposition.

In Proposition 5.5.7, we note that for every $n \in \mathbb{N}$, there is a unital normal embedding of the type I_n von Neumann algebra, $M_n\big(\ell^\infty(\mathbb{N})\big)$, into a type II_1 von Neumann algebra. This, along with the functoriality of the Jordan-Chevalley-Dunford decomposition established in Corollary 5.5.5, suggests that any meaningful Jordan-Chevalley-Dunford decomposition for operators in type II_1 von Neumann algebras will necessarily involve affiliated operators, as noted in Remark 5.5.8.

In summary, this thesis develops new insights into the Jordan–Chevalley decomposition, its analogues in infinite dimensions, and explores its connection with the convergence of normalized power sequences. The chapters that follow give detailed proofs and a thorough discussion of these results.

Chapter 1

Preliminaries

This chapter presents the essential background required to follow the developments in the rest of the thesis. We have aimed to keep the exposition fairly self-contained, and have included all the key concepts and results that will be used in later chapters. Notations that are not explicitly introduced here may be found in the section 'Notations & abbreviations'. As we move forward, we will often return to this chapter for the foundational results collected here for ready references.

1.1 Linear algebra

Definition 1.1.1 (Similarity)**.** Let $\mathbb{K}$ be an arbitrary field. Two matrices $A, B \in M_n(\mathbb{K})$ are said to be *similar* in $M_n(\mathbb{K})$ if there exists an invertible matrix S in $GL_n(\mathbb{K})$ such that $S^{-1}AS = B$. We use the notation $A \sim_{\text{sim}} B$, to indicate that A and B are similar.

It is easily verified that the relation $\sim_{\text{sim}}$ is an equivalence relation on $M_n(\mathbb{K})$. We say that S in $GL_n(\mathbb{K})$ *implements* the similarity $A \sim_{\text{sim}} B$ in $M_n(\mathbb{K})$ if $S^{-1}AS = B$. By the phrase *conjugation* with an invertible matrix $S \in GL_n(\mathbb{K})$, we mean the mapping $A \mapsto SAS^{-1}$ on $M_n(\mathbb{K})$.

Definition 1.1.2 (Similarity invariants)**.** A function f defined on $M_n(\mathbb{K})$ is said to be a *similarity invariant* if the equation $f(A) = f(S^{-1}AS)$ holds for all $S \in GL_n(\mathbb{K})$. In other words, we have $f(A) = f(B)$ whenever $A \sim_{\text{sim}} B$ in $M_n(\mathbb{K})$.

Let $\mathscr{V}$ be an n-dimensional $\mathbb{K}$-space and $T : \mathscr{V} \to \mathscr{V}$ be a $\mathbb{K}$-linear transformation. If f is a similarity invariant on $M_n(\mathbb{K})$, then we may unambiguously define $f(T) := f(T_{\mathcal{B}})$, where $T_{\mathcal{B}}$ is the matrix representation of T with respect to a basis $\mathcal{B}$ for $\mathscr{V}$. The trace, determinant, rank, and characteristic polynomial are some examples of similarity invariants, which may thus be defined for any $\mathbb{K}$-linear transformation on $\mathscr{V}$.

Lemma 1.1.3. *Let $\mathbb{K}$ be a field and $\mathscr{V}$ be a finite-dimensional vector space over $\mathbb{K}$. Let $T : \mathscr{V} \to \mathscr{V}$ be a $\mathbb{K}$-linear mapping such that the characteristic polynomial of T splits over $\mathbb{K}$. If $\mathscr{V}'$ is a subspace of $\mathscr{V}$ invariant under T, then the characteristic polynomial of the restriction of T to $\mathscr{V}'$, $T|_{\mathscr{V}'} : \mathscr{V}' \to \mathscr{V}'$, splits over $\mathbb{K}$.*

Proof. Let the dimension of $\mathscr{V}$ be n, and the dimension of $\mathscr{V}'$ be m. Let $\mathcal{B}' := \{v_1, \dots, v_m\}$ be a basis for $\mathscr{V}'$ which is extended to a basis $\mathcal{B} := \{v_1, \dots, v_n\}$ for $\mathscr{V}$. The matrix of T with respect to the basis $\mathcal{B}$ is then of the block-triangular form,

$$\begin{bmatrix} A & C \\ \mathbf{0}_{n-m,m} & B \end{bmatrix} \in M_n(\mathbb{K})$$

where $A \in M_m(\mathbb{K})$ is the matrix of $T|_{\mathscr{V}'}$ with respect to the basis $\mathcal{B}'$. Thus the characteristic polynomial of T is given by $\chi_T(x) = \det(xI_m - A)\det(xI_{n-m} - B)$ and the characteristic polynomial of $T|_{\mathscr{V}'}$ is given by $\chi_{T|_{\mathscr{V}'}}(x) = \det(xI_m - A)$. Clearly $\chi_{T|_{\mathscr{V}'}}(x)$ divides $\chi_T(x)$ and hence $\chi_{T|_{\mathscr{V}'}}(x)$ splits over $\mathbb{K}$. □

Lemma 1.1.4 (Simultaneous triangularization)**.** *Let $\mathbb{K}$ be a field and A, B be commuting matrices in $M_n(\mathbb{K})$ such that the characteristic polynomials of both A and B split over $\mathbb{K}$. Then there is a matrix $S \in GL_n(\mathbb{K})$ such that $S^{-1}AS$ and $S^{-1}BS$ are both upper-triangular matrices in $M_n(\mathbb{K})$.*

Proof. We prove the result by induction on n. For $n = 1$ there is nothing to show. Thus, we may assume that $n \geq 2$. Let $\lambda \in \mathbb{K}$ be an eigenvalue of A and $\mathscr{V}_\lambda \subseteq \mathbb{K}^n$ be the corresponding eigenspace. Since A and B commute, B leaves $\mathscr{V}_\lambda$ invariant. By Lemma 1.1.3, $B|_{\mathscr{V}_\lambda}$ has an eigenvalue in $\mathbb{K}$. In other words, $B|_{\mathscr{V}_\lambda}$, and thus B, has an eigenvector in $\mathscr{V}_\lambda$. In summary, A and B have a common eigenvector $v \in \mathbb{K}^n$, say with respective eigenvalues λ and μ.

Let S_0 be an invertible matrix in $M_n(\mathbb{K})$ which maps the standard basis vector $\mathbf{e}_1 \in \mathbb{K}^n$ to v. Then $\mathbf{e}_1$ is a common eigenvector for the matrices $S_0^{-1}AS_0$ and $S_0^{-1}BS_0$ with eigenvalues λ and μ, respectively, so that

$$S_0^{-1}AS_0 = \begin{bmatrix} \lambda & X \\ \mathbf{0}_{n-1,1} & A' \end{bmatrix}, \quad S_0^{-1}BS_0 = \begin{bmatrix} \mu & Y \\ \mathbf{0}_{n-1,1} & B' \end{bmatrix},$$

for some $X, Y \in M_{1,n-1}(\mathbb{K})$ and $A', B' \in M_{n-1}(\mathbb{K})$. Note that this proves the result in the case of $n = 2$. Since A and B commute, so do $S_0^{-1}AS_0$ and $S_0^{-1}BS_0$ and a quick computation shows that A' and B' also commute. Assuming that the result holds for matrices in $M_{n-1}(\mathbb{K})$, there is a matrix S' in $GL_{n-1}(\mathbb{K})$ such that $S'^{-1}A'S'$ and $S'^{-1}B'S'$ are both upper-triangular matrices in $M_{n-1}(\mathbb{K})$.

Setting $S := S_0(1 \oplus S')$, it is clear that $S^{-1}AS$ and $S^{-1}BS$ are both upper-triangular matrices in $M_n(\mathbb{K})$. □

***Remark* 1.1.5.** With $B = I_n$ in Lemma 1.1.4, we note that every matrix A in $M_n(\mathbb{K})$ whose characteristic polynomial splits over $\mathbb{K}$, is similar to an upper-triangular matrix in $M_n(\mathbb{K})$.

Lemma 1.1.6. *Let $\mathbb{K}$ be an algebraically closed field and $\mathscr{V}$ be a finite-dimensional vector space over $\mathbb{K}$. If $T, T' : \mathscr{V} \to \mathscr{V}$ are commuting $\mathbb{K}$-linear maps, then*

$$\mathrm{sp}(T + T') \subseteq \mathrm{sp}(T) + \mathrm{sp}(T').$$

Proof. Let A and A' be matrix representations of the linear transformations T and T', respectively, with respect to some basis of $\mathscr{V}$. From Lemma 1.1.4, without loss of generality, we may assume that both A and A' are upper-triangular matrices. Clearly, $A + A'$ is also upper-triangular. Since the eigenvalues of an upper-triangular matrix are precisely its diagonal entries, it follows that $\mathrm{sp}(A + A') \subseteq \mathrm{sp}(A) + \mathrm{sp}(A')$. Hence $\mathrm{sp}(T + T') \subseteq \mathrm{sp}(T) + \mathrm{sp}(T')$. □

1.2 Field theory

Let $\mathbb{K}$ be a field. A *field extension* of $\mathbb{K}$ is a field $\mathbb{L}$ containing $\mathbb{K}$ as a subfield. We denote this by $\mathbb{L}/\mathbb{K}$. For the sake of completeness, we recall the definitions of standard field extensions.

Definition 1.2.1 (Finite extension)**.** A field extension $\mathbb{L}/\mathbb{K}$ is said to be *finite* if $\mathbb{L}$ is a finite-dimensional vector space over $\mathbb{K}$.

Definition 1.2.2 (Algebraic extension)**.** Let $\mathbb{L}/\mathbb{K}$ be a field extension. An element $\alpha \in \mathbb{L}$ is said to be *algebraic* over $\mathbb{K}$ if there exists a non-zero polynomial $p(x) \in \mathbb{K}[x]$ such that $p(\alpha) = 0$. In such a case, the unique monic irreducible polynomial $m(x) \in \mathbb{K}[x]$ of lowest degree such that $m(\alpha) = 0$ is called the minimal polynomial of α in $\mathbb{L}/\mathbb{K}$.

If every element of $\mathbb{L}$ is algebraic over $\mathbb{K}$, the extension $\mathbb{L}/\mathbb{K}$ is called an *algebraic extension*.

Definition 1.2.3 (Splitting field extension)**.** Let $p(x) \in \mathbb{K}[x]$ be a non-zero polynomial. A field extension $\mathbb{L}$ is called the *splitting field* of $p(x)$ over $\mathbb{K}$, if $\mathbb{L}$ is the smallest field extension of $\mathbb{K}$ such that $p(x)$ splits completely in $\mathbb{L}[x]$, that is, factors into linear terms in $\mathbb{L}[x]$.

Definition 1.2.4 (Normal Extension)**.** An algebraic extension $\mathbb{L}/\mathbb{K}$ is said to be *normal* if every irreducible polynomial $p(x) \in \mathbb{K}[x]$ that has at least one root in $\mathbb{L}$ splits completely over $\mathbb{L}$, that is, all its roots lie in $\mathbb{L}$.

Definition 1.2.5 (Separable Extension)**.** An algebraic extension $\mathbb{L}/\mathbb{K}$ is called *separable* if the minimal polynomial of every $\alpha \in \mathbb{L}$ over $\mathbb{K}$ has distinct roots in its splitting field.

Definition 1.2.6 (Galois Extension)**.** A field extension $\mathbb{L}/\mathbb{K}$ is called a *Galois extension* if it is both normal and separable.

For our discussion in Chapter 2, we mainly need the following definitions.

Definition 1.2.7 (Fixed field of a field extension)**.** Let $\mathbb{L}$ be a field extension of $\mathbb{K}$. Then the set of all field automorphisms of $\mathbb{L}$ which fixes every element of $\mathbb{K}$ is denoted by $\mathrm{Aut}(\mathbb{L}/\mathbb{K})$. The set $\mathbb{F}$ consisting of elements of $\mathbb{L}$ fixed by every automorphism in $\mathrm{Aut}(\mathbb{L}/\mathbb{K})$ is a field, with $\mathbb{K} \subseteq \mathbb{F} \subseteq \mathbb{L}$, and is called the *fixed field* of $\mathrm{Aut}(\mathbb{L}/\mathbb{K})$.

If $\mathbb{L}/\mathbb{K}$ is a Galois extension, then $\mathrm{Aut}(\mathbb{L}/\mathbb{K})$ forms a group, referred to as the *Galois group* of the extension.

Definition 1.2.8 (Perfect field (see [DF04, pg. 549], [Lan05, pg. 252]))**.** A field $\mathbb{K}$ is said to be *perfect* if either $\mathbb{K}$ has characteristic 0, or when $\mathbb{K}$ has characteristic p the Frobenius endomorphism $x \mapsto x^p$ is an automorphism of $\mathbb{K}$, that is, every element of $\mathbb{K}$ is a p^{th} power in $\mathbb{K}$.

There are several equivalent definitions of perfect fields in the literature. For instance, $\mathbb{K}$ is perfect if any one of the following equivalent conditions holds:

(i) Every irreducible polynomial over $\mathbb{K}$ is separable.

(ii) Every finite extension of $\mathbb{K}$ is separable.

(iii) Every algebraic extension of $\mathbb{K}$ is separable.

***Remark* 1.2.9.** The key fact about perfect fields that we use (in Theorem 2.4.9), is that every splitting field extension of a perfect field is a Galois extension; in particular, if $\mathbb{L}$ is a splitting field extension of a polynomial in $\mathbb{K}[x]$, then the fixed field of $\mathrm{Aut}(\mathbb{L}/\mathbb{K})$ is $\mathbb{K}$ (see [Lan05, Chapter 6]).

***Example* 1.2.10.** Some commonly encountered perfect fields include :

(i) Fields of characteristic zero; in particular, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, the p-adics.

(ii) Algebraically closed fields.

(iii) Finite fields, $\mathbb{F}_{p^n}$.

1.3 Operator theory

We invoke four basic operator inequalities labelled (OI1), (OI2), (OI3), (OI4) repeatedly in our discussion in Chapter 3 and 4, which we list below.

(i) ([KR83, Proposition 4.2.3-(ii)]) If A is a self-adjoint operator in $\mathscr{B}(\mathscr{H})$, then

$$-\|A\|I \;\le\; A \;\le\; \|A\|I. \tag{OI1}$$

Moreover, $||A||$ is the least non-negative real number such that the above inequality holds (see [KR83, Exercise 2.8.11]).

(ii) ([Bha96, Proposition V.1.6]) If H is a positive invertible operator in $\mathscr{B}(\mathscr{H})$, then

$$\left|\left|H^{-1}\right|\right|^{-1} I \le H \le ||H||\, I. \tag{OI2}$$

(iii) ([KR83, Corollary 4.2.7]) If A and B are self-adjoint operators in $\mathscr{B}(\mathscr{H})$ such that $A \leq B$, then

$$T^*AT \;\leq\; T^*BT \quad \text{for all} \;\; T \in \mathscr{B}(\mathscr{H}). \tag{OI3}$$

(iv) ([Bha96, Proposition V.1.9]) If H and K are positive operators in $\mathscr{B}(\mathscr{H})$ such that $0 \leq H \leq K$, then

$$0 \;\leq\; H^{\frac{1}{k}} \;\leq\; K^{\frac{1}{k}} \quad \text{for all} \;\; k \in \mathbb{N}. \tag{OI4}$$

Apart from these inequalities, for $T \in \mathscr{B}(\mathscr{H})$, we also make frequent use of the C^*-identity, $||T^*T|| = ||T||^2 = ||TT^*||$, along with the spectral radius formula $\lim_{k\in\mathbb{N}} \|T^k\|^{\frac{1}{k}} = \mathrm{r}(T)$.

Theorem 1.3.1 (**Fuglede's theorem** [Fug55, Theorem I])**.** Let $M \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then, for an operator $T \in \mathscr{B}(\mathscr{H})$, M commutes with T if and only if every spectral projection of M commutes with T. Consequently, $MT = TM$ if and only if $M^*T = TM^*$.

Lemma 1.3.2. *Let H be a positive operator in $\mathscr{B}(\mathscr{H})$. Then the following are equivalent:*

(i) $\mathrm{ran}(H)$ *is a closed subspace of* $\mathscr{H}$*.*

(ii) Either H is invertible, or 0 *is an isolated point of* $\mathrm{sp}\,(H)$*.*

(iii) $\lim_{k\to\infty} H^{\frac{1}{k}} = \mathbf{R}(H)$ *in norm.*

Proof. Let $E := \mathbf{R}(H)$. Since H is self-adjoint, H and E commute.

(i)$\Longleftrightarrow$(ii) : Note that $\mathrm{ran}(H) = \mathrm{ran}(E)$ if and only if $\mathrm{ran}(H)$ is a closed subspace of $\mathscr{H}$. By the Douglas factorization lemma (see [Dou66]), $\mathrm{ran}(H) = \mathrm{ran}(E)$ if and only if there are positive real numbers $0 < \lambda < \mu$ such that $\lambda^2 E \leq H^2 \leq \mu^2 E$. From (OI4) for $k = 2$, we have $\lambda E \leq H \leq \mu E$. Using the continuous function calculus in the context of the commutative C^*-algebra generated by H and E, we observe that this is equivalent to the spectrum of H being contained in $\{0\} \cup [\lambda, \mu]$.

(ii)$\Longleftrightarrow$(iii) : Note that the commutative C^*-algebra generated by H may be viewed as the space of complex-valued continuous functions on $\mathrm{sp}\,(H)$. Let $f_k : \mathrm{sp}\,(H) \to \mathbb{R}_{\geq 0}$ be the continuous function given by $x \mapsto x^{\frac{1}{k}}$. We observe that the sequence of continuous functions, $\{f_k\}_{k\in\mathbb{N}}$, converges uniformly in $C(\mathrm{sp}\,(H))$ if and only if either $0 \notin \mathrm{sp}\,(H)$ or 0 is an isolated point of $\mathrm{sp}\,(H)$; moreover, in that case, f_k converges uniformly to the indicator function of $\mathrm{sp}\,(H) \setminus \{0\}$. At the level of operators, this is equivalent to the assertion that $\lim_{k\to\infty} H^{\frac{1}{k}} = \mathbf{R}(H)$ in norm. □

Lemma 1.3.3. *Let H be a positive operator in $\mathscr{B}(\mathscr{H})$, and $\alpha \geq 0$. Then*

$$(H^k + \alpha^k I)^{\frac{1}{k}} \leq H + \alpha I.$$

Proof. Since $H^k + \alpha^k I \leq H^k + \alpha^k I + \sum_{m=1}^{k-1} \binom{k}{m} \alpha^{k-m} H^m = (H + \alpha I)^k$, the assertion follows from inequality (OI4). □

Lemma 1.3.4. *Let H, K be positive operators in $\mathscr{B}(\mathscr{H})$ such that $0 \leq H \leq K$. Then $\mathbf{R}(H) \leq \mathbf{R}(K)$.*

Proof. From inequality (OI4), we observe that $0 \leq H^{\frac{1}{k}} \leq K^{\frac{1}{k}}$. Using [KR83, Lemma 5.1.5] and taking SOT-limits as $k \to \infty$, we get the desired result. □

***Remark* 1.3.5.** Let $T \in \mathscr{B}(\mathscr{H})$, and let S, S' be invertible operators in $\mathscr{B}(\mathscr{H})$. Then $\mathrm{ran}(STS') = \mathrm{ran}(ST) = S\,\mathrm{ran}(T)$. Thus, T has closed range if and only if STS' has closed range. In particular, for a projection $E \in \mathscr{B}(\mathscr{H})$, it follows that S^*ES has closed range, and

$$\mathbf{R}(S^*ES) = \mathbf{R}(S^*E).$$

Lemma 1.3.6. *Let E be a projection and S be an invertible operator in $\mathscr{B}(\mathscr{H})$. Then,*

$$\mathbf{R}(S^*ES) = I - \mathbf{R}(S^{-1}(I - E)S).$$

Proof. Since $S^{-1}(I - E)S$ is a closed-range operator, and S^*ES is a positive closed-range operator, it suffices to show that $\mathrm{ran}(S^{-1}(I-E)S) = \ker(S^*ES)$. Clearly, $(S^*ES)\left(S^{-1}(I - E)S\right) = 0$, whence it follows that

$$\mathrm{ran}(S^{-1}(I - E)S) \subseteq \ker(S^*ES).$$

Conversely, let $x \in \ker(S^*ES)$. Since S^* is invertible, $Sx \in \ker(E)$. In other words, $ESx = 0$, whence $S^{-1}(I - E)Sx = x$, that is, $x \in \mathrm{ran}(S^{-1}(I - E)S)$. Thus,

$$\ker(S^*ES) \subseteq \mathrm{ran}(S^{-1}(I - E)S).$$ □

Definition 1.3.7 (Resolution of the identity (see [KR83, §5.2]))**.** A family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of projections in $\mathscr{B}(\mathscr{H})$ indexed by $\mathbb{R}$ is said to be a *resolution of the identity* on $\mathscr{H}$ if it satisfies :

(i) $\bigwedge_{\lambda\in\mathbb{R}} E_\lambda = 0$ and $\bigvee_{\lambda\in\mathbb{R}} E_\lambda = I$;

(ii) $E_\lambda \leq E_{\lambda'}$ when $\lambda \leq \lambda'$;

(iii) $E_\lambda = \bigwedge_{\lambda'>\lambda} E_{\lambda'}$.

If there is a constant $a \in \mathbb{R}$ such that $E_\lambda = 0$ when $\lambda < -a$ and $E_\lambda = I$ when $a < \lambda$, then $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is said to be a *bounded* resolution of the identity.

Theorem 1.3.8 (Spectral resolution of a self-adjoint operator. (see [KR83, Theorem 5.2.2]))**.** *If A is a self-adjoint operator in $\mathscr{B}(\mathscr{H})$, then there is a resolution of the identity, $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, on $\mathscr{H}$, such that* *(i)* $E_\lambda = 0$ *when* $\lambda < -\|A\|$, *and* $E_\lambda = I$ *when* $\|A\| \leq \lambda$ *;*

(ii) $AE_\lambda \leq \lambda E_\lambda$ *and* $\lambda(I - E_\lambda) \leq A(I - E_\lambda)$ *for each* λ *;*

(iii) $A = \int_{-\|A\|}^{\|A\|} \lambda \, \mathrm{d}E_\lambda$ *in the sense of norm convergence of approximating Riemann sums.*

Moreover, $\lambda_0 \notin \mathrm{sp}\,(A)$ *if and only if there is a neighborhood of* λ_0 *where* E_λ*, as a function of* λ*, is constant.*

The family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ so determined, is called *the spectral resolution of* A.

***Remark* 1.3.9.** *Let* H *be a positive operator in* $\mathscr{B}(\mathscr{H})$ *with spectral resolution* $\{E_\lambda\}_{\lambda \in \mathbb{R}}$*. Then* $E_\lambda = 0$ *for* $\lambda < 0$ *and* $E_\lambda = I$ *for* $\lambda \geq \|H\|$*, and* $H = \int_0^{\|H\|} \lambda \, \mathrm{d}E_\lambda$*. For* $\lambda_0 > 0$*, the spectral resolution of the positive operator* $(I - E_{\lambda_0})H$ *is given by* $\left\{E_{\lambda_0} + (I - E_{\lambda_0})E_\lambda\right\}_{\lambda \geq 0}$*.*

The notion of spectral resolution may be defined for a more general class of operators called *spectral operators*.

1.3.1 Spectral operators

We rely on [DS88] as the main reference for our discussion on spectral operators. For the following definitions, we refer to [Dun58].

Definition 1.3.10 (Idempotent-valued spectral resolution of an operator (see [Dun58, §1])**.** Let $T \in \mathscr{B}(\mathfrak{X})$ and $\mathfrak{B}$ be the σ-algebra of Borel sets in the complex plane. Let $\mathcal{E} : \mathfrak{B} \to \mathscr{B}(\mathfrak{X})$ be an idempotent-valued map such that the following properties are satisfied :

(i) $T\mathcal{E}(\boldsymbol{\beta}) = \mathcal{E}(\boldsymbol{\beta})T, \quad \mathrm{sp}\left(T|_{\mathrm{ran}(\mathcal{E}(\boldsymbol{\beta}))}\right) \subseteq \overline{\boldsymbol{\beta}}, \quad \forall\, \boldsymbol{\beta} \in \mathfrak{B}.$

(ii) $\mathcal{E}(\varnothing) = 0, \quad \mathcal{E}(\mathbb{C}) = I,$ and $\mathcal{E}\,(\mathbb{C} \setminus \boldsymbol{\beta}) = I - \mathcal{E}(\boldsymbol{\beta})$ for $\boldsymbol{\beta} \in \mathfrak{B}$.

(iii) For $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in \mathfrak{B}$,

$$\mathcal{E}(\boldsymbol{\beta}_1 \cap \boldsymbol{\beta}_2) = \mathcal{E}(\boldsymbol{\beta}_1) \wedge \mathcal{E}(\boldsymbol{\beta}_2) = \mathcal{E}(\boldsymbol{\beta}_1)\mathcal{E}(\boldsymbol{\beta}_2),$$
$$\mathcal{E}(\boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2) = \mathcal{E}(\boldsymbol{\beta}_1) \vee \mathcal{E}(\boldsymbol{\beta}_2) = \mathcal{E}(\boldsymbol{\beta}_1) + \mathcal{E}(\boldsymbol{\beta}_2) - \mathcal{E}(\boldsymbol{\beta}_1)\mathcal{E}(\boldsymbol{\beta}_2).$$

(iv) $||\mathcal{E}(\boldsymbol{\beta})|| \leq \boldsymbol{c} \;\; \forall\, \boldsymbol{\beta} \in \mathfrak{B}$, for some constant $\boldsymbol{c} \geq 0$.

(v) $\mathcal{E}(\boldsymbol{\beta})$ is countably additive with respect to the strong operator topology (SOT), that is, for every sequence, $\{\boldsymbol{\beta}_k\}_{k \in \mathbb{N}}$, of disjoint Borel sets, we have

$$\mathcal{E}\left(\bigcup_{k=1}^{\infty} \boldsymbol{\beta}_k\right) x = \sum_{k=1}^{\infty} \mathcal{E}\left(\boldsymbol{\beta}_k\right) x, \quad \text{for every } x \in \mathfrak{X}.$$

If such a mapping $\boldsymbol{\beta} \mapsto \mathcal{E}(\boldsymbol{\beta})$ exists, then it is uniquely determined by T, and is called *the* idempotent-valued resolution of the identity for T, or *the* idempotent-valued spectral resolution of T. We denote the idempotent-valued spectral resolution of T by $\mathcal{E}_T$.

Definition 1.3.11 (Spectral operator)**.** An operator $T \in \mathscr{B}(\mathfrak{X})$ which has an idempotent-valued spectral resolution is called a *spectral* operator.

Definition 1.3.12 (Scalar-type operator)**.** A spectral operator $D \in \mathscr{B}(\mathfrak{X})$ is said to be *scalar-type* if

$$D = \int_{\mathbb{C}} \lambda \, \mathrm{d}\mathcal{E}_D(\lambda),$$

where $\mathcal{E}_D$ is the idempotent-valued spectral resolution of D.

We now outline some fundamental properties of spectral operators below. For the sake of brevity, the proofs of the theorems are omitted.

Definition 1.3.13 (Analytic extension of $R(\mu;T)x$)**.** Let $T \in \mathscr{B}(\mathfrak{X})$ and let $R(\mu;T)$ denote the resolvent, $(\mu I - T)^{-1}$, of T corresponding to the point μ in the resolvent set $\mathbb{C} \setminus \mathrm{sp}\,(T)$. For $x \in \mathfrak{X}$, an $\mathfrak{X}$ valued function f defined and analytic on an open subset $\mathrm{dom}(f) \subset \mathbb{C}$ containing $\mathbb{C} \setminus \mathrm{sp}\,(T)$ is said to be an *analytic extension* of $R(\mu;T)x$ if

$$(\mu I - T)f(\mu) = x \ ; \ \mu \in \mathrm{dom}(f).$$

It is clear that for such an extension f,

$$f(\mu) = R(\mu;T)x \ ; \ \mu \in \mathbb{C} \setminus \mathrm{sp}\,(T)\,.$$

Definition 1.3.14 (Single valued extension property)**.** Let $T \in \mathscr{B}(\mathfrak{X})$ and let $x \in \mathfrak{X}$. The function $R(\mu;T)x$ is said to have a *single valued extension property* if every pair f, g of analytic extensions of $R(\mu;T)x$ coincide on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. That is, $f(\mu) = g(\mu) \ \forall \ \mu \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$.

Definition 1.3.15. (Local spectrum and resolvent) Let $T \in \mathscr{B}(\mathfrak{X})$ and let $x \in \mathfrak{X}$ be such that $R(\mu;T)x$ has the single value extension property . Then the *resolvent* of x, denoted by $\rho_T(x)$, is defined as the union of all the sets $\mathrm{dom}(f)$ as f varies over all analytic extensions of $R(\mu;T)x$. The *spectrum* $\sigma_T(x)$ of x is then defined to be the complement of $\rho_T(x)$.

Theorem 1.3.16 ([DS88, Theorem XV.3.2])**.** *Let $T \in \mathscr{B}(\mathfrak{X})$ be a spectral operator. Then for each $x \in \mathfrak{X}$, the function $R(\mu;T)x$ has the single valued extension property.*

It follows from the above theorem that for a spectral operator $T \in \mathscr{B}(\mathfrak{X})$ and $x \in \mathfrak{X}$, the function $R(\mu;T)x$ has a maximal analytic extension defined on $\rho_T(x)$. We denote the maximal extension of $R(\mu;T)x$ by $x_T(\mu)$.

Corollary 1.3.17 ([DS88, Theorem XV.3.3])**.** *Let $T \in \mathscr{B}(\mathfrak{X})$ be a spectral operator. Then the spectrum $\sigma_T(x)$ of $x \in \mathfrak{X}$ is empty if and only if $x = 0$.*

With the help of the notions defined above, a concrete description of *the* idempotent-valued spectral resolution of a spectral operator can be given.

Theorem 1.3.18 ([DS88, Theorem XV.3.4]). *Let $T \in \mathscr{B}(\mathfrak{X})$ be a spectral operator with spectral resolution $\mathcal{E}_T$ and let $\mathcal{C}$ be a closed set of complex numbers. Then*

$$\mathcal{E}_T(\mathcal{C})\mathfrak{X} = \{x \in \mathfrak{X} : \sigma_T(x) \subseteq \mathcal{C}\}.$$

Corollary 1.3.19 ([DS88, Corollary XV.3.7]). *Let T be a spectral operator in $\mathscr{B}(\mathfrak{X})$, and let $\mathcal{E}_T$ be the idempotent-valued spectral resolution of T. If $A \in \mathscr{B}(\mathfrak{X})$ commutes with T, then A commutes with $\mathcal{E}_T(\boldsymbol{\beta})$ for every Borel subset $\boldsymbol{\beta} \subseteq \mathbb{C}$. Moreover, $\sigma_T(Ax) \subseteq \sigma_T(x)$ for all $x \in \mathfrak{X}$.*

Theorem 1.3.20 ([DS88, Theorem XV.3.10]). *Let T be an operator in $\mathscr{B}(\mathfrak{X})$ and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be idempotents in $\mathscr{B}(\mathfrak{X})$ such that each $\mathcal{E}_i$ commutes with T and $\mathcal{E}_1 + \cdots + \mathcal{E}_n = I$. Then T is a spectral operator if and only if each restriction $T|_{\mathcal{E}_i\mathfrak{X}}$ is a spectral operator. Moreover, if T is a spectral operator then the idempotent-valued spectral resolution of the restriction $T|_{\mathcal{E}_i\mathfrak{X}}$ is the corresponding restriction of the idempotent-valued spectral resolution of T.*

Theorem 1.3.21 (see [DS88, Theorem XV.5.6]). *Let T be a spectral operator in $\mathscr{B}(\mathfrak{X})$ and let $\mathcal{E}_T$ be the idempotent-valued spectral resolution of T. If $f : \mathbb{C} \to \mathbb{C}$ is an analytic function, then $f(T)$ is a spectral operator whose idempotent-valued spectral resolution is given by*

$$\mathcal{E}_{f(T)}(\boldsymbol{\beta}) = \mathcal{E}_T\left(f^{-1}(\boldsymbol{\beta})\right) \text{ for a Borel subset } \boldsymbol{\beta}.$$

1.3.2 Compact operators

Definition 1.3.22. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces and $T \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$. Then the following conditions are equivalent:

(i) The image of the unit ball of $\mathfrak{X}$ under T is relatively compact in $\mathfrak{Y}$;

(ii) The image of any bounded subset of $\mathfrak{X}$ under T is relatively compact in $\mathfrak{Y}$;

(iii) The image of any bounded subset of $\mathfrak{X}$ under T is totally bounded in $\mathfrak{Y}$;

(iv) For any bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathfrak{X}$, the sequence $\{Tx_n\}_{n\in\mathbb{N}}$ contains a converging subsequence.

Any $T \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ satisfying any (and hence, all) of the above properties is said to be a *compact operator.*

Below we discuss the Fredholm alternative, and define the Riesz idempotents associated with a compact operator in $\mathscr{B}(\mathscr{H})$.

Theorem 1.3.23 (Fredholm Alternative (see [Dou98, Theorem 5.22]))**.** *Let $C \in \mathscr{B}(\mathscr{H})$ be a compact operator. Then $\operatorname{sp}(C) \setminus \{0\}$ is countable with 0 as the only possible limit point, and each $\lambda \in \operatorname{sp}(C) \setminus \{0\}$ is an eigenvalue of C with finite multiplicity and $\overline{\lambda}$ is an eigenvalue of C^* with the same multiplicity. Moreover, the generalized eigenspace of C corresponding to λ is finite-dimensional and has the same dimension as the generalized eigenspace of C^* corresponding to $\overline{\lambda}$.*

Definition 1.3.24. Let $C \in \mathscr{B}(\mathscr{H})$ be a compact operator, and let $\operatorname{sp}(C) \setminus \{0\} \neq \varnothing$. Then, for $\lambda \in \operatorname{sp}(C) \setminus \{0\}$ the operator $\mathcal{E}_\lambda$, defined using the holomorphic functional calculus,

$$\mathcal{E}_\lambda = \frac{1}{2\pi i} \int_\gamma (\mu I - C)^{-1} \, \mathrm{d}\mu,$$

where γ is a Jordan contour that encloses only λ from $\operatorname{sp}(C)$, is an idempotent in $\mathscr{B}(\mathscr{H})$ such that $\mathcal{E}_\lambda$ commutes with C and $\operatorname{sp}(C|_{\mathcal{E}_\lambda \mathscr{H}}) = \{\lambda\}$. It is referred to as the *Riesz idempotent* of C corresponding to λ.

***Remark* 1.3.25.** Note that if $\mathcal{E}_\lambda$ denotes the Riesz idempotent of C corresponding to $\lambda \in \operatorname{sp}(C) \setminus \{0\}$, then the generalized eigenspace of C corresponding to λ is precisely $\mathcal{E}_\lambda \mathscr{H}$, and the generalized eigenspace of C^* corresponding to $\overline{\lambda}$ is precisely $\mathcal{E}_\lambda^* \mathscr{H}$.

1.3.3 Unbounded operators

To facilitate our discussion in Chapter 4, and Chapter 5, we collect a few basic definitions concerning unbounded operators acting on an infinite-dimensional Hilbert space $\mathscr{H}$.

By an unbounded operator T acting on a Hilbert space $\mathscr{H}$ we mean a linear operator with $\operatorname{dom}(T), \operatorname{ran}(T) \subseteq \mathscr{H}$; if $\operatorname{dom}(T)$ is dense in $\mathscr{H}$, we say that T is densely-defined. Note that the word 'unbounded' is meant as 'not necessarily bounded' rather than 'not bounded'.

Definition 1.3.26. Two unbounded operators S, T acting on $\mathscr{H}$ are equal, written as $S = T$, if $\operatorname{dom}(S) = \operatorname{dom}(T)$ and $Sx = Tx$ for all $x \in \operatorname{dom}(S)$.

Definition 1.3.27. For unbounded operators S, T acting on $\mathscr{H}$, we say that T is an extension of S, symbolically written as $S \subseteq T$, if $\operatorname{dom}(S) \subseteq \operatorname{dom}(T)$ and $Sx = Tx$ for all $x \in \operatorname{dom}(S)$.

Definition 1.3.28 (Closed operator)**.** An unbounded operator T acting on $\mathscr{H}$ is said to be closed if the graph of T, $\mathscr{G}(T) = \{(x, Tx) : x \in \operatorname{dom}(T)\}$, is a closed subset of $\mathscr{H} \bigoplus \mathscr{H}$. T is said to be pre-closed or closable if the closure of $\mathscr{G}(T)$ is the graph of an operator; this operator is referred to as the closure of T and is denoted by $\overline{T}$.

***Remark* 1.3.29.** Interpreting the closure of $\mathcal{G}(T)$ in limit terms, we see that T is closed if and only if convergence of the sequence $\{x_n\} \subseteq \operatorname{dom}(T)$ to x and $\{Tx_n\}$ to y implies $x \in \operatorname{dom}(T)$ and $Tx = y$; T is preclosed if and only convergence of the sequence $\{x_n\} \subseteq \operatorname{dom}(T)$ to 0 and $\{Tx_n\}$ to y implies $y = 0$.

Definition 1.3.30. Let S, T be unbounded operators acting on $\mathscr{H}$. Then the operators $S+T$ and ST are defined as follows :

$$\begin{aligned} \mathrm{dom}(S+T) &:= \mathrm{dom}(S) \cap \mathrm{dom}(T) \;, \quad (S+T)x := Sx + Tx, \text{ for all } x \in \mathrm{dom}(S+T); \\ \mathrm{dom}(ST) &:= \{x \in \mathrm{dom}(T) : Tx \in \mathrm{dom}(S)\}, \ (ST)x := S(Tx), \text{ for all } x \in \mathrm{dom}(ST). \end{aligned}$$

If S, T are closed operators on $\mathscr{H}$ such that $S+T$ (ST, respectively) is pre-closed, then its closure $\overline{S+T}$ ($\overline{ST}$, respectively) is called the strong-sum (strong-product, respectively) of S and T, and is denoted by $S \hat{+} T$ ($S \hat{\cdot} T$, respectively).

Definition 1.3.31 (Core)**.** Let T be a closed linear operator acting on $\mathscr{H}$. A linear subspace $\mathcal{D} \subseteq \mathrm{dom}(T)$ is called a core for T if the closure of the restriction of T to $\mathcal{D}$ is equal to T. In terms of the operator graph, $\mathcal{D}$ is a core for T if and only if $\overline{\mathcal{G}(T|_{\mathcal{D}})} = \mathcal{G}(T)$

Definition 1.3.32. Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces and T be a linear transformation with $\mathrm{dom}(T)$ dense in $\mathscr{H}$, and $\mathrm{ran}(T) \subseteq \mathscr{K}$. Then the adjoint operator T^* is defined as follows:

$$\mathrm{dom}(T^*) := \{y \in \mathscr{K} : \exists z_y \in \mathscr{H} \text{ with } \langle x, z_y \rangle = \langle Tx, y \rangle \ \ \forall \ x \in \mathrm{dom}(T)\},$$

and $T^* y = z_y$ for $y \in \mathrm{dom}(T^*)$.
A densely-defined linear operator T acting on $\mathscr{H}$ is said to be self-adjoint if $T^* = T$.

From [KR83, Theorem 2.7.8], when T is closed, T^*T is a densely defined self-adjoint operator.

1.4 Topology

Below we note two elementary results from point-set topology, before moving on to Stonean spaces.

Lemma 1.4.1. *Let Ω be a topological space.*

(i) If O_1, O_2 are disjoint open subsets of Ω, then $O_1 \cap \overline{O_2} = \varnothing$.

(ii) If Q is a clopen subset of Ω, then for every subset $\mathcal{S} \subseteq \Omega$,

$$Q \cap \overline{\mathcal{S}} = \overline{Q \cap \mathcal{S}}.$$

Proof. (i) Since $O_1 \cap O_2 = \varnothing$, O_2 is a subset of the closed set $\Omega \backslash O_1$ so that $\overline{O_2} \subseteq \Omega \backslash O_1$, that is, $O_1 \cap \overline{O_2} = \varnothing$.

(ii) Note that $Q \cap \overline{\mathcal{S}}$ is a closed set containing $Q \cap \mathcal{S}$. Hence $\overline{Q \cap \mathcal{S}} \subseteq Q \cap \overline{\mathcal{S}}$. For the other direction, take $x \in Q \cap \overline{\mathcal{S}}$. For every open neighbourhood O of x, $Q \cap O$ is an open neighbourhood of x. Since $x \in \overline{\mathcal{S}}$, we must have $(Q \cap \mathcal{S}) \cap O = (Q \cap O) \cap \mathcal{S} \neq \varnothing$, so that $x \in \overline{Q \cap \mathcal{S}}$. Thus, $Q \cap \overline{\mathcal{S}} = \overline{Q \cap \mathcal{S}}$. ◻

Theorem 1.4.2. *Let Ω, Ω' be topological spaces. A function $f : \Omega \to \Omega'$ is continuous if for a collection of open sets O_α of Ω with $\bigcup_\alpha O_\alpha = \Omega$, the restriction of f to O_α is continuous for each α.*

Proof. Let O' be an open subset of Ω'. Note that $f^{-1}(O') \cap O_\alpha = f|_{O_\alpha}^{-1}(O')$. Since $f|_{O_\alpha}$ is continuous, $f|_{O_\alpha}^{-1}(O')$ is open in O_α and hence open in Ω. Thus $f^{-1}(O') = \bigcup_\alpha \big(f^{-1}(O') \cap O_\alpha\big)$ is open in Ω. □

1.4.1 Stonean spaces

Stonean spaces arose from the work of Stone in [Sto40], where he introduced such spaces, relating them to their algebras of continuous functions, and also alluded to the possibility of dealing with unbounded continuous functions.

Definition 1.4.3 (Stonean space)**.** A compact Hausdorff space $\mathcal{X}$ is said to be a *Stonean space* if the closure of each open set in $\mathcal{X}$ is open in $\mathcal{X}$; such spaces are also known as *extremally disconnected spaces* in the literature.

In [Kad86], Kadison described the structure of unbounded continuous functions on Stonean spaces, to fully understand the algebra of unbounded normal operators containing an abelian von Neumann algebra and a relation between the two.

Definition 1.4.4 (see [Kad86, Definition 1.1])**.** Let $\mathcal{X}$ be a Stonean space. A (finite) complex-valued function f defined and continuous on $\mathcal{X} \setminus \mathcal{Z}$, where $\mathcal{Z}$ is a closed nowhere-dense subset of $\mathcal{X}$, is said to be a *normal function* (on $\mathcal{X}$) when, given a point $z \in \mathcal{Z}$ and a positive α, there is an open set O in $\mathcal{X}$ containing z such that $|f(y)| \geq \alpha$ for each $y \in O \setminus \mathcal{Z}$. If f is real-valued, we say that f is a *self-adjoint function* (on $\mathcal{X}$).
We denote by $\mathcal{N}(\mathcal{X})$ and $\mathcal{S}(\mathcal{X})$, respectively, the family of normal functions on $\mathcal{X}$ and the family of self-adjoint functions on $\mathcal{X}$.

Lemma 1.4.5. *Let $\mathcal{X}$ be a Stonean space, and $O_1 \supseteq O_2$ be open subsets of $\mathcal{X}$. Then*

$$\overline{O_1} \backslash \overline{O_2} = \overline{O_1 \backslash \overline{O_2}}.$$

Proof. Since $\mathcal{X} \backslash \overline{O_2}$ is clopen, from Lemma 1.4.1-(ii), we observe that,

$$\overline{O_1} \backslash \overline{O_2} = \overline{O_1} \cap (\mathcal{X} \backslash \overline{O_2}) = \overline{O_1 \cap (\mathcal{X} \backslash \overline{O_2})} = \overline{O_1 \backslash \overline{O_2}}.$$ □

Theorem 1.4.6 (Extension theorem)**.** *Let $\mathcal{X}$ be a Stonean space, O be an open subset of $\mathcal{X}$, and Z be a compact Hausdorff space. Then every continuous map $f : O \to Z$ has a unique continuous extension $\widetilde{f} : \overline{O} \to Z$. (Note that, by the universal property of the Stone-Čech compactification, the closure $\overline{O}$ must then be homeomorphic to βO.)*

Proof. This may be found in standard textbooks such as [Wal74, Exercise 2J.4], [GJ60, Problem 1H.6]. □

It is well known that every infinite compact F-space (see the definition in [GJ60, §14.24]) contains a copy of $\beta\mathbb{N}$ (see [GJ60, Problem 14M.5]). Using Theorem 1.4.6, below we prove this result in the special case of infinite Stonean spaces for the convenience of the reader.

Proposition 1.4.7. *A Stonean space $\mathcal{X}$ with infinitely many points contains a copy of $\beta\mathbb{N}$, that is, there is a continuous injection $\beta\mathbb{N} \hookrightarrow \mathcal{X}$. (Since $\beta\mathbb{N}$ is compact, it is homeomorphic to its image in $\mathcal{X}$.)*

Proof. As $\mathcal{X}$ is Hausdorff, there is a non-empty open subset of $\mathcal{X}$ which is not dense in $\mathcal{X}$, and considering its closure we have a proper non-empty clopen subset Q of $\mathcal{X}$. Since $\mathcal{X}$ is infinite, either Q or $\mathcal{X}\backslash Q$ must be infinite. Thus, there is an infinite clopen subset $Q_1 \subsetneq \mathcal{X}$. Note that Q_1 itself is an infinite Stonean space. Inductively, we get a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of infinite subsets of $\mathcal{X}$, such that

$$\mathcal{X} =: Q_0 \supsetneq Q_1 \supsetneq \cdots \supsetneq Q_n \supsetneq \cdots,$$

and each Q_n is a clopen subset of Q_{n-1}, and thus a clopen subset of $\mathcal{X}$.

For $n \in \mathbb{N}$, we define $O_n := Q_{n-1}\backslash Q_n$. Clearly O_n's are mutually disjoint non-empty (cl)open subsets of $\mathcal{X}$. For every $n \in \mathbb{N}$, we choose a point $x_n \in O_n$. It is clear that the set $\mathbb{A} := \{x_n : n \in \mathbb{N}\}$ is an infinite discrete subset of $\mathcal{X}$, so that the mapping from $\mathbb{N}$ to $\mathbb{A}$ given by $n \mapsto x_n$ is a homeomorphism.

Let Z be a compact Hausdorff space, and $f : \mathbb{A} \to Z$ be a function; note that f is automatically continuous as $\mathbb{A}$ is a discrete subset of $\mathcal{X}$. Let $F : \bigcup_{n\in\mathbb{N}} O_n \to Z$ be the function defined as $F(x) = f(x)$ if $x \in O_n$. Since O_n's are pairwise disjoint open sets, by Theorem 1.4.2, F is a well-defined continuous mapping on the open set $\mathcal{O} := \bigcup_{n\in\mathbb{N}} O_n$. By Theorem 1.4.6, F extends uniquely to a continuous map $\widetilde{F}$ on $\overline{\mathcal{O}}$. As $\mathbb{A}$ is dense in $\overline{\mathbb{A}}$, there is at most one continuous extension of f on $\overline{\mathbb{A}}$, and thus the restriction of $\widetilde{F}$ to $\overline{\mathbb{A}}$ is the unique continuous extension of f to $\overline{\mathbb{A}}$.

By the universal property of Stone–Čech compactification (see [GJ60, Theorem 6.5]), $\overline{\mathbb{A}} \subseteq \mathcal{X}$ is homeomorphic to $\beta\mathbb{A}$. Since $\mathbb{N}$ and $\mathbb{A}$ are homeomorphic, so are $\beta\mathbb{N}$ and $\beta\mathbb{A}$. Hence, $\beta\mathbb{N}$ is homeomorphic to $\overline{\mathbb{A}} \subseteq \mathcal{X}$. □

1.4.2 Partially ordered sets and Scott topology

The discussion in this subsection will be useful in Lemma 5.2.11, which facilitates the 'gluing' of combinatorial matrix properties, for continuously varying matrices parametrized by a Stonean space $\mathcal{X}$.

Definition 1.4.8 (Partially ordered set)**.** A set $\mathfrak{P}$ with a binary relation, $\leq$, that is reflexive, antisymmetric, and transitive, (called a partial order) is said to be a *poset* (short for partially ordered set). For $x, y \in \mathfrak{P}$, we write, $x < y$, when $x \neq y$ and $x \leq y$. Below we note the example of a finite poset which is the most relevant to us.

***Example* 1.4.9.** Let $\mathscr{P}_n$ denote the set of all partitions of $[n]$. For $\pi_1, \pi_2 \in \mathscr{P}_n$ we write $\pi_1 \leq \pi_2$ if π_2 is a refinement of π_1, that is, each element of π_2 is a subset of some element of π_1. It may be easily verified that $\leq$ defines a partial order on $\mathscr{P}_n$. Note that $\mathscr{P}_n$ has a unique maximal element, the partition $\big\{\{1\}, \{2\}, \ldots, \{n\}\big\}$, and a unique minimal element, the partition $\{[n]\}$.

Definition 1.4.10 (upper set)**.** Let $(\mathfrak{P}, \leq)$ be a partially ordered set. A subset $U \subseteq \mathfrak{P}$ is said to be an upper set if $y \in U$ whenever $x \leq y$ for some $x \in U$. The upper set,

$$x^{\uparrow} := \{y \in \mathfrak{P} : x \leq y\},$$

is called the *principal upper set* generated by $x \in \mathfrak{P}$.

Definition 1.4.11 (The Scott topology)**.** Let $(\mathfrak{P}, \leq)$ be a partially ordered set. A subset U of $\mathfrak{P}$ is said to be *Scott-open* if it is an upper set which is inaccessible by directed joins, that is, any directed set with supremum in U has a non-empty intersection with U. If $\mathfrak{P}$ is a finite set, then each upper set is Scott open, and the set of upper sets forms a base for the Scott topology.

Definition 1.4.12. Let $\vec{v} = (v_1, \ldots, v_n)$ be a vector in $\mathbb{C}^n$, and let $\sim_{\vec{v}}$ be a binary relation on $[n]$ defined by $i \sim_{\vec{v}} j$ if and only if $v_i = v_j$. It is straightforward to see that $\sim_{\vec{v}}$ is an equivalence relation on $[n]$, so that the set of equivalence classes for $\sim_{\vec{v}}$ forms a partition of $[n]$. We denote this partition of $[n]$ by $\mathcal{P}(\vec{v})$. Note that the partition $\mathcal{P}(\vec{v})$ of $[n]$ groups together the coordinate indices of $\vec{v}$ with the same coordinate value.

Lemma 1.4.13. *Let $(\mathscr{P}_n, \leq)$ denote the poset of all partitions of $[n]$, partially ordered via refinement and equipped with the Scott topology. Then the mapping from $\mathbb{C}^n$ to $\mathscr{P}_n$ given by $\vec{v} \mapsto \mathcal{P}(\vec{v})$, is continuous.*

Proof. For a vector $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{C}^n$, define $\varepsilon(\vec{v}) := \min_{i \nsim_{\vec{v}} j}\{|v_i - v_j|\}$. If $\vec{v}$ is a constant vector we stipulate $\varepsilon(\vec{v}) := +\infty$. Note that $|v_i - v_j| < \varepsilon(\vec{v})$ implies that $v_i = v_j$. Let $\vec{w}$ be a vector in the open ball of radius $\frac{\varepsilon(\vec{v})}{2}$ centred at $\vec{v}$ with respect to the sup-norm, that is, $|v_k - w_k| < \frac{\varepsilon(\vec{v})}{2}$ for every $k \in \mathbb{N}$. Thus, for every pair of indices $i, j \in [n]$ with $w_i = w_j$, we have

$$|v_i - v_j| \leq |v_i - w_i| + |w_i - w_j| + |v_j - w_j| < \varepsilon(\vec{v}),$$

whence $v_i = v_j$. This implies that $\mathcal{P}(\vec{w})$ is a refinement of $\mathcal{P}(\vec{v})$. Equivalently, $\mathcal{P}(\vec{w}) \in \mathcal{P}(\vec{v})^{\uparrow}$.

Let $\vec{u}, \vec{v} \in \mathbb{C}^n$ be vectors such that $\mathcal{P}(\vec{u}) \in \mathcal{P}(\vec{v})^{\uparrow}$. From the discussion in the above paragraph, for every vector $\vec{w}$ in the open $\frac{\varepsilon(\vec{u})}{2}$-ball centred at $\vec{u}$, we note that $\mathcal{P}(\vec{w}) \in \mathcal{P}(\vec{u})^{\uparrow} \subseteq \mathcal{P}(\vec{v})^{\uparrow}$, that is, $\mathcal{P}(\vec{w}) \in \mathcal{P}(\vec{v})^{\uparrow}$. In summary, the inverse image of the principal upper set $\mathcal{P}(\vec{v})^{\uparrow}$ is open in $\mathbb{C}^n$. Since the principal upper sets form a base for the Scott topology on $\mathscr{P}_n$, continuity of the mapping, $\vec{v} \mapsto \mathcal{P}(\vec{v})$, follows. $\square$

1.5 Type I_n AW^*-algebras

A von Neumann algebra or W^*-algebra is a unital $*$-algebra of bounded linear operators on a Hilbert space that is closed in the weak operator topology (or equivalently in the strong operator topology).

An AW^*-algebra is an algebraic generalization of the notion of a von Neumann algebra, introduced by Kaplansky (see [Kap51]).

Definition 1.5.1. A C^*-algebra $\mathfrak{A}$ is said to be an AW^*-algebra if the set of orthogonal projections in $\mathfrak{A}$ forms a complete lattice, and each maximal commutative C^*-subalgebra is monotone-complete (that is, every increasing net of self-adjoint elements which is bounded above, has a least upper bound).

Commutative AW^*-algebras correspond to complete Boolean algebras, and are of the form $C(\mathcal{X})$ where $\mathcal{X}$ is a Stonean space (see [Kap51, §2]).

Many structural results for von Neumann algebras extend naturally to AW^*-algebras. For instance, AW^*-algebras can be classified based on the behaviour of their projections and admit a type decomposition analogous to that of von Neumann algebras [Ber72]. While not all aspects of the coarse structure theory carry over, the case of finite algebras of type I is particularly well-behaved and carries over satisfactorily.

For our purposes, we essentially require the structure theory of type I_n AW^*-algebras, which we summarize in the following remarks.

Theorem 1.5.2 (see [KR97, Theorem 6.6.5] and [KR83, Theorem 5.2.1])**.** *Every type I_n von Neumann algebra is of the form $M_n(\mathscr{A})$ for an abelian von Neumann algebra $\mathscr{A}$ $*$-isomorphic to the center of the von Neumann algebra. Since the maximal ideal space of an abelian von Neumann algebra is extremally disconnected, $\mathscr{A}$ is of the form $C(\mathcal{X})$ for some (hyper)-Stonean space $\mathcal{X}$. Thus, every type I_n von Neumann algebra is of the form $M_n\big(C(\mathcal{X})\big)$ for some (hyper)-Stonean space $\mathcal{X}$.*

***Remark* 1.5.3.** Every type I_n AW^*-algebra is of the form $M_n(\mathfrak{A})$ for an abelian AW^*-algebra $\mathfrak{A}$, and thus is of the form $M_n\big(C(\mathcal{X})\big)$, for a Stonean space $\mathcal{X}$.

***Remark* 1.5.4.** Since a matrix in $M_n\big(C(\mathcal{X})\big)$ may be viewed as a continuous $M_n(\mathbb{C})$-valued function on $\mathcal{X}$, there is a natural $*$-isomorphism between $M_n\big(C(\mathcal{X})\big)$ and $C\big(\mathcal{X}; M_n(\mathbb{C})\big)$ allowing us to view them interchangeably.

1.6 Murray von Neumann algebras

For the basic concepts from the theory of unbounded operators, we refer to §1.3.3, and further to [KR83, §2.7].

Definition 1.6.1 ([KR83, Definition 5.6.2])**.** Let $\mathscr{R}$ be a von Neumann algebra acting on the Hilbert space $\mathscr{H}$. A closed densely-defined linear operator A with $\mathrm{dom}(A)$, $\mathrm{ran}(A) \subseteq \mathscr{H}$, is said to be *affiliated* with $\mathscr{R}$, if $V^*AV = A$ for each unitary operator V in the commutant of $\mathscr{R}$. We denote the set of all such affiliated operators by $\mathrm{Aff}(\mathscr{R})$.

Definition 1.6.2 (see [KL14, Definition 6.14])**.** Let $\mathscr{N}$ be a finite von Neumann algebra acting on the Hilbert space $\mathscr{H}$. For $A, B \in \mathrm{Aff}(\mathscr{N})$, the operators $A+B, AB$, are pre-closed and densely-defined (see [KL14, Proposition 6.8]). The two binary operations $\hat{+}, \hat{\cdot}$, defined by $A \mathbin{\hat{+}} B = \overline{A+B}$, $A \mathbin{\hat{\cdot}} B = \overline{AB}$, along with the operator adjoint, $(\cdot)^*$, as involution, endow $\mathrm{Aff}(\mathscr{N})$ with the structure of a $*$-algebra (see [KL14, Theorem 6.13]). With this $*$-algebraic structure, $\mathrm{Aff}(\mathscr{N})$ is called the *Murray-von Neumann algebra* associated with $\mathscr{N}$.

If $\mathscr{N}$ is of type I_n $(n \in \mathbb{N})$, then we say that $\mathrm{Aff}(\mathscr{N})$ is a type I_n Murray-von Neumann algebra. Similarly, if $\mathscr{N}$ is of type II_1, then we say that $\mathrm{Aff}(\mathscr{N})$ is a type II_1 Murray-von Neumann algebra.

For example, let $\mathcal{Y}$ be a locally compact Hausdorff space and μ be a Radon measure on $\mathcal{Y}$. Then, for the abelian von Neumann algebra $L^\infty(\mathcal{Y};\mu)$, $\mathrm{Aff}\big(L^\infty(\mathcal{Y};\mu)\big)$ is $*$-isomorphic to $L^0(\mathcal{Y};\mu)$, the $*$-algebra of μ-measurable functions on $\mathcal{Y}$ (see [Nay21, Remark 5.2]). In particular, equipping $\mathbb{N}$ with the counting measure, we have $\mathrm{Aff}\big(\ell^\infty(\mathbb{N})\big) = \mathbb{C}^{\mathbb{N}}$, the $*$-algebra of complex-valued functions on $\mathbb{N}$.

For a finite von Neumann algebra $\mathscr{N}$, the $*$-algebra $\mathrm{Aff}(\mathscr{N})$ may be defined intrinsically, independent of the representation of $\mathscr{N}$ on a Hilbert space (see [Nel74, Theorem 4] and [Nay21, Theorem 4.3]). This is achieved by realizing $\mathrm{Aff}(\mathscr{N})$ as the completion of $\mathscr{N}$ in the $\mathfrak{m}$-topology (defined below). For topological and order-theoretic aspects of Murray-von Neumann algebras, we refer to [Nay21].

Definition 1.6.3 (see [Nay21, Definition 3.3])**.** Let $\mathscr{N}$ be a finite von Neumann algebra. For $\varepsilon, \delta > 0$ and a normal tracial state τ on $\mathscr{N}$, we define,

$$O(\tau,\varepsilon,\delta) := \{A \in \mathscr{N} : \text{ there is a projection } E \in \mathscr{N} \text{ with } \tau(I_{\mathscr{N}} - E) \le \delta, \text{ and } \|AE\| \le \varepsilon\}.$$

The translation-invariant topology generated by the fundamental system of neighbourhoods of $0_{\mathscr{N}}$, $\{O(\tau,\varepsilon,\delta)\}$, is called the $\mathfrak{m}$-topology of $\mathscr{N}$. By [Nay21, Theorem 3.13], this defines a Hausdorff topology.

The $\mathfrak{m}$-topology defines closeness between operators based on their uniform closeness on subspaces determined by "large" projections, effectively capturing the notion of local convergence in measure. In our discussion, terms such as '$\mathfrak{m}$-convergence', '$\mathfrak{m}$-limit', etc., are to be understood with their obvious meanings derived from the $\mathfrak{m}$-topology.

Theorem 1.6.4 (see [Nay21, Theorem 3.12])**.** *Let $\mathscr{N}$ be a finite von Neumann algebra. Then the maps,*

$$A \mapsto A^* : \mathscr{N} \to \mathscr{N},$$
$$(A, B) \mapsto A + B : \mathscr{N} \times \mathscr{N} \to \mathscr{N},$$
$$(A, B) \mapsto AB : \mathscr{N} \times \mathscr{N} \to \mathscr{N},$$

are Cauchy continuous with respect to the $\mathfrak{m}$-topology and therefore admit unique extensions to the $\mathfrak{m}$-completion of $\mathscr{N}$, endowing it with the structure of a $*$-algebra.

***Remark* 1.6.5.** Let $\mathscr{N}$ be a finite von Neumann algebra. From [Nay21, Theorem 4.3], it follows that the $\mathfrak{m}$-completion of $\mathscr{N}$ is naturally $*$-isomorphic to $\mathrm{Aff}(\mathscr{N})$.

The set of positive operators in $\mathrm{Aff}(\mathscr{N})$ defines a proper cone on the self-adjoint part of $\mathrm{Aff}(\mathscr{N})$, $\mathrm{Aff}(\mathscr{N})_{sa}$, which is used to endow it with an order structure. By [Nay21, Proposition 4.21], $\mathrm{Aff}(\mathscr{N})_{sa}$ is monotone-complete, that is, every bounded increasing net in $\mathrm{Aff}(\mathscr{N})_{sa}$ has a least upper bound in $\mathrm{Aff}(\mathscr{N})_{sa}$.

Definition 1.6.6 (cf. [Nay21, Theorem 4.9], [GN24, Definition 4.12])**.** Let $\mathscr{N}, \mathscr{N}'$, respectively, be finite von Neumann algebras acting on the Hilbert spaces $\mathscr{H}, \mathscr{H}'$, respectively, and $\Phi : \mathscr{N} \to \mathscr{N}'$ be a unital normal $*$-homomorphism. By [Nay21, Theorem 4.9], Φ is Cauchy continuous with respect to the $\mathfrak{m}$-topologies on $\mathscr{N}$ and $\mathscr{N}'$, and uniquely extends to an $\mathfrak{m}$-continuous unital $*$-homomorphism, $\Phi_{\mathrm{aff}} : \mathrm{Aff}(\mathscr{N}) \to \mathrm{Aff}(\mathscr{N}')$.

By [Nay21, Theorem 4.3], Φ_{aff} preserves $\hat{+}, \hat{\cdot}, *$, and by [Nay21, Theorem 4.24], Φ_{aff} is equivalently a unital normal $*$-homomorphism. In fact, following the steps in the proof of [GN24, Theorem 5.8], we note that Φ_{aff} is the unique extension of Φ that preserves $\hat{+}, \hat{\cdot}, *$.

We shall return to the study of affiliated operators and Murray–von Neumann algebras in §5.4, with particular focus on their behavior under direct sums, which will be taken up in §5.4.1.

Chapter 2

Jordan-Chevalley decomposition of matrices over arbitrary fields

2.1 Introduction

The beginnings of linear algebra may be traced to the quest for systematic methods for solving systems of linear equations. It is no exaggeration to claim that Gaussian elimination is the foundational algorithm on top of which most of linear algebra rests. Loosely speaking, Gaussian elimination involves the transformation of a system of linear equations into an equivalent one via successive elementary row operations on its augmented matrix leading to a modified augmented matrix, ultimately arriving at the so-called *row echelon form*. From there on, understanding the solution space becomes a much more manageable task, both theoretically and computationally.

Let $\mathbb{K}$ be a field, $\mathscr{V}$ be a finite-dimensional vector space over $\mathbb{K}$ with $\dim \mathscr{V} = n$, and let $T : \mathscr{V} \to \mathscr{V}$ be a $\mathbb{K}$-linear transformation. One is naturally motivated to identify a basis for $\mathscr{V}$ in which the matrix of T has a fairly simple form, making it easily interpretable and amenable to computations. Or equivalently, given a matrix $A \in M_n(\mathbb{K})$, one wishes to find a simple-looking matrix in $M_n(\mathbb{K})$ similar to A from which the fundamental properties of A may be gleaned. This line of investigation leads to the topic of *canonical forms*, the most popular of which are the *Jordan canonical form* and the *rational canonical form*.

If T has a basis of eigenvectors, then the matrix of T with respect to this basis has a particularly simple, diagonal form which gives us a quick qualitative idea about the behaviour of T; such a linear transformation is said to be *diagonalizable*. Unfortunately, not all linear transformations are diagonalizable. In this context, the Jordan-Chevalley decomposition is a cornerstone result which describes T as the sum of a potentially-diagonalizable transformation (see Definition 2.4.3) D and a nilpotent transformation N commuting with each other.

The primary goal of this chapter is to present elementary computational proofs of the existence and uniqueness of the Jordan-Chevalley decomposition and the Jordan canonical form for square matrices over $\mathbb{K}$. We provide a mathematically rigorous, simplified and general version of the sketch outlined by Howland and Vaillancourt in [HV92] for complex matrices. The main strategy involves transforming the matrix to an appropriate block-diagonal form via similarity transformations obtained from Roth's removal rule ([Rot52, Theorem II]) as the main technical tool. The connection with Gaussian elimination is more than an idle analogy as we may interpret Roth's removal rule as capturing in a condensed form, successive conjugation with matrices representing certain elementary row operations.

2.2 Sylvester's equation over arbitrary fields

In this section, we discuss Sylvester's equation $AX - XB = Y$ (cf. [Syl84], [Ros56]) in the context of matrices over arbitrary fields, which is useful in setting up Roth's removal rule (see Lemma 2.2.3). The complex analogue of the main theorem in this section, Theorem 2.2.2, may be found in [HJ13, Theorem 2.4.4.1]. Although the proof therein applies to the case of arbitrary fields without major changes, it uses the Cayley-Hamilton theorem in an essential way; since we intend to apply our tools to prove the Cayley-Hamilton theorem (see Theorem 2.4.2), we employ a different strategy for proving Theorem 2.2.2 in order to avoid circular reasoning.

Proposition 2.2.1. *Let $\mathbb{K}$ be an algebraically closed field, and $A \in M_m(\mathbb{K})$, $B \in M_n(\mathbb{K})$. Let $\mathcal{L}_A, \mathcal{R}_B : M_{m,n}(\mathbb{K}) \to M_{m,n}(\mathbb{K})$ denote the operators of left-multiplication by A, right-multiplication by B, respectively, that is, the linear mappings given by $X \mapsto AX$, $X \mapsto XB$, respectively. Then*

$$\mathrm{sp}(\mathcal{L}_A - \mathcal{R}_B) \subseteq \mathrm{sp}(A) - \mathrm{sp}(B).$$

Proof. Clearly, the operators $\mathcal{L}_A$ and $\mathcal{R}_B$ commute with each other for every choice of $A \in M_m(\mathbb{K})$ and $B \in M_n(\mathbb{K})$. From Lemma 1.1.6 we have $\mathrm{sp}(\mathcal{L}_A - \mathcal{R}_B) \subseteq \mathrm{sp}(\mathcal{L}_A) - \mathrm{sp}(\mathcal{R}_B)$. Note that, for $\lambda \in \mathbb{K}$, $A - \lambda I_m$ is invertible if and only if $\mathcal{L}_{A-\lambda I_m} = \mathcal{L}_A - \lambda \mathcal{I}$ is invertible, where $\mathcal{I}$ denotes the identity operator on $M_{m,n}(\mathbb{K})$. Thus, $\mathrm{sp}(\mathcal{L}_A) = \mathrm{sp}(A)$. Similarly, $\mathrm{sp}(\mathcal{R}_B) = \mathrm{sp}(B)$. Hence $\mathrm{sp}(\mathcal{L}_A - \mathcal{R}_B) \subseteq \mathrm{sp}(A) - \mathrm{sp}(B)$. ☐

Theorem 2.2.2 (cf. [HJ13, Theorem 2.4.4.1])**.** *Let $\mathbb{K}$ be a field. Let $A \in M_m(\mathbb{K})$ and $B \in M_n(\mathbb{K})$. If the characteristic polynomials of A and B are coprime, then the equation $AX - XB = Y$ has a unique solution $X \in M_{m,n}(\mathbb{K})$ for every $Y \in M_{m,n}(\mathbb{K})$.*

Proof. Let $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$. We may view A and B as matrices over the field extension $\overline{\mathbb{K}}$. Since the characteristic polynomials of A and B are coprime in $\mathbb{K}[x]$, they are coprime in $\overline{\mathbb{K}}[x]$. Thus $\mathrm{sp}(A) \cap \mathrm{sp}(B) = \varnothing$, so that $0 \notin \mathrm{sp}(A) - \mathrm{sp}(B)$. By Proposition 2.2.1, since $\mathrm{sp}(\mathcal{L}_A - \mathcal{R}_B) \subseteq \mathrm{sp}(A) - \mathrm{sp}(B)$, we observe that 0 does not belong to $\mathrm{sp}(\mathcal{L}_A - \mathcal{R}_B)$. Thus $\mathcal{L}_A - \mathcal{R}_B : M_{m,n}(\overline{\mathbb{K}}) \to M_{m,n}(\overline{\mathbb{K}})$ is invertible; Moreover, it leaves $M_{m,n}(\mathbb{K})$ invariant. Since $\mathcal{L}_A - \mathcal{R}_B : M_{m,n}(\mathbb{K}) \to M_{m,n}(\mathbb{K})$ is a $\mathbb{K}$-linear injective mapping, it must be surjective (by the rank-nullity theorem). Rephrasing the equation, $AX - XB = Y$, as $(\mathcal{L}_A - \mathcal{R}_B)X = Y$, we get the desired conclusion. □

Lemma 2.2.3 (cf. [Rot52, Theorem II]). *Let $\mathbb{K}$ be a field. Let $A \in M_m(\mathbb{K}), B \in M_n(\mathbb{K})$ and $C, C' \in M_{m,n}(\mathbb{K})$. If the matrix equation $AX - XB = C - C'$ has a solution in $M_{m,n}(\mathbb{K})$, then we have the matrix similarity,*

$$\begin{bmatrix} A & C \\ \mathbf{0}_{n,m} & B \end{bmatrix} \sim_{sim} \begin{bmatrix} A & C' \\ \mathbf{0}_{n,m} & B \end{bmatrix} \text{ in } M_{m+n}(\mathbb{K}).$$

Proof. Let $X \in M_{m,n}(\mathbb{K})$ be such that $AX - XB = C - C'$. Then the assertion follows from the following matrix computation,

$$\begin{bmatrix} I_m & X \\ \mathbf{0}_{n,m} & I_n \end{bmatrix} \begin{bmatrix} A & C \\ \mathbf{0}_{n,m} & B \end{bmatrix} \begin{bmatrix} I_m & -X \\ \mathbf{0}_{n,m} & I_n \end{bmatrix} = \begin{bmatrix} A & C + XB - AX \\ \mathbf{0}_{n,m} & B \end{bmatrix} = \begin{bmatrix} A & C' \\ \mathbf{0}_{n,m} & B \end{bmatrix}. \qquad \square$$

2.3 The block-diagonal form

In this section, we deduce Theorem 2.3.2 which is the main technical tool of the chapter, generalizing [HJ13, Theorem 2.4.6.1] from the field of complex numbers to arbitrary fields.

Lemma 2.3.1. *Let $\mathbb{K}$ be a field. Let $A_1, A_2, \ldots, A_m$ be matrices in $M_{n_1}(\mathbb{K})$, $M_{n_2}(\mathbb{K}), \ldots,$ $M_{n_m}(\mathbb{K})$, respectively, such that their characteristic polynomials are pairwise coprime, and let $n = n_1 + n_2 + \cdots + n_m$. Then for any choice of matrices $A_{i,j} \in M_{n_i,n_j}(\mathbb{K})$ $(i < j \in [m])$, the two matrices,*

$$T = \begin{bmatrix} A_1 & A_{1,2} & \cdots & A_{1,m} \\ \mathbf{0}_{n_2,n_1} & A_2 & \cdots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n_m,n_1} & \mathbf{0}_{n_m,n_2} & \cdots & A_m \end{bmatrix}, \quad D = \begin{bmatrix} A_1 & \mathbf{0}_{n_1,n_2} & \cdots & \mathbf{0}_{n_1,n_m} \\ \mathbf{0}_{n_2,n_1} & A_2 & \cdots & \mathbf{0}_{n_2,n_m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n_m,n_1} & \mathbf{0}_{n_m,n_2} & \cdots & A_m \end{bmatrix},$$

are similar in $M_n(\mathbb{K})$. (Note that T is block-triangular and D is block-diagonal and both have the same diagonal blocks.)

Proof. The proof is by induction on m. For $m = 1$, there is nothing to show. We begin with the non-trivial step corresponding to $m = 2$. Since the characteristic polynomials of A_1 and A_2 are coprime, from Theorem 2.2.2, there is a unique matrix $X \in M_{n_1,n_2}(\mathbb{K})$ such that $A_1 X - X A_2 = A_{1,2}$. From Lemma 2.2.3, we conclude that $T \sim_{\text{sim}} D$ in $M_n(\mathbb{K})$ with $n = n_1 + n_2$.

Next, we assume that the statement is true for $m-1$; $m \geq 2$ and define

$$T_{1,2} := \begin{bmatrix} A_{1,2} & A_{1,3} & \cdots & A_{1,m} \end{bmatrix},$$

$$T' := \begin{bmatrix} A_2 & A_{2,3} & \cdots & A_{2,m} \\ \mathbf{0}_{n_3,n_2} & A_3 & \cdots & A_{3,m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n_m,n_2} & \mathbf{0}_{n_m,n_3} & \cdots & A_m \end{bmatrix}, \; D' := \begin{bmatrix} A_2 & \mathbf{0}_{n_2,n_3} & \cdots & \mathbf{0}_{n_2,n_m} \\ \mathbf{0}_{n_3,n_2} & A_3 & \cdots & \mathbf{0}_{n_3,n_m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n_m,n_2} & \mathbf{0}_{n_m,n_3} & \cdots & A_m \end{bmatrix}.$$

Since the characteristic polynomial of T' is the product of the characteristic polynomials of $A_2, \ldots, A_m$, the characteristic polynomial of A_1 and the characteristic polynomial of T' are coprime. From the base case of $m = 2$, we have that,

$$T = \begin{bmatrix} A_1 & T_{1,2} \\ \mathbf{0}_{n-n_1,n_1} & T' \end{bmatrix} \sim_{\text{sim}} \begin{bmatrix} A_1 & \mathbf{0}_{n_1,n-n_1} \\ \mathbf{0}_{n-n_1,n_1} & T' \end{bmatrix} \text{ in } M_n(\mathbb{K}). \tag{2.3.1}$$

By the induction hypothesis, $T' \sim_{\text{sim}} D'$ in $M_{n-n_1}(\mathbb{K})$ so that

$$\begin{bmatrix} A_1 & \mathbf{0}_{n_1,n-n_1} \\ \mathbf{0}_{n-n_1,n_1} & T' \end{bmatrix} \sim_{\text{sim}} D = \begin{bmatrix} A_1 & \mathbf{0}_{n_1,n-n_1} \\ \mathbf{0}_{n-n_1,n_1} & D' \end{bmatrix} \text{ in } M_n(\mathbb{K}). \tag{2.3.2}$$

Note that the similarity in (2.3.2) is implemented by the matrix $I_{n_1} \oplus S \in GL_n(\mathbb{K})$, where $S \in GL_{n-n_1}(\mathbb{K})$ implements the similarity $T' \sim_{\text{sim}} D'$. From the matrix similarities in (2.3.1) and (2.3.2), we conclude that $T \sim_{\text{sim}} D$ in $M_n(\mathbb{K})$. □

Theorem 2.3.2 (cf. [HJ13, Theorem 2.4.6.1]). *Let $\mathbb{K}$ be a field, and let $A \in M_n(\mathbb{K})$ be such that the characteristic polynomial of A splits over $\mathbb{K}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{K}$ be the distinct roots of $\chi_A(x)$ with algebraic multiplicities $n_1, \ldots, n_m$, respectively, so that $n = n_1 + \cdots + n_m$. Then A is similar to a block-diagonal matrix $\bigoplus_{i=1}^m A_i$, where A_i is an upper-triangular matrix in $M_{n_i}(\mathbb{K})$ all of whose diagonal entries are equal to λ_i.*

Proof. From Remark 1.1.5, there is an upper-triangular matrix T such that $A \sim_{\text{sim}} T$ in $M_n(\mathbb{K})$. Without loss of generality, we may assume that T is such that the eigenvalues of A appear on the diagonal of T according to the enumeration $\lambda_1, \lambda_2, \ldots, \lambda_m$, so that all equal eigenvalues are grouped together along the diagonal of T. Thus, T may be viewed as a block-triangular matrix of the form,

$$T = \begin{bmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,m} \\ \mathbf{0}_{n_2,n_1} & T_{2,2} & \cdots & T_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{n_m,n_1} & \mathbf{0}_{n_m,n_2} & \cdots & T_{m,m} \end{bmatrix} \; ; \; T_{i,j} \in M_{n_i,n_j}(\mathbb{K}) \; , \; i \leq j \in [m]$$

such that each diagonal block $T_{i,i}$ is an upper-triangular matrix in $M_{n_i}(\mathbb{K})$, all of whose diagonal entries are equal to λ_i.

Note that the characteristic polynomial of $T_{i,i}$ is $(x-\lambda_i)^{n_i}$. Since the λ_i's are distinct, the characteristic polynomials of the $T_{i,i}$'s ($i \in [m]$) are pairwise coprime. By Lemma 2.3.1, T, and thus A, is similar to the block-diagonal matrix $\bigoplus_{i=1}^m T_{i,i}$. Setting $A_i := T_{i,i}$ completes the proof. □

2.4 The Jordan-Chevalley decomposition

In this section, we illustrate how the concrete block-diagonal form for a matrix given in Theorem 2.3.2 may be fruitfully used to infer some fundamental results about linear transformations. We begin with a short proof of the Cayley-Hamilton theorem. As the main application, we show the existence and uniqueness of the Jordan-Chevalley decomposition of a square matrix over a field $\mathbb{K}$. Note that the matrices in the decomposition may be over a larger field in general; the details are provided in Corollary 2.4.8.

Lemma 2.4.1. *Let $\mathbb{K}$ be a field and $A \in M_n(\mathbb{K})$ be such that the characteristic polynomial of A splits over $\mathbb{K}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{K}$ be the distinct roots of $\chi_A(x)$ with algebraic multiplicities $n_1, \ldots, n_m$, respectively, so that $n = n_1 + \cdots + n_m$. Then we have the following:*

(i) $\mathbb{K}^n = \bigoplus_{i=1}^m \ker\big((A - \lambda_i I)^{n_i}\big)$,

(ii) $\ker\big((A - \lambda_i I)^{n_i}\big) = \ker\big((A - \lambda_i I)^{n}\big)$ *for* $i \in [m]$.

Proof. Note that for every polynomial $p(x) \in \mathbb{K}[x]$ and invertible matrix $S \in GL_n(\mathbb{K})$, we have

$$\ker\big(p(S^{-1}AS)\big) = \ker\big(S^{-1}p(A)S\big) = S^{-1}\ker\big(p(A)\big).$$

Thus, it follows that the assertions in (i) and (ii) hold for A if and only if they hold for some matrix similar to A. Using Theorem 2.3.2, without loss of generality we may assume that A is in block-diagonal form, $\bigoplus_{i=1}^m A_i$, where A_i is an upper-triangular matrix in $M_{n_i}(\mathbb{K})$ with all of its diagonal entries equal to λ_i. Viewing $\mathbb{K}^n$ as the direct sum $\mathbb{K}^{n_1} \oplus \mathbb{K}^{n_2} \oplus \cdots \bigoplus \mathbb{K}^{n_m}$ via concatenation of coordinates, the block-diagonal form makes it straightforward to verify that

$$\ker\left((A-\lambda_i I)^{n_i}\right) = \{0\}_{\mathbb{K}^{n_1}} \bigoplus \cdots \bigoplus \mathbb{K}^{n_i} \bigoplus \cdots \bigoplus \{0\}_{\mathbb{K}^{n_m}} = \ker\left((A-\lambda_i I)^{n_i+\ell}\right) \;\; \forall \ell \in \mathbb{N},$$

from which (i) and (ii) follow immediately. □

We now present a brief, self-contained proof of the classical Cayley–Hamilton theorem, which states that every square matrix satisfies its own characteristic polynomial.

Theorem 2.4.2 (Cayley-Hamilton theorem)**.** *Let $\mathbb{K}$ be a field, A be a matrix in $M_n(\mathbb{K})$, and $\chi_A(x) \in \mathbb{K}[x]$ be the characteristic polynomial of A. Then $\chi_A(A) = 0$ in $M_n(\mathbb{K})$.*

Proof. Let $\mathbb{L}$ be the splitting field of $\chi_A(x)$ and $\lambda_1, \ldots, \lambda_m$ be distinct roots of χ_A in $\mathbb{L}$ with multiplicities $n_1, \ldots, n_m$, respectively, so that $n = n_1 + \cdots + n_m$. Viewing A as a matrix in $M_n(\mathbb{L})$, $\chi_A(x) = \prod_{i=1}^m (x-\lambda_i)^{n_i} \in \mathbb{L}[x]$. Note that $\ker\left(\chi_A(A)\right) = \ker\left(\prod_{i=1}^m (A-\lambda_i I)^{n_i}\right)$ is a subspace of $\mathbb{L}^n$ which contains $\ker\left((A-\lambda_i I)^{n_i}\right)$ $\forall\, i \in [m]$. From Lemma 2.4.1-(i), we conclude that $\ker\left(\chi_A(A)\right) = \mathbb{L}^n$ which is equivalent to saying that $\chi_A(A) = 0$ in $M_n(\mathbb{L})$. Since $\chi_A(x)$ is a polynomial in $\mathbb{K}[x]$, the equality $\chi_A(A) = 0$ holds in $M_n(\mathbb{K})$. □

Definition 2.4.3. A matrix A in $M_n(\mathbb{K})$ is said to be *diagonalizable* if there is a diagonal matrix $D \in M_n(\mathbb{K})$ such that $A \sim_{\text{sim}} D$ in $M_n(\mathbb{K})$. It is said to be *potentially-diagonalizable* if there is a field extension $\mathbb{L}/\mathbb{K}$ such that A is diagonalizable when viewed as a matrix in $M_n(\mathbb{L})$.

Definition 2.4.4. Let $\mathbb{K}$ be a field and $A \in M_n(\mathbb{K})$. Then a *Jordan-Chevalley decomposition* of A in $M_n(\mathbb{K})$ is an expression of the form, $A = D + N$, involving a commuting pair of matrices $D, N \in M_n(\mathbb{K})$ such that D is potentially-diagonalizable and N is nilpotent.

Theorem 2.4.5. *Let $\mathbb{K}$ be a field and A be a matrix in $M_n(\mathbb{K})$ such that the characteristic polynomial of A splits over $\mathbb{K}$. Then A has a unique Jordan-Chevalley decomposition in $M_n(\mathbb{K})$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the distinct eigenvalues of A with algebraic multiplicities $n_1, n_2, \ldots, n_m$, respectively, so that $n = n_1 + n_2 + \cdots + n_m$. From Theorem 2.3.2, there exists an invertible matrix $S \in GL_n(\mathbb{K})$ such that $S^{-1}AS = \bigoplus_{i=1}^m A_i$ where A_i is an upper-triangular matrix in $M_{n_i}(\mathbb{K})$ all of whose diagonal entries are equal to λ_i. In other words, each A_i is of the form $A_i = \lambda_i I_{n_i} + N_i$ where $N_i \in M_{n_i}(\mathbb{K})$ is strictly upper-triangular. Let $D' := \bigoplus_{i=1}^m \lambda_i I_{n_i}$ and $N' := \bigoplus_{i=1}^m N_i$. Then D' is a diagonal matrix, and N' is strictly upper-triangular, hence nilpotent, in $M_n(\mathbb{K})$ such that

$$\bigoplus_{i=1}^m A_i = D' + N' \text{ and } D'N' = N'D'.$$

Setting $D := SD'S^{-1}$, $N := SN'S^{-1}$, we have,

$$A = D + N \text{ and } DN = ND.$$

Clearly D, N are diagonalizable, nilpotent, respectively, in $M_n(\mathbb{K})$. Thus, $D+N$ is a Jordan-Chevalley decomposition of A.

Uniqueness. Let $A = D_0 + N_0$ be a Jordan-Chevalley decomposition where D_0 is potentially-diagonalizable, and N_0 is nilpotent. Let $\mathbb{L}$ be a field extension of $\mathbb{K}$ such that D_0 is diagonalizable when viewed as a matrix in $M_n(\mathbb{L})$. Let $\overline{\mathbb{L}}$ denote the algebraic closure of $\mathbb{L}$. For the rest of the proof, we view A, D, N, S, D_0, N_0 as matrices in $M_n(\overline{\mathbb{L}})$. Since D is diagonalizable in the context of $M_n(\mathbb{K})$, *a fortiori* it is diagonalizable in the context of $M_n(\overline{\mathbb{L}})$, and the same holds for D_0. Thus $A = D + N$ and $A = D_0 + N_0$ are both Jordan-Chevalley decompositions of A in $M_n(\overline{\mathbb{L}})$.

Since $D_0 N_0 = N_0 D_0$ and the spectrum of a nilpotent matrix is $\{0\}$, from Lemma 1.1.6 we have

$$\mathrm{sp}(A) = \mathrm{sp}(D_0 + N_0) \subseteq \mathrm{sp}(D_0) + \mathrm{sp}(N_0) = \mathrm{sp}(D_0).$$

Conversely, since A and N_0 commute, we have

$$\mathrm{sp}(D_0) = \mathrm{sp}\left(A + (-N_0)\right) \subseteq \mathrm{sp}(A) + \mathrm{sp}(-N_0) = \mathrm{sp}(A).$$

Thus $\mathrm{sp}(D_0) = \mathrm{sp}(A)$, that is, D_0 and A have identical sets of eigenvalues.

The block-diagonal form of A, $S^{-1}AS = \bigoplus_{i=1}^m A_i$, and the corresponding diagonal form of D, $S^{-1}DS = \bigoplus_{i=1}^m \lambda_i I_{n_i}$, makes it clear that the λ_i-eigenspace of D in $\overline{\mathbb{L}}^n$ is equal to $\ker\left((A - \lambda_i I)^{n_i}\right)$.

Let $\mathscr{V}_i$ be the λ_i-eigenspace of D_0 in $\overline{\mathbb{L}}^n$ so that $D_0|_{\mathscr{V}_i} = \lambda_i I|_{\mathscr{V}_i}$. Thus $(A - D_0)|_{\mathscr{V}_i} = (A - \lambda_i I)|_{\mathscr{V}_i}$. Since N_0 is a nilpotent matrix in $M_n(\overline{\mathbb{L}})$, from Lemma 2.4.1-(i) we have $N_0^n = 0$ which implies that $(A - \lambda_i I)^n|_{\mathscr{V}_i} = (A - D_0)^n|_{\mathscr{V}_i} = \mathbf{0}$. Thus $\mathscr{V}_i$ is contained in $\ker\left((A - \lambda_i I)^n\right) = \ker\left((A - \lambda_i I)^{n_i}\right)$ (see Lemma 2.4.1-(ii)) for $i \in [m]$.

From the diagonalizability of D and Lemma 2.4.1, we have

$$\overline{\mathbb{L}}^n = \bigoplus_{i=1}^m V_i \subseteq \bigoplus_{i=1}^m \ker\left((A - \lambda_i I)^{n_i}\right) = \overline{\mathbb{L}}^n.$$

Thus the λ_i-eigenspace of D_0, $\mathscr{V}_i$, is in fact equal to $\ker\left((A - \lambda_i I)^{n_i}\right)$. Since both D_0 and D are diagonalizable over $M_n(\overline{\mathbb{L}})$ with identical spectrum and λ_i-eigenspaces for $i \in [m]$, we must have $D_0 = D$ in $M_n(\overline{\mathbb{L}})$ and $N_0 = A - D_0 = A - D = N$. This establishes the uniqueness of the Jordan-Chevalley decomposition of A. □

***Remark* 2.4.6.** Note that the above proof shows that a matrix $A \in M_n(\mathbb{K})$ admits a unique Jordan-Chevalley decomposition in $M_n(\mathbb{L})$, where $\mathbb{L}$ is the splitting field of the characteristic polynomial of A. In particular, every potentially-diagonalizable matrix in $M_n(\mathbb{K})$ is diagonalizable in $M_n(\overline{\mathbb{K}})$.

***Remark* 2.4.7.** For the discussion in this remark, we retain the notation used in the proof of Theorem 2.4.5. Since $(A_i - \lambda_i I_{n_i})^{n_i} = 0$, it is easily verified that for any polynomial $p(x) \in \mathbb{K}[x]$ with $p(x) \equiv \lambda_i \pmod{(x-\lambda_i)^{n_i}}$, we have $p(A_i) = \lambda_i I_{n_i}$. Thus in order to find a polynomial $p(x)$ such that $p(\oplus_{i=1}^m A_i) = \oplus_{i=1}^m \lambda_i I_{n_i}$, it suffices to find a solution to the system of congruence equations,

$$p(x) \equiv \lambda_i \pmod{(x-\lambda_i)^{n_i}}, \quad (i \in [m]).$$

The ring-theoretic version of the Chinese remainder theorem (see [Art91, pg. 383]) assures us of the existence of such a polynomial $p(x)$. In that case, $p(A) = D$ and for the polynomial $q(x) = x - p(x)$ we have $N = q(A)$; the potentially-diagonalizable and nilpotent parts of A are polynomials in A.

The starting point of the standard proof(s) of Theorem 2.4.5 (see [Hum80, pg. 18]) is the above ring-theoretic setup. The advantage of our proof of Theorem 2.4.5 is that no knowledge of the Chinese remainder theorem is necessary; in fact, the above remark may serve as a motivation to pursue its study.

Corollary 2.4.8. *Let $\mathbb{K}$ be a field and $A \in M_n(\mathbb{K})$. Let $\mathbb{L}$ be the splitting field of the characteristic polynomial of A. Let $\mathbb{F}$ be the fixed field of* $\mathrm{Aut}(\mathbb{L}/\mathbb{K})$ *so that $\mathbb{K} \subseteq \mathbb{F} \subseteq \mathbb{L}$. Then the Jordan-Chevalley decomposition of A, viewed as a matrix in $M_n(\mathbb{L})$, in fact holds in $M_n(\mathbb{F})$.*

Proof. Since the characteristic polynomial of A splits over $\mathbb{L}$, by Theorem 2.4.5, A has a Jordan-Chevalley decomposition in $M_n(\mathbb{L})$, explicitly, $A = D + N$ for a diagonalizable matrix $D \in M_n(\mathbb{L})$ and a nilpotent matrix $N \in M_n(\mathbb{L})$ such that D and N commute.

Let $\phi \in \mathrm{Aut}(\mathbb{L}/\mathbb{K})$. Note that the mapping $\Lambda_\phi : M_n(\mathbb{L}) \to M_n(\mathbb{L})$ defined via entry-wise application of ϕ gives a ring homomorphism which leaves $M_n(\mathbb{K})$ invariant. It is straightforward to see that $\Lambda_\phi(D)$ is diagonalizable, $\Lambda_\phi(N)$ is nilpotent and $\Lambda_\phi(D)$ and $\Lambda_\phi(N)$ commute.

Since $A = \Lambda_\phi(A) = \Lambda_\phi(D) + \Lambda_\phi(N)$, by the uniqueness of Jordan-Chevalley decomposition, we conclude that $\Lambda_\phi(D) = D, \Lambda_\phi(N) = N$; In other words, every $\phi \in \mathrm{Aut}(\mathbb{L}/\mathbb{K})$ leaves the entries of D, N invariant. Thus all entries of D and N lie in $\mathbb{F}$ so that $D, N \in M_n(\mathbb{F})$. □

Theorem 2.4.9. *Let $\mathbb{K}$ be a perfect field and $A \in M_n(\mathbb{K})$. Then A has a unique Jordan-Chevalley decomposition in $M_n(\mathbb{K})$.*

Proof. Since $\mathbb{K}$ is a perfect field, the fixed field of $\mathrm{Aut}(\mathbb{L}/\mathbb{K})$ is $\mathbb{K}$ (Remark 1.2.9) and the assertion is an immediate consequence of Corollary 2.4.8. □

2.5 The Jordan canonical form

In this section, we prove the existence and uniqueness of the Jordan canonical form of square matrices over a field whose characteristic polynomial splits over the field. A standard proof of the existence of Jordan canonical form of A uses generalized eigenvectors, which naturally lead to the consideration of shift operators and corresponding Jordan matrices. Since we eschew that path in favour of a computational approach, how are we to stumble upon the central role played by Jordan matrices in the discussion of similarity classes of a square matrix? The answer lies in our proof of Proposition 2.5.3. Although it is similar in spirit to the proof in [HJ13, Theorem 3.1.5] for complex matrices, we believe that the individual steps in our proof are more natural and a little less mysterious.

Definition 2.5.1 (Jordan matrix)**.** For $\lambda \in \mathbb{K}$, the Jordan matrix $J_m(\lambda)$ is defined to be the $m \times m$ upper-triangular matrix which has all of its diagonal entries equal to λ, all of its super-diagonal entries equal to 1, and the rest of the entries equal to 0. For example,

$$J_1(\lambda) = \begin{bmatrix} \lambda \end{bmatrix}, J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, J_3(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Lemma 2.5.2. *Let $\mathbb{K}$ be a field. If $n_1 \geq n_2 \geq \cdots \geq n_m$ are positive integers with $n_1 \geq 2$, then for $\alpha_2, \alpha_3, \ldots, \alpha_m \in \mathbb{K}$, we have the matrix similarity,*

$$\begin{bmatrix} J_{n_1}(0) & \begin{pmatrix} \alpha_2 \mathbf{e}_{n_1,n_2} & \cdots & \alpha_m \mathbf{e}_{n_1,n_m} \end{pmatrix} \\ \mathbf{0}_{n_2+\cdots+n_m,n_1} & \bigoplus_{i=2}^m J_{n_i}(0) \end{bmatrix} \sim_{sim} \bigoplus_{i=1}^m J_{n_i}(0) \text{ in } M_{n_1+\cdots+n_m}(\mathbb{K}).$$

Proof. Let $\ell \geq n$ be positive integers with $\ell \geq 2$, and X_1 be the matrix in $M_{\ell,n}(\mathbb{K})$ whose $(i, i-1)$–entry is 1 for $2 \leq i \leq \ell$, and the rest of the entries are 0. We may verify via direct computation that the $X = X_1$ is a solution to the matrix equation,

$$J_\ell(0)X - XJ_n(0) = \mathbf{e}_{\ell,n}. \tag{2.5.1}$$

For $2 \leq i \leq m$, from the above discussion in the context of $\ell = n_1, n = n_i$, there exist matrices $X_i \in M_{n_1,n_i}$ satisfying the matrix equation(s),

$$J_{n_1}(0)X_i - X_iJ_{n_i}(0) = \mathbf{e}_{n_1,n_i}.$$

Since the matrix $\begin{bmatrix} \alpha_2 X_2 & \cdots & \alpha_m X_m \end{bmatrix} \in M_{n_1, n_2+\cdots+n_m}(\mathbb{K})$ is a solution to the matrix equation,

$$J_{n_1}(0)X - X\left(\bigoplus_{i=2}^{m} J_{n_i}(0)\right) = \begin{bmatrix} \alpha_2 \mathbf{e}_{n_1,n_2} & \cdots & \alpha_m \mathbf{e}_{n_1,n_m} \end{bmatrix},$$

the result follows from Lemma 2.2.3. □

The appearance of Jordan matrices in Lemma 2.5.2 may seem prescient, but the lemma is merely an organizational tool (as are most lemmas) that we isolate to facilitate the use of Roth's removal rule in Proposition 2.5.3 below.

Proposition 2.5.3 (cf. [HJ13, Theorem 3.1.5])**.** *Let $\mathbb{K}$ be a field and A be a strictly upper-triangular matrix in $M_n(\mathbb{K})$. Then there is a unique collection of positive integers $n_1 \geq n_2 \geq \ldots \geq n_m$ satisfying $n_1 + \cdots + n_m = n$ such that A is similar to $\bigoplus_{i=1}^{m} J_{n_i}(0)$ in $M_n(\mathbb{K})$.*

Proof. Existence. We proceed via induction on n. The case of $n = 1$ corresponds to $A = [0]$, proof of which is trivial. For the case of $n = 2$, note that every strictly upper-triangular matrix in $M_2(\mathbb{K})$ is of the form $\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ for some $\alpha \in \mathbb{K}$. If $\alpha = 0$, then it is already of the form $J_1(0) \oplus J_1(0)$. If $\alpha \neq 0$, then by conjugation with the invertible matrix $1 \oplus \alpha$ in $GL_2(\mathbb{K})$, we get $J_2(0)$.

Let us assume that the assertion is true for strictly upper-triangular matrices in $M_{n-1}(\mathbb{K})$. Let $A = \begin{bmatrix} 0 & A_{1,2} \\ \mathbf{0}_{n-1,1} & A_{2,2} \end{bmatrix}$ where $A_{1,2} = \begin{bmatrix} a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} \end{bmatrix}$ and

$$A_{2,2} = \begin{bmatrix} 0 & a_{2,3} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in M_{n-1}(\mathbb{K}).$$

By the induction hypothesis, $A_{2,2} \sim_{\text{sim}} \bigoplus_{i=1}^{m} J_{n_i}(0)$ in $M_{n-1}(\mathbb{K})$ for some $m \in \mathbb{N}$ and positive integers $n_1 \geq n_2 \geq \ldots \geq n_m$ with $n_1 + \cdots + n_m = n - 1$.

Let S be an invertible matrix in $GL_{n-1}(\mathbb{K})$ which implements the similarity between $A_{2,2}$ and $\bigoplus_{i=1}^{m} J_{n_i}(0)$. A direct computation tells us that the invertible matrix $1 \oplus S \in GL_n(\mathbb{K})$

implements a similarity of A with a matrix in $M_n(\mathbb{K})$ of the form,

$$\begin{bmatrix} 0 & \begin{pmatrix} r_1 & r_2 & \cdots & r_{n-1} \end{pmatrix} \\ \mathbf{0}_{n-1,1} & \bigoplus_{i=1}^m J_{n_i}(0) \end{bmatrix}, \text{ for some } r_1, r_2, \dots, r_{n-1} \in \mathbb{K}.$$

Observation 1 : *There are* $\alpha_1, \dots, \alpha_m \in \{0,1\}$ *such that,*

$$A \sim_{sim} \begin{bmatrix} 0 & \begin{pmatrix} \alpha_1 \mathbf{e}_{1,n_1} & \cdots & \alpha_m \mathbf{e}_{1,n_m} \end{pmatrix} \\ \mathbf{0}_{n-1,1} & \bigoplus_{i=1}^m J_{n_i}(0) \end{bmatrix} \text{ in } M_n(\mathbb{K}). \tag{2.5.2}$$

Proof of Observation 1. Let q_i denote the cumulative sum $n_1 + \cdots + n_i$ with the convention that $q_0 = 0$. For the row vector $X = \begin{pmatrix} r_1 & r_2 & \cdots & r_{n-1} \end{pmatrix} \in M_{1,n-1}(\mathbb{K})$, from a straightforward computation, we observe that

$$\begin{aligned} & 0 \cdot X - X \left(\bigoplus_{i=1}^m J_{n_i}(0)\right) \\ = & \begin{pmatrix} r_1 \mathbf{e}_{1,n_1} & r_{q_1+1}\mathbf{e}_{1,n_2} & \cdots & r_{q_{m-1}+1}\mathbf{e}_{1,n_m} \end{pmatrix} - \begin{pmatrix} r_1 & r_2 & \cdots & r_{n-1} \end{pmatrix}. \end{aligned}$$

Using Lemma 2.2.3 (Roth's removal rule), we have,

$$A \sim_{\text{sim}} \begin{bmatrix} 0 & \begin{pmatrix} r_1 & r_2 & \cdots & r_{n-1} \end{pmatrix} \\ \mathbf{0}_{n-1,1} & \bigoplus_{i=1}^m J_{n_i}(0) \end{bmatrix} \tag{2.5.3}$$

$$\sim_{\text{sim}} \begin{bmatrix} 0 & \begin{pmatrix} r_1 \mathbf{e}_{1,n_1} & r_{q_1+1}\mathbf{e}_{1,n_2} & \cdots & r_{q_{m-1}+1}\mathbf{e}_{1,n_m} \end{pmatrix} \\ \mathbf{0}_{n-1,1} & \bigoplus_{i=1}^m J_{n_i}(0) \end{bmatrix}. \tag{2.5.4}$$

Consider the function $f : \mathbb{K} \to \mathbb{K}$ which maps 0 to 1 and a non-zero element of $\mathbb{K}$ to itself. By conjugating the matrix in (2.5.4) with the invertible diagonal matrix

$$1 \bigoplus \left(\bigoplus_{i=1}^m f(r_{q_{i-1}+1}) I_{n_i} \right) \text{ in } GL_n(\mathbb{K}),$$

the non-zero entries in the first row transform to 1 with the remaining entries left unchanged. Thus A is similar to a matrix of the form given in (2.5.2) where

$$\alpha_i = \begin{cases} 1 & \text{if } r_{q_{i-1}+1} \neq 0 \\ 0 & \text{if } r_{q_{i-1}+1} = 0. \end{cases} \qquad \square$$

In the matrix similarity (2.5.2), if $\alpha_i = 0$ for all $i \in [m]$, then $A \sim_{\text{sim}} \bigoplus_{i=1}^{m+1} J_{n_i}(0)$ with $n_{m+1} := 1$, and we are done. If not all of the α_i's are zero, then the Observation 2 below completes the proof of existence of the Jordan canonical form.

Observation 2 : *Let ℓ be the smallest number in $[m]$ such that $\alpha_\ell = 1$. Then*

$$A \sim_{\text{sim}} \left(\bigoplus_{i=1}^{\ell-1} J_{n_i}(0) \right) \oplus J_{n_\ell+1}(0) \oplus \left(\bigoplus_{i=\ell+1}^{m} J_{n_i}(0) \right) \text{ in } M_n(\mathbb{K}),$$

where the first summand is interpreted as being non-existent when $\ell = 1$

Proof of Observation 2. Let σ be the permutation on $[n]$ consisting of one non-trivial cycle given by $\big(q_{\ell-1}+1, \; q_{\ell-1}, \; q_{\ell-1}-1, \; \cdots, \; 2, \; 1\big)$; recall that q_i denotes the cumulative sum $n_1 + \cdots + n_i$ with $q_0 = 0$. Let S_σ be the matrix obtained from I_n by permuting its columns according to σ, that is, the i^{th} column of S_σ is $\mathbf{e}_{\sigma(i)}$. Conjugating the matrix in (2.5.2) with S_σ, we observe that,

$$\begin{aligned} A \; &\sim_{\text{sim}} \; \left(\bigoplus_{i=1}^{\ell-1} J_{n_i}(0) \right) \oplus \begin{bmatrix} 0 & \begin{pmatrix} \alpha_\ell \mathbf{e}_{1,n_\ell} & \cdots & \alpha_m \mathbf{e}_{1,n_m} \end{pmatrix} \\ \mathbf{0}_{n_\ell+\cdots+n_m,1} & \bigoplus_{i=\ell}^{m} J_{n_i}(0) \end{bmatrix} \\ &\sim_{\text{sim}} \; \left(\bigoplus_{i=1}^{\ell-1} J_{n_i}(0) \right) \oplus J_{n_\ell+1}(0) \oplus \left(\bigoplus_{i=\ell+1}^{m} J_{n_i}(0) \right), \end{aligned}$$

where the second similarity follows from Lemma 2.5.2. □

Uniqueness. Let $n_1 \geq n_2 \geq \ldots \geq n_m$ be positive integers satisfying $n_1 + \cdots + n_m = n$ such that $A \sim_{\text{sim}} \bigoplus_{i=1}^{m} J_{n_i}(0)$ in $M_n(\mathbb{K})$. Let S be an invertible matrix in $GL_n(\mathbb{K})$ such that $S^{-1}AS = \bigoplus_{i=1}^{m} J_{n_i}(0)$. Then for every positive integer k, we have $S^{-1}A^kS = \bigoplus_{i=1}^{m} J_{n_i}(0)^k$ so that $A^k \sim_{\text{sim}} \bigoplus_{i=1}^{m} J_{n_i}(0)^k$.

It is straightforward to verify via direct computation that $\text{rank}(J_m(0)^k) = (m-k)^+$ for $m, k \in \mathbb{N}$. Since rank is a similarity invariant, with the convention $A^0 := I$, for every positive integer k, we have

$$\begin{aligned} \text{rank}(A^{k-1}) - \text{rank}(A^k) &= \sum_{i=1}^{m} \text{rank}(J_{n_i}(0)^{k-1}) - \sum_{i=1}^{m} \text{rank}(J_{n_i}(0)^k) \\ &= \sum_{i=1}^{m} \left((n_i - k + 1)^+ - (n_i - k)^+ \right). \end{aligned}$$

Since $(n_i - k + 1)^+ - (n_i - k)^+$ is equal to 1 if and only if $k \leq n_i$, and 0 otherwise, the quantity $\sum_{i=1}^{m} \left((n_i - k + 1)^+ - (n_i - k)^+ \right)$ counts how many of the Jordan blocks in $\bigoplus_{i=1}^{m} J_{n_i}(0)$ are of size at least k. Thus the number of Jordan blocks in $\bigoplus_{i=1}^{m} J_{n_i}(0)$ having size exactly k is given by the formula,

$$\left(\operatorname{rank}(A^{k-1}) - \operatorname{rank}(A^k)\right) - \left(\operatorname{rank}(A^k) - \operatorname{rank}(A^{k+1})\right). \tag{2.5.5}$$

This proves the uniqueness of the Jordan decomposition. □

***Remark* 2.5.4.** Let $\mathbb{K}$ be a field. For a matrix $A \in M_n(\mathbb{K})$ and $\lambda \in \mathbb{K}$, the sequence of integers,

$$\operatorname{rank}\left((A - \lambda I)^{k-1}\right) - \operatorname{rank}\left((A - \lambda I)^k\right),$$

is called the *Weyr characteristic* of A associated with λ (see [HJ13, pg. 168, 170]).

Theorem 2.5.5 (Jordan canonical form)**.** *Let $A \in M_n(\mathbb{K})$ and let $\mathbb{L} \supseteq \mathbb{K}$ be the splitting field of the characteristic polynomial of A. Let $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{L}$ be the distinct roots of $\chi_A(x)$ with algebraic multiplicities $n_1, \ldots, n_m$, respectively, so that $n = n_1 + \cdots + n_m$. Then for each $i \in [m]$, there are positive integers $\ell_i \leq n_i$ such that A is similar to a unique matrix of the form $\bigoplus_{i=1}^m \left(\bigoplus_{j=1}^{\ell_i} J_{n_{i,j}}(\lambda_i)\right)$ with $n_{i,1} \geq n_{i,2} \geq \cdots \geq n_{i,\ell_i}$ and $\sum_{j=1}^{\ell_i} n_{i,j} = n_i$.*

Proof. By Theorem 2.3.2, the matrix A is similar to a block-diagonal matrix $\bigoplus_{i=1}^m A_i$ in $M_n(\mathbb{L})$, where A_i is an upper-triangular matrix in $M_{n_i}(\mathbb{L})$ all of whose diagonal entries are equal to λ_i. Each A_i is of the form $\lambda_i I_{n_i} + N_i$ where $N_i \in M_{n_i}(\mathbb{L})$ is a strictly upper-triangular matrix. By Proposition 2.5.3, each N_i is similar to a matrix of the form $\bigoplus_{j=1}^{\ell_i} J_{n_{i,j}}(0)$ with $n_{i,1} \geq n_{i,2} \geq \cdots \geq n_{i,\ell}$ and $\sum_{j=1}^{\ell_i} n_{i,j} = n_i$. Thus each A_i is similar to a matrix of the form $\bigoplus_{j=1}^{\ell_i} J_{n_{i,j}}(\lambda_i)$ which implies that A is similar to a matrix of the form $\bigoplus_{i=1}^m \left(\bigoplus_{j=1}^{\ell_i} J_{n_{i,j}}(\lambda_i)\right)$.

To deduce the uniqueness of the Jordan canonical form, we note from the formula in (2.5.5) that for every $i \in [m]$ the integers $n_{i,j}$ are completely determined by the sequence, $\{\operatorname{rank}(A - \lambda_i I)^k\}_{k \in \mathbb{N}}$. □

2.6 Unboundedness of the Jordan-Chevalley decomposition

For a matrix $A \in M_n(\mathbb{C})$, let $A = D(A) + N(A)$ be its Jordan-Chevalley decomposition. In this section, we establish one of the key results of this thesis, namely that that the mapping $A \mapsto D(A)$ is norm-unbounded on the unit ball of $M_n(\mathbb{C})$ for $n \geq 3$.

Proposition 2.6.1. *For every matrix $A \in M_2(\mathbb{C})$, we have $\|D(A)\| \leq \|A\|$. Thus, the image of the unit ball of $M_2(\mathbb{C})$ is norm-bounded under the mapping $A \mapsto D(A)$.*

Proof. Let $A \in M_2(\mathbb{C})$. If A has two distinct eigenvalues, then A is diagonalizable. Thus we have $D(A) = A$ and $N(A) = 0$, and in this case, $\|D(A)\| = \|A\|$. If A has only one eigenvalue $\lambda \in \mathbb{C}$ (with multiplicity 2), then $D(A)$ must be similar to the scalar matrix λI_2, whence $D(A) = \lambda I_2$. Thus, in this case, $\|D(A)\| = \|\lambda I_2\| = |\lambda| \leq \|A\|$. □

***Remark* 2.6.2.** From the uniqueness of Jordan-Chevalley decomposition, it is easily verified that for a matrix $A \in M_n(\mathbb{C})$ and an invertible matrix $S \in GL_n(\mathbb{C})$, we have $D(SAS^{-1}) = SD(A)S^{-1}$. Since $\mathrm{sp}(A) = \mathrm{sp}(D(A))$, and for $\lambda \in \mathbb{C}$, λI_n is the only diagonalizable matrix in $M_n(\mathbb{C})$ with spectrum $\{\lambda\}$, we see that if $\mathrm{sp}(A) = \{\lambda\}$, then $D(A) = \lambda I_n$.

Proposition 2.6.3. *Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be distinct complex numbers. Let $A \in M_m(\mathbb{C})$ and $B \in M_n(\mathbb{C})$ be complex matrices with $\mathrm{sp}\,(A) = \{\lambda_1\}$ and $\mathrm{sp}\,(B) = \{\lambda_2\}$, and $C \in M_{m,n}(\mathbb{C})$. Let $X \in M_{m,n}(\mathbb{C})$ be the unique matrix satisfying $AX - XB = C$. Then the diagonalizable part of the matrix*

$$\begin{bmatrix} A & C \\ \mathbf{0}_{n,m} & B \end{bmatrix},$$

in its Jordan-Chevalley decomposition, is given by

$$\begin{bmatrix} \lambda_1 I_m & (\lambda_1 - \lambda_2)X \\ \mathbf{0}_{n,m} & \lambda_2 I_n \end{bmatrix}.$$

Proof. Since $\mathrm{sp}\,(A) \cap \mathrm{sp}\,(B) = \varnothing$, from Theorem 2.2.2, there is a unique matrix X in $M_{m,n}(\mathbb{C})$ solving the Sylvester equation $AX - XB = C$. Let

$$M := \begin{bmatrix} A & C \\ \mathbf{0}_{n,m} & B \end{bmatrix}, \ M' := \begin{bmatrix} A & \mathbf{0}_{m,n} \\ \mathbf{0}_{n,m} & B \end{bmatrix}, \text{ and } S := \begin{bmatrix} I_m & X \\ \mathbf{0}_{n,m} & I_n \end{bmatrix}.$$

Note that,

$$SMS^{-1} = \begin{bmatrix} I_m & X \\ \mathbf{0}_{n,m} & I_n \end{bmatrix} \begin{bmatrix} A & C \\ \mathbf{0}_{n,m} & B \end{bmatrix} \begin{bmatrix} I_m & -X \\ \mathbf{0}_{n,m} & I_n \end{bmatrix} = \begin{bmatrix} A & \mathbf{0}_{m,n} \\ \mathbf{0}_{n,m} & B \end{bmatrix} = M'.$$

From Remark 2.6.2, it follows that the diagonalizable part of M' is

$$D(M') := \begin{bmatrix} \lambda_1 I_m & \mathbf{0}_{m,n} \\ \mathbf{0}_{n,m} & \lambda_2 I_n \end{bmatrix},$$

and the diagonalizable part of M is

$$D(M) = D(S^{-1}M'S) = S^{-1}D(M')S = \begin{bmatrix} I_m & -X \\ \mathbf{0}_{n,m} & I_n \end{bmatrix} \begin{bmatrix} \lambda_1 I_m & \mathbf{0}_{m,n} \\ \mathbf{0}_{n,m} & \lambda_2 I_n \end{bmatrix} \begin{bmatrix} I_m & X \\ \mathbf{0}_{n,m} & I_n \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 I_m & (\lambda_1 - \lambda_2)X \\ \mathbf{0}_{n,m} & \lambda_2 I_n \end{bmatrix}.$$
□

Theorem 2.6.4. *For $n \geq 3$, the image of the unit ball of $M_n(\mathbb{C})$ under the mapping $A \mapsto D(A)$ is norm-unbounded.*

Proof. First we prove the result for $n = 3$. Let $\lambda \in \mathbb{C} \setminus \{0\}$, $A_\lambda := \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \in M_2(\mathbb{C})$, and $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in M_{2,1}(\mathbb{C})$. Then the equation $A_\lambda X = v$ has a unique solution, given by

$$X_\lambda = A_\lambda^{-1} v = \begin{bmatrix} \frac{1}{\lambda} & -\frac{1}{\lambda^2} \\ 0 & \frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\lambda^2} \\ \frac{1}{\lambda} \end{bmatrix}.$$

Let $T_\lambda \in M_3(\mathbb{C})$ be defined by

$$T_\lambda := \begin{bmatrix} A_\lambda & v \\ \mathbf{0}_{1,2} & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $A_\lambda X_\lambda - X_\lambda \cdot 0 = v$, from Proposition 2.6.3 the diagonalizable part of T_λ in its Jordan-Chevalley decomposition is given by

$$D(T_\lambda) := \begin{bmatrix} \lambda I_2 & \lambda X_\lambda \\ \mathbf{0}_{1,2} & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & -\frac{1}{\lambda} \\ 0 & \lambda & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the Jordan-Chevalley decomposition of T_λ is given by,

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & -\frac{1}{\lambda} \\ 0 & \lambda & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & \frac{1}{\lambda} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.6.1}$$

With $\lambda_k := \frac{1}{k}$ for $k \in \mathbb{N}$, we note that the sequence $\{T_{\lambda_k}\}_{k\in\mathbb{N}}$ converges in norm to a Jordan matrix (and hence norm-bounded), while the sequence $\{D(T_{\lambda_k})\}_{k\in\mathbb{N}}$ is unbounded in norm as the $(1,3)$–entry of $D(T_{\lambda_k})$ is $-k$.

For $n > 3$, considering the sequence $\{T_{\lambda_k} \oplus \mathbf{0}_{n-3}\}_{k\in\mathbb{N}}$ in $M_n(\mathbb{C})$, a similar conclusion may be drawn. □

2.7 Concluding remarks

While the Jordan-Chevalley decomposition and the Jordan canonical form are classical results, our approach offers new proofs. Conventional proofs of the Jordan–Chevalley decomposition (such as in [HK71, Chapter 7]) typically rely on the structure theorem for finitely generated modules over a principal ideal domain (PID); the relevant PID being the univariate polynomial ring $\mathbb{K}[x]$, and the finitely generated $\mathbb{K}[x]$-module being $\mathbb{K}^n$ via actions of polynomials in $A \in M_n(\mathbb{K})$.

Apart from the above pedagogical application, we must emphasize that when dealing with operators on an infinite-dimensional Hilbert space, a hands-on computational approach may be more helpful as illustrated in our treatment of the Jordan-Chevalley decomposition of matrices over $C(\mathcal{X})$ for $\mathcal{X}$ a Stonean space; this line of reasoning continues in subsequent chapters on the convergence of normalized power sequences.

Chapter 3

Dunford decomposition and convergence of normalized power sequences

3.1 Introduction

Let $\mathscr{H}$ be a complex Hilbert space, and T be a bounded operator on $\mathscr{H}$. The spectral radius formula,

$$\lim_{k\to\infty} \|T^k\|^{\frac{1}{k}} = \mathrm{r}(T),$$

indicates that the asymptotic growth-rate of the sequence $\left\{\|T^k\|\right\}_{k\in\mathbb{N}}$, is governed by, and thus carries information about the spectral radius of T. This suggests that the asymptotic behaviour of the power sequence, $\{T^k\}_{k\in\mathbb{N}}$, may yield finer information about the spectral properties of T. In this vein, the spectral radius formula was considerably generalized by Yamamoto as follows.

Yamamoto's theorem (see [Yam67, Theorem 1]) *Let A be a matrix in $M_n(\mathbb{C})$ and $|\lambda_j|(A)$ denote the j^{th}-largest number in the list of modulus of eigenvalues of A (counted with multiplicity). Then*

$$\lim_{k\to\infty} s_j(A^k)^{\frac{1}{k}} = |\lambda_j|(A),$$

where $s_j(A^k)$ denotes the j^{th}-largest singular value of A^k.

Nayak significantly strengthened this result by proving a spatial version of Yamamoto's theorem, showing that for any matrix $A \in M_n(\mathbb{C})$, the normalized power sequence $|A^k|^{1/k}{}_{k\in\mathbb{N}}$ converges in norm. In fact, an explicit description of the limiting positive-semidefinite matrix is provided in terms of the diagonalizable part in the Jordan-Chevalley decomposition of the matrix, as follows.

Nayak's theorem (see [Nay23, Theorem 3.8]). *Let $A \in M_n(\mathbb{C})$ and $\{a_1, \dots, a_m\}$ be the set of modulus of eigenvalues of A such that $0 \le a_1 < a_2 < \cdots < a_m$. Let $A = D + N$ be the Jordan-Chevalley decomposition of A into its commuting diagonalizable and nilpotent parts (D, N, respectively). For $1 \le j \le m$, let E_j be the orthogonal projection onto the* *subspace of $\mathbb{C}^n$ spanned by the eigenvectors of D corresponding to eigenvalues with modulus less than or equal to a_j, and set $E_0 := 0$. Then the following assertions hold:*

(i) The normalized power sequence of A converges in norm to the positive-semidefinite matrix,

$$\sum_{j=1}^{m} a_j(E_j - E_{j-1}).$$

(ii) A non-zero vector $x \in \mathbb{C}^n$ is in $\operatorname{ran}(E_j)\backslash\operatorname{ran}(E_{j-1})$ if and only if $\lim_{k\to\infty} \|A^k x\|^{\frac{1}{k}} = a_j$.

The above result has been generalized to the context of real semisimple Lie groups, in [HT24], by Huang and Tam. In [BB24], Bhat and Bala have shown that the normalized power sequence of a compact operator on acting on a complex separable Hilbert space is norm-convergent, making essential use of the above result for matrices.

The main goal of this chapter is to prove a generalization of [Nay23, Theorem 3.8] to the context of spectral operators in $\mathscr{B}(\mathscr{H})$ (see Definition 1.3.11). We note the relevant result below.

Theorem 3.4.2 *Let A be a spectral operator in $\mathscr{B}(\mathscr{H})$, and $\mathcal{E}_A$ be the idempotent-valued spectral resolution of A. For $\lambda \in \mathbb{R}$, let $F_\lambda := \mathbf{R}(\mathcal{E}_A(\mathbb{D}_\lambda))$. Then the following assertions hold:*

(i) $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity and

$$\lim_{k\to\infty} |A^k|^{\frac{1}{k}} = \int_0^{\mathrm{r}(A)} \lambda \, \mathrm{d}F_\lambda, \text{ in norm.}$$

Moreover, the spectrum of the limiting positive operator, $\int_0^{\mathrm{r}(A)} \lambda \, \mathrm{d}F_\lambda$, is,

$$|\mathrm{sp}\,(A)\,| := \{|\lambda| : \lambda \in \mathrm{sp}\,(A)\},$$

the modulus of the spectrum of A.

(ii) For every vector $x \in \mathscr{H}$, there is a smallest non-negative real number λ_x such that x lies in the range of the spectral idempotent $\mathcal{E}_A(\mathbb{D}_{\lambda_x})$, which may be obtained as the following limit,

$$\lim_{k\to\infty} \|A^k x\|^{\frac{1}{k}} = \lambda_x.$$

Furthermore, in Theorem 3.4.4, as a consequence of Theorem 3.4.2, we obtain some results pertaining to the asymptotic behaviour of one-parameter groups in $\mathscr{B}(\mathscr{H})$ whose infinitesimal generator is a spectral operator. Our techniques in proving Theorem 3.4.2 are necessarily different from those used in [Nay23], where the trace in $M_n(\mathbb{C})$ plays an important role in inferring spectral properties of A via the computation of its moments.

Since there are spectral operators which are **not** trace-class (for example, invertible scalar-type operators when $\mathscr{H}$ is infinite-dimensional), a direct imitation of the proof strategy in [Nay23] is destined to fail.

It is worth pointing out that although in [Nay23], the spectral radius formula is never used, and in fact, follows as a consequence of [Nay23, Theorem 3.8], whereas in our proofs make generous use of the spectral radius formula in our arguments. Fuglede's theorem (see Theorem 1.3.1) helps us in splitting the problem into two parts, one involving invertible spectral operators, and another involving spectral operators whose spectrum is contained in a small disc centred at the origin of $\mathbb{C}$. The argument is based on a careful analysis of the interplay of these parts.

Let $\mathscr{M}$ be a II_1 factor. In [HS09, Theorem 8.1], Haagerup and Schultz showed that for an operator in $\mathscr{M}$, its normalized power sequence converges in the strong-operator topology and the spectral resolution of the limiting positive operator is described in terms of the so-called Haagerup-Schultz projections. An operator in $\mathscr{M}$ is said to be SOT-quasinilpotent if its normalized power sequence converges to 0 in the strong-operator topology. In this context, Dykema and Krishnaswamy-Usha ([DKU21]) identified an appropriate modification of the notion of spectrality called the UNZA property (uniformly non-zero angles property). An operator in $\mathscr{M}$ is said to have the UNZA property if the angles between its Haagerup–Schultz projections are uniformly bounded away from zero. In [DKU21, Theorem 4.7-(ii)], it is noted that every operator in $\mathscr{M}$ with the UNZA property has a decomposition of the form $D+N$ where D is a scalar-type operator and N is a SOT-quasinilpotent operator such that $DN = ND$. In [DKU21, Example 5.1], an example is given of an operator in $\mathscr{M}$ which is **not** spectral in the sense of Dunford.

Interestingly, the path in [DKU21] is in the opposite direction to the one in our discussion. There, one starts with the Haagerup-Schultz theorem about SOT-convergence of the normalized power sequence of operators in $\mathscr{M}$, and uses its consequences to establish the 'spectrality' of a large class of operators in $\mathscr{M}$ (the operators with UNZA property), whereas we start with the spectrality of an operator and use it to prove the norm-convergence of its normalized power sequence.

3.2 Dunford decomposition : Spectral operators

We have discussed the notion of spectral operators and some of their properties in §1.3.1. The following theorems give a canonical reduction of bounded spectral operators, which we shall refer to as the *Dunford decomposition* of a spectral operator.

Theorem 3.2.1 (see [DS88, Theorem XV.4.5])**.** *An operator $A \in \mathscr{B}(\mathfrak{X})$ is spectral if and only if there is a scalar-type operator D and a quasinilpotent operator N, in $\mathscr{B}(\mathfrak{X})$ such that $DN = ND$ and $A = D + N$. Furthermore, this decomposition is unique, and A and D have identical spectra and identical idempotent-valued spectral resolutions.*

Theorem 3.2.2 (cf. [DS88, Theorem XV.6.4])**.** *An operator $D \in \mathscr{B}(\mathscr{H})$ is a scalar-type operator if and only if there is a normal operator M and an invertible operator S, in $\mathscr{B}(\mathscr{H})$ such that $D = S^{-1}MS$.*

It follows that the spectral projection of M corresponding to the Borel set $\boldsymbol{\beta}$ is given by

$$E(\boldsymbol{\beta}) = S\mathcal{E}_D(\boldsymbol{\beta})S^{-1},$$

where $\mathcal{E}_D$ denotes the idempotent-valued spectral resolution of D.

***Remark* 3.2.3.** In view of the above theorem, the Dunford decomposition of spectral operators in $\mathscr{B}(\mathscr{H})$ may be regarded as a generalization of the Jordan–Chevalley decomposition for matrices, since quasinilpotent operators in $\mathscr{B}(\mathscr{H})$ generalize nilpotent matrices, and scalar-type operators in $\mathscr{B}(\mathscr{H})$ generalize diagonalizable ones.

Since the property of being a quasinilpotent operator is preserved under similarity, and the commutation relation is preserved under similarity implemented by the same invertible operator, we make the following remark.

***Remark* 3.2.4.** An operator $A \in \mathscr{B}(\mathscr{H})$ is spectral if and only if it is similar to a sum of a normal operator and a quasinilpotent operator in $\mathscr{B}(\mathscr{H})$, which commute with each other, that is, there is a normal operator M, a quasinilpotent operator N, and an invertible operator S, in $\mathscr{B}(\mathscr{H})$ such that $A = S^{-1}(M+N)S$.

In Lemma 3.2.6, we note an approximation result concerning the terms in the normalized power sequence of spectral operators of the form $M+N$, where M and N are commuting normal and quasinilpotent operators, respectively, in $\mathscr{B}(\mathscr{H})$. This result encapsulates the norm convergence of the NPS for such spectral operators, with the limit being the positive operator $|M|$.

Lemma 3.2.5. *Let $T \in \mathscr{B}(\mathscr{H})$, and N be a quasinilpotent in $\mathscr{B}(\mathscr{H})$, such that $TN = NT$. Then,*

(i) TN is quasinilpotent.

(ii) $\mathrm{sp}\,(T+N) = \mathrm{sp}\,(T)$. In particular, $T+N$ is invertible if and only if T is invertible.

Proof. Note that $\mathrm{sp}\,(N) = \{0\}$, as N is quasinilpotent. Since $TN = NT$, from [KR83, Proposition 3.2.10], we observe that,

$$\mathrm{sp}\,(TN) \subseteq \mathrm{sp}\,(T)\,\mathrm{sp}\,(N) = \{0\}, \tag{3.2.1}$$

$$\mathrm{sp}\,(T+N) \subseteq \mathrm{sp}\,(T) + \mathrm{sp}\,(N) = \mathrm{sp}\,(T)\,. \tag{3.2.2}$$

(i) Since the spectrum of an operator is non-empty, from (3.2.1), we note that $\mathrm{sp}\,(TN) = \{0\}$. Thus, TN is quasinilpotent.

(ii) Since $T + N$ and $-N$ commute, as an application of (3.2.2), we also have the opposite inclusion,

$$\operatorname{sp}(T) = \operatorname{sp}((T+N)+(-N)) \subseteq \operatorname{sp}(T+N) + \operatorname{sp}(-N) = \operatorname{sp}(T+N).$$

In particular, $0 \notin \operatorname{sp}(T+N)$ if and only if $0 \notin \operatorname{sp}(T)$; equivalently, $T+N$ is invertible if and only if T is invertible. □

Lemma 3.2.6. *Let M be a normal operator and N be a quasinilpotent in $\mathscr{B}(\mathscr{H})$, such that $NM = MN$. Let $\varepsilon > 0$, and E_ε and $E'_\varepsilon = I - E_\varepsilon$ be the spectral projections of M corresponding to $\mathbb{D}_\varepsilon$ and $\mathbb{C}\backslash\mathbb{D}_\varepsilon$, respectively. Then, there is a positive integer $n(\varepsilon) \in \mathbb{N}$ such that for all $k \geq n(\varepsilon)$,*

$$(1-\varepsilon)^{2k}\,(M^*M)^k E'_\varepsilon \leq \left((M+N)^k\right)^*(M+N)^k \leq (1+\varepsilon)^{2k}\,(M^*M)^k E'_\varepsilon + (2\varepsilon)^{2k}\,E_\varepsilon. \quad (3.2.3)$$

Proof. To begin with, we derive operator inequalities (see (3.2.4) and (3.2.7)) in two cases based on simplifying assumptions on the normal operator N, and eventually combine both to arrive at inequality (3.2.3).

Observation 1 : *If M is invertible, then there is a positive integer $n(\varepsilon) \in \mathbb{N}$ such that for all $k \geq n(\varepsilon)$,*

$$(1-\varepsilon)^{2k}(M^*M)^k \leq \left((M+N)^k\right)^*(M+N)^k \leq (1+\varepsilon)^{2k}(M^*M)^k. \quad (3.2.4)$$

Proof of Observation 1. Let $S := I + M^{-1}N$. Since $MN = NM$, we have $M + N = MS = SM$. Since M^{-1} commutes with N, by Lemma 3.2.5, we note that $M^{-1}N$ is quasinilpotent and that S is an invertible operator. For each $k \in \mathbb{N}$, we have the operator inequality

$$\|S^{-k}\|^{-2} I \;\leq\; (S^k)^* S^k \;\leq\; \|S^k\|^2 I.$$

From inequality (OI3), we have

$$\|S^{-k}\|^{-2} M^{k^*} M^k \;\leq\; M^{k^*} S^{k^*} S^k M^k = \left((M+N)^k\right)^*(M+N)^k \;\leq\; \|S^k\|^2 M^{k^*} M^k. \quad (3.2.5)$$

By Lemma 3.2.5-(ii), $\operatorname{sp}(S) = \operatorname{sp}(I) = \{1\}$ so that $\operatorname{sp}\left(S^{-1}\right) = \operatorname{sp}(S)^{-1} = \{1\}$. Hence, from the spectral radius formula, there is a positive integer $n(\varepsilon) \in \mathbb{N}$ such that

$$\|S^k\| \leq (1+\varepsilon)^k \;, \text{ and } \quad \|S^{-k}\| \leq (1+\varepsilon)^k, \text{ for all } \; k \geq n(\varepsilon). \quad (3.2.6)$$

Keeping in mind that $(1-\varepsilon) \leq (1+\varepsilon)^{-1}$, and combining the inequalities in (3.2.5) and (3.2.6), we get inequality (3.2.4). □

Observation 2 : *If* $\mathrm{sp}\,(M) \subseteq \mathbb{D}_\varepsilon$, *then there is a positive integer* $n(\varepsilon) \in \mathbb{N}$ *such that for all* $k \geq n(\varepsilon)$,

$$0 \leq \big((M+N)^k\big)^*(M+N)^k \leq (2\varepsilon)^{2k} I. \tag{3.2.7}$$

Proof of Observation 2. Since M and Q commute, by Lemma 3.2.5, note that $\mathrm{sp}\,(M) = \mathrm{sp}\,(M+N) \subseteq \mathbb{D}_\varepsilon$. From the spectral radius formula, for large enough n, we have

$$\|(M+N)^k\| \leq (2\varepsilon)^k,$$

whence inequality (3.2.7) follows from (OI1). □

Now that the result has been established in the above two special cases, we proceed towards the case of general M. Since N commutes with M, by Fuglede's theorem, N commutes with all spectral projections of M. In particular, M, M^*, N, N^* commute with the projections E_ε and E'_ε.

Note that $E'_\varepsilon M$, $E'_\varepsilon N$ may be viewed as an invertible normal operator, a quasinilpotent operator, respectively, in $\mathscr{B}\big(E'_\varepsilon(\mathscr{H})\big)$, and they commute with each other. By inequality (3.2.4) in Observation 1, there is a positive integer $n_1(\varepsilon)$ such that for all $k \geq n_1(\varepsilon)$,

$$(1-\varepsilon)^{2k}(M^*M)^k E'_\varepsilon \leq \big((M+N)^k\big)^*(M+N)^k E'_\varepsilon \leq (1+\varepsilon)^{2k}(M^*M)^k E'_\varepsilon. \tag{3.2.8}$$

Similarly, we view $E_\varepsilon M$ as a normal operator and $E_\varepsilon N$ as a quasinilpotent operator, in $\mathscr{B}\big(E_\varepsilon(\mathscr{H})\big)$. Clearly, ME_ε and NE_ε commute with each other, and $\mathrm{sp}\,(E_\varepsilon) \subseteq \mathbb{D}_\varepsilon$. By inequality (3.2.7) in Observation 2, there is a positive integer $n_2(\varepsilon)$ such that for all $k \geq n_1(\varepsilon)$,

$$0 \leq \big((M+N)^k\big)^*(M+N)^k E_\varepsilon \leq (2\varepsilon)^{2k} E_\varepsilon. \tag{3.2.9}$$

Let $n(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Keeping in mind the orthogonal sum $E_\varepsilon + E'_\varepsilon = I$, and adding the inequalities in (3.2.8) and (3.2.9) for $k \geq n(\varepsilon)$, we get our desired inequality (3.2.3). □

We note that Lemma 3.2.6 is crucially used in our proof of Theorem 3.4.2. Thus it may be useful to keep in mind that our approach essentially splits the problem into two parts, one involving invertible spectral operators, and another involving spectral operators whose spectrum is contained in a small disc centred at the origin.

3.3 A preparatory analysis

In this section, our main goal is to prove the norm-convergence and find the norm-limit of the sequence $\{(S^*H^kS)^{\frac{1}{k}}\}_{k\in\mathbb{N}}$, for S an invertible operator and H a positive operator in $\mathscr{B}(\mathscr{H})$. We first establish this for the case where H has finite spectrum, and subsequently extend the result via an approximation argument to obtain the result for the general case.

This is a crucial step for transition from the special case of spectral operators, those of the form $M + N$, where M and N are commuting normal and quasinilpotent operators, respectively, in $\mathscr{B}(\mathscr{H})$, to the general case of spectral operators in $\mathscr{B}(\mathscr{H})$ in the context of Remark 3.2.4.

Lemma 3.3.1. *Let $H_1, H_2, \ldots, H_m$ be positive closed-range operators in $\mathscr{B}(\mathscr{H})$. Then, for sequences $\{a_{1,k}\}_{k\in\mathbb{N}}, \{a_{2,k}\}_{k\in\mathbb{N}}, \ldots, \{a_{m,k}\}_{k\in\mathbb{N}}$ of non-negative real numbers, the limit*

$$\lim_{k\to\infty} \big(\sum_{i=1}^{m} a_{i,k}\ H_i\big)^{\frac{1}{k}}$$

exists in norm if and only if the limit

$$\lim_{k\to\infty} \big(\sum_{i=1}^{m} a_{i,k}\ \mathbf{R}(H_i)\big)^{\frac{1}{k}}$$

exists in norm. Moreover, when the limits exist, they coincide with each other.

Proof. Let $h_{-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be the function defined by $h_{-1}(x) = x^{-1}$ for $x \in \mathbb{R}_{>0}$, and $h_{-1}(0) = 1$, and $h_{\frac{1}{2}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denote the square root function, $x \mapsto \sqrt{x}$.

For each $i \in [m]$, since the range of H_i is closed, by Lemma 1.3.2, $\mathrm{sp}\,(H_i) \setminus \{0\}$ is a compact set, whence h_{-1} is an invertible continuous function on $\mathrm{sp}\,(H_i)$. Note that the restriction of the function $h_{\frac{1}{2}} h_{-1} h_{\frac{1}{2}}$ to $\mathrm{sp}\,(H_i)$ is the indicator function on $\mathrm{sp}\,(H_i)$ whose support is the closed set $\mathrm{sp}\,(H_i) \setminus \{0\}$. Using the continuous function calculus for H_i, we observe that the operator $S_i := h_{-1}(H_i)$ is positive and invertible in $\mathscr{B}(\mathscr{H})$, satisfying

$$H_i^{\frac{1}{2}}\, S_i\, H_i^{\frac{1}{2}} = h_{\frac{1}{2}}(H_i)\, h_{-1}(H_i)\, h_{\frac{1}{2}}(H_i) = (h_{\frac{1}{2}} h_{-1} h_{\frac{1}{2}})(H_i) = \mathbf{R}(H_i).$$

Let $\alpha := \min_{i\in[m]}\{\|S_i^{-1}\|^{-1}\}$, and $\beta := \max_{i\in[m]}\{\|S_i\|\}$. Then, using (OI2), we have

$$\alpha I \ \leq\ S_i \ \leq\ \beta I \quad ;\ i \in [m].$$

It follows from (OI3) that,

$$\alpha H_i \ \leq\ H_i^{\frac{1}{2}} S_i H_i^{\frac{1}{2}} = \mathbf{R}(H_i) \ \leq\ \beta H_i \quad ;\ i \in [m].$$

Since $a_{i,k}$'s are non-negative real numbers, for each $k \in \mathbb{N}$, we have

$$\alpha \sum_{i=1}^{m} a_{i,k}\ H_i \ \leq\ \sum_{i=1}^{m} a_{i,k}\ \mathbf{R}(H_i) \ \leq\ \beta \sum_{i=1}^{m} a_{i,k}\ H_i.$$

It follows from (OI4) that,

$$\alpha^{\frac{1}{k}} \Big(\sum_{i=1}^{m} a_{i,k} \ H_i \Big)^{\frac{1}{k}} \ \leq \ \Big(\sum_{i=1}^{m} a_{i,k} \ \mathbf{R}(H_i) \Big)^{\frac{1}{k}} \ \leq \ \beta^{\frac{1}{k}} \Big(\sum_{i=1}^{m} a_{i,k} \ H_i \Big)^{\frac{1}{k}},$$

and by simple algebraic manipulation,

$$\beta^{-\frac{1}{k}} \Big(\sum_{i=1}^{m} a_{i,k} \ \mathbf{R}(H_i) \Big)^{\frac{1}{k}} \ \leq \ \Big(\sum_{i=1}^{m} a_{i,k} \ H_i \Big)^{\frac{1}{k}} \ \leq \ \alpha^{-\frac{1}{k}} \Big(\sum_{i=1}^{m} a_{i,k} \ \mathbf{R}(H_i) \Big)^{\frac{1}{k}}.$$

Since $0 < \alpha < \beta$, we have $\lim_{k \to \infty} \alpha^{\frac{1}{k}} = \lim_{k \to \infty} \beta^{\frac{1}{k}} = 1$, and the result follows from the sandwich theorem for limits. □

Proposition 3.3.2. *Let $a_1 < \cdots < a_m$ be non-negative real numbers and $H_1, \ldots, H_m$ be positive operators in $\mathscr{B}(\mathscr{H})$ such that for each $i \in [m]$, the reverse cumulative sum, $\sum_{j=i}^{m} H_j$, is a closed-range positive operator. Define $G_i := \mathbf{R}(\sum_{j=i}^{m} H_j)$ for $i \in [m]$, with the convention that $G_{m+1} := 0$. Then,*

$$\lim_{k \to \infty} \Big(\sum_{i=1}^{m} a_i^k \ H_i \Big)^{\frac{1}{k}} = \sum_{i=1}^{m} a_i (G_i - G_{i+1}), \ \textit{in norm.}$$

Proof. For $i \in [m]$, let us denote the reverse cumulative sums by,

$$K_i := \sum_{j=i}^{m} H_j,$$

so that $G_i = \mathbf{R}(K_i)$ for $i \in [m]$, and $G_{k+1} = 0$ as stipulated in the hypothesis of the theorem. Since $K_1 \geq K_2 \geq \cdots \geq K_m$, using Lemma 1.3.4, we have

$$G_1 \geq G_2 \geq \cdots \geq G_m.$$

Thus $\{G_i - G_{i+1} : i \in [m]\}$ consists of mutually orthogonal projections, and using Abel's summation by parts, we have

$$\Big(\sum_{i=1}^{m} a_i \big(G_i - G_{i+1} \big) \Big)^k = \sum_{i=1}^{m} a_i^k \ \big(G_i - G_{i+1} \big) = \ a_1^k \ G_1 + \sum_{i=1}^{m-1} \ (a_{i+1}^k - a_i^k) \ G_{i+1},$$

Hence, for all $k \in \mathbb{N}$, we have

$$\Big(a_1^k \ G_1 + \sum_{i=1}^{m-1} \ (a_{i+1}^k - a_i^k) \ G_{i+1} \Big)^{\frac{1}{k}} = \sum_{i=1}^{m} a_i \big(G_i - G_{i+1} \big). \tag{3.3.1}$$

Again using Abel's summation by parts, we get

$$\sum_{i=1}^{m} a_i^k \ H_i \ = \ a_1^k \ K_1 + \sum_{i=1}^{k-1} (a_{i+1}^k - a_i^k) \ K_{i+1} \ . \tag{3.3.2}$$

Since $\{a_1^k\}_{k\in\mathbb{N}}, \{a_2^k - a_1^k\}_{k\in\mathbb{N}}, \ldots, \{a_m^k - a_{m-1}^k\}_{k\in\mathbb{N}}$ are sequences of non-negative real numbers, and by our hypothesis K_i's are positive closed-range operators, using Lemma 3.3.1 and equation (3.3.1),

$$\begin{aligned}\lim_{k\to\infty} \Big(\sum_{i=1}^{m} a_i^k \ H_i\Big)^{\frac{1}{k}} &= \lim_{k\to\infty} \Big(a_1^k \ K_1 + \sum_{i=1}^{m-1} (a_{i+1}^k - a_i^k) \ K_{i+1}\Big)^{\frac{1}{k}} \\ &= \lim_{k\to\infty} \Big(a_1^k \ G_1 + \sum_{i=1}^{m-1} (a_{i+1}^k - a_i^k) \ G_{i+1}\Big)^{\frac{1}{k}} \\ &= \sum_{i=1}^{m} a_i\big(G_i - G_{i+1}\big),\end{aligned}$$

where the above limits are taken in the norm topology. □

Corollary 3.3.3. *Let $a_1 < \ldots < a_m$ be non-negative real numbers. Let $E_1, \ldots, E_m \in \mathscr{B}(\mathscr{H})$ be mutually orthogonal projections such that $E_1 + \cdots + E_m = I$, and S be an invertible operator in $\mathscr{B}(\mathscr{H})$. For $i \in [m]$, let $F_i := \mathbf{R}(S^{-1}(\sum_{j=1}^{i} E_i)S)$ with $F_0 := 0$. then,*

$$\lim_{k\to\infty} \Big(\sum_{i=1}^{m} a_i^k \ S^* E_i S\Big)^{\frac{1}{k}} = \sum_{i=1}^{m} a_i(F_i - F_{i-1}), \quad \textit{in norm.}$$

Proof. Clearly for each $i \in [m]$, the operator $S^* E_i S$ is positive, and $\sum_{j=i}^{m} E_j$ is a projection, being a sum of mutually orthogonal projections. From Remark 1.3.5, the reverse cumulative sum $\sum_{j=i}^{m} S^* E_j S = S^*(\sum_{j=i}^{m} E_j)S$, is a positive closed-range operator.

Let $G_i := \mathbf{R}(S^*(\sum_{j=i}^{m} E_j)S)$ for $i \in [m]$, with $G_{k+1} := 0$. From Lemma 1.3.6, for $i \in [m]$ we observe that

$$G_i = I - \mathbf{R}\Big(S^{-1}(I - \sum_{j=i}^{m} E_j)S\Big) = I - \mathbf{R}\Big(S^{-1}(\sum_{j=1}^{i-1} E_j)S\Big) = I - F_{i-1}.$$

Hence, keeping in mind that $F_m = I$, we have $G_i - G_{i+1} = F_i - F_{i-1}$, for each $i \in [m]$. The result then follows from Theorem 3.3.2. □

Lemma 3.3.4. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be a bounded resolution of the identity on a Hilbert space $\mathscr{H}$, and S be an invertible operator in $\mathscr{H}$. Then the family $\big\{\mathbf{R}(S^{-1}E_\lambda S)\big\}_{\lambda\in\mathbb{R}}$ defines a bounded resolution of the identity on $\mathscr{H}$.*

Proof. Let S_0 be an invertible operator in $\mathscr{B}(\mathscr{H})$. For $\lambda \in \mathbb{R}$, let $F_\lambda := \mathbf{R}(S_0^* E_\lambda S_0)$. We first show that $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity. From Remark 1.3.5,

$F_\lambda = \mathbf{R}(S_0^* E_\lambda)$. Since S_0^* is invertible, note that $S_0^* \operatorname{ran}(E_\lambda)$ is a closed subspace of $\mathscr{H}$ and $\operatorname{ran}(F_\lambda) = S_0^* \operatorname{ran}(E_\lambda)$.

(i) Since $\bigcap_{\lambda\in\mathbb{R}} \operatorname{ran}(E_\lambda) = \{0\}_{\mathscr{H}}$ and $\bigcup_{\lambda\in\mathbb{R}} \operatorname{ran}(E_\lambda)$ is dense in $\mathscr{H}$, from the invertibility of S_0^* we have

$$\bigcap_{\lambda\in\mathbb{R}} \operatorname{ran}(F_\lambda) = S_0^* \bigcap_{\lambda\in\mathbb{R}} \operatorname{ran}(E_\lambda)\big) = \{0\}_{\mathscr{H}},$$
$$\bigcup_{\lambda\in\mathbb{R}} \operatorname{ran}(F_\lambda) = S_0^* \big(\bigcup_{\lambda\in\mathbb{R}} \operatorname{ran}(E_\lambda)\big) \text{ is dense in } \mathscr{H}.$$

Thus, $\bigwedge_{\lambda\in\mathbb{R}} F_\lambda = 0$, and $\bigvee_{\lambda\in\mathbb{R}} F_\lambda = I$.

(ii) If $\lambda \le \lambda'$, then

$$\begin{aligned} E_\lambda \le E_{\lambda'} &\implies S_0^* E_\lambda S_0 \le S_0^* E_{\lambda'} S_0 \\ &\implies \mathbf{R}(S_0^* E_\lambda S_0) \le \mathbf{R}(S_0^* E_{\lambda'} S_0) \\ &\implies F_\lambda \le F_{\lambda'}. \end{aligned}$$

(iii) Since $E_\lambda = \bigwedge_{\lambda'>\lambda} E_{\lambda'}$, we observe that $\operatorname{ran}(E_\lambda) = \bigcap_{\lambda'>\lambda} \operatorname{ran}(E_{\lambda'})$. Thus,

$$\bigcap_{\lambda'>\lambda} \operatorname{ran}(F_{\lambda'}) = S_0^* \big(\bigcap_{\lambda'>\lambda} \operatorname{ran}(E_{\lambda'})\big) = S_0^* \operatorname{ran}(E_\lambda) = \operatorname{ran}(F_\lambda),$$

which implies that $\bigwedge_{\lambda'>\lambda} F_{\lambda'} = F_\lambda$.

From the above, we conclude that $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a resolution of the identity. Note that $F_\lambda = 0$, $F_\lambda = I$, respectively, if and only if $E_\lambda = 0, E_\lambda = I$, respectively. Thus $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity.

Let us choose S_0 to be the invertible operator $(S^{-1})^*$. Using Remark 1.3.5, we have

$$\mathbf{R}(S^{-1}E_\lambda S) = \mathbf{R}(S^{-1}E_\lambda) = \mathbf{R}(S_0^* E_\lambda) = \mathbf{R}(S_0^* E_\lambda S_0).$$

From the preceding discussion, we conclude that $\{\mathbf{R}(S^{-1}E_\lambda S)\}_{\lambda\in\mathbb{R}}$ defines a bounded resolution of the identity on $\mathscr{H}$. □

Theorem 3.3.5. *Let $H \in \mathscr{B}(\mathscr{H})$ be a positive operator with spectral resolution $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, and S be an invertible operator in $\mathscr{B}(\mathscr{H})$. For $\lambda \in \mathbb{R}$, let $F_\lambda := R(S^{-1}E_\lambda S)$. Then $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity and*

$$\lim_{k\to\infty} (S^* H^k S)^{\frac{1}{k}} = \int_0^{\|H\|} \lambda \, \mathrm{d}F_\lambda, \text{ in norm.}$$

In addition, the spectrum of the limiting positive operator, $\int_0^{\|H\|} \lambda \, \mathrm{d}F_\lambda$, coincides with the spectrum of H.

Proof. From Lemma 3.3.4, $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity. Since $F_\lambda - F_{\lambda'} = 0$ if and only if $E_\lambda - E_{\lambda'} = 0$, from Theorem 1.3.8 it follows that the spectrum of the positive operator $\int_0^{\|H\|} \lambda \, \mathrm{d}F_\lambda$, coincides with the spectrum of H.

Without loss of generality (by replacing H with $H/\|H\|$ if necessary), we may assume that $\|H\| = 1$, so that $\mathrm{sp}\,(H) \subseteq [0,1]$. Below we set up some notation. Let $K := \int_0^{\|H\|} \lambda \; \mathrm{d}F_\lambda$. For $m \in \mathbb{N}$, we partition $[0,1]$ uniformly into intervals $\left[\frac{i-1}{m}, \frac{i}{m}\right]$; $i \in [m]$, and let $E_{i,m}$ and $F_{i,m}$ denote the spectral projections for H and K, respectively, corresponding to the interval $\left[0, \frac{i}{m}\right]$. The lower and upper approximations to H are defined as follows :

$$H_m := \sum_{i=1}^{m} \tfrac{i-1}{m}(E_{i,m} - E_{i-1,m}) \ , \quad H'_m := \sum_{i=1}^{m} \tfrac{i}{m}(E_{i,m} - E_{i-1,m}).$$

Similarly, the lower and upper approximations to K are defined as follows :

$$K_m := \sum_{i=1}^{m} \tfrac{i-1}{m}\left(F_{i,m} - F_{i-1,m}\right) \ , \quad K'_m := \sum_{i=1}^{m} \tfrac{i}{m}\left(F_{i,m} - F_{i-1,m}\right).$$

Our main interest is in the norm-convergence of the sequence $T_k := (S^*H^kS)^{\frac{1}{k}}$. The approximation argument involves lower and upper approximations of T_k by

$$T_{m,k} := (S^*H_m^kS)^{\frac{1}{k}} \ , \quad T'_{m,k} := (S^*{H'_m}^kS)^{\frac{1}{k}}, \text{ for } m,k \in \mathbb{N}.$$

By Corollary 3.3.3, for every $m \in \mathbb{N}$, we have

$$\lim_{k\to\infty} T_{m,k} = K_m \ , \quad \lim_{k\to\infty} T'_{m,k} = K'_m.$$

Given $\varepsilon > 0$, fix a positive integer m_ε such that $\frac{1}{m_\varepsilon} < \frac{\varepsilon}{2}$. There is an $n(\varepsilon) \in \mathbb{N}$ such for all $k \geq n(\varepsilon)$, we have

$$\|T_{m_\varepsilon,k} - K_{m_\varepsilon}\| < \frac{\varepsilon}{2} \ , \quad \|T'_{m_\varepsilon,k} - K'_{m_\varepsilon}\| < \frac{\varepsilon}{2}.$$

By inequality (OI1), we see that,

$$-\tfrac{\varepsilon}{2}I \ \leq \ T_{m_\varepsilon,k} - K_{m_\varepsilon} \ \leq \ \tfrac{\varepsilon}{2}I \ , \quad -\tfrac{\varepsilon}{2}I \ \leq \ T'_{m_\varepsilon,k} - K'_{m_\varepsilon} \ \leq \ \tfrac{\varepsilon}{2}I \quad \forall\ k \geq n(\varepsilon). \tag{3.3.3}$$

Since $H_m^k \leq H^k \leq {H'_m}^k$ for all $m,k \in \mathbb{N}$, we have

$$\begin{aligned} & S^*H_m^kS \ \leq \ S^*H^kS \ \leq \ S^*{H'_m}^kS, && \text{by inequality (OI3)} \\ \Longrightarrow \quad & T_{m,k} \ \leq \ T_k \ \leq \ T'_{m,k}. && \text{by inequality (OI4)} \end{aligned}$$

Combining with (3.3.3), we see that for all $k \geq n(\varepsilon)$,

$$\begin{aligned} & -\tfrac{\varepsilon}{2}I + K_{m_\varepsilon} \;\leq\; T_{m_\varepsilon,k} \;\leq\; T_k \;\leq\; T'_{m_\varepsilon,k} \;\leq\; K'_{m_\varepsilon} + \tfrac{\varepsilon}{2}I, \\ \Longrightarrow \quad & -\tfrac{\varepsilon}{2}I + K_{m_\varepsilon} - K \;\leq\; T_k - K \;\leq\; K'_{m_\varepsilon} - K + \tfrac{\varepsilon}{2}I. \end{aligned}$$

Since $\|K_{m_\varepsilon} - K\| \leq \frac{1}{m_\varepsilon} \leq \frac{\varepsilon}{2}$ and $\|K'_{m_\varepsilon} - K\| \leq \frac{1}{m_\varepsilon} \leq \frac{\varepsilon}{2}$, for all $k \geq n(\varepsilon)$ we have,

$$-\varepsilon I \;\leq\; T_k - K \;\leq\; \varepsilon I.$$

In summary, for every $\varepsilon > 0$, there is a positive integer $n(\varepsilon)$ such that for all $k \geq n(\varepsilon)$, we have $\|T_k - K\| \leq \varepsilon$. Thus, $\lim_{k\to\infty} T_k = \lim_{k\to\infty}(S^* H^k S)^{\frac{1}{k}} = K$. □

3.4 Norm convergence of the normalized power sequence of spectral operators

In this section, we prove the main result of this chapter, which asserts the norm-convergence of the normalized power sequence of a spectral operator, providing an explicit description of the limiting positive operator in terms of its idempotent-valued spectral resolution. We recall that for $r \geq 0$, $\mathbb{D}_r$ denotes the disc of radius r in $\mathbb{C}$ centred at the origin, and for $r < 0$, we stipulate that $\mathbb{D}_r = \varnothing$; in addition, for $r \in \mathbb{R}$, $\mathbb{H}_r$ denotes the closed half-plane of complex numbers with real part less than or equal to r.

The following lemma serves as a technical tool which facilitates the use of Theorem 3.3.5 in Theorem 3.4.2.

Lemma 3.4.1. *Let $T \in \mathscr{B}(\mathscr{H})$ and S be an invertible operator in $\mathscr{B}(\mathscr{H})$. Then the NPS (normalized power sequence) of $S^{-1}TS$ converges in norm if and only if the sequence $\{(S^*{T^k}^*T^kS)^{\frac{1}{2k}}\}_{k\in\mathbb{N}}$ converges in norm. Moreover, when the limits exist, they coincide with each other.*

Proof. Since S is invertible, we observe that

$$\begin{aligned} & \|S\|^{-2}I \leq {S^{-1}}^*S^{-1} \leq \|S^{-1}\|^2 I, && \text{by inequality (OI2)} \\ \Longrightarrow \quad & \|S\|^{-2}S^*{T^k}^*T^kS \leq S^*{T^k}^*{S^{-1}}^*S^{-1}T^kS \leq \|S^{-1}\|^2 S^*{T^k}^*T^kS. && \text{by inequality (OI3)} \end{aligned}$$

Thus from inequality (OI4), it follows that,

$$\|S\|^{-\frac{1}{k}}(S^*{T^k}^*T^kS)^{\frac{1}{2k}} \leq |(S^{-1}TS)^k|^{\frac{1}{k}} \leq \|S^{-1}\|^{\frac{1}{k}}(S^*{T^k}^*T^kS)^{\frac{1}{2k}}. \tag{3.4.1}$$

By basic algebraic manipulation, from inequality (3.4.1), we get

$$\|S^{-1}\|^{-\frac{1}{k}}|(S^{-1}TS)^k|^{\frac{1}{k}} \leq (S^*{T^k}^*T^kS)^{\frac{1}{2k}} \leq \|S\|^{\frac{1}{k}}|(S^{-1}TS)^k|^{\frac{1}{k}}. \tag{3.4.2}$$

Since $\lim_{k\to\infty}\|S\|^{\frac{1}{k}} = \lim_{k\to\infty}\|S^{-1}\|^{\frac{1}{k}} = 1$, the result follows from inequalities (3.4.1) and (3.4.2), using the sandwiching of limits. $\square$

Theorem 3.4.2. *Let A be a spectral operator in $\mathscr{B}(\mathscr{H})$, and $\mathcal{E}_A$ be the idempotent-valued spectral resolution of A. For $\lambda \in \mathbb{R}$, let $F_\lambda := \mathbf{R}(\mathcal{E}_A(\mathbb{D}_\lambda))$. Then the following assertions hold :*

(i) $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity and

$$\lim_{k\to\infty} |A^k|^{\frac{1}{k}} = \int_0^{\mathrm{r}(A)} \lambda \; \mathrm{d}F_\lambda, \text{ in norm.}$$

Moreover, the spectrum of the limiting positive operator, $\int_0^{\mathrm{r}(A)} \lambda \; \mathrm{d}F_\lambda$, is,

$$|\mathrm{sp}\,(A)\,| := \{|\lambda| : \lambda \in \mathrm{sp}\,(A)\},$$

the modulus of the spectrum of A.

(ii) For every vector $x \in \mathscr{H}$, there is a smallest non-negative real number λ_x such that x lies in the range of the spectral idempotent $\mathcal{E}_A(\mathbb{D}_{\lambda_x})$, which may be obtained as the following limit

$$\lim_{k\to\infty} \|A^k x\|^{\frac{1}{k}} = \lambda_x.$$

Proof. Let the unique Dunford decomposition of A be given by $D+N$, where D is scalar-type, and N is a quasinilpotent operator commuting with D. From Theorem 3.2.2, there is a normal operator M and an invertible operator S, in $\mathscr{B}(\mathscr{H})$ such that $D = S^{-1}MS$. Without loss of generality, we may assume that $\|S\| = 1$, by replacing S with $S/\|S\|$ if necessary. Let $N' := SNS^{-1}$. Note that

$$\mathrm{sp}\left(N'\right) = \mathrm{sp}\left(SNS^{-1}\right) = \mathrm{sp}\,(N) = \{0\},$$

so that N' is a quasinilpotent operator. Since $DN = ND$, clearly $M = SDS^{-1}$ and $N' = SNS^{-1}$ commute with each other, and $A = S^{-1}(M+N')S$.

(i) For a Borel set $\boldsymbol{\beta} \in \mathcal{B}$, note that the spectral projection of M corresponding to $\boldsymbol{\beta}$ is $E(\boldsymbol{\beta}) := S\mathcal{E}_A(\boldsymbol{\beta})S^{-1}$. For $\lambda \in \mathbb{R}$, we define $E_\lambda := E(\mathbb{D}_\lambda) = S\mathcal{E}_A(\mathbb{D}_\lambda)S^{-1}$; recall that $\mathcal{E}_A(\mathbb{D}_\lambda) = \mathcal{E}_A(\varnothing) = 0$ when $\lambda < 0$. It is not difficult to see that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity.

Let $0 < \varepsilon < 1$, and $E'_\varepsilon := S\mathcal{E}_A(\mathbb{C}\backslash\mathbb{D}_\varepsilon)S^{-1} = I - E_\varepsilon$. Then, from Lemma 3.2.6, there is an $n_0(\varepsilon) \in \mathbb{N}$ such that for all $k \ge n_0(\varepsilon)$,

$$\begin{aligned}(1-\varepsilon)^{2k}\,(M^*M)^k E'_\varepsilon &\le ((M+N')^k)^*(M+N')^k \\ &\le (1+\varepsilon)^{2k}\,(M^*M)^k E'_\varepsilon + (2\varepsilon)^{2k}\,E_\varepsilon.\end{aligned} \tag{3.4.3}$$

Clearly, $E'_\varepsilon M^{n*} M^k = |E'_\varepsilon M|^{2k}$. For the sake of convenience, we introduce the following notation:

$$H_\varepsilon := |E'_\varepsilon M| = (I - E_\varepsilon)|M| \ , \quad H := |M|, \quad T_k := (S^*(M+N')^{n*}(M+N')^k S)^{\frac{1}{2k}}.$$

Note that H is a positive operator with spectral resolution $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, and from Remark 1.3.9, the spectral resolution of the positive operator H_ε is given by

$$\begin{aligned} E_\lambda^{(\varepsilon)} := E_\varepsilon + (I - E_\varepsilon)E_\lambda &= S\,\mathcal{E}_A(\mathbb{D}_\varepsilon)\,S^{-1} + (S\,\mathcal{E}_A(\mathbb{C}\backslash\mathbb{D}_\varepsilon)\,S^{-1})(S\,\mathcal{E}_A(\mathbb{D}_\lambda)\,S^{-1}) \\ &= S\,\mathcal{E}_A\left(\mathbb{D}_\varepsilon \cup (\mathbb{D}_\lambda\backslash\mathbb{D}_\varepsilon)\right)\,S^{-1}. \end{aligned} \tag{3.4.4}$$

Clearly, $H_\varepsilon \leq H$ and recall that $\|S\| = 1$. Using inequality (OI3) in inequality (3.4.3), for all $k \geq n_0(\varepsilon)$ we have,

$$\begin{aligned} (1-\varepsilon)^{2k}\; S^* H_\varepsilon^{2k} S \;\leq\; T_k^{2k} \;&\leq\; (1+\varepsilon)^{2k}\; S^* H_\varepsilon^{2k} S + (2\varepsilon)^{2k}\; S^* E_\varepsilon S \\ &\leq\; (1+\varepsilon)^{2k}\; S^* H_\varepsilon^{2k} S + (2\varepsilon)^{2k}\; \|S\|^2\; I \\ &\leq\; (1+\varepsilon)^{2k}\; S^* H^{2k} S + (2\varepsilon)^{2k} I. \end{aligned}$$

Consequently, from inequality (OI4) and Lemma 1.3.3, we observe that for all $k \geq n_0(\varepsilon)$,

$$(1-\varepsilon)\;\left(S^* H_\varepsilon^{2k} S\right)^{\frac{1}{2k}} \;\leq\; T_k \;\leq\; (1+\varepsilon)\;\left(S^* H^{2k} S\right)^{\frac{1}{2k}} + 2\varepsilon I. \tag{3.4.5}$$

For $\lambda \in \mathbb{R}$, let
$$F_\lambda := \mathbf{R}(S^{-1}E_\lambda S) = \mathbf{R}\big(\mathcal{E}_A(\mathbb{D}_\lambda)\big) \text{ and}$$
$$F_\lambda^{(\varepsilon)} := \mathbf{R}\big(S^{-1}E_\lambda^{(\varepsilon)} S\big) = \mathbf{R}\big(\mathcal{E}_A\big(\mathbb{D}_\varepsilon \cup (\mathbb{D}_\lambda\backslash\mathbb{D}_\varepsilon)\big)\big)$$

From Lemma 3.3.4, we observe that that F_λ, $F_\lambda^{(\varepsilon)}$ are bounded resolutions of the identity. Let

$$K := \int_0^{\|H\|} \lambda\,\mathrm{d}F_\lambda \ , \quad K_\varepsilon := \int_0^{\|H_\varepsilon\|} \lambda\,\mathrm{d}F_\lambda^{(\varepsilon)}.$$

Since $F_\lambda = F_\lambda^{(\varepsilon)}$ for $\lambda \geq \varepsilon$, we note that

$$\|K_\varepsilon - K\| \leq 2\varepsilon.$$

From Proposition 3.3.5,

$$\lim_{k\to\infty} (S^* H^{2k} S)^{\frac{1}{2k}} = K \ , \quad \lim_{k\to\infty} (S^* H_\varepsilon^{2k} S)^{\frac{1}{2k}} = K_\varepsilon, \quad \text{in} \quad \text{norm} \ , \tag{3.4.6}$$

with $\mathrm{sp}\,(K) = \mathrm{sp}\,(H) = |\mathrm{sp}\,(M)\,| = |\mathrm{sp}\,(A)\,|$, and $\mathrm{sp}\,(K_\varepsilon) = \mathrm{sp}\,(H_\varepsilon) = |\mathrm{sp}\,(E'_\varepsilon M)\,|$.

From the norm-convergence in (3.4.6), there is a positive integer $n(\varepsilon) \geq n_0(\varepsilon)$ such that for all $k \geq n(\varepsilon)$, we have

$$\|(S^* H_\varepsilon^{2k} S)^{\frac{1}{2k}} - K_\varepsilon\| \;\leq\; \varepsilon \quad \text{and} \quad \|(S^* H^{2k} S)^{\frac{1}{2k}} - K\| \;\leq\; \varepsilon,$$

which, using (OI1), yields the following operator inequalities,

$$-\varepsilon I \leq (S^* H_\varepsilon^{2k} S)^{\frac{1}{2k}} - K_\varepsilon \;\leq\; \varepsilon I \quad \text{and} \quad -\varepsilon I \;\leq\; (S^* H^{2k} S)^{\frac{1}{2k}} - K \;\leq\; \varepsilon I. \tag{3.4.7}$$

Using inequality (3.4.7) in combination with inequality (3.4.5), for all $k \geq n(\varepsilon)$, we have

$$(1-\varepsilon)(K_\varepsilon - \varepsilon I) \;\leq\; T_k \;\leq\; (1+\varepsilon)(K + \varepsilon I) + 2\varepsilon I. \tag{3.4.8}$$

Recall that $\varepsilon < 1$. Since $K \leq \|K\| I$, we get the following upper bound for $T_k - K$ from inequality (3.4.8),

$$T_k - K \leq \varepsilon I + \varepsilon K + \varepsilon^2 I + 2\varepsilon I \leq (4 + \|K\|)\varepsilon I, \text{ for } k \geq n(\varepsilon).$$

Since $\|K - K_\varepsilon\| \;\leq\; 2\varepsilon$, clearly

$$-2\varepsilon I \;\leq\; K - K_\varepsilon \;\leq\; 2\varepsilon I, \quad \|K_\varepsilon\| \leq \|K\| + 2\varepsilon \leq \|K\| + 2,$$

and we get the following lower bound for $T_k - K$ from inequality (3.4.8),

$$\begin{aligned} T_k - K &\geq (-K + K_\varepsilon) - \varepsilon I - \varepsilon(K_\varepsilon - \varepsilon I) \geq -2\varepsilon I - \varepsilon I - \|K_\varepsilon\|\varepsilon I \\ &\geq -3\varepsilon I - (\|K\| + 2)\varepsilon I \geq -(5 + \|K\|)\varepsilon I, \text{ for } k \geq n(\varepsilon). \end{aligned}$$

Thus, for $k \geq n(\varepsilon)$, we have

$$-(5 + \|K\|)\varepsilon I \leq T_k - K \leq (4 + \|K\|)\varepsilon I,$$

which implies that $\|T_k - K\| \;\leq\; (5 + \|K\|)\varepsilon$. We conclude that the sequence $\{T_k\}_{k\in\mathbb{N}}$, converges in norm to K. Finally, from Lemma 3.4.1, it follows that the normalized power sequence of $A = S^{-1}(M + N')S$ converges in norm to K.

Let $\lambda_0 \in \mathbb{R}$. Note that as a function of λ, E_λ is constant in a neighbourhood of λ_0 if and only if $\mathcal{E}_A(\mathbb{D}_\lambda)$ is constant in a neighbourhood of λ_0 if and only if F_λ is constant in a neighbourhood of λ_0. From Theorem 1.3.8, we observe that $\lambda_0 \notin \mathrm{sp}\,(K)$ if and only if $\lambda_0 \notin \mathrm{sp}\,(|M|)$, from which it follows that,

$$\mathrm{sp}\,(K) = \mathrm{sp}\,(|M|) = |\mathrm{sp}\,(M)\,| = |\mathrm{sp}\,(D)\,| = |\mathrm{sp}\,(A)\,|.$$

(ii) For a vector $x \in \mathscr{H}$, let $\Lambda_x := \{\lambda \geq 0 : x \in \mathrm{ran}\,(\mathcal{E}_A(\mathbb{D}_\lambda))\}$. Since $\mathrm{ran}\big(\mathcal{E}_A(\mathbb{D}_{\mathrm{r}(A)})\big) = \mathscr{H}$, clearly $\mathrm{r}(A) \in \Lambda_x$ so that Λ_x is a non-empty subset of $\mathbb{R}_{\geq 0}$. Let λ_x denote the infimum of Λ_x. Note that $\{\mathbf{R}(\mathcal{E}_A(\mathbb{D}_\lambda))\}_{\lambda \in \mathbb{R}}$ is a bounded resolution of the identity as shown in part (i), and from the monotonicity and right-continuity of resolutions of the identity (see Definition 1.3.7), one readily verifies that $\lambda_x \in \Lambda_x$.

For $\mu \geq 0$, let $\mathscr{V}_\mu := \mathrm{ran}\,(\mathcal{E}_A(\mathbb{D}_\mu))$ and $\mathscr{W}_\mu := \mathrm{ran}\,(\mathcal{E}_A(\mathbb{C}\backslash\mathbb{D}_\mu)) = \ker\,(\mathcal{E}_A(\mathbb{D}_\mu))$. The idempotent $\mathcal{E}_A(\mathbb{C}\backslash\mathbb{D}_\mu)$ is the projection onto the subspace $\mathscr{W}_\mu$ along the complementary subspace $\mathscr{V}_\mu$. In particular, for every pair of vectors $v \in \mathscr{V}_\mu,\ w \in \mathscr{W}_\mu$, we have $\mathcal{E}_A(\mathbb{C}\backslash\mathbb{D}_\mu)(v+w) = w$. Since $\mathscr{V}_\mu$ and $\mathscr{W}_\mu$ are invariant under A, for every $n \in \mathbb{N}$ we have,

$$\mathcal{E}_A(\mathbb{C}\backslash\mathbb{D}_\mu)(A^k v + A^k w) = A^k w. \tag{3.4.9}$$

Observation 1 : *For $\mu \geq 0$ and a vector x in $\mathscr{V}_\mu$, we have*

$$\limsup_{k\to\infty} \|A^k x\|^{\frac{1}{k}} \leq \mu.$$

Proof of Observation 1. Note that $\mathrm{sp}\left(A|_{\mathscr{V}\mu}\right) \subseteq \mathbb{D}_\mu$. For every vector x in $\mathrm{ran}\big(\mathcal{E}_A(\mathbb{D}_\mu)\big)$, using the spectral radius formula we have,

$$\begin{aligned}
\limsup_{k\to\infty} \|A^k x\|^{\frac{1}{k}} &= \limsup_{k\to\infty} \|(A|_{\mathscr{V}\mu})^k x\|^{\frac{1}{k}} \leq \limsup_{k\to\infty} \left(\|(A|_{\mathscr{V}\mu})^k\|^{\frac{1}{k}} \|x\|^{\frac{1}{k}} \right) \\
&= \left(\lim_{k\to\infty} \|(A|_{\mathscr{V}\mu})^k\|^{\frac{1}{k}} \right) \left(\lim_{k\to\infty} \|x\|^{\frac{1}{k}} \right) \\
&\leq \mu.
\end{aligned}$$

□

Observation 2 : *For $0 \leq \mu < \mathrm{r}(A)$ and a vector x in $\mathscr{H}\backslash\mathscr{V}\mu$, we have*

$$\liminf_{k\to\infty} \|A^k x\|^{\frac{1}{k}} \geq \mu.$$

Proof of Observation 2. Since the assertion is trivially true for $\mu = 0$, we may assume that $\mu > 0$. Note that $\mathrm{sp}\left(A|_{\mathscr{W}\mu}\right)$ is contained in the closed annulus $\overline{\mathbb{D}_{\mathrm{r}(A)}\backslash\mathbb{D}_\mu}$, so that $A|_{\mathscr{W}\mu}$ is invertible. From the spectral mapping theorem, it follows that $\mathrm{sp}((A|_{\mathscr{W}\mu})^{-1})$ is contained in the closed annulus $\overline{\mathbb{D}_{\mu^{-1}}\backslash\mathbb{D}_{\mathrm{r}(A)^{-1}}}$. Using the spectral radius formula, we observe that $\lim_{k\to\infty} \|(A|_{\mathscr{W}\mu})^{-k}\|^{\frac{1}{k}} \leq \mu^{-1}$, which implies $\lim_{k\to\infty} \|(A|_{\mathscr{W}\mu})^{-k}\|^{-\frac{1}{k}} \geq \mu$. For a non-zero vector x in $\mathscr{W}\mu$, we get the following inequality,

$$\begin{aligned}
\liminf_{k\to\infty} \|A^k x\|^{\frac{1}{k}} &= \liminf_{k\to\infty} \|(A|_{\mathscr{W}\mu})^k x\|^{\frac{1}{k}} \geq \liminf_{k\to\infty} \left(\|(A|_{\mathscr{W}\mu})^{-n}\|^{-1} \|x\| \right)^{\frac{1}{k}} \\
&= \left(\lim_{k\to\infty} \|(A|_{\mathscr{W}\mu})^{-k}\|^{-\frac{1}{k}} \right) \left(\lim_{k\to\infty} \|x\|^{\frac{1}{k}} \right) \\
&\geq \mu.
\end{aligned}$$

More generally, for a vector x in $\mathscr{H}\backslash\mathscr{V}\mu$, there is a unique pair of vectors $v \in \mathscr{V}\mu, w \in \mathscr{W}\mu$ such that $x = v + w$ and $w \neq 0$. Note that $\mathcal{E}_A(\mathbb{C}\backslash\mathbb{D}_\mu) \neq 0$ as $\mu < \mathrm{r}(A)$, so that

$\|\mathcal{E}_A(\mathbb{C}\backslash\mathbb{D}_\mu)\| > 0$. For $\alpha := \|\mathcal{E}_A(\mathbb{C}\backslash\mathbb{D}_\mu)\|^{-1} > 0$ and $n \in \mathbb{N}$, using (3.4.9) we have the inequality, $\|A^k v + A^k w\| \geq \alpha\|A^k w\|$. It now follows from the first part of the proof that

$$\begin{aligned}\liminf_{k\to\infty} \|A^k x\|^{\frac{1}{k}} &= \liminf_{k\to\infty} \|A^k v + A^k w\|^{\frac{1}{k}} \geq \liminf_{k\to\infty} \left(\alpha^{\frac{1}{k}} \|A^k w\|^{\frac{1}{k}}\right) \\ &= \left(\liminf_{k\to\infty} \|A^k w\|^{\frac{1}{k}}\right) \left(\lim_{k\to\infty} \alpha^{\frac{1}{k}}\right) \\ &\geq \mu. \end{aligned}$$

□

For a vector $x \in \mathscr{H}$ and $\varepsilon > 0$, from Observation 1, Observation 2, and the definition of λ_x, it follows that

$$\lambda_x - \varepsilon \leq \liminf_{k\to\infty} \|A^k x\|^{\frac{1}{k}} \leq \limsup_{k\to\infty} \|A^k x\|^{\frac{1}{k}} \leq \lambda_x.$$

Thus, for every vector x in $\mathscr{H}$, the limit of the sequence $\{\|A^k x\|^{\frac{1}{k}}\}_{k\in\mathbb{N}}$ exists, and is equal to λ_x. □

***Remark* 3.4.3.** We record two immediate consequences of Theorem 3.4.2. Firstly, note that the norm-limit of the normalized power sequence of a spectral operator does **not** depend on the quasi-nilpotent part in its Dunford decomposition. Secondly, since every matrix in $M_m(\mathbb{C})$ is a spectral operator, [Nay23, Theorem 3.8] follows as an immediate corollary of Theorem 3.4.2.

As a natural corollary of Theorem 3.4.2, below we derive some results on the asymptotic behaviour of a one-parameter dynamical system with state space $\mathscr{H}$ whose infinitesimal generator is a spectral operator.

Theorem 3.4.4. *Let A be a spectral operator in $\mathscr{B}(\mathscr{H})$, and $\mathcal{E}_A$ be the idempotent-valued spectral resolution of A. For $\lambda \in \mathbb{R}$, let $G_\lambda := \mathbf{R}(\mathcal{E}_A(\mathbb{H}_\lambda))$. Then the following assertions hold:*

(i) $\{G_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity and

$$\lim_{t\to\infty} |\exp(tA)|^{\frac{1}{t}} = \int_{-\infty}^{\mathrm{r}(A)} \exp(\lambda)\ \mathrm{d}G_\lambda, \text{ in norm.}$$

Moreover, the spectrum of the limiting positive operator, $\int_{-\infty}^{\mathrm{r}(A)} \exp(\lambda)\ \mathrm{d}G_\lambda$, is,

$$\exp\left(\Re\left(\mathrm{sp}\left(A\right)\right)\right) := \left\{\exp(\Re\lambda) : \lambda \in \mathrm{sp}\left(A\right)\right\}.$$

(ii) For every non-zero vector $x \in \mathscr{H}$, there is a smallest real number γ_x such that x lies in the range of the spectral idempotent $\mathcal{E}_A(\mathbb{H}_{\gamma_x})$, which may be obtained as the following limit,

$$\lim_{t\to\infty} \frac{\log\left(\|\exp(tA)\, x\|\right)}{t} = \gamma_x.$$

Proof. Let A be a spectral operator, and $\mathcal{E}_A$ be the idempotent-valued spectral resolution of A. Since the scalar function $z \mapsto \exp(z)$ is analytic on $\mathrm{sp}\,(A)$, it follows from Theorem 1.3.21 that $\exp(A)$ is a spectral operator, whose idempotent-valued resolution of the identity, $\mathcal{E}_{\exp(A)}$, is given by $\mathcal{E}_{\exp(A)}(\boldsymbol{\beta}) = \mathcal{E}_A(\exp^{-1}\boldsymbol{\beta})$ for a Borel set $\boldsymbol{\beta}$.

For $\lambda \in \mathbb{R}$, let $F_\lambda := \mathbf{R}\big(\mathcal{E}_{\exp(A)}\,(\mathbb{D}_\lambda)\big)$. Note that $\mathcal{E}_{\exp(A)}\left(\mathbb{D}_{\exp(\lambda)}\right) = \mathcal{E}_A(\mathbb{H}_\lambda)$ for all $\lambda \in \mathbb{R}$, whence $G_\lambda = F_{\exp(\lambda)}$ for all $\lambda \in \mathbb{R}$. Since $\exp|_{\mathbb{R}}$ is a strictly increasing continuous function and $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity, it follows that $\{G_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity. Moreover, the projection-valued measure corresponding to $\{G_\lambda\}_{\lambda\in\mathbb{R}}$ is the pushforward of the projection-valued measure corresponding to $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ under $\exp|_{\mathbb{R}}$. Thus, using Theorem 3.4.2-(i) and noting from the spectral mapping theorem that $\mathrm{r}(\exp(A)) = \exp(\mathrm{r}(A))$, we conclude from a change of variables (cf. [Bog07, Theorem 3.6.1]) that

$$\lim_{k\to\infty} |\exp(kA)|^{\frac{1}{k}} = \int_0^{\mathrm{r}(\exp(A))} \lambda \;\mathrm{d}F_\lambda = \int_{-\infty}^{\mathrm{r}(A)} \exp(\lambda)\,\mathrm{d}G_\lambda, \text{ in norm,}$$

and the spectrum of the limit is,

$$\begin{aligned} |\,\mathrm{sp}\,(\exp(A))\,| &= \{|\lambda| : \lambda \in \mathrm{sp}\,(\exp(A))\} = \{|\exp(\lambda)| : \lambda \in \mathrm{sp}\,(A)\} \\ &= \{\exp(\Re\lambda) : \lambda \in \mathrm{sp}\,(A)\} = \exp\left(\Re\,(\mathrm{sp}\,(A))\right). \end{aligned}$$

From Theorem 3.4.2-(ii), for every vector $x \in \mathscr{H}$, there is a smallest non-negative real number λ_x such that x lies in the range of the spectral idempotent $\mathcal{E}_{\exp(A)}(\mathbb{D}_{\lambda_x})$, which may be obtained as the following limit,

$$\lim_{\substack{k\to\infty\\ k\in\mathbb{N}}} \|\exp(kA)\,x\|^{\frac{1}{k}} = \lambda_x.$$

Let $x \in \mathscr{H}$ be a non-zero vector. Since $\exp(A)$, being invertible, has trivial nullspace, we observe that $\Lambda_x := \{\lambda \geq 0 : x \in \mathrm{ran}\left(\mathcal{E}_{\exp(A)}(\mathbb{D}_\lambda)\right)\}$ is bounded below by $\left(\mathrm{r}(\exp(A)^{-1})\right)^{-1}$. In particular, $\lambda_x = \min \Lambda_x > 0$ for all $x \neq 0$.

Let $\gamma_x := \ln \lambda_x$. Since $\mathcal{E}_A(\mathbb{H}_\lambda) = \mathcal{E}_{\exp(A)}\left(\mathbb{D}_{\exp(\lambda)}\right)$ for all $\lambda \in \mathbb{R}$ and $\exp(\lambda)$ is a strictly increasing function, it follows that γ_x is the smallest real number such that x lies in the range of the spectral idempotent $\mathcal{E}_A(\mathbb{H}_{\gamma_x})$. Hence,

$$\begin{aligned} &\lim_{\substack{k\to\infty\\ k\in\mathbb{N}}} \|\exp(kA)\,x\|^{\frac{1}{k}} = \exp(\gamma_x) \\ \Rightarrow &\lim_{\substack{k\to\infty\\ k\in\mathbb{N}}} \frac{\log\left(\|\exp(kA)\,x\|\right)}{k} = \gamma_x. \end{aligned}$$

Imitating the steps in the proof of [Nay23, Theorem 4.1], we conclude that

$$\lim_{\substack{t\to\infty\\ t\in\mathbb{R}}} |\exp(tA)|^{\frac{1}{t}} = \lim_{\substack{k\to\infty\\ k\in\mathbb{N}}} |\exp(kA)|^{\frac{1}{k}},$$

and that

$$\lim_{\substack{t\to\infty\\ t\in\mathbb{R}}} \|\exp(tA)\,x\|^{\frac{1}{t}} = \lim_{\substack{k\to\infty\\ k\in\mathbb{N}}} \|\exp(kA)\,x\|^{\frac{1}{k}} = \exp(\gamma_x) \text{ for every } x\in\mathscr{H}. \qquad \square$$

3.5 Normalized power sequences of weighted shift operators

In [HS09, Example 8.4], an example of a weighted shift operator due to Voiculescu is given whose normalized power sequence does **not** converge in SOT, and *a fortiori*, does **not** converge in norm. In [BB24, Example 3.12], Bhat and Bala consider yet another example of a weighted shift operator that does **not** converge in WOT, and *a fortiori*, does **not** converge in norm. In Proposition 3.5.2 below, we characterize all unilateral weighted shift operators for which the normalized power sequence converges in norm.

Definition 3.5.1. Let $w = \{w_k\}_{k\in\mathbb{N}}$ be a bounded sequence of complex numbers, with the convention that $w_0 := 0$. Let $\{\delta_k\}_{k\in\mathbb{N}}$ denote the standard orthonormal basis of $\ell^2(\mathbb{N})$, with the convention $\delta_0 := 0$. The bounded operators $F_w, B_w : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined on the standard orthonormal basis as $F_w(\delta_m) = w_m\delta_{m+1}, B_w(\delta_m) = w_m\delta_{m-1}$, respectively, for $m\in\mathbb{N}$, are said to be the unilateral *weighted forward-shift operator, weighted backward-shift operator*, respectively, with weight w.

Proposition 3.5.2. *Let $w = \{w_k\}_{k\in\mathbb{N}}$ be a bounded sequence of complex numbers.*

(i) For $m, k \in \mathbb{N}$, let

$$\alpha_{m,k} := \left(\prod_{i=0}^{k-1} |w_{m+i}|\right)^{\frac{1}{k}}.$$

Then the normalized power sequence of F_w converges in norm if and only if there is a non-negative real number α such that $\lim_{k\to\infty}\alpha_{m,k} = \alpha$ uniformly in m; in this case, $\lim_{k\to\infty}|F_w^k|^{\frac{1}{k}} = \alpha I$ in norm.

(ii) The normalized power sequence of B_w converges in norm if and only if $\lim_{k\to\infty} w_k = 0$; in this case, $\lim_{k\to\infty}|B_w^k|^{\frac{1}{k}} = 0$ in norm.

Proof. We use the convention $w_m := 0$ for $m\in\mathbb{Z}\backslash\mathbb{N}$. Note that $F_w^*(\delta_m) = \overline{w_{m-1}}\,\delta_{m-1}$ and $B_w^*(\delta_m) = \overline{w_{m+1}}\delta_{m+1}$ for $m\in\mathbb{N}$. Let E_{δ_m} denote the projection onto the span of δ_m for $m\in\mathbb{N}$. A straightforward computation tells us that $(F_w^k)^*F_w^k = \sum_{m\in\mathbb{N}}(\prod_{i=0}^{k-1}|w_{m+i}|^2)E_{\delta_m}$ and $(B_w^k)^*B_w^k = \sum_{m\in\mathbb{N}}(\prod_{i=0}^{k-1}|w_{m-i}|^2)E_{\delta_m}$.

(i) Since

$$|F_w^k|^{\frac{1}{k}} = \sum_{m\in\mathbb{N}} \Big(\prod_{i=0}^{k-1} |w_{m+i}|\Big)^{\frac{1}{k}} E_{\delta_m} = \sum_{m\in\mathbb{N}} \alpha_{m,k}\, E_{\delta_m},$$

the assertion follows.

(ii) Note that $|B_w^k|^{\frac{1}{k}} \to 0$ in SOT. So the only possibility for its norm-limit is 0. From above, we have

$$|B_w^k|^{\frac{1}{k}} = \sum_{m\in\mathbb{N}} \Big(\prod_{i=0}^{k-1} |w_{m-i}|\Big)^{\frac{1}{k}} E_{\delta_m}.$$

Observe that for each $m \in \mathbb{N}$, the limit $\lim_{k\to\infty} \prod_{i=0}^{k-1} |w_{m-i}| = 0$, and this convergence is uniform in m if and only if $\lim_{k\to\infty} |w_k| = 0$ if and only if $\lim_{k\to\infty} w_k = 0$. Thus the assertion follows.

□

3.6 Concluding remarks

From [HS09], [DKU21] and our discussion, there appears to be an intimate connection between the notion of spectrality of an operator (and its variants) with the convergence of its normalized power sequence in an appropriate operator topology (such as norm-topology, SOT, WOT, etc.). Although the normalized power sequence of every compact operator on $\mathscr{H}$ converges in norm (see [BB24]), there are compact operators which are **not** spectral in the sense of Dunford (see Examples 4.3.1, 4.3.2). Might some modification of the notion of spectrality account for all compact operators? Or is there a phenomenon broader than spectrality which governs the convergence (or lack thereof) of the normalized power sequence of operators? In view of these mysteries, one may pose a general question.

Question : *Characterize all operators $A \in \mathscr{B}(\mathscr{H})$ for which the normalized power sequence converges in norm/* SOT/ WOT*.*

Chapter 4

Dunford-type decomposition for compact operators

4.1 Introduction

Compact operators are often viewed as infinite-dimensional analogues of matrices, serving as a natural bridge between the finite and infinite dimensions. Beyond their approximation by finite-rank operators, compact operators also exhibit deep spectral behaviour reminiscent of spectral operators. While compact operators are not, in general, spectral operators, they still enjoy a notable spectral theory : each non-zero spectral value of a compact operator is isolated, and admits a Riesz idempotent, which projects onto the corresponding generalized eigenspace. These Riesz idempotents, though not uniformly bounded, play a role analogous to the idempotent-valued spectral resolutions available for spectral operators. This resemblance is more than superficial; it points toward a deeper structural affinity between compact and spectral operators.

Further evidence of this affinity arises from the behaviour of normalized power sequences. For a compact operator acting on a complex separable Hilbert space, Bhat and Bala showed in [BB24] that the sequence $\{|C^k|^{\frac{1}{k}}\}_{k\in\mathbb{N}}$ converges in norm. This further motivates the quest to understand compact operators through the lens of spectral-type decompositions. The question of whether a Dunford-type decomposition exist for compact operators guides our investigation in this chapter. In Theorem 4.4.10, we establish a Dunford-type decomposition for compact operators in $\mathscr{B}(\mathscr{H})$. While the components of this decomposition need not be bounded, they retain enough structure to justify the terminology of a Dunford-type decomposition; the scalar-type and quasinilpotent parts are shown to be simultaneously quasi-similar to a bounded normal operator and a bounded quasinilpotent operator, which commute with each other. In the context of Remark 3.2.4, this decomposition generalizes the Dunford decomposition, by relaxing the notion of similarity to that of quasi-similarity.

We begin this chapter by revisiting the norm convergence of normalized power sequences for compact operators and prove a more general version of the result of Bhat and Bala.

We then construct two examples of non-spectral compact operators acting on the infinite-dimensional complex separable Hilbert space $\bigoplus_{k\in\mathbb{N}} \mathbb{C}^3$, before moving on to the final section, which is devoted to the main result of this chapter.

4.2 Norm convergence of the normalized power sequence of compact operators

In this section, motivated by the techniques used in Chapter 3, we provide an alternative proof of [BB24, Theorem 3.10], which asserts the norm-convergence of the normalized power sequence for compact operators acting on a complex separable infinite-dimensional Hilbert space. Note that our proof in Theorem 4.2.4 is valid for non-separable Hilbert spaces as well. It is motivated by the fact that compact operators are not too far from being spectral, and may admit an idempotent-valued spectral resolution if the condition of uniform boundedness, Definition 1.3.10-(iv), is dropped; the natural candidates for the spectral idempotents being the associated Riesz idempotents. Our approach also yields an explicit description of the limiting operator in terms of the Riesz idempotents associated with the compact operator.

Lemma 4.2.1. *Let $\mathscr{H}$ be a complex Hilbert space, and $\mathcal{E}$ be a bounded idempotent in $\mathscr{B}(\mathscr{H})$. Then,*

$$|\mathcal{E}|^2 + |I-\mathcal{E}|^2 = \mathcal{E}^*\mathcal{E} + (I-\mathcal{E})^*(I-\mathcal{E}),$$

is a positive invertible operator.

Proof. Since $\mathcal{E}^*$ is an idempotent in $\mathscr{B}(\mathscr{H})$, $\operatorname{ran}(\mathcal{E}^*) + \operatorname{ran}(I-\mathcal{E}^*) = \mathscr{H}$, so that the angle between $\operatorname{ran}(\mathcal{E}^*)$ and $\ker(\mathcal{E}^*)$ is non-zero. By a corollary of Douglas lemma ([Dou66]), $\operatorname{ran}\big(\sqrt{|\mathcal{E}|^2 + |I-\mathcal{E}|^2}\big) = \operatorname{ran}(\mathcal{E}^*) + \operatorname{ran}(I-\mathcal{E}^*) = \mathscr{H}$. Thus $|\mathcal{E}|^2| + |I-\mathcal{E}|^2$ is a positive operator with full range, whence it is invertible. □

Lemma 4.2.2. *Let $C \in \mathscr{B}(\mathscr{H})$ be a compact operator, and $0 < \mu < 1$. Let $\mathcal{E}_\mu$ be the Riesz idempotent of C corresponding to $\operatorname{sp}(C) \cap \mathbb{D}_\mu$. Then there is a positive integer $n(\mu)$ such that for all $k \ge n(\mu)$, we have*

$$(1-\mu)\,\big|\big(C(I-\mathcal{E}_\mu)\big)^k\big|^{\frac{1}{k}} \le |C^k|^{\frac{1}{k}} \le (1+\mu)\,\big|\big(C(I-\mathcal{E}_\mu)\big)^k\big|^{\frac{1}{k}} + 4\mu I.$$

Proof. Note that C and $\mathcal{E}_\mu$ commute with each other, and $\operatorname{sp}(C\mathcal{E}_\mu) \subseteq \mathbb{D}_\mu$. From the spectral radius formula, there is an $n_1(\mu) \in \mathbb{N}$ such that $\|(C\mathcal{E}_\mu)^k\|^{\frac{1}{k}} \le 2\mu$ for all $k \ge n_1(\mu)$. It follows from (OI1) that

$$(C^k)^*({\mathcal{E}_\mu}^*\mathcal{E}_\mu)C^k \;\le\; (2\mu)^{2k} I, \quad \text{ for all } k \ge n_1(\mu). \tag{4.2.1}$$

Let $\mathcal{E}'_\mu := I - \mathcal{E}_\mu$. Adding the positive operator ${C^k}^*{\mathcal{E}'_\mu}^*\mathcal{E}'_\mu C^k$ both the sides in (4.2.1), for all $k \ge n_1(\mu)$, we have

$${C^k}^*{\mathcal{E}'_\mu}^*\mathcal{E}'_\mu C^k \;\le\; {C^k}^*({\mathcal{E}'_\mu}^*\mathcal{E}'_\mu + \mathcal{E}_\mu^*\mathcal{E}_\mu)C^k \;\le\; {C^k}^*{\mathcal{E}'_\mu}^*\mathcal{E}'_\mu C^k + (2\mu)^{2k} I. \tag{4.2.2}$$

Let $H_\mu := {\mathcal{E}'_\mu}^* \mathcal{E}'_\mu + \mathcal{E}_\mu^* \mathcal{E}_\mu$. By Lemma 4.2.1, H_μ is an invertible positive operator. Thus from (OI2), we have the operator inequality,

$$\|H_\mu^{-1}\|^{-1} I \;\le\; H_\mu \;\le\; \|H_\mu\| I,$$

and from (OI3), we have

$$\|H_\mu^{-1}\|^{-1} {C^k}^* C^k \;\le\; {C^k}^* H_\mu C^k \;\le\; \|H_\mu\| {C^k}^* C^k. \tag{4.2.3}$$

Combining the LHS of (4.2.3) with RHS of (4.2.2), for all $k \ge n_1(\mu)$, we have

$$\|H_\mu^{-1}\|^{-1} {C^k}^* C^k \le {C^k}^* H_\mu C^k \le {C^k}^* {\mathcal{E}'_\mu}^* \mathcal{E}'_\mu C^k + (2\mu)^{2k} I.$$

Using (OI4), we have

$$\|H_\mu^{-1}\|^{-\frac{1}{2k}} \, |C^k|^{\frac{1}{k}} \;\le\; \left({C^k}^* {\mathcal{E}'_\mu}^* \mathcal{E}'_\mu C^k + (2\mu)^{2k} I\right)^{\frac{1}{2k}} \;\le\; |(C\mathcal{E}'_\mu)^k|^{\frac{1}{k}} + (2\mu) I \quad \text{for all } k \ge n_1(\mu).$$

Since $\lim_{k\to\infty} \|H_\mu^{-1}\|^{\frac{1}{2k}} = 1 = \lim_{k\to\infty} \|H_\mu\|^{-\frac{1}{2k}}$, we note that there is an $n_2(\mu) \in \mathbb{N}$ such that $\|H_\mu^{-1}\|^{\frac{1}{2k}} \le 1+\mu$ and $\|H_\mu\|^{-\frac{1}{2k}} \ge 1-\mu$ for all $k \ge n_2(\mu)$. Let $n(\mu) = \max\{n_1(\mu), n_2(\mu)\}$. Then,

$$|C^k|^{\frac{1}{k}} \le (1+\mu)\; |(C\mathcal{E}'_\mu)^k|^{\frac{1}{k}} + 4\mu I \quad \text{for all } k \ge n(\mu). \tag{4.2.4}$$

From the LHS of (4.2.2), we have

$$|(C\mathcal{E}'_\mu)^k|^2 \le |C^k|^2 \|H_\mu\| \quad \text{for all } k \ge n(\mu),$$

and it follows from (OI4), that

$$\begin{aligned} \|H_\mu\|^{-\frac{1}{2k}} |(C\mathcal{E}'_\mu)^k|^{\frac{1}{k}} \;&\le\; |C^k|^{\frac{1}{k}} \\ \implies (1-\mu)|(C\mathcal{E}'_\mu)^k|^{\frac{1}{k}} \;&\le\; |C^k|^{\frac{1}{k}} \quad \text{for all } k \ge n(\mu) \end{aligned} \tag{4.2.5}$$

The assertion follows by combining the inequalities in (4.2.5) and (4.2.4). □

We now restrict the spectral operator version of the result to finite-rank operators, before extending it to compact operators.

Proposition 4.2.3. *Let A be a finite-rank operator in $\mathscr{B}(\mathscr{H})$ with non-zero eigenvalues having modulus strictly greater than $\varepsilon > 0$. For $\lambda \ge 0$, let $\mathcal{E}_\lambda$ denote the Riesz idempotent of A corresponding to $\mathbb{D}_\lambda$. For $\lambda \ge 0$, let $F_\lambda := \mathbf{R}(\mathcal{E}_\lambda)$ and for $\lambda < 0$, define $F_\lambda = 0$. Then F_λ is a bounded resolution of the identity and we have*

$$\lim_{k\to\infty} |A^k|^{\frac{1}{k}} = \int_{(\varepsilon, \mathrm{r}(A)]} \lambda \, \mathrm{d}F_\lambda, \text{ in norm.}$$

Moreover, the spectrum of the limiting positive operator is $|\mathrm{sp}\,(A)\,|$, *the modulus of the spectrum of* A.

Proof. Since A is finite-rank, A^* is finite-rank (see [Con13, Theorem 4.4]). Let $\mathscr{V}$ be the span of $\mathrm{ran}(A)$ and $\mathrm{ran}(A^*)$, then $\mathscr{V}$ is a finite dimensional subspace of $\mathscr{H}$. Note that $\mathscr{V}^\perp \subseteq \ker(A) \cap \ker(A^*)$.

Let $E_{\mathscr{V}}$ denote the orthogonal projection onto the subspace $\mathscr{V} \subseteq \mathscr{H}$. Then $E_{\mathscr{V}}$ commutes with both A and A^*, and $A = AE_{\mathscr{V}}$. Thus, we may view A as an operator in $\mathscr{B}(\mathscr{V})$. Since $\mathscr{V}$ is finite-dimensional, A is spectral. Observe that the family of Riesz idempotents $\{\mathcal{E}_\lambda\}$ satisfies the properties listed in Definition 1.3.10, and hence coincides with the unique idempotent-valued spectral resolution of A for spectral operators, that is, $\mathcal{E}_A(\mathbb{D}_\lambda) = \mathcal{E}_\lambda$ for all $\lambda \geq 0$. The assertion then follows by an application of Theorem 3.4.2. □

Theorem 4.2.4. *Let* $C \in \mathscr{B}(\mathscr{H})$ *be a compact operator, and for* $\lambda > 0$, *let* $\mathcal{E}_\lambda$ *denote the Riesz idempotent of* C *corresponding to* $\mathbb{D}_\lambda$. *Let* $F_\lambda := \mathbf{R}(\mathcal{E}_\lambda)$ *for* $\lambda > 0$; $F_0 := \bigwedge_{\lambda'>0} F_{\lambda'}$; *and* $F_\lambda = 0$ *for* $\lambda < 0$. *Then* $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ *is a bounded resolution of the identity and we have*

$$\lim_{k\to\infty} |C^k|^{\frac{1}{k}} = \int_0^{\mathrm{r}(C)} \lambda \, \mathrm{d}F_\lambda, \text{ in norm.}$$

Moreover, the spectrum of the limiting positive operator, $\int_0^{\mathrm{r}(C)} \lambda \ \mathrm{d}F_\lambda$, *is,*

$$|\mathrm{sp}\,(C)\,| := \{|\lambda| : \lambda \in \mathrm{sp}\,(C)\},$$

the modulus of the spectrum of C.

Proof. First we prove that $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a bounded resolution of the identity. Clearly $\bigwedge_{\lambda\in\mathbb{R}} F_\lambda = 0$, and since $\mathcal{E}_\lambda = I$ when $\lambda \geq \mathrm{r}(C)$, we have $\bigvee_{\lambda\in\mathbb{R}} F_\lambda = I$. If $0 < \lambda \leq \lambda'$, then $\mathbb{D}_\lambda \subseteq \mathbb{D}_{\lambda'}$, so $\mathrm{ran}(\mathcal{E}_\lambda) \subseteq \mathrm{ran}(\mathcal{E}_{\lambda'})$ and hence $F_\lambda \leq F_{\lambda'}$. Since $\mathrm{sp}\,(C) \setminus \{0\}$ is a discrete subset of $\mathbb{C}$, we observe that $|\mathrm{sp}\,(C)\,| \setminus \{0\}$ is a discrete subset of $\mathbb{R}$. If $\lambda > 0$ and $\delta = \mathrm{d}(\lambda, |\mathrm{sp}\,(C)\,| \setminus \{\lambda\})$, then for $\lambda < \lambda' < \lambda + \delta$ we have $\mathcal{E}_{\lambda'} = \mathcal{E}_\lambda$ so that $F_{\lambda'} = F_\lambda$. Thus, clearly $F_\lambda = \bigwedge_{\lambda'>\lambda} F_{\lambda'}$ for $\lambda > 0$. For $\lambda \leq 0$, by definition $F_\lambda = \bigwedge_{\lambda'>\lambda} F_{\lambda'}$. This shows that $\{F_\lambda\}$ is a bounded resolution of the identity.

Let $0 < \varepsilon < 1$, and define

$$H_\varepsilon := \int_{(\varepsilon, \mathrm{r}(C)]} \lambda \, \mathrm{d}F_\lambda, \quad \text{and} \quad H := \int_{[0,\mathrm{r}(C)]} \lambda \ dF_\lambda.$$

Let $\mathcal{E}'_\varepsilon := I - \mathcal{E}_\varepsilon$. Note that $\mathrm{ran}(\mathcal{E}'_\varepsilon)$ is finite-dimensional, so that $C\mathcal{E}'_\varepsilon$ is a finite-rank operator. Moreover, $\mathrm{sp}\,(C\mathcal{E}'_\varepsilon) \subseteq (\mathbb{D}_{\mathrm{r}(C)} \setminus \mathbb{D}_\varepsilon) \cup \{0\}$. From Proposition 4.2.3,

$$\lim_{k\to\infty} |(C\mathcal{E}'_\varepsilon)^k|^{\frac{1}{k}} = H_\varepsilon, \text{ in norm.}$$

Then there is an $n_0(\varepsilon) \in \mathbb{N}$ such that $\left\| \, |(C\mathcal{E}'_\varepsilon)^k|^{\frac{1}{k}} - H_\varepsilon \right\| \leq \varepsilon$ for all $k \geq n_0(\varepsilon)$. From Lemma 4.2.2, there is a positive integer $n(\varepsilon) \geq n_0(\varepsilon)$ such that

$$(1-\varepsilon)|(C\mathcal{E}'_\varepsilon)^k|^{\frac{1}{k}} \leq |C^k|^{\frac{1}{k}} \leq (1+\varepsilon)|(C\mathcal{E}'_\varepsilon)^k|^{\frac{1}{k}} + 4\varepsilon I \quad \text{for all } k \geq n(\varepsilon). \tag{4.2.6}$$

Clearly $\|H - H_\varepsilon\| \leq \varepsilon$, so that from (OI1), we have $-\varepsilon I \leq H - H_\varepsilon \leq \varepsilon I$. Then, from the RHS inequality of (4.2.6), for all $k \geq n(\varepsilon)$,

$$\begin{aligned} |C^k|^{\frac{1}{k}} - H &\leq (1+\varepsilon)(|(C\mathcal{E}'_\varepsilon)^k|^{\frac{1}{k}} - H_\varepsilon) + (1+\varepsilon)(H_\varepsilon - H) + \varepsilon H + 4\varepsilon I \\ &\leq 2\varepsilon(1+\varepsilon)I + \varepsilon\|H\|I + 4\varepsilon I \\ &\leq (8 + \|H\|)\varepsilon I. \end{aligned}$$

Similarly, from the LHS inequality of (4.2.6), for all $k \geq n(\varepsilon)$,

$$\begin{aligned} |C^k|^{\frac{1}{k}} - H &\geq (1-\varepsilon)(|(C\mathcal{E}'_\varepsilon)^k|^{\frac{1}{k}} - H_\varepsilon) + (1-\varepsilon)(H_\varepsilon - H) - \varepsilon H \\ &\geq -2\varepsilon(1-\varepsilon)I - \varepsilon\|H\|I \\ &\geq -(2 + \|H\|)\varepsilon I. \end{aligned}$$

Hence, for all $k \geq n(\varepsilon)$, we have $\left\| |C^k|^{\frac{1}{k}} - H \right\| \leq (8 + \|H\|)\varepsilon$. In summary,

$$\lim_{k\to\infty} |C^k|^{\frac{1}{k}} = H, \text{ in norm.}$$

Let $\lambda_0 \in \mathbb{R}$. Note that as a function of λ, $\mathcal{E}_\lambda$ is constant in a neighbourhood of λ_0 if and only if F_λ is constant in a neighbourhood of λ_0. From Theorem 1.3.8, we observe that $\lambda_0 \notin \operatorname{sp}(H)$ if and only if $\lambda_0 \notin |\operatorname{sp}(C)|$, which completes the proof. □

4.3 Examples of non-spectral compact operators

In this section, we show that a bounded compact operator acting on an infinite-dimensional Hilbert space is not necessarily spectral.

***Example* 4.3.1.** *For $k \in \mathbb{N}$, consider the 3×3 complex matrix,*

$$A_k := \begin{pmatrix} \frac{1}{k} & \left(\frac{1}{k}\right)^{\frac{1}{4}} & 0 \\ 0 & \frac{1}{k} & \left(\frac{1}{k}\right)^{\frac{1}{4}} \\ 0 & 0 & 0 \end{pmatrix},$$

viewed as an operator acting on the Hilbert space $\mathbb{C}^3$ with the standard inner product. Then the operator defined by the orthogonal sum $\bigoplus_{k\in\mathbb{N}} A_k$ is a compact operator acting on the separable Hilbert space $\bigoplus_{k\in\mathbb{N}}\mathbb{C}^3$, which is **not** *spectral in the sense of Dunford.*

Proof. Let $\mathscr{H}$ denote the separable Hilbert space $\oplus_{k\in\mathbb{N}}\mathbb{C}^3$. Since $\|A_k\| \leq \|A_1\|$ for all $k \in \mathbb{N}$, A is a bounded linear operator on $\mathscr{H}$. Furthermore, since $\|A_k\| \longrightarrow 0$ as $k \to \infty$, A is seen to be the norm-limit of the sequence of finite-rank operators $\left\{\bigoplus_{k=1}^{m} A_k\right\}_{m\in\mathbb{N}}$, so that A is a compact operator in $\mathscr{B}(\mathscr{H})$. It is easy to see that the spectrum of A is given by $\mathrm{sp}\,(A) = \{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\}$.

Suppose, if possible, that A is a spectral operator, and let $\mathcal{E}_A$ be the idempotent-valued spectral resolution of A. For $k \in \mathbb{N}$, define the 3×3 matrix,

$$\mathcal{E}_k := \begin{pmatrix} 1 & 0 & -k^{\frac{3}{2}} \\ 0 & 1 & k^{\frac{3}{4}} \\ 0 & 0 & 0 \end{pmatrix}.$$

A straightforward computation shows that for each $k \in \mathbb{N}$, $\mathcal{E}_k$ is an idempotent such that $A_k\mathcal{E}_k = \mathcal{E}_kA_k = A_k$ with $\mathrm{sp}\left(A_k|_{\mathcal{E}_k\mathbb{C}^3}\right) = \{\frac{1}{k}\}$, and $\mathrm{sp}\left(A_k|_{(I-\mathcal{E}_k)\mathbb{C}^3}\right) = \{0\}$. Indeed, if $\mathcal{E}_{A_k}$ denotes the idempotent-valued spectral resolution of the spectral operator A_k, then $\mathcal{E}_k = \mathcal{E}_{A_k}(\{\frac{1}{k}\})$.

For $k \in \mathbb{N}$, define $\mathcal{E}'_k := \bigoplus_{m\in\mathbb{N}}\delta_{km}\mathcal{E}_k$, where δ_{km} denotes the Kronecker delta. Then $\mathcal{E}'_k$ is an idempotent in $\mathscr{B}(\mathscr{H})$.

Claim : *For each* $k \in \mathbb{N}$, $\mathcal{E}'_k = \mathcal{E}_A(\{\frac{1}{k}\})|_{\mathcal{E}'_k\mathscr{H}}$.

Proof of Claim. Let $A'_k := \bigoplus_{m\in\mathbb{N}}\delta_{km}A_k$. Since $\mathcal{E}_{A_k}$ is the idempotent-valued spectral resolution of A_k, from the uniqueness of the idempotent-valued spectral resolution, $\bigoplus_{m\in\mathbb{N}}\delta_{km}\mathcal{E}_{A_k}$ must be the idempotent-valued spectral resolution of the finite-rank operator $\bigoplus_{m\in\mathbb{N}}\delta_{km}A_k$. On the other hand, since A is a spectral operator, from Theorem 1.3.20, $A|_{\mathcal{E}'_k\mathscr{H}} = A'_k$ is a spectral operator whose idempotent-valued spectral resolution, $\mathcal{E}_{A'_k}$, is given by the restriction of $\mathcal{E}_A$ on $\mathcal{E}'_k\mathscr{H}$. Thus,

$$\mathcal{E}'_k = \bigoplus_{m\in\mathbb{N}}\delta_{km}\mathcal{E}_k = \bigoplus_{m\in\mathbb{N}}\delta_{km}\mathcal{E}_{A_k}(\{\tfrac{1}{k}\}) = \mathcal{E}_{A'_k}(\{\tfrac{1}{k}\}) = \mathcal{E}_A(\{\tfrac{1}{k}\})|_{\mathcal{E}'_k\mathscr{H}}. \qquad \square$$

Observe that $\|\mathcal{E}'_k\| = \|\mathcal{E}_k\|$. From the above claim, it is clear that $\|\mathcal{E}'_k\| \leq \|\mathcal{E}_A(\{\frac{1}{k}\})\|$. Since $\|\mathcal{E}_k\| \to \infty$ as $k \to \infty$, $\|\mathcal{E}_A(\{\frac{1}{k}\})\|$ blows up as $k \to \infty$, which contradicts the property (iv) of Definition 1.3.10. Hence, A cannot be a spectral operator. $\square$

Example 4.3.2. *For* $k \in \mathbb{N}$, *consider the* 3×3 *complex matrix,*

$$B_k := \begin{pmatrix} \frac{1}{k} & (\frac{1}{k})^{\frac{1}{2}} & 0 \\ 0 & \frac{1}{k} & (\frac{1}{k})^{\frac{1}{2}} \\ 0 & 0 & 0 \end{pmatrix},$$

viewed as an operator acting on the Hilbert space $\mathbb{C}^3$ with the standard inner product. Then the operator $B := \bigoplus_{k\in\mathbb{N}} B_k$ is a compact operator acting on the separable Hilbert space, $\mathscr{H} = \bigoplus_{k\in\mathbb{N}}\mathbb{C}^3$, with the following two properties:

(i) There is a pair D, N of commuting operators in $\mathscr{B}(\mathscr{H})$ such that D is quasi-similar to a bounded normal operator, N is a nilpotent operator, and $B = D + N$.

(ii) The operators D, N do not lie in $\mathcal{K}(\mathscr{H}) + \mathbb{C}I$, where $\mathcal{K}(\mathscr{H})$ denotes the ideal of compact operators in $\mathscr{B}(\mathscr{H})$.

Proof. Since $\|B_k\| \le \|B_1\|\ \forall\ k \in \mathbb{N}$, B is a bounded linear operator on $\mathscr{H}$. Since each B_k is a finite-rank operator and $\|B_k\| \to 0$ as $k \to \infty$, B is a compact operator in $\mathscr{B}(\mathscr{H})$. Note that $\mathrm{sp}(B) = \{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\}$.

(i) For $k \in \mathbb{N}$, let

$$D_k := \begin{pmatrix} \frac{1}{k} & 0 & -1 \\ 0 & \frac{1}{k} & (\frac{1}{k})^{\frac{1}{2}} \\ 0 & 0 & 0 \end{pmatrix}, \quad N_k := \begin{pmatrix} 0 & (\frac{1}{k})^{\frac{1}{2}} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_k := \begin{pmatrix} 1 & 0 & -k \\ 0 & 1 & k^{\frac{1}{2}} \\ 0 & 0 & 1 \end{pmatrix}.$$

For each $k \in \mathbb{N}$, note from Theorem 2.6.4 that D_k and N_k are commuting diagonalizable and nilpotent parts, respectively, in the Jordan-Chevalley decomposition of B_k, and S_k is an invertible matrix such that $S_k D_k S_k^{-1} = \mathrm{diag}\{0, \frac{1}{k}, \frac{1}{k}\}$.

Let $D := \bigoplus_{k\in\mathbb{N}} D_k$, and $N := \bigoplus_{k\in\mathbb{N}} N_k$. Then, D and N are bounded operators on $\mathscr{H}$ as $||D|| \le ||D_1||$ and $||N|| \le ||N_1||$. It is clear that D and N commute with each other, $B = D + N$, and N is a nilpotent with $N^3 = 0$.

Let $M := \bigoplus_{k\in\mathbb{N}} \mathrm{diag}\{0, \frac{1}{k}, \frac{1}{k}\}, S := \bigoplus_{k\in\mathbb{N}} \frac{1}{k} S_k$, and $S' := \bigoplus_{k\in\mathbb{N}} \frac{1}{k} S_k^{-1}$. Then M is a normal operator, and S, S' are quasi-invertible operators, in $\mathscr{B}(\mathscr{H})$.

It is straightforward to verify that $SD = MS$, and $DS' = S'M$, which shows that D and M are quasi-similar operators.

(ii). Let $\{\mathbf{e}_k : k \in \mathbb{N}\}$ denote the standard orthonormal basis of the Hilbert space $\mathscr{H}$. Note that $N\mathbf{e}3k + 1 = \mathbf{e}3k$ for $k \in \mathbb{N}_0$, so the image of this bounded sequence under N has no convergent subsequence. This shows that N is not a compact operator. Moreover, for a non-zero scalar $\alpha \in \mathbb{C}$, $\mathrm{sp}\,(N - \alpha I) = \{-\alpha\}$ so that $N - \alpha I$ is invertible, and thus cannot be compact.

Thus, N does not lie in $\mathcal{K}(\mathscr{H}) + \mathbb{C}I$. Consequently, since $B = D + N$ is a compact operator, D does not lie in $\mathcal{K}(\mathscr{H}) + \mathbb{C}I$. □

***Remark* 4.3.3.** We note, by imitating the proof in Example 4.3.1, that B is another example of a non-spectral compact operator. Alternatively, one can show that if B is spectral, then D must be the scalar-type operator in the Dunford-decomposition of B. Example 4.3.2-(ii) then contradicts [DS88, Corollary XV.7.3] (The scalar-type component and the quasinilpotent component of a compact spectral operator are both compact), proving that B cannot be spectral.

The non-spectral compact operators A and B, encountered in Example 4.3.1 and Example 4.3.2 differ significantly : the candidate for the scalar-type part of A (in Example 4.3.1) is unbounded, whereas for B (in Example 4.3.2), it is bounded.

4.4 Dunford-type decomposition for compact operators

We have seen that a compact operator need not be spectral, and thus may not admit a Dunford decomposition. Nevertheless, compact operators share some properties with spectral operators, as listed below.

Proposition 4.4.1. *Let $C \in \mathscr{B}(\mathscr{H})$ be a compact operator. Then for each $x \in \mathscr{H}$, the function $R(\mu; C)x$ has the single valued extension property.*

Proof. Let $x \in \mathscr{H}$, and let f, g be two analytic extensions of $R(\mu; C)x$. Define,

$$h(\mu) := f(\mu) - g(\mu) \ ; \ \mu \in \mathrm{dom}(f) \cap \mathrm{dom}(g).$$

If $h(\mu_0) \neq 0$ for some $\mu_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, then there is a neighbourhood, $O_{\mu_0} \subseteq \mathrm{dom}(f) \cap \mathrm{dom}(g)$, of μ_0, such that

$$h(\mu) \neq 0 \ , \ (\mu I - C)h(\mu) = 0 \quad \forall \ \mu \in O_{\mu_0}.$$

This shows that each $\mu \in O_{\mu_0}$ is an eigenvalue of C, which is a contradiction as a compact operator has countably many eigenvalues at the most. □

Thus, using the Definition 1.3.15 for a compact operator $C \in \mathscr{B}(\mathscr{H})$, we may define the local spectrum $\sigma_C(x)$ for each $x \in \mathscr{H}$.

Corollary 4.4.2. *If C is a compact operator in $\mathscr{B}(\mathscr{H})$, then the local spectrum, $\sigma_C(x)$, of x is empty if and only if $x = 0$.*

Proof. It is immediate that $\sigma_C(0) = \varnothing$. Conversely, let $\sigma_C(x) = \varnothing$. Then, the maximal extension $x_C(\mu)$ of $R(\mu; C)x$ is an everywhere defined single valued function, hence entire. Note that for each continuous linear functional ϕ on $\mathscr{H}$, we have

$$\lim_{\mu \to 0} \phi\big(x_C(\mu)\big) = \lim_{\substack{\mu \to 0 \\ \mu \notin \mathrm{sp}(C)}} \phi\big(R(\mu; C)x\big) = 0,$$

which, from Liouville's theorem, implies that $\phi\big(x_C(\mu)\big) = 0$ for all $\mu \in \mathbb{C}$. Since $\phi\big(x_C(\mu)\big) = 0$ for each continuous linear functional ϕ on $\mathscr{H}$, by a corollary of Hahn-Banach separation theorem (see [KR83, Corollary 1.2.11]), $x_C(\mu) = 0$ and thus $x = (\mu I - C)x_C(\mu) = 0$. □

Proposition 4.4.3. *Let $C \in \mathscr{B}(\mathscr{H})$ be a compact operator with $\mathrm{sp}\,(C) \setminus \{0\} \neq \varnothing$. Let $\mathcal{E}_\lambda$ denote the Riesz idempotent of C corresponding to $\lambda \in \mathrm{sp}\,(C) \setminus \{0\}$. Then,*

(i) $\mathcal{E}_\lambda \mathscr{H} = \big\{x \in \mathscr{H} : \sigma_C(x) \subseteq \{\lambda\}\big\}$,

(ii) $(I - \mathcal{E}_\lambda)\mathscr{H} = \big\{x \in \mathscr{H} : \sigma_C(x) \subseteq \mathrm{sp}\,(C) \setminus \{\lambda\}\big\}$.

Proof. Let C_λ and C'_λ be the restriction of C to $\mathcal{E}_\lambda \mathscr{H}$ and $(I - \mathcal{E}_\lambda)\mathscr{H}$, respectively. Since $\mathcal{E}_\lambda \mathscr{H}$ is the generalized eigenspace of C corresponding to the eigenvalue $\lambda \neq 0$, observe that $\mathrm{sp}\,(C_\lambda) = \{\lambda\}$ and $\mathrm{sp}\,(C'_\lambda) = \mathrm{sp}\,(C) \setminus \{\lambda\}$.

(i) Let $x \in \mathcal{E}_\lambda \mathscr{H}$, so that $\mathcal{E}_\lambda x = x$. It is seen from the relation

$$R(\mu; C_\lambda)x = R(\mu; C)\mathcal{E}_\lambda x = R(\mu; C)x,$$

that $R(\mu; C_\lambda)x$ is an analytic extension of $R(\mu; C)x$ to the open set $\mathbb{C} \setminus \mathrm{sp}\,(C_\lambda)$. Thus $\rho_C(x) \supseteq \mathbb{C} \setminus \mathrm{sp}\,(C_\lambda)$ and $\sigma_C(x) \subseteq \mathrm{sp}\,(C_\lambda) = \{\lambda\}$.

Conversely, assume that $\sigma_C(x) \subseteq \{\lambda\}$. Note that $R(\mu; C'_\lambda)(I - \mathcal{E}_\lambda)x$ is an analytic extension of $R(\mu; C)(I - \mathcal{E}_\lambda)x$ to $\mathbb{C} \setminus \mathrm{sp}\,(C'_\lambda)$. Moreover, if $x_C(\mu)$ denotes the maximal analytic extension of $R(\mu; C)x$, then $(I - \mathcal{E}_\lambda)x_C(\mu)$ is an analytic extension of $R(\mu; C)(I - \mathcal{E}_\lambda)x$ to $\rho_C(x)$. Thus, $R(\mu; C)(I - \mathcal{E}_\lambda)x$ has an analytic extension to $\rho_C(x) \cup \mathbb{C} \setminus \mathrm{sp}\,(C'_\lambda) = \mathbb{C}$. This implies that $\sigma_C\big((I - \mathcal{E}_\lambda)x\big) = \varnothing$. From Corollary 4.4.2, $(I - \mathcal{E}_\lambda)x = 0$ and hence $\mathcal{E}_\lambda x = x$.

(ii) Let $x \in (I - \mathcal{E}_\lambda)\mathscr{H}$. Since $R(\mu; C_{\lambda'})x$ is an analytic extension of $R(\mu; C)x$ to the open set $\mathbb{C} \setminus \mathrm{sp}\,(C'_\lambda)$, $\rho_C(x) \supseteq \mathbb{C} \setminus \mathrm{sp}\,(C'_\lambda)$ and $\sigma_C(x) \subseteq \mathrm{sp}\,(C'_\lambda) = \mathrm{sp}\,(C) \setminus \{\lambda\}$.

Conversely, assume that $\sigma_C(x) \subseteq \mathrm{sp}\,(C) \setminus \{\lambda\}$. Since $R(\mu; C_\lambda)\mathcal{E}_\lambda x$ is an analytic extension of $R(\mu; C)\mathcal{E}_\lambda x$ to $\mathbb{C} \setminus \mathrm{sp}\,(C_\lambda)$, and $\mathcal{E}_\lambda x_C(\mu)$ is an analytic extension of $R(\mu; C)\mathcal{E}_\lambda x$ to $\rho_C(x)$, it follows that $R(\mu; C)\mathcal{E}_\lambda x$ has an analytic extension to $\mathbb{C} \setminus \mathrm{sp}\,(C_\lambda) \cup \rho_C(x) = \mathbb{C}$. From Corollary 4.4.2, $\mathcal{E}_\lambda x = 0$ and hence $(I - \mathcal{E}_\lambda)x = x$. □

Proposition 4.4.4. *Let $C \in \mathscr{B}(\mathscr{H})$ be a compact operator with $\mathrm{sp}\,(C) \setminus \{0\} \neq \varnothing$, and $\mathcal{E}_\lambda$ denote the Riesz idempotent of C corresponding to $\lambda \in \mathrm{sp}\,(C) \setminus \{0\}$. If A is a linear operator acting on $\mathscr{H}$ such that $CA \subseteq AC$ and $\mathcal{E}_\lambda \mathscr{H} \subseteq \mathrm{dom}(A)$, then $\mathcal{E}_\lambda A \subseteq A\mathcal{E}_\lambda$. Moreover, $\sigma_C(Ax) \subseteq \sigma_C(x)$ for every $x \in \mathcal{E}_\lambda \mathscr{H}$.*

Proof. Let $\lambda \in \mathrm{sp}\,(C) \setminus \{0\}$ be such that $\mathcal{E}_\lambda \mathscr{H} \subseteq \mathrm{dom}(A)$.

Claim : *For $x \in \mathcal{E}_\lambda \mathscr{H}$, the range of an analytic extension of $R(\mu; C)x$ is contained in $\mathcal{E}_\lambda \mathscr{H}$.*

Proof of Claim. Let $f(\mu)$ be an analytic extension of $R(\mu; C)x$, and let $\mu_0 \in \mathrm{dom}(f)$, then it is enough to show that $f(\mu_0) \in \mathcal{E}_\lambda \mathscr{H}$.

Note that $\overline{\mathbb{C} \setminus \mathrm{sp}\,(C)} = \mathbb{C}$, as C is compact. In particular, for μ_0, there is a sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathbb{C} \setminus \mathrm{sp}\,(C)$ such that

$$\mu_k \to \mu_0 \text{ as } k \to \infty.$$

Since $f(\mu)$ is an analytic extension of $R(\mu; C)x$, we have

$$f(\mu_k) \to f(\mu_0) \text{ as } k \to \infty, \tag{4.4.1}$$

and
$$f(\mu_k) = R(\mu_k; C)x = (\mu_k I - C)^{-1}x = \sum_{i=0}^{\infty} \frac{1}{\mu_k^{i+1}} C^i x \text{ with } C^0 := I.$$

Since C leaves the subspace $\mathcal{E}_\lambda \mathscr{H}$ invariant, it follows that if $x \in \mathcal{E}_\lambda \mathscr{H}$, then $R(\mu_k; C)x$ is contained in $\mathcal{E}_\lambda \mathscr{H}$ for each n. Thus, from (4.4.1), $f(\mu_0) \in \overline{\mathcal{E}_\lambda \mathscr{H}} = \mathcal{E}_\lambda \mathscr{H}$. □

For $x \in \mathcal{E}_\lambda \mathscr{H}$, let $x_C(\mu)$ be the maximal analytic extension of $R(\mu; C)x$. From the above claim, the range of $x_C(\mu)$ is contained in $\mathcal{E}_\lambda \mathscr{H}$. Then, it is clear from the equation

$$Ax = A(\mu I - C)x_C(\mu) = (\mu I - C)Ax_C(\mu), \tag{4.4.2}$$

that $Ax_C(\mu)$ is an analytic extension of $R(\mu; C)Ax$ to $\rho_C(x)$. Hence $\rho_C(Ax) \supseteq \rho_C(x)$, so that $\sigma_C(Ax) \subseteq \sigma_C(x)$. Using Proposition 4.4.3,

$$A\mathcal{E}_\lambda \mathscr{H} \subseteq \mathcal{E}_\lambda \mathscr{H} \quad \text{and} \quad A(I - \mathcal{E}_\lambda)\mathrm{dom}(A) \subseteq (I - \mathcal{E}_\lambda)\mathscr{H}.$$

This shows that $\mathcal{E}_\lambda A(I - \mathcal{E}_\lambda)\mathrm{dom}(A) = 0$, and for each $x \in \mathrm{dom}(A)$,

$$\mathcal{E}_\lambda Ax = \mathcal{E}_\lambda A(I - \mathcal{E}_\lambda + \mathcal{E}_\lambda)x = \mathcal{E}_\lambda A\mathcal{E}_\lambda x = A\mathcal{E}_\lambda x. \tag{4.4.3}$$

Since $\mathrm{dom}(\mathcal{E}_\lambda A) = \mathrm{dom}(A) \subseteq \mathscr{H} = \mathrm{dom}(A\mathcal{E}_\lambda)$, it is clear from equation (4.4.3) that $\mathcal{E}_\lambda A \subseteq A\mathcal{E}_\lambda$. □

***Remark* 4.4.5.** Note that in the above proof, we only used the weaker relation $(CA)|_{\mathcal{E}_\lambda \mathscr{H}} = (AC)|_{\mathcal{E}_\lambda \mathscr{H}}$ instead of $CA \subseteq AC$, as captured in equation (4.4.2), to show that $(\mathcal{E}_\lambda A)|_{\mathcal{E}_\lambda \mathscr{H}} = (A\mathcal{E}_\lambda)|_{\mathcal{E}_\lambda \mathscr{H}}$.

***Remark* 4.4.6.** In particular, the above proposition shows that if A is a bounded linear operator on $\mathscr{H}$, then $AC = CA \implies A\mathcal{E}_\lambda = \mathcal{E}_\lambda A$ for all $\lambda \in \mathrm{sp}\,(C) \setminus \{0\}$.

We now proceed to show that all compact operators in $\mathscr{B}(\mathscr{H})$ admit a Dunford-type decomposition, where both the scalar-type part and the quasinilpotent part are closed linear operators (possibly unbounded) acting on $\mathscr{H}$. Analogous to the case of bounded operators acting on a Hilbert spaces, we define the notions of similarity and quasi-similarity in the context of closed linear operators, as follows.

Definition 4.4.7. (Similarity) Let S, T be closed linear operators acting on Hilbert spaces $\mathscr{H}, \mathscr{K}$ respectively. We say that S and T are *similar* if there is an invertible operator $X \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, such that $\operatorname{dom}(S) = X \operatorname{dom}(T)$ and $Su = XTX^{-1}u$ for all $u \in \operatorname{dom}(S)$, $Tv = X^{-1}SXv$ for all $v \in \operatorname{dom}(T)$. In other words, $S \subseteq XTX^{-1}$ and $T \subseteq X^{-1}SX$.

Note from the above definition that if an operator acting on $\mathscr{H}$ is similar to a bounded operator, then it must be bounded.

Definition 4.4.8. (Quasi-similarity) Let S, T be closed linear operators acting on Hilbert spaces $\mathscr{H}, \mathscr{K}$ respectively. We say that S and T are *quasi-similar* if there are quasi-invertible operators X, Y in $\mathscr{B}(\mathscr{K}, \mathscr{H}), \mathscr{B}(\mathscr{H}, \mathscr{K})$ respectively, such that $\operatorname{dom}(S) \supseteq X \operatorname{dom}(T)$, $\operatorname{dom}(T) \supseteq Y \operatorname{dom}(S)$ and $SXv = XTv$ for all $v \in \operatorname{dom}(T)$, $YSu = TYu$ for all $u \in \operatorname{dom}(S)$. In other words, $XT \subseteq SX$ and $YS \subseteq TY$.

Note that in the above definition, it is possible for an unbounded operator to be quasi-similar to a bounded operator; this extends the notion of similarity for bounded operators to a suitable notion of similarity of unbounded operators with bounded ones, without forcing boundedness.

Lemma 4.4.9. *Let $C \in \mathscr{B}(\mathscr{H})$ be a compact operator, and let $\mathcal{E}_\lambda$ denote the Riesz idempotent of C corresponding to $\lambda \in \operatorname{sp}(C) \setminus \{0\}$. Then the subspaces $\mathscr{V}$ and $\mathscr{W}$ defined by*

$$\mathscr{V} := \begin{cases} \operatorname{span}\big(\bigcup_{\lambda \in \operatorname{sp}(C)\setminus\{0\}} \operatorname{ran}(\mathcal{E}_\lambda)\big) & \text{if } \operatorname{sp}(C) \setminus \{0\} \neq \varnothing \\ \{\mathbf{0}\} & \text{if } \operatorname{sp}(C) = \{0\} \end{cases}$$

$$\mathscr{W} := \begin{cases} \bigcap_{\lambda \in \operatorname{sp}(C)\setminus\{0\}} \ker(\mathcal{E}_\lambda) & \text{if } \operatorname{sp}(C) \setminus \{0\} \neq \varnothing \\ \mathscr{H} & \text{if } \operatorname{sp}(C) = \{0\} \end{cases}$$

are invariant under C with $\mathscr{V} \cap \mathscr{W} = \{0\}$ and $\mathscr{V} + \mathscr{W}$ is a dense subspace of $\mathscr{H}$.

Proof. If $\operatorname{sp}(C) = \{0\}$, then the result holds trivially. Thus, without loss of generality we may assume that $\operatorname{sp}(C) \neq \{0\}$. Since for each $\lambda \in \operatorname{sp}(C) \setminus \{0\}$, the subspace $\mathcal{E}_\lambda \mathscr{H}$ is invariant under C, it follows that $\mathscr{V}$ is an invariant subspace of C. Moreover, $\overline{\mathscr{V}}$ is also invariant subspace of C. Indeed, for $x \in \overline{\mathscr{V}}$, let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathscr{V}$ such that $x_k \to x$ as $k \to \infty$, so that $Cx_k \to Cx$ as $k \to \infty$. Since $\{Cx_k\}_{k \in \mathbb{N}} \subseteq \mathscr{V}$, it follows that $Cx \in \overline{\mathscr{V}}$. Thus $\overline{\mathscr{V}}$ is an invariant subspace of C.

Using Remark 1.3.25 for the compact operator C^*, it follows from the equation

$$\mathscr{W} = \bigcap_{\lambda \in \operatorname{sp}(C)\setminus\{0\}} \ker(\mathcal{E}_\lambda) = \Bigg(\bigcup_{\lambda \in \operatorname{sp}(C)\setminus\{0\}} \operatorname{ran}(\mathcal{E}_\lambda^*)\Bigg)^{\perp},$$

that $\mathscr{W}^{\perp}$ is invariant under C^*. In other words, $\mathscr{W}$ is invariant under C.

Since $\mathcal{E}_\lambda$'s are idempotents, $\operatorname{ran}(\mathcal{E}_\lambda)$ and $\ker(\mathcal{E}_\lambda)$ are complementary subspaces, that is, $\operatorname{ran}(\mathcal{E}_\lambda) \cap \ker(\mathcal{E}_\lambda) = \{0\}$ and $\operatorname{ran}(\mathcal{E}_\lambda) + \ker(\mathcal{E}_\lambda) = \mathscr{H}$, from which it follows that $\mathscr{V} \cap \mathscr{W} = \{0\}$ and $\overline{\mathscr{V} + \mathscr{W}} = \mathscr{H}$. □

Theorem 4.4.10. *Let $\mathscr{H}$ be an infinite-dimensional complex Hilbert space, $C \in \mathscr{B}(\mathscr{H})$ be a compact operator, and $\mathscr{V}$, $\mathscr{W}$ be subspaces of $\mathscr{H}$ as defined in Lemma 4.4.9. Then there exist unique closed linear operators D, N acting on $\mathscr{H}$ such that*

(i) $\mathscr{V} + \mathscr{W}$ is a core for both D and N;

(ii) $CD \subseteq DC$ and $CN \subseteq NC$;

(iii) D and N are simultaneously quasi-similar to a commuting pair of a bounded normal operator and a bounded quasinilpotent operator respectively; and

(iv) $C = D \hat{+} N$.

Proof. Without loss of generality we may assume that $\operatorname{sp}(C) \neq \{0\}$, as in that case C is quasinilpotent (hence spectral), and the result follows trivially from the Dunford-decomposition of such operators.

Recall from Lemma 4.4.9 that $\mathscr{V}$ is an invariant subspace of K, and $\mathscr{W}$ is a closed invariant subspace of K such that $\mathscr{V} \cap \mathscr{W} = \{0\}$ and $\mathscr{V} + \mathscr{W}$ is a dense subspace of $\mathscr{H}$. We divide the proof into three exhaustive cases based on the subspace $\mathscr{V}$.

Case 1 : $\mathscr{V}$ is finite-dimensional.

Then, $\operatorname{sp}(C)$ is finite. Set $\mathcal{E} := \sum_{\lambda \in \operatorname{sp}(C)\setminus\{0\}} \mathcal{E}_\lambda$. Then $\mathcal{E}$ being a sum of finitely-many orthogonal idempotents, is an idempotent. Clearly $\mathcal{E}\mathscr{H} = \mathscr{V}$, and $\mathcal{E}$ commutes with C.

Note that $C\mathcal{E}$ is a finite rank operator, hence spectral. Let D_0 and N_0 be the commuting scalar-type and quasinilpotent operators in the Dunford-decomposition of $C\mathcal{E}$.

Set $D := D_0\mathcal{E}$ and $N := N_0\mathcal{E} + C(I - \mathcal{E})$. Note that $\mathcal{E}$ commutes with D_0 and N_0, so that from the uniqueness of the Dunford decomposition of $C\mathcal{E}$, $D_0 = D_0\mathcal{E} = D$ is a scalar-type operator, $N_0 = N_0\mathcal{E}$ is a quasinilpotent operator. From Lemma 3.2.5, N being a sum of two commuting quasinilpotents is a quasinilpotent operator. Moreover,

$$\begin{aligned} C &= C\mathcal{E} + C(I - \mathcal{E}) = D_0\mathcal{E} + N_0\mathcal{E} + C(I - \mathcal{E}) = D + N, \text{ and} \\ DN &= D_0\mathcal{E}N_0\mathcal{E} = D_0N_0 = N_0D_0 = N_0\mathcal{E}D_0\mathcal{E} = ND. \end{aligned}$$

Thus, $C = D + N$ is the Dunford-decomposition of C, and the result follows.

Case 2 : $\mathscr{V}$ is infinite-dimensional, and $\overline{\mathscr{V}} = \mathscr{H}$.

In this case, note from Lemma 4.4.9 that $\mathscr{W} = \{0\}$. Since $\mathscr{V}$ is infinite-dimensional, $\operatorname{sp}(C)$ must be an infinite set. Let $\{\lambda_k\}$ be an enumeration of $\operatorname{sp}(C) \setminus \{0\}$, so that $\lambda_k \to 0$ as $k \to \infty$.

Existence : Recall that for each $\lambda \in \mathrm{sp}\,(C) \setminus \{0\}$, $\mathcal{E}_\lambda \mathscr{H}$ is a finite-dimensional subspace of $\mathscr{H}$ (see theorem 1.3.23), invariant under C. Let C_λ denote the restriction $C|_{\mathcal{E}_\lambda \mathscr{H}}$. Then being a linear transformation of a finite-dimensional space, C_λ admits a Jordan-Chevalley decomposition. Let D_λ and N_λ be the diagonalizable part and the nilpotent part, respectively, in the Jordan-Chevalley decomposition of C_λ. Let D, N be linear operators acting on $\mathscr{H}$ as follows:

$$\mathrm{dom}(D) := \{x \in \mathscr{H} : \sum_{k\in\mathbb{N}} D_{\lambda_k}\mathcal{E}_{\lambda_k}x \text{ converges in } \mathscr{H}\}$$
$$\mathrm{dom}(N) := \{x \in \mathscr{H} : \sum_{k\in\mathbb{N}} N_{\lambda_k}\mathcal{E}_{\lambda_k}x \text{ converges in } \mathscr{H}\}, \text{ and}$$

$$Dx = \sum_{k\in\mathbb{N}} D_{\lambda_k}\mathcal{E}_{\lambda_k}x \;\forall\; x \in \mathrm{dom}(D)\,, \quad Nx = \sum_{k\in\mathbb{N}} N_{\lambda_k}\mathcal{E}_{\lambda_k}x \;\forall\; x \in \mathrm{dom}(N)$$

Since $\mathcal{E}_{\lambda_k}\mathcal{E}_{\lambda_\ell} = 0$ whenever $k \neq \ell$, note that $\mathcal{E}_\lambda \mathscr{H} \subseteq \mathrm{dom}(D) \cap \mathrm{dom}(N)$ for all $\lambda \in \mathrm{sp}\,(C) \setminus \{0\}$. Thus $\mathscr{V} \subseteq \mathrm{dom}(D) \cap \mathrm{dom}(N)$, so that D, N are densely-defined linear operators on $\mathscr{H}$.

Claim : *D and N are pre-closed operators.*

Proof of Claim. We prove that D is a pre-closed operator, the proof for N follows via a similar reasoning. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathrm{dom}(D)$ such that $x_n \to 0$ and $Dx_n \to y \in \mathscr{H}$. In view of Remark 1.3.29 it is sufficient to show that $y = 0$.

Since each $\mathcal{E}_\lambda$ is bounded, we have $\lim_{n\to\infty} \mathcal{E}_\lambda D x_n = \mathcal{E}_\lambda y$ for all $\lambda \in \mathrm{sp}\,(C) \setminus \{0\}$. On the other hand, $\mathcal{E}_\lambda D x_n = \mathcal{E}_\lambda \sum_{k\in\mathbb{N}} D_{\lambda_k}\mathcal{E}_{\lambda_k}x_n = \mathcal{E}_\lambda D_\lambda \mathcal{E}_\lambda x_n = D_\lambda \mathcal{E}_\lambda x_n \to 0$ as $n \to \infty$ because $\lim_{n\to\infty} x_n = 0$ and $D_\lambda\mathcal{E}_\lambda$'s are bounded operators. Consequently, $\mathcal{E}_\lambda y = 0$ for all $\lambda \in \mathrm{sp}\,(C) \setminus \{0\}$. In other words, $y \in \mathscr{W} = \{0\}$. □

By passing to the closures of D and N, we henceforth assume that both D and N are closed operators.

(i) It is clear from the definition of D and N that $\mathscr{V} + \mathscr{W} = \mathscr{V} \subseteq \mathrm{dom}(D) \cap \mathrm{dom}(N)$, and $\mathscr{V}$ is a core for both D and N.

(ii) For $x \in \mathscr{V}$, observe that

$$\begin{aligned} CDx = C\sum_{k\in\mathbb{N}} D_{\lambda_k}\mathcal{E}_{\lambda_k}x = \sum_{k\in\mathbb{N}} CD_{\lambda_k}\mathcal{E}_{\lambda_k}x &= \sum_{k\in\mathbb{N}} C_{\lambda_k}D_{\lambda_k}\mathcal{E}_{\lambda_k}x \\ &= \sum_{k\in\mathbb{N}} D_{\lambda_k}C_{\lambda_k}\mathcal{E}_{\lambda_k}x \quad (\because C_{\lambda_k}D_{\lambda_k} = D_{\lambda_k}C_{\lambda_k}) \\ &= \sum_{k\in\mathbb{N}} D_{\lambda_k}C\mathcal{E}_{\lambda_k}x = \sum_{k\in\mathbb{N}} D_{\lambda_k}\mathcal{E}_{\lambda_k}Cx = DCx. \end{aligned}$$

Passing to limits (whenever exists), this shows that $\mathrm{dom}(D)$ is invariant under C and $CD|_{\mathrm{dom}(D)} = DC|_{\mathrm{dom}(D)}$. Equivalently, $CD \subseteq DC$. Likewise, $CQ \subseteq QC$.

(iii) Let n_λ be the dimension of the subspace $\mathcal{E}_\lambda\mathscr{H}$. Then the operators $C_\lambda, D_\lambda, N_\lambda$ can be viewed as complex matrices in $M_{n_\lambda}(\mathbb{C})$. Since $\mathrm{sp}\,(C_\lambda) = \{\lambda\}$, from Remark 2.6.2, $D_\lambda = \lambda I_{n_\lambda}$, viewed as a matrix in $M_{n_\lambda}(\mathbb{C})$. Let $J(N_\lambda) \in M_{n_\lambda}(\mathbb{C})$ denote a Jordan canonical form of N_λ. Note that $J(N_\lambda) \sim_{\text{sim}} \lambda J(N_\lambda)$ in $M_{n_\lambda}(\mathbb{C})$, whence there exists an invertible operator $S_\lambda \in \mathscr{B}(\mathcal{E}_\lambda\mathscr{H})$ such that $S_\lambda N_\lambda S_\lambda^{-1}$, viewed as a matrix in $M_{n_\lambda}(\mathbb{C})$ (with respect to the standard orthonormal basis of $\mathbb{C}^{n_\lambda}$) is $\lambda J(N_\lambda)$.
Define the linear operators $X : \mathscr{H} \to \bigoplus_{k\in\mathbb{N}}\mathcal{E}_{\lambda_k}\mathscr{H}$ and $Y : \bigoplus_{k\in\mathbb{N}}\mathcal{E}_{\lambda_k}\mathscr{H} \to \mathscr{H}$ by

$$Xx = \bigoplus_{k\in\mathbb{N}}\frac{1}{2^k}\frac{S_{\lambda_k}\mathcal{E}_{\lambda_k}x}{\|S_{\lambda_k}\mathcal{E}_{\lambda_k}\|} \quad , \quad Y\bigoplus_{k\in\mathbb{N}}x_k = \sum_{k\in N}\frac{1}{2^k}\frac{S_{\lambda_k}^{-1}x_k}{\|S_{\lambda_k}^{-1}\|} \quad .$$

It is easy to see that X and Y are well-defined linear maps. Since $\|Xx\|^2 \le \sum_{k\in\mathbb{N}}\frac{1}{4^k}\|x\|^2$ and $\|Y\bigoplus x_k\|^2 \le \big(\sum_{k\in\mathbb{N}}\frac{1}{2^k}\|x_k\|\big)^2 \le \big(\sum_{k\in\mathbb{N}}\frac{1}{4^k}\big)\big(\sum_{k\in\mathbb{N}}\|x_k\|^2\big)$, X and Y are bounded. If $Xx = 0$, then $S_{\lambda_k}\mathcal{E}_{\lambda_k}x = \mathcal{E}_{\lambda_k}x = 0$ for all $k \in \mathbb{N}$ which implies $\mathscr{W} \ni x = 0$. In other words, X is injective. Observe that every vector of the form $x_1 \oplus x_2 \oplus \cdots \oplus x_n \oplus 0 \oplus 0 \oplus \cdots$, where $x_k \in \mathcal{E}_{\lambda_k}\mathscr{H}$ for $k, n \in \mathbb{N}$, lies in $\mathrm{ran}(X)$ so that $\mathrm{ran}(X)$ is dense in $\bigoplus_{k\in\mathbb{N}}\mathcal{E}_{\lambda_k}\mathscr{H}$. A similar argument using invertibility of S_{λ_k}'s shows that Y is an injective map, and noting $\mathscr{V} \subseteq \mathrm{ran}(Y)$ follows that $\mathrm{ran}(Y)$ is dense in $\mathscr{H}$. Hence X, Y are quasi-invertible. Define the linear operators $M, N_0 : \bigoplus_{k\in\mathbb{N}}\mathcal{E}_{\lambda_k}\mathscr{H} \to \bigoplus_{k\in\mathbb{N}}\mathcal{E}_{\lambda_k}\mathscr{H}$ by

$$M = \bigoplus_{k\in\mathbb{N}}S_{\lambda_k}D_{\lambda_k}{S_{\lambda_k}}^{-1} \quad \text{and} \quad N_0 = \bigoplus_{k\in\mathbb{N}}S_{\lambda_k}N_{\lambda_k}{S_{\lambda_k}}^{-1}.$$

Observe that $\max_{k\in\mathbb{N}}\|S_{\lambda_k}D_{\lambda_k}{S_{\lambda_k}}^{-1}\| = \max_{k\in\mathbb{N}}\|S_{\lambda_k}N_{\lambda_k}{S_{\lambda_k}}^{-1}\| = \max_{k\in\mathbb{N}}|\lambda_k| < \infty$, thus M and N_0 are bounded linear operators. Viewing M and N_0 as elements of $\bigoplus_{k\in\mathbb{N}}M_{n_{\lambda_k}}(\mathbb{C})$, it is easy to see that $M = \oplus_{k\in\mathbb{N}}\lambda_k I_{n_{\lambda_k}}$ is a compact normal operator, and a straightforward calculation shows that $\mathrm{sp}\,(N_0) = \mathrm{sp}\,(\oplus_{k\in\mathbb{N}}\lambda_k J(N_{\lambda_k})) = \{0\}$, that is, N_0 is a quasinilpotent operator. Clearly, M and N_0 commute with each other.

By virtue of the construction it is readily verified that $\mathrm{ran}(Y) \subseteq \mathrm{dom}(D) \cap \mathrm{dom}(N)$; $XD \subseteq MX$, $DY = YM$ and $XN \subseteq N_0X$, $NY = YN_0$. In particular, D and N are simultaneously quasi-similar to M and N_0, respectively.

(iv) For $x \in \mathscr{V}$, observe that

$$(D+N)x = \sum_{k\in\mathbb{N}}D_{\lambda_k}\mathcal{E}_{\lambda_k}x + \sum_{k\in\mathbb{N}}N_{\lambda_k}\mathcal{E}_{\lambda_k}x = \sum_{k\in\mathbb{N}}C_{\lambda_k}\mathcal{E}_{\lambda_k}x = C\sum_{k\in\mathbb{N}}\mathcal{E}_{\lambda_k}x = Cx.$$

Passing to limits, it follows that $D + N \subseteq C$. Let $\{x_n\}$ be a sequence in $\mathrm{dom}(D) \cap \mathrm{dom}(N)$ such that $x_n \to 0$ and $(D + N)x_n \to y$ as $n \to \infty$. Then $y = \lim_{n\to\infty} Cx_n = 0$, whence $D + N$ is a pre-closed operator from Remark 1.3.29.

Next, let $x \in \mathscr{H}$. If $\{x_n\}$ is a sequence in $\mathrm{dom}(D) \cap \mathrm{dom}(N)$ converging to x, then $(D + N)x_n = Cx_n \to Cx$ as $n \to \infty$. This implies that $x \in \mathrm{dom}(\overline{D + N})$. In other words $\overline{D + N}$ is bounded. Since $\overline{D + N}$ coincides with C on a dense subspace, we must have

$C = \overline{D + N} = D \hat{+} N.$

Uniqueness : Let D' and N' be closed linear operators acting on $\mathscr{H}$ such that the conditions $(i) - (iv)$ are satisfied.

It follows from (iv) that $\mathscr{V}' := \mathrm{dom}(D') \cap \mathrm{dom}(N')$ is a dense subspace of $\mathscr{H}$. Using (ii) along with the fact that $\mathcal{E}_\lambda \mathscr{H} \subseteq \mathscr{V} \subseteq \mathrm{dom}(D')$, from Proposition 4.4.4 we have

$$\mathcal{E}_\lambda D' \subseteq D' \mathcal{E}_\lambda \quad \text{and} \quad \mathcal{E}_\lambda N' \subseteq N' \mathcal{E}_\lambda \,, \tag{4.4.4}$$

whence both D' and N' leaves each $\mathcal{E}_\lambda \mathscr{H}$ invariant.

From (iii), for some Hilbert space $\mathscr{K}$, let $M', N_0' \in \mathscr{B}(\mathscr{K})$ be normal and quasinilpotent operators, respectively, and $X' : \mathscr{H} \to \mathscr{K}, Y' : \mathscr{K} \to \mathscr{H}$ be quasi-invertible operators such that $\mathscr{V}' \supseteq \mathrm{ran}(Y)$ and,

$$X'D' \subseteq M'X' \,,\; D'Y' = Y'M' \quad \text{and} \quad X'N' \subseteq N_0'X' \,,\; N'Y' = Y'N_0' \,. \tag{4.4.5}$$

Since $\mathcal{E}_\lambda \mathscr{H}$ is finite-dimensional, and X' is quasi-invertible, $X'|_{\mathcal{E}_\lambda \mathscr{H}}$ is invertible. Note from (4.4.4) and (4.4.5) that $X'\mathcal{E}_\lambda \mathscr{H}$ is invariant under the action of M' and the restrictions $D'|_{\mathcal{E}_\lambda \mathscr{H}}$ and $M'|_{X'\mathcal{E}_\lambda \mathscr{H}}$ are similar. Indeed,

$$X'|_{\mathcal{E}_\lambda \mathscr{H}} \; D'|_{\mathcal{E}_\lambda \mathscr{H}} \; X'|_{\mathcal{E}_\lambda \mathscr{H}}^{-1} = M'|_{X'\mathcal{E}_\lambda \mathscr{H}}.$$

Likewise, $X'\mathcal{E}_\lambda \mathscr{H}$ is invariant under the action of N_0' and the restrictions $N'|_{\mathcal{E}_\lambda \mathscr{H}}$ and $N_0'|_{X'\mathcal{E}_\lambda \mathscr{H}}$ are similar, with

$$X'|_{\mathcal{E}_\lambda \mathscr{H}} \; N'|_{\mathcal{E}_\lambda \mathscr{H}} \; X'|_{\mathcal{E}_\lambda \mathscr{H}}^{-1} = N_0'|_{X'\mathcal{E}_\lambda \mathscr{H}}.$$

Thus, $D'|_{\mathcal{E}_\lambda \mathscr{H}}$ and $N'|_{\mathcal{E}_\lambda \mathscr{H}}$ are scalar-type and quasinilpotent operators respectively. From (ii), it follows that both $D'|_{\mathcal{E}_\lambda \mathscr{H}}$ and $N'|_{\mathcal{E}_\lambda \mathscr{H}}$ commutes with C_λ. Thus, $C_\lambda = D'|_{\mathcal{E}_\lambda \mathscr{H}} + N'|_{\mathcal{E}_\lambda \mathscr{H}}$ is a Dunford-decomposition of C_λ. From the uniqueness of the Dunford-decomposition, we must have $D_\lambda = D'|_{\mathcal{E}_\lambda \mathscr{H}}$ and $N_\lambda = N'|_{\mathcal{E}_\lambda \mathscr{H}}$ for all $\lambda \in \mathrm{sp}\,(C) \setminus \{0\}$. Since $\mathscr{V}$ is a core for the linear operators D' and N', this shows that $D' = D$ and $N' = N$.

Moreover, it follows from (iv) that $\mathrm{dom}(D) = \mathrm{dom}(N)$. Indeed, since $\mathscr{V}$ is a core for D, for any $x \in \mathrm{dom}(D)$ we may choose a sequence $\{x_n\} \subseteq \mathscr{V}$ such that $x_n \to x$ and $Dx_n \to Dx$ as $n \to \infty$. Consequently $Nx_n = Cx_n - Dx_n \to Cx - Dx$, which (from the closedness of N) implies that $x \in \mathrm{dom}(N)$ and $Nx = Cx - Dx$. Thus $\mathrm{dom}(D) \subseteq \mathrm{dom}(N)$, and a similar argument shows the reverse inclusion.

Case 3 : $\mathscr{V}$ is infinite-dimensional, and $\overline{\mathscr{V}}$ is a proper subspace of $\mathscr{H}$.

Since $\overline{\mathscr{V}}$ and $\mathscr{W}$ are complementary subspaces of $\mathscr{H}$, there is an idempotent $\mathcal{E}$ in $\mathscr{B}(\mathscr{H})$ such that $\mathcal{E}\overline{\mathscr{V}} = \overline{\mathscr{V}}$ and $\mathcal{E}\mathscr{W} = \{0\}$ (see [KR83, Theorem 1.1.8]). Since $\overline{\mathscr{V}}$ and $\mathscr{W}$ are invariant under C (Lemma 4.4.9), note that C commutes with $\mathcal{E}$. From Case 2, the result holds for the restriction $C|_{\overline{\mathscr{V}}} \in \mathscr{B}(\overline{\mathscr{V}})$. Let $D_{\mathscr{V}}$ and $N_{\mathscr{V}}$ be a pair of closed operators acting on $\overline{\mathscr{V}}$ such that conditions (i)-(iv) hold for $C|_{\overline{\mathscr{V}}}$.

Let D, N be linear operators acting on $\mathscr{H}$, given by:

$$\mathrm{dom}(D) := \mathrm{dom}(D_{\mathscr{V}}) + \mathscr{W}\ , \quad \mathrm{dom}(N) := \mathrm{dom}(N_{\mathscr{V}}) + \mathscr{W}\ , \text{ and}$$

$$Dx = D_{\mathscr{V}}\mathcal{E}x \ \forall\ x \in \mathrm{dom}(D)\,, \quad Nx = N_{\mathscr{V}}\mathcal{E}x + C(I-\mathcal{E})x \ \forall\ x \in \mathrm{dom}(N)$$

It is easy to see that D and N are densely-defined linear operators on $\mathscr{H}$.

Claim : *D and N are closed operators.*

Proof of Claim. Let $\{x_n\}$ be a sequence of vectors in $\mathrm{dom}(D)$ such that $x_n \to x$ and $Dx_n \to y$ as $n \to \infty$. Since $\mathcal{E}$ is bounded we have $\mathcal{E}x_n \to \mathcal{E}x$, and $Dx_n = D_{\mathscr{V}}\mathcal{E}x_n \to y$. The closedness of $D_{\mathscr{V}}$ implies that $\mathcal{E}x \in \mathrm{dom}(D_{\mathscr{V}})$ and $D_{\mathscr{V}}\mathcal{E}x = y$. Consequently, $x = \mathcal{E}x + (I-\mathcal{E})x \in \mathrm{dom}(D_{\mathscr{V}}) + \mathscr{W} = \mathrm{dom}(D)$ and $Dx = y$. Thus, D is a closed operator. A similar argument using closedness of $N_{\mathscr{V}}$ and boundedness of $C(I-\mathcal{E})$ shows that N is closed. □

(i) From Case 2, since $\mathscr{V}$ is a core for both $D_{\mathscr{V}}$ and $N_{\mathscr{V}}$, it is clear from the definition of D and N that $\mathscr{V} + \mathscr{W} \subseteq \mathrm{dom}(D) \cap \mathrm{dom}(N)$, and $\mathscr{V} + \mathscr{W}$ is a core for both D and N.

(ii) For $x \in \mathrm{dom}(D)$, observe that

$$CDx = CD_{\mathscr{V}}\mathcal{E}x = C|_{\overline{\mathscr{V}}}D_{\mathscr{V}}\mathcal{E}x = D_{\mathscr{V}}C|_{\overline{\mathscr{V}}}\mathcal{E}x = D_{\mathscr{V}}\mathcal{E}Cx = DCx;$$

likewise, for $x \in \mathrm{dom}(N)$, we have

$$CNx = CN_{\mathscr{V}}\mathcal{E}x + C^2(I-\mathcal{E})x = N_{\mathscr{V}}\mathcal{E}Cx + C(I-\mathcal{E})Cx = NCx.$$

This shows that both $\mathrm{dom}(D)$, $\mathrm{dom}(N)$ are invariant under C and $CD \subseteq DC$, $CN \subseteq NC$.

(iii) For some Hilbert space $\mathscr{W}_0$, let $X_{\mathscr{V}}$, $Y_{\mathscr{V}}$ be quasi-invertible operators in $\mathscr{B}(\overline{\mathscr{V}}, \mathscr{W}_0)$, $\mathscr{B}(\mathscr{W}_0, \overline{\mathscr{V}})$ respectively and, $M_{\mathscr{V}}$ and $N_{0\mathscr{V}}$ be commuting normal and quasinilpotent operators, respectively in $\mathscr{B}(\mathscr{W}_0)$ be such that $\mathrm{ran}(Y_{\mathscr{V}}) \subseteq \mathrm{dom}(D_{\mathscr{V}}) \cap \mathrm{dom}(N_{\mathscr{V}})$, and

$$X_{\mathscr{V}}D_{\mathscr{V}} \subseteq M_{\mathscr{V}}X_{\mathscr{V}}\ ,\ D_{\mathscr{V}}Y_{\mathscr{V}} = Y_{\mathscr{V}}M_{\mathscr{V}} \quad \text{and} \quad X_{\mathscr{V}}N_{\mathscr{V}} \subseteq N_{0\mathscr{V}}X_{\mathscr{V}}\ ,\ N_{\mathscr{V}}Y_{\mathscr{V}} = Y_{\mathscr{V}}N_{0\mathscr{V}}.$$

Define the linear operators $X : \mathscr{H} \to \mathscr{W}_0 \bigoplus \mathscr{W}$ and $Y : \mathscr{W}_0 \bigoplus \mathscr{W} \to \mathscr{H}$ by

$$Xx = X_{\mathscr{V}}\mathcal{E}x \oplus (I-\mathcal{E})x\ , \quad Y(w_0 \oplus w) = Y_{\mathscr{V}}w_0 + w. \tag{4.4.6}$$

Clearly, X and Y are well-defined bounded linear operators. Since $X_{\mathscr{V}}$, $Y_{\mathscr{V}}$ are quasi-invertible and $I - \mathcal{E}$ is identity on $\mathscr{W}$, it is not difficult to see that X and Y are quasi-invertible.

Let $M, N_0 : \mathscr{W}_0 \oplus \mathscr{W} \to \mathscr{W}_0 \oplus \mathscr{W}$ be the bounded linear operators defined by

$$M = M_{\mathscr{V}} \bigoplus 0 \quad \text{and} \quad N_0 = N_{0\mathscr{V}} \bigoplus C|_{\mathscr{W}} \tag{4.4.7}$$

Since $M_{\mathscr{V}}$ is a compact normal operator in $\mathscr{B}(\mathscr{W}_0)$, M is a compact normal operator in $\mathscr{B}(\mathscr{W}_0 + \mathscr{W})$. Moreover, $\operatorname{sp}(N_0) = \operatorname{sp}(N_{0\mathscr{V}}) \cup \operatorname{sp}(C|_{\mathscr{W}}) = \{0\}$ shows that N_0 is a quasinilpotent operator. Noting that. $M_{\mathscr{V}}$ and $N_{0\mathscr{V}}$ commute with each other, it follows easily that M and N_0 commute with each other.

Invoking the corresponding results from Case 2, it is straightforward to verify that $\operatorname{dom}(D) \cap \operatorname{dom}(N) \supseteq \operatorname{ran}(Y)$; $XD \subseteq MX$, $DY = YM$ and $XN \subseteq N_0X$, $NY = YN_0$, proving that D and N are simultaneously quasi-similar to M and N_0, respectively.

(iv) For $x \in \operatorname{dom}(D) \cap \operatorname{dom}(N)$, note that

$$(D + N)x = D_{\mathscr{V}}\mathcal{E}x + N_{\mathscr{V}}\mathcal{E}x + C(I - \mathcal{E})x = C|_{\overline{\mathscr{V}}}\mathcal{E}x + C(I - \mathcal{E})x = C\mathcal{E}x + C(I - \mathcal{E})x = Cx.$$

Following the argument detailed in Case 2-(iv), we conclude that $C = \overline{D + N} = D \hat{+} N$.

Uniqueness : Let D' and N' be closed operators acting on $\mathscr{H}$ such that the conditions $(i) - (iv)$ are satisfied.

By reasoning analogous to the uniqueness argument in Case 2, specifically arriving at (4.4.4), we observe that the subspaces $\mathscr{V}$ and $\mathscr{W}$ are invariant under both D' and N'. Consequently, the uniqueness of the Dunford-type decomposition for C reduces to the analysis of its restrictions, $C|_{\overline{\mathscr{V}}}$ and $C|_{\mathscr{W}}$. The result then follows by applying the uniqueness criteria established in Cases 2 and 1, respectively. □

***Remark* 4.4.11.** Note that the closed operators D and N, facilitating the Dunford-type-decomposition of the compact operator $C \in \mathscr{B}(\mathscr{H})$ in Theorem 4.4.10, commute with each other on a dense subspace of $\mathscr{H}$. In particular, the core $\mathscr{V} + \mathscr{W}$ is preserved under both D and N and

$$DNx = NDx \qquad \text{for all } x \in \mathscr{V} + \mathscr{W}.$$

Corollary 4.4.12. *Let $C \in \mathscr{B}(\mathscr{H})$ be a compact operator. Then for some Hilbert space $\mathscr{K}$ there is a compact spectral operator $C' \in \mathscr{B}(\mathscr{K})$, and quasi-invertible operators $X \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and $Y \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, such that $XC = C'X$ and $CY = YC'$.*

Proof. Let M and N_0 be bounded normal and quasinilpotent operators, respectively, as defined in (4.4.7). Then it is not difficult to see that $C' := M + N_0$ is a compact spectral operator.

Let X, Y be quasi-invertible operators as defined in (4.4.6). Then it is straightforward to verify that $XC = C'X$ and $CY = YC'$. □

4.5 Concluding remarks

Through the examples and the results discussed in this chapter, we have highlighted both the limitations and the potential for generalization of the Dunford decomposition for compact operators operators acting on complex Hilbert spaces. It remains an open and intriguing area to investigate how these generalized decompositions interact with broader classes of operators.

On another note, it is easy to see that the class of spectral operators is closed under similarity transformations. In the light of Corollary 4.4.12, and Theorem 4.2.4, one may pose the following question.

Question : *Does the normalized power sequence of operators in the quasi-similarity orbit of spectral operators converge in norm?*

Chapter 5

The Jordan-Chevalley-Dunford decomposition for operators in type I Murray-von Neumann algebras

5.1 Introduction

In the previous chapters, we have discussed infinite-dimensional analogues of the Jordan-Chevalley decomposition, in the context of spectral and compact operators. In this chapter, we take a conscious step back and carefully examine the Jordan-Chevalley decomposition in the setting of type I_n AW^*-algebras (for $n \in \mathbb{N}$), before specializing to the type I_n von Neumann algebra case and then piecing the results together for type I finite von Neumann algebras. Note that every type I_n AW^*-algebra is algebraically of the form $M_n\big(C(\mathcal{X})\big)$ for a Stonean space $\mathcal{X}$ (see Remark 1.5.3).

For a finite von Neumann algebra $\mathscr{N}$, we denote the Murray-von Neumann algebra (see Definition 1.6.2) of densely-defined closed operators affiliated with $\mathscr{N}$ by $\mathrm{Aff}(\mathscr{N})$. For $n \in \mathbb{N}$, let $\mathscr{M}_n$ be a type I_n von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Note that every type I_n von Neumann algebra is of the form $M_n(\mathscr{A})$ for an abelian von Neumann algebra $\mathscr{A}$, and $\mathscr{A}$ is $*$-isomorphic to $C(\mathcal{X})$ for a (hyper-)Stonean space $\mathcal{X}$ (see Theorem 1.5.2). Using [Nay21, Theorem 4.15], we may identify $\mathrm{Aff}\big(M_n(\mathscr{A})\big)$ with $M_n\big(\mathrm{Aff}(\mathscr{A})\big)$ as unital ordered complex topological $*$-algebras; where the order structure refers to the self-adjoint part of these $*$-algebras.

In [Kad86], drawing on the works of Stone, Fell and Kelley (see [Sto49], [FK52]), Kadison discusses complex-valued normal functions on a Stonean space with the intention of providing a function representation for affiliated operators (see Definition 1.6.1) for abelian von Neumann algebras. The space of complex-valued normal functions on a Stonean space $\mathcal{X}$ (though their domains are in fact open dense subsets of $\mathcal{X}$) is denoted by $\mathcal{N}(\mathcal{X})$ (see Definition 1.4.4). Using the remarks in [Kad86, §3, §4], we may identify $\mathrm{Aff}\big(C(\mathcal{X})\big)$ with $\mathcal{N}(\mathcal{X})$ as monotone-complete ordered $*$-algebras, in a natural manner. Thus, every type I_n Murray von Neumann algebra is of the form $M_n\big(\mathcal{N}(\mathcal{X})\big)$ for some (hyper)-Stonean space $\mathcal{X}$.

In §5.2, for a locally compact Hausdorff space $\mathcal{Y}$, we define the notion of $\mathcal{Y}$-valued normal functions on a Stonean space, with primary interest in the case of $\mathcal{Y} = M_n(\mathbb{C})$. In Remark 5.2.9, we identify $M_n(\mathbb{C})$-valued normal functions on a Stonean space $\mathcal{X}$ with matrices in $M_n\big(\mathcal{N}(\mathcal{X})\big)$ in a natural manner. A key result of this chapter is Theorem 5.3.11, which asserts the existence and uniqueness of the Jordan-Chevalley decomposition for matrices in $M_n\big(\mathcal{N}(\mathcal{X})\big)$.

Theorem 5.3.11 immediately yields a version of the Jordan-Chevalley decomposition in the context of $\mathrm{Aff}(\mathscr{M}_n)$, which we record in Proposition 5.5.1-(i). An operator in a Murray-von Neumann algebra is said to be $\mathfrak{u}$-*scalar-type* if it can be transformed into an (unbounded) normal operator via an (unbounded) similarity transformation; note that the word 'unbounded' is meant as 'not necessarily bounded' rather than 'not bounded'. In Proposition 5.5.1-(i), we observe that every operator in $\mathrm{Aff}(\mathscr{M}_n)$ may be uniquely decomposed as the strong-sum of a $\mathfrak{u}$-scalar-type operator and a nilpotent operator in $\mathrm{Aff}(\mathscr{M}_n)$ which commute (in strong-product) with each other. In Proposition 5.5.1-(ii), making essential use of Theorem 3.4.2, we show that the normalized power sequence of an operator in $\mathrm{Aff}(\mathscr{M}_n)$ converges in the $\mathfrak{m}$-topology (see Definition 1.6.3).

The passage from the type I_n case (for $n \in \mathbb{N}$) to the setting of type I finite von Neumann algebras via infinite direct sums introduces some subtleties. Specifically, while the property of being $\mathfrak{u}$-scalar-type is preserved under infinite direct sums, it is not guaranteed that nilpotency will be preserved (see Remark 5.5.2). The notion of $\mathfrak{m}$-quasinilpotence, which we introduce in this chapter, emerges as the appropriate substitute for nilpotence. An operator A in a Murray-von Neumann algebra is termed $\mathfrak{m}$-quasinilpotent if its normalized power sequence, $\{|A^k|^{\frac{1}{k}}\}_{k\in\mathbb{N}}$, converges to 0 in the $\mathfrak{m}$-topology. Let $\mathscr{M}$ be a type I finite von Neumann algebra. In Theorem 5.5.3, which is the main result of this chapter, we show that every operator in $\mathrm{Aff}(\mathscr{M})$ may be uniquely decomposed as the strong-sum of a commuting pair consisting of a $\mathfrak{u}$-scalar-type operator and an $\mathfrak{m}$-quasinilpotent operator, which we call its *Jordan-Chevalley-Dunford decomposition*. Moreover, the normalized power sequence of any operator in $\mathrm{Aff}(\mathscr{M})$ converges in the $\mathfrak{m}$-topology.

We emphasize that the natural home for a Jordan-Chevalley-Dunford decomposition of operators in $\mathscr{M}$ is in $\mathrm{Aff}(\mathscr{M})$. Using Theorem 2.6.4, in Remark 5.5.6 we note an example of an operator in the type I_3 von Neumann algebra, $M_3\big(\ell^\infty(\mathbb{N})\big)$, whose diagonalizable and nilpotent parts **do not** lie in $M_3\big(\ell^\infty(\mathbb{N})\big)$.

In Proposition 5.5.7, we note that for every $n \in \mathbb{N}$ and a type II_1 von Neumann algebra $\mathscr{L}$, there is a unital normal embedding of the type I_n von Neumann algebra, $M_n\big(\ell^\infty(\mathbb{N})\big)$, into $\mathscr{L}$. Taking into account the preceding observations and the functorial nature of Murray-von Neumann algebras (as detailed in [Nay21, §4], [GN24, §4]), we are strongly inclined to believe that any meaningful Jordan-Chevalley-Dunford decomposition for operators in type II_1 von Neumann algebras will fundamentally rely on affiliated operators (see Remark 5.5.8).

It is worthwhile noting that our path to the proof of Theorem 5.3.11 yields insights that may be more broadly applicable. An important class of examples of Stonean spaces is given by the maximal ideal space of $L^\infty(\mathcal{Y};\mu)$, where $\mathcal{Y}$ is a locally compact Hausdorff space equipped with a Radon measure μ. Thus in studying the Jordan-Chevalley decomposition of matrices in $M_n\big(L^\infty(\mathcal{Y};\mu)\big)$, one naturally expects challenges of a measure-theoretic nature, which may be translated into a topological language using Stonean spaces; for example, as is the case in the topological proof of the spectral theorem due to Stone (see [Sto32], [KR83, §5.2]). In this vein, Lemma 5.2.11 facilitates the 'gluing' of combinatorial matrix properties (such as rank, multiplicity of eigenvalues, invariant factors, etc.) for continuously varying matrices parametrized by a Stonean space $\mathcal{X}$. We anticipate that it will be generally applicable to transferring results about matrices in $M_n(\mathbb{C})$ to the context of matrices in $M_n\big(C(\mathcal{X})\big)$ and $M_n\big(\mathcal{N}(\mathcal{X})\big)$ and eventually to type I finite von Neumann algebras. We also aspire for this work to offer a modest contribution towards a general principle for extending results from von Neumann factors to general von Neumann algebras, potentially providing a more user-friendly and accessible alternative to the established direct integral approach (see [KR97, Chapter 14]).

5.2 Vector-valued normal functions on Stonean spaces

In this section, we broaden the scope of the notion of normal functions on Stonean spaces (see [Sto49], [FK52], [Kad86]) to encompass functions with values in a finite-dimensional C^*-algebra $\mathfrak{A}$, and explore their basic properties. A key finding is that the set of these $\mathfrak{A}$-valued normal functions defined on a Stonean space carries a natural $*$-algebra structure.

Theorem 5.2.1. *Let $\mathcal{X}$ be a Stonean space, $\mathcal{O}$ be an open dense subset of $\mathcal{X}$, and $\mathcal{Y}$ be a locally compact Hausdorff space. Then, every continuous map $f:\mathcal{O}\to\mathcal{Y}$ has a unique maximal continuous extension, $f_{\mathcal{N}}$, relative to $\mathcal{X}$, and the domain of $f_{\mathcal{N}}$ is an open dense subset of $\mathcal{X}$.*

Proof. Let $\alpha\mathcal{Y}$ denote the one-point compactification of $\mathcal{Y}$. By Theorem 1.4.6, there is a unique continuous function $\widetilde{f}:\mathcal{X}\to\alpha\mathcal{Y}$ which extends f. Since $\mathcal{Y}$ is an open subset of $\alpha\mathcal{Y}$, we note that $\widetilde{f}^{-1}(\mathcal{Y})$ is an open subset of $\mathcal{X}$; furthermore, $\widetilde{f}^{-1}(\mathcal{Y})$ is a dense subset of $\mathcal{X}$ as it contains $\mathcal{O}$. Let $f_{\mathcal{N}}$ be the restriction of $\widetilde{f}$ to $\widetilde{f}^{-1}(\mathcal{Y})$, so that $\mathrm{dom}(f_{\mathcal{N}})=\widetilde{f}^{-1}(\mathcal{Y})$. Since $\mathcal{O}$ is dense in $\mathcal{X}$, any continuous extension g of f with $\mathrm{dom}(g)\subseteq\mathcal{X}$, must be equal to the restriction of $\widetilde{f}$ to $\mathrm{dom}(g)$, so that $\mathrm{dom}(g)\subseteq\mathrm{dom}(f_{\mathcal{N}})$. This shows that $f_{\mathcal{N}}$ is the unique maximal continuous extension of f relative to $\mathcal{X}$. ☐

Definition 5.2.2. Let $\mathcal{X}$ be a Stonean space and $\mathcal{Y}$ be a locally compact Hausdorff space. Let $\mathcal{O}$ be an open dense subset of $\mathcal{X}$ and $f:\mathcal{O}\to\mathcal{Y}$ be a continuous function. We call the extension $f_{\mathcal{N}}$ of f, given in Theorem 5.2.1, as the *normal extension* of f relative to $\mathcal{X}$. When the ambient space $\mathcal{X}$ is clear from the context, we omit the phrase 'relative to $\mathcal{X}$'.

The function f is said to be a *normal function* on $\mathcal{X}$ (though $\mathrm{dom}(f) = \mathcal{O}$ is really only dense in $\mathcal{X}$) if f is its own normal extension, that is, $f_{\mathcal{N}} = f$. In order to emphasize the codomain space $\mathcal{Y}$, we will often refer to such f as a *$\mathcal{Y}$-valued normal function.*

We denote the set of all $\mathcal{Y}$-valued normal functions on $\mathcal{X}$ by $\mathcal{N}(\mathcal{X}; \mathcal{Y})$.

The usage of the adjective 'normal' stems from its traditional usage in the context of $\mathcal{Y} = \mathbb{C}$ in the study of the spectral theorem for unbounded normal operators on Hilbert space (see [KR83, §5.6]).

Proposition 5.2.3. *Let $\mathcal{X}$ be a Stonean space, $(\mathcal{B}, \| \cdot \|)$ be a finite-dimensional normed linear space, and f be a $\mathcal{B}$-valued continuous function defined on an open dense subset of $\mathcal{X}$. For $m \in \mathbb{N}$, let*

$$O_m := \{x \in \mathrm{dom}(f) : \|f(x)\| < m\}.$$

Then $f \in \mathcal{N}(\mathcal{X}; \mathcal{B})$ if and only if $\mathrm{dom}(f) = \bigcup_{m\in\mathbb{N}} \overline{O_m}$.

Proof. Note that in a finite-dimensional normed space, closed balls are compact, so that $\mathcal{B}$ is a locally compact Hausdorff space, and $\mathcal{N}(\mathcal{X}; \mathcal{B})$ makes sense. It is clear that $\mathrm{dom}(f) \subseteq \bigcup_{m\in\mathbb{N}} \overline{O_m}$. Below we show $f \in \mathcal{N}(\mathcal{X}; \mathcal{B})$ if and only if $\bigcup_{m\in\mathbb{N}} \overline{O_m} \subseteq \mathrm{dom}(f)$.

($\Longrightarrow$) Let $B(0, m)$ denote the open ball of radius m centred at 0 in $\mathcal{B}$. Clearly $O_m = f^{-1}\big(B(0, m)\big)$ is open in $\mathrm{dom}(f)$ and hence in $\mathcal{X}$. Since $\overline{B(0, m)}$ is a compact Hausdorff space, by Theorem 1.4.6, there is a unique continuous extension of $f|_{O_m}$, $\widetilde{f_m}$, to the clopen subset $\overline{O_m}$ taking values in $\overline{B(0, m)}$. From the uniqueness, it is clear that for $k < m$, $\widetilde{f_k}$ is the restriction of $\widetilde{f_m}$ to $\overline{O_k}$. Thus, there is a well-defined continuous function $\widetilde{f}$ on $\bigcup_{m\in\mathbb{N}} \overline{O_m}$ defined as $\widetilde{f}(x) = \widetilde{f_m}(x)$ for $x \in \overline{O_m}$. Clearly, $\mathrm{dom}(f) = \bigcup_{m\in\mathbb{N}} O_m \subseteq \bigcup_{m\in\mathbb{N}} \overline{O_m} = \mathrm{dom}(\widetilde{f})$. Thus, $\widetilde{f}$ is a continuous extension of f, and as f is a normal function, we have $\mathrm{dom}(f) \supseteq \mathrm{dom}(\widetilde{f}) \supseteq \bigcup_{m\in\mathbb{N}} \overline{O_m}$.

($\Longleftarrow$) Let g be a continuous extension of f and $x \in \mathrm{dom}(g)$. Let k be a positive integer strictly larger than $\|g(x)\|$. There is a neighborhood U_x of x in $\mathcal{X}$ such that $\|g(x)\| < k$ for all $x \in U_x \cap \mathrm{dom}(g)$. Thus, $U_x \cap (\mathrm{dom}(f) \backslash \overline{O_k}) \subseteq U_x \cap (\mathrm{dom}(g) \backslash \overline{O_k}) = \varnothing$. From Lemma 1.4.5, we note that the closure of the open set $\mathrm{dom}(f) \backslash \overline{O_k}$ is $\mathcal{X} \backslash \overline{O_k}$. Thus, from Lemma 1.4.1-(i), $U_x \cap (\mathcal{X} \backslash \overline{O_k}) = \varnothing$, so that $U_x \subseteq \overline{O_k} \subseteq \mathrm{dom}(f)$. In particular $x \in \mathrm{dom}(f)$, whence $\mathrm{dom}(g) = \mathrm{dom}(f)$. In particular, $f = f_{\mathcal{N}}$ and hence $f \in \mathcal{N}(\mathcal{X}; \mathcal{B})$. ☐

***Remark* 5.2.4.** In view of Proposition 5.2.3 above, one may verify that the definition of normal functions (see Definition 1.4.4) is equivalent to our definition in the case when $\mathcal{B} = \mathbb{C}$ with the absolute value, $| \cdot |$, as the norm, that is, $\mathcal{N}(\mathcal{X}; \mathbb{C}) = \mathcal{N}(\mathcal{X})$.

Let $\mathfrak{A}$ be a finite-dimensional C^*-algebra. Clearly, $\mathfrak{A}$ is a locally compact Hausdorff space. Let $\mathcal{O}(\mathcal{X}; \mathfrak{A})$ be the set of $\mathfrak{A}$-valued continuous functions whose domains are open dense subsets of $\mathcal{X}$; note that $\mathcal{N}(\mathcal{X}; \mathfrak{A}) \subseteq \mathcal{O}(\mathcal{X}; \mathfrak{A})$. Since the intersection of two open dense subsets of a topological space is again an open dense subset, we may endow $\mathcal{O}(\mathcal{X};\mathfrak{A})$ with two binary operations $+, \cdot$, and an involution, $*$, as follows:

(i) For $f_1, f_2 \in \mathcal{O}(\mathcal{X};\mathfrak{A})$, $\mathrm{dom}(f_1 + f_2) = \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$ and $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ for $x \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$;

(ii) For $f_1, f_2 \in \mathcal{O}(\mathcal{X};\mathfrak{A})$, $\mathrm{dom}(f_1 \cdot f_2) = \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$ and $(f_1 \cdot f_2)(x) = f_1(x)f_2(x)$ for $x \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$;

(iii) For $f \in \mathcal{O}(\mathcal{X};\mathfrak{A})$, $\mathrm{dom}(f^*) = \mathrm{dom}(f)$ and $f^*(x) = f(x)^*$ for $x \in \mathrm{dom}(f)$;

(iv) For $f \in \mathcal{O}(\mathcal{X};\mathfrak{A})$ and $\lambda \in \mathbb{C}$, $\mathrm{dom}(\lambda f) = \mathrm{dom}(f)$ and $(\lambda f)(x) = \lambda f(x)$ for $x \in \mathrm{dom}(f)$.

It is easily verified that the binary operations, $+, \cdot$ are commutative and associative.

Definition 5.2.5. Let $f, g \in \mathcal{O}(\mathcal{X};\mathfrak{A})$. We write $f \sim_{\mathcal{O}} g$ if and only if f and g have identical normal extensions, that is, $f_{\mathcal{N}} = g_{\mathcal{N}}$. It is easy to see that $\sim_{\mathcal{O}}$ defines an equivalence relation on $\mathcal{O}(\mathcal{X};\mathfrak{A})$.

Lemma 5.2.6. *Let $\mathcal{X}$ be a Stonean space, and $\mathfrak{A}$ be a finite-dimensional C^*-algebra. Let $f_1, f_2 \in \mathcal{O}(\mathcal{X};\mathfrak{A})$. Then $f_1 \sim_{\mathcal{O}} f_2$ if and only if the restrictions of f_1, f_2, to the open dense subset, $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$, are identical.*

Proof. Let $f_1, f_2 \in \mathcal{O}(\mathcal{X};\mathfrak{A})$, with $\mathcal{O}_1 := \mathrm{dom}(f_1)$ and $\mathcal{O}_2 := \mathrm{dom}(f_2)$. If $f_1 \sim_{\mathcal{O}} f_2$, then by definition they have a common normal extension, f, and hence

$$f_1|_{\mathcal{O}_1 \cap \mathcal{O}_2} = f|_{\mathcal{O}_1 \cap \mathcal{O}_2} = f_2|_{\mathcal{O}_1 \cap \mathcal{O}_2}.$$

Conversely, let $g := f_1|_{\mathcal{O}_1 \cap \mathcal{O}_2} = f_2|_{\mathcal{O}_1 \cap \mathcal{O}_2}$. By the uniqueness of normal extension (see Theorem 5.2.1), $(f_1)_{\mathcal{N}} = g_{\mathcal{N}} = (f_2)_{\mathcal{N}}$. In other words, $f_1 \sim_{\mathcal{O}} f_2$. □

Corollary 5.2.7. *Let $\mathcal{X}$ be a Stonean space, and $\mathfrak{A}$ be a finite-dimensional C^*-algebra. Let $f_1, f_2, g_1, g_2 \in \mathcal{O}(\mathcal{X};\mathfrak{A})$ such that $f_1 \sim_{\mathcal{O}} f_2$ and $g_1 \sim_{\mathcal{O}} g_2$. Then we have*

(i) $f_1 + g_1 \sim_{\mathcal{O}} f_2 + g_2$; (ii) $f_1 \cdot g_1 \sim_{\mathcal{O}} f_2 \cdot g_2$; (iii) ${f_1}^* \sim_{\mathcal{O}} {f_2}^*$;

(iv) $\lambda f_1 \sim_{\mathcal{O}} \lambda f_2$; (v) $f_1 + (-f_1) \sim_{\mathcal{O}} \mathbf{0}$.

Thus, $\mathcal{O}(\mathcal{X};\mathfrak{A})/\sim_{\mathcal{O}}$ forms a unital $*$-algebra with the everywhere-defined constant functions $\mathbf{0}, \mathbf{1}$ serving as the zero, multiplicative identity, respectively.

Proof. The assertions follow from straightforward applications of Lemma 5.2.6. □

***Remark* 5.2.8.** From Corollary 5.2.7, it follows that $\mathcal{O}(\mathcal{X};\mathfrak{A})/\sim_{\mathcal{O}}$ is a $*$-algebra. Using the natural one-to-one correspondence between normal functions in $\mathcal{N}(\mathcal{X};\mathfrak{A})$ and $\sim_{\mathcal{O}}$ equivalence classes in $\mathcal{O}(\mathcal{X};\mathfrak{A})$ (via normal extensions), we may identify $\mathcal{N}(\mathcal{X};\mathfrak{A})$ with $\mathcal{O}(\mathcal{X};\mathfrak{A})/\sim_{\mathcal{O}}$ and also treat it as a $*$-algebra. The set of elements in $\mathcal{N}(\mathcal{X};\mathfrak{A})$ with domain $\mathcal{X}$ is precisely $C(\mathcal{X};\mathfrak{A})$.

***Remark* 5.2.9.** Let $\mathcal{X}$ be a Stonean space. Let $A \in \mathcal{O}\big(\mathcal{X}; M_n(\mathbb{C})\big)$, so that $\mathrm{dom}(A)$ is an open dense subset of $\mathcal{X}$. For $x \in \mathrm{dom}(A)$ and $i, j \in [n]$, let $a_{ij}(x)$ be the $(i,j)^{\text{th}}$ entry of the matrix $A(x) \in M_n(\mathbb{C})$. Clearly $a_{ij} : \mathrm{dom}(A) \to \mathbb{C}$, defined by $x \mapsto a_{ij}(x)$, is a continuous function, and thus belongs to $\mathcal{O}(\mathcal{X}; \mathbb{C})$. This gives us a natural mapping

$$\varphi : \mathcal{O}\big(\mathcal{X}; M_n(\mathbb{C})\big) \to M_n\big(\mathcal{O}(\mathcal{X}; \mathbb{C})\big) \text{ defined by } A \mapsto [a_{ij}]_{i,j=1}^n.$$

Note that φ respects $+, \cdot, *$. Moreover, if $A \sim_{\mathcal{O}} A'$ in $\mathcal{O}\big(\mathcal{X}; M_n(\mathbb{C})\big)$, then

$$a_{ij} = [\varphi(A)]_{ij} \sim_{\mathcal{O}} [\varphi(A')]_{ij} = a'_{ij},$$

in $\mathcal{O}(\mathcal{X}; \mathbb{C})$ for $1 \le i, j \le n$. Passing to the equivalence classes (see Remark 5.2.8), we get a unital $*$-homomorphism

$$\widetilde{\varphi} : \mathcal{N}\big(\mathcal{X}; M_n(\mathbb{C})\big) \to M_n\big(\mathcal{N}(\mathcal{X})\big).$$

In the other direction, consider a matrix $[a_{ij}]_{i,j=1}^n$ in $M_n\big(\mathcal{O}(\mathcal{X}; \mathbb{C})\big)$, with $a_{ij} \in \mathcal{O}(\mathcal{X}; \mathbb{C})$, and let $\mathcal{O}' = \bigcap_{i,j=1}^n \mathrm{dom}(a_{ij})$. Note that $\mathcal{O}'$ is an open dense subset of $\mathcal{X}$. Clearly, $A : \mathcal{O}' \to M_n(\mathbb{C})$, defined by $A(x) = [a_{ij}(x)]_{i,j=1}^n$, is a continuous function, and thus belongs to $\mathcal{O}\big(\mathcal{X}; M_n(\mathbb{C})\big)$. Thus, we have a natural mapping

$$\psi : M_n\big(\mathcal{O}(\mathcal{X}; \mathbb{C})\big) \to \mathcal{O}\big(\mathcal{X}; M_n(\mathbb{C})\big) \text{ defined by } [a_{ij}]_{i,j=1}^n \mapsto A.$$

Note that ψ respects $+, \cdot, *$. Moreover, for $1 \le i, j \le n$, if a'_{ij} is a collection of functions in $\mathcal{O}(\mathcal{X}; \mathbb{C})$ such that $a_{ij} \sim_{\mathcal{O}} a'_{ij}$ in $\mathcal{O}(\mathcal{X}; \mathbb{C})$ for all $i, j \in [n]$, then

$$A = \psi\big([a_{ij}]_{i,j=1}^n\big) \sim_{\mathcal{O}} \psi\big([a'_{ij}]_{i,j=1}^n\big) = A',$$

in $\mathcal{O}\big(\mathcal{X}; M_n(\mathbb{C})\big)$. Passing to the equivalence classes, we get a unital $*$-homomorphism,

$$\widetilde{\psi} : \mathcal{N}\big(\mathcal{X}; M_n(\mathbb{C})\big) \to M_n\big(\mathcal{N}(\mathcal{X})\big).$$

It can be easily verified that $\widetilde{\varphi}$ and $\widetilde{\psi}$ are in fact inverses of each other. We conclude that $\mathcal{N}\big(\mathcal{X}; M_n(\mathbb{C})\big)$ and $M_n\big(\mathcal{N}(\mathcal{X})\big)$ are $*$-isomorphic in a natural manner. We will often view these $*$-algebras interchangeably.

5.2.1 Transference results

In [DP63, Theorem 2], Deckard and Pearcy proved that for a Stonean space $\mathcal{X}$, every matrix in $M_n\big(C(\mathcal{X})\big)$ may be transformed into upper-triangular form via unitary conjugation. Making essential use of this result, below we prove that every matrix in $M_n\big(\mathcal{N}(\mathcal{X})\big)$ may be transformed into upper-triangular form via unitary conjugation. The core of the proof demonstrates a method of transferring results from $M_n\big(C(\mathcal{X})\big)$ to $M_n\big(\mathcal{N}(\mathcal{X})\big)$, via normal extensions; a recurrence of this method will be encountered in the proof of Proposition 5.5.1-(ii).

Theorem 5.2.10. *Let $\mathcal{X}$ be a Stonean space, and $A \in M_n\big(\mathcal{N}(\mathcal{X})\big)$. Then there is a unitary matrix $V \in M_n\big(C(\mathcal{X})\big)$ such that $B = V^*AV$ lies in $UT_n\big(\mathcal{N}(\mathcal{X})\big)$.*

Proof. With Remark 5.2.9 in mind, we view A as an element of $\mathcal{N}\big(\mathcal{X}; M_n(\mathbb{C})\big)$. For $m \in \mathbb{N}_0$, let

$$O_m := \{x \in \mathrm{dom}(A) : \|A(x)\| < m\}.$$

By Proposition 5.2.3, $\mathrm{dom}(A) = \bigcup_{m\in\mathbb{N}} \overline{O_m}$. For $m \in \mathbb{N}$, let $\mathcal{X}_m := \overline{O_m} \backslash \overline{O_{m-1}}$. Then $\mathcal{X}_m$ is a family of mutually disjoint clopen subsets of $\mathcal{X}$ with $\bigcup_{m\in\mathbb{N}} \mathcal{X}_m = \mathrm{dom}(A)$; in particular, the $\mathcal{X}_m$'s are Stonean spaces. From [DP63, Theorem 2], and identifying the AW^* algebras $M_n\big(C(\mathcal{X}_m)\big)$ and $C\big(\mathcal{X}_m; M_n(\mathbb{C})\big)$ (see Remark 1.5.4), there is a unitary element $V_m \in C\big(\mathcal{X}_m; M_n(\mathbb{C})\big)$ such that $B_m = V_m^* A|_{\mathcal{X}_m} V_m$ lies in $C\big(\mathcal{X}_m; UT_n(\mathbb{C})\big)$. From Theorem 1.4.2, the mapping,

$$V : \mathrm{dom}(A) \to M_n(\mathbb{C}) \text{ given by } V(x) = V_m(x) \text{ for } x \in \mathcal{X}_m,$$

is continuous. Since $V(x)$ is a unitary matrix for each $x \in \mathrm{dom}(A)$, the range of V lies in the compact subset of unitary matrices in $M_n(\mathbb{C})$. Thus, by Theorem 1.4.6, there is a unique continuous extension, $\widetilde{V}$, of V on $\overline{\mathrm{dom}(A)} = \mathcal{X}$. In other words, $\widetilde{V} \in C\big(\mathcal{X}; M_n(\mathbb{C})\big)$. Clearly, $B = \widetilde{V}^* A \widetilde{V}$ lies in $\mathcal{N}\big(\mathcal{X}; UT_n(\mathbb{C})\big)$, and thus may be viewed as an element of $UT_n\big(\mathcal{N}(\mathcal{X})\big)$ using Remark 5.2.9. ☐

The following lemma offers a transfer of discrete matrix invariants such as rank, eigenvalue multiplicities etc. from $M_n(\mathbb{C})$ to $M_n\big(\mathcal{N}(\mathcal{X})\big)$, by identifying a suitable family of disjoint open subsets of $\mathcal{X}$ with union dense in $\mathcal{X}$, such that on each of the open subsets, the desired matrix result (captured in the choice of the poset) may be obtained continuously. In the context of Jordan-Chevalley decomposition, the poset of all partitions of $[n]$ ordered via refinement, emerges as an appropriate choice. Utilizing Lemma 5.2.11 in Proposition 5.3.10, we achieve the primary goal of this chapter in Theorem 5.3.11.

Lemma 5.2.11. *Let $\mathcal{X}$ be a Stonean space, $\mathcal{O}$ be an open dense subset of $\mathcal{X}$, and $(\mathfrak{P}, \leq)$ be a finite partially ordered set equipped with the Scott topology. Let $\phi : \mathcal{O} \to (\mathfrak{P}, \leq)$ be a continuous mapping, and for $p \in \mathfrak{P}$, consider the subset $\mathcal{X}_p := \phi^{-1}\{p\}$ of $\mathcal{O}$. Then there is a collection $\{O_p\}_{p\in\mathfrak{P}}$ of mutually disjoint open subsets of $\mathcal{X}$ such that $O_p \subseteq \mathcal{X}_p$ and $\bigcup_{p\in\mathfrak{P}} O_p \subseteq \mathcal{O}$ is dense in $\mathcal{X}$.*

Proof. For $p \in \mathfrak{P}$, we define $\Lambda_p := \bigcup_{p\leq q} \mathcal{X}_q$, $\Lambda'_p := \bigcup_{p<q} \mathcal{X}_q$, and $O_p := \Lambda_p \backslash \overline{\Lambda'_p}$, with the convention that $\Lambda'_p := \varnothing$ if p is a maximal element in $\mathfrak{P}$. Since every principal upper-set of $\mathfrak{P}$ is open in the Scott topology (see Definition 1.4.11), for every $p \in \mathfrak{P}$, the subset $\{q \in \mathfrak{P} : p < q\} = \bigcup_{p<q} q^{\uparrow}$ is an open subset of $\mathfrak{P}$.

Let p be any element of $\mathfrak{P}$. From the continuity of ϕ, the sets

$$\Lambda_p = \bigcup_{p \le q} \mathcal{X}_q = \bigcup_{p \le q} \phi^{-1}\{q\} = \phi^{-1}\left(p^{\uparrow}\right),$$
$$\Lambda'_p = \bigcup_{p < q} \mathcal{X}_q = \bigcup_{p < q} \phi^{-1}\{q\} = \phi^{-1}(\{q \in \mathfrak{P} : p < q\}) = \phi^{-1}\left(\bigcup_{p<q} q^{\uparrow}\right),$$

are open subsets of $\mathcal{O}$, and hence of $\mathcal{X}$. Note that O_p is an open subset of $\mathcal{X}$, and since $\mathcal{X}_p = \Lambda_p \backslash \Lambda'_p$, we have $O_p \subseteq \mathcal{X}_p$. Moreover, since the $\mathcal{X}_p$'s are mutually disjoint, the collection $\{O_p\}_{p \in \mathfrak{P}}$ consists of mutually disjoint open subsets of $\mathcal{X}$, with

$$\bigcup_{p \in \mathfrak{P}} O_p \subseteq \bigcup_{p \in \mathfrak{P}} \mathcal{X}_p = \mathcal{O}.$$

Note that $Q := \overline{\bigcup_{p \in \mathfrak{P}} O_p}$ is a clopen subset of $\mathcal{X}$. Below we show that $Q = \mathcal{X}$, from which the desired assertion follows.

Let, if possible, Q be a proper clopen subset of $\mathcal{X}$, so that $\mathcal{O} \backslash Q$ is a dense open subset of the non-empty clopen set $\mathcal{X} \backslash Q$. Since $\mathfrak{P}$ is a finite poset, we may choose $x \in \mathcal{O} \backslash Q$ such that $\phi(x)$ is a maximal element of $\phi(\mathcal{O} \backslash Q) \subseteq \mathfrak{P}$. Clearly $x \in \mathcal{X}_{\phi(x)} \subseteq \Lambda_{\phi(x)}$. At the same time, since $x \notin Q$, we have $x \notin O_{\phi(x)} = \Lambda_{\phi(x)} \backslash \overline{\Lambda'_{\phi(x)}}$. Thus, x must be contained in $\overline{\Lambda'_{\phi(x)}}$. In particular, $\overline{\Lambda'_{\phi(x)}} \cap (\mathcal{O} \backslash Q) \neq \varnothing$, whence from Lemma 1.4.1-(i), we have $\Lambda'_{\phi(x)} \cap (\mathcal{O} \backslash Q) \neq \varnothing$. For any $y \in \Lambda'_{\phi(x)} \cap (\mathcal{O} \backslash Q)$, we have $\phi(y) \in \phi(\Lambda'_{\phi(x)}) = \bigcup_{\phi(x) < q} q^{\uparrow}$, which implies that $\phi(x) < \phi(y)$, contradicting the maximality of $\phi(x)$ in $\phi(\mathcal{O} \backslash Q)$. We conclude that $Q = \mathcal{X}$. $\square$

5.3 Jordan-Chevalley decomposition for matrices over normal functions on a Stonean space

In this section, we prove a key result of this chapter, Theorem 5.3.11, which asserts that for a Stonean space $\mathcal{X}$, a matrix in $M_n(\mathcal{N}(\mathcal{X}))$ admits a unique Jordan-Chevalley decomposition (see Definition 5.3.3). Moreover, we note that for an $n \times n$ matrix over the subring $C(\mathcal{X})$ of $\mathcal{N}(\mathcal{X})$, the diagonalizable and nilpotent parts need not lie in $M_n(C(\mathcal{X}))$.

Before we begin, we note the following definitions for matrices over a unital commutative ring $\mathcal{R}$, although we are particularly interested in the case of $\mathcal{R} = C(\mathcal{X})$, $\mathcal{N}(\mathcal{X})$.

Definition 5.3.1. Two matrices $A, B \in M_n(\mathcal{R})$ are said to be *similar* in $M_n(\mathcal{R})$ if there is an invertible matrix $S \in GL_n(\mathcal{R})$ such that $B = SAS^{-1}$. It is straightforward to verify that *similarity* is an equivalence relation on $M_n(\mathcal{R})$. We refer to the corresponding equivalence classes as *similarity orbits*.

Definition 5.3.2. We say that a matrix $A \in M_n(\mathcal{R})$ is *diagonalizable* in $M_n(\mathcal{R})$ if A is similar to a diagonal matrix in $M_n(\mathcal{R})$. We say that a matrix N is *nilpotent* if $N^k = \mathbf{0}_n$ for some $k \in \mathbb{N}$.

Definition 5.3.3. A matrix $A \in M_n(\mathcal{R})$ is said to have a *Jordan-Chevalley decomposition* in $M_n(\mathcal{R})$ if there is a commuting pair of matrices $D, N \in M_n(\mathcal{R})$ such that D is diagonalizable in $M_n(\mathcal{R})$, N is nilpotent, and $A = D + N$. The decomposition $A = D + N$ is said to be a Jordan-Chevalley decomposition of A.

Note that for $\mathcal{R} = \mathbb{C}$, every matrix in $M_n(\mathbb{C})$ has a unique Jordan-Chevalley decomposition (see [HK71, Chapter 7]). In this context, we use the notation $D(A), N(A)$ for the diagonalizable, nilpotent parts, respectively, of $A \in M_n(\mathbb{C})$.

Let $n_1, \ldots, n_m$ be positive integers such that $n_1 + \cdots + n_m = n$, and let $\lambda_1, \ldots, \lambda_m$ be complex numbers. For $1 \le i \le m$, let N_i be a strictly upper-triangular matrix in $M_{n_i}(\mathbb{C})$, which is clearly nilpotent. It is straightforward to see that $A_i := \lambda_i I_{n_i} + N_i$ is an upper-triangular matrix in $M_{n_i}(\mathbb{C})$ whose diagonalizable and nilpotent parts (in its Jordan-Chevalley decomposition) are $\lambda_i I_{n_i}$ and N_i, respectively. Thus, the diagonalizable and nilpotent parts of the matrix $\bigoplus_{i=1}^m A_i$ in $M_n(\mathbb{C})$ are given by $\bigoplus_{i=1}^m \lambda_i I_{n_i}$ and $\bigoplus_{i=1}^m N_i$, respectively. For such matrices, the Jordan-Chevalley decomposition is readily available without the need for any computation.

Definition 5.3.4. We say that an upper-triangular matrix $A = [a_{ij}]_{i,j=1}^n$ in $M_n(\mathbb{C})$ is *good* if $a_{ij} = 0$ whenever $a_{ii} \neq a_{jj}$.

For example, the block-diagonal matrix $\bigoplus_{i=1}^m (\lambda_i I_{n_i} + N_i)$ is a good upper-triangular matrix.

In Lemma 5.3.5 below, we note that the Jordan-Chevalley decomposition of a good upper-triangular matrix can be easily read off. In the course of our discussion in this section, we see that every matrix in $M_n(\mathbb{C})$ is similar to a good upper-triangular matrix, hence reducing the problem of ascertaining Jordan-Chevalley decomposition of a matrix to the much simpler setting of good upper-triangular matrices.

Lemma 5.3.5. *Let $A = [a_{ij}]_{i,j=1}^n$ be a good upper-triangular matrix in $M_n(\mathbb{C})$, that is, $a_{ij} = 0$ whenever $a_{ii} \neq a_{jj}$. Then the diagonalizable part of A is $D := \mathrm{diag}(a_{11}, \ldots, a_{nn})$, which is obtained by replacing the off-diagonal entries of A by 0, and the nilpotent part of A is $N := A - D$, which is obtained by replacing the diagonal entries of A by 0.*

Proof. Observe that N is a strictly upper-triangular matrix, hence nilpotent. Using the uniqueness of the Jordan-Chevalley decomposition, it is enough to show that D and N commute with each other. A straightforward calculation shows that,

$$DN - ND = [a_{ij}(a_{ii} - a_{jj})]_{i,j=1}^n.$$

Since $a_{ij} = 0$ whenever $a_{ii} \neq a_{jj}$, it follows that $DN = ND$. □

Definition 5.3.6. We define a total ordering called the *twisted lexicographical ordering* on $[n] \times [n]$ as follows: For two elements $(i_1, j_1), (i_2, j_2)$ in $[n] \times [n]$, we stipulate

$$(i_1, j_1) <_{\text{tl}} (i_2, j_2),$$

if either $i_1 < i_2$, or $i_1 = i_2$ and $j_1 > j_2$.

Note that the total ordering, $<_{\text{tl}}$, on $[n] \times [n]$ has $(1, n)$ as its minimum and $(n, 1)$ as its maximum element. This ordering can be visualized by traversing an $n \times n$ grid as follows:

1. Begin at the top-right cell (corresponding to $(1, n)$);
2. Move leftward across the current row.
3. Upon reaching the left edge, proceed to the rightmost cell of the row below and continue moving left.
4. The traversal ends at the bottom-left cell (corresponding to $(n, 1)$).

Lemma 5.3.7 below is the main tool in Proposition 5.3.9 which we use to identify a good upper-triangular matrix in the similarity orbit of a given upper-triangular matrix.

Lemma 5.3.7. *Let A be an upper-triangular matrix in $M_n(\mathbb{C})$ and $i < j$ be fixed indices in $[n]$. For $\lambda \in \mathbb{C}$, let $A_\lambda := (I_n + \lambda E_{ij})A(I_n - \lambda E_{ij})$. Then A_λ is an upper-triangular matrix similar to A such that the $(k, \ell)^{\text{th}}$ entry of A_λ coincides with the $(k, \ell)^{\text{th}}$ entry of A whenever $k = \ell$ or $(i, j) <_{\text{tl}} (k, \ell)$.*

Proof. Note that $(I_n + \lambda E_{ij})(I_n - \lambda E_{ij}) = I_n - \lambda^2 E_{ij}^2 = I_n$, and hence $(I_n + \lambda E_{ij})^{-1} = I_n - \lambda E_{ij}$. Since $i < j$, E_{ij} is a strictly upper-triangular matrix, so that $A_\lambda := (I_n + \lambda E_{ij})A(I_n - \lambda E_{ij})$ is an upper-triangular matrix similar to A. Clearly, the diagonal of A_λ coincides with the diagonal of A.

For $k, \ell \in [n]$, let $a_{k\ell}$ denote the $(k, \ell)^{\text{th}}$ entry of A. Since A is upper-triangular, and $i < j$, we have $a_{ji} = 0$ and,

$$\begin{aligned} A_\lambda = (I_n + \lambda E_{ij})A(I_n - \lambda E_{ij}) &= A + \lambda E_{ij}A - \lambda A E_{ij} - \lambda^2 E_{ij} A E_{ij} \\ &= A + \lambda E_{ij}A - \lambda A E_{ij} - \lambda^2 a_{ji} E_{ij} \\ &= A + \lambda [E_{ij}, A]. \end{aligned} \tag{5.3.1}$$

It is straightforward to check that for any matrix $A \in M_n(\mathbb{C})$, the $(k, \ell)^{\text{th}}$ entry of the commutator $[E_{ij}, A]$, is given by

$$[E_{ij}, A]_{k\ell} = a_{j\ell}\delta_{ki} - a_{ki}\delta_{j\ell}.$$

Note that $\delta_{ki} = 0 = a_{ki}$ when $i < k$, and $\delta_{j\ell} = 0 = a_{j\ell}$ when $j > \ell$. We conclude that if $(i, j) <_{\text{tl}} (k, \ell)$, then $[E_{ij}, A]_{k\ell} = a_{j\ell}\delta_{ki} - a_{ki}\delta_{j\ell} = 0$.

Thus, from equation (5.3.1), it follows that the $(k,\ell)^{\text{th}}$ entry of A_λ is equal to $a_{k,\ell}$ whenever $(i,j) <_{\text{tl}} (k,\ell)$. □

Definition 5.3.8. Let $\mathscr{P}_n$ denote the set of all partitions of $[n]$. For $\pi \in \mathscr{P}_n$, let $\sim_\pi$ be a binary relation on $[n]$ defined by $i \sim_\pi j$ if and only if i, j belong to the same subset of $[n]$ from the partition π.

For $\vec{v} \in \mathbb{C}^n$, recall from Definition 1.4.12 that, $\sim_{\vec{v}}$ is an equivalence relation on $[n]$, which gives rise to the partition $\mathcal{P}(\vec{v})$ on $[n]$. Clearly, $\sim_{\vec{v}}$ and $\sim_{\mathcal{P}(\vec{v})}$ define the same equivalence relation on $[n]$. For a partition π of $[n]$, we define the following sets

$$\begin{aligned} UT(\pi) &: = \big\{A \in UT_n(\mathbb{C}) : \mathcal{P}(\text{dvec}(A)) = \pi\big\}. \\ GUT(\pi) &: = \big\{A = [a_{ij}]_{i,j=1}^n \in UT(\pi) : a_{ij} = 0 \text{ whenever } i \nsim_\pi j\big\}. \end{aligned}$$

Note that $UT(\pi)$ is the set of all upper-triangular matrices whose principal diagonal, by grouping together the diagonal indices corresponding to the same eigenvalue, induces the partition π, and $GUT(\pi)$ is the collection of all good upper-triangular matrices in $UT(\pi)$.

Proposition 5.3.9. *Let Ω be a topological space, and π be a partition of $[n]$. Let $A : \Omega \to UT_n(\mathbb{C})$ be a continuous function with range in $UT(\pi)$. Then there is a continuous function $S : \Omega \to UT_n(\mathbb{C}) \cap GL_n(\mathbb{C})$ such that SAS^{-1} has range in $GUT(\pi)$. In other words, every upper-triangular matrix over $C(\Omega)$ is similar to a good upper-triangular matrix over $C(\Omega)$, with the similarity implemented by an invertible upper-triangular matrix over $C(\Omega)$.*

Proof. Note that the $*$-algebras $C\big(\Omega; M_n(\mathbb{C})\big)$ and $M_n\big(C(\Omega)\big)$, are naturally $*$-isomorphic, allowing us to view them interchangeably. Let $A = [a_{ij}]_{i,j=1}^n$ with $a_{ij} \in C(\Omega)$. Consider the sets,

$$\begin{aligned} F_\pi &:= \big\{(i,j) \in [n] \times [n] : i < j \text{ and } i \nsim_\pi j\big\} \subseteq [n] \times [n], \\ \mathbb{O}(A) &:= \big\{SAS^{-1} : S \in UT_n\big(C(\Omega)\big) \cap GL_n\big(C(\Omega)\big) \text{ with } \text{dvec}(S) = (\mathbf{1}, \ldots, \mathbf{1})\big\}. \end{aligned}$$

Observe that $\mathbb{O}(A)$ consists of upper-triangular matrices similar to A in $M_n\big(C(\Omega)\big)$, and the diagonal of every matrix in $\mathbb{O}(A)$ coincides with the diagonal of A. Below we show that there is a matrix $B \in \mathbb{O}(A)$ all of whose F_π-entries are $\mathbf{0}$. Clearly, for such a B, $\text{ran}(B) \subseteq GUT(\pi)$, which proves the result.

Suppose, if possible, every matrix $B = [b_{ij}]_{i,j\in[n]}$ in $\mathbb{O}(A)$ has a non-zero entry for some index in F_π. Let

$$(k,\ell) := \min_{B \in \mathbb{O}(A)} \big\{\max\{(i,j) \in F_\pi : b_{ij} \neq \mathbf{0}\}\big\}, \tag{5.3.2}$$

where the min and max are taken with respect to the twisted lexicographical ordering on $[n] \times [n]$. Since F_π is a totally ordered finite set, there is a matrix $B' = [b'_{ij}]_{i,j\in[n]}$ in $\mathbb{O}(A)$ such that

$$\max\{(i,j) \in F_\pi : b'_{ij} \neq \mathbf{0}\} = (k,\ell).$$

Note that $k < \ell$, $b'_{k\ell} \neq \mathbf{0}$,and $b'_{ij} = \mathbf{0}$ for all $(i, j) \in F_\pi$ such that $(k, \ell) <_{\text{tl}} (i, j)$.

Since $\text{ran}(A) \subseteq UT(\pi)$, it follows that $a_{ii}(x) \neq a_{jj}(x)$ for all $(i, j) \in F_\pi$, and $x \in \Omega$. In other words, $a_{kk} - a_{\ell\ell}$ is an invertible function in $C(\Omega)$. Define

$$S := I_n + (a_{kk} - a_{\ell\ell})^{-1} b'_{k\ell} E_{k\ell}.$$

Clearly $S \in UT_n\big(C(\Omega)\big)$, and $\text{dvec}(S) = (\mathbf{1}, \ldots, \mathbf{1})$. It follows from the computation in Lemma 5.3.7 that S is invertible in $UT_n\big(C(\Omega)\big)$ with $S^{-1} = I_n - (a_{kk} - a_{\ell\ell})^{-1} b'_{k\ell} E_{k\ell}$, so that $SB'S^{-1}$ lies in $\mathbb{O}(A)$, and the $(k, \ell)^{\text{th}}$ entry of $SB'S^{-1}$ is given by

$$\begin{aligned}
[SB'S^{-1}]_{k\ell} &= b'_{k\ell} + (a_{kk} - a_{\ell\ell})^{-1} b'_{k\ell}(b'_{\ell\ell} - b'_{kk}) \\
&= b'_{k\ell} + b'_{k\ell}(a_{kk} - a_{\ell\ell})^{-1}(a_{\ell\ell} - a_{kk}) \qquad (\text{ as } b'_{ii} = a_{ii} \ \forall i \in [n]) \\
&= \mathbf{0}.
\end{aligned}$$

Again, from Lemma 5.3.7, if $(k, \ell) <_{\text{tl}} (i, j)$, then the $(i, j)^{\text{th}}$ entry of $SB'S^{-1}$ is equal to b'_{ij}. In particular, if the index $(i, j) \in F_\pi$ satisfies $(k, \ell) <_{\text{tl}} (i, j)$, then $[SB'S^{-1}]_{ij} = b'_{ij} = \mathbf{0}$. Thus,

$$\max\{(i, j) \in F_\pi : [SB'S^{-1}]_{ij} \neq \mathbf{0}\} <_{\text{tl}} (k, \ell).$$

This contradicts the minimality of (k, ℓ) in (5.3.2), and the assertion follows. □

Proposition 5.3.10. *Let $\mathcal{X}$ be a Stonean space, and $A \in \mathcal{N}\big(\mathcal{X}; M_n(\mathbb{C})\big)$. Then there is an open dense subset $\mathcal{O}$ of $\mathcal{X}$, which is contained in* $\text{dom}(A)$*, and $S \in C\big(\mathcal{O}; GL_n(\mathbb{C})\big)$ such that $S(x)A(x)S(x)^{-1}$ is a good upper-triangular matrix (see Definition 5.3.4) for every $x \in \mathcal{O}$.*

Proof. From the proof of Theorem 5.2.10, there is a unitary element $V \in C\big(\mathcal{X}; M_n(\mathbb{C})\big)$ such that $B = V^*AV$ lies in $\mathcal{N}\big(\mathcal{X}; UT_n(\mathbb{C})\big)$. Let $\mathcal{O}' := \text{dom}(B)$, and $\phi : \mathcal{O}' \to \mathscr{P}_n$ be the map defined by the following commutative diagram,

$$\begin{array}{ccc}
\mathcal{O}' & \xrightarrow{\ \phi\ } & \mathscr{P}_n \\
{\scriptstyle B}\big\downarrow & & \big\uparrow{\scriptstyle \mathcal{P}} \\
UT_n(\mathbb{C}) & \xrightarrow[\text{dvec}]{} & \mathbb{C}^n
\end{array}$$

that is, $\phi(x) = \mathcal{P}\big(\text{dvec}(B(x))\big)$ for $x \in \mathcal{O}'$. Clearly, B and $T \mapsto \text{dvec}(T)$ are continuous maps, and the continuity of the map $\vec{v} \mapsto \mathcal{P}(\vec{v})$ follows from Lemma 1.4.13. Thus, ϕ is continuous.

From Lemma 5.2.11, there is a collection $\{O_\pi\}_{\pi \in \mathscr{P}_n}$ of disjoint open subsets of $\mathcal{O}'$ such that $O_\pi \subseteq \phi^{-1}(\pi) = \big\{x \in \mathcal{X} : \mathcal{P}(\text{dvec}(B(x))) = \pi\big\}$ and $\bigcup_{\pi \in \mathscr{P}_n} O_\pi \subseteq \mathcal{O}'$ is dense in $\mathcal{X}$. It is immediate from their very definitions that for each partition π of $[n]$, the continuous function $B|_{O_\pi} : O_\pi \to UT_n(\mathbb{C})$ has range in $UT(\pi)$. From Proposition 5.3.9, there is a continuous function $S'_\pi : O_\pi \to UT_n(\mathbb{C}) \cap GL_n(\mathbb{C})$ such that $S'_\pi B|_{O_\pi} {S'_\pi}^{-1}$ has range in $GUT(\pi)$.

Define $\mathcal{O} := \bigcup_{\pi\in\mathscr{P}_n} O_\pi$. Since O_π's are mutually disjoint open sets, Theorem 1.4.2 tells us that the map

$$S' : \mathcal{O} \to UT_n(\mathbb{C}) \cap GL_n(\mathbb{C}) \text{ given by } S'(x) = S'_\pi(x) \text{ if } x \in O_\pi,$$

is a well-defined continuous map. Define the map $S : \mathcal{O} \to M_n(\mathbb{C})$ by $S(x) = S'(x)V(x)^*$ for $x \in \mathcal{O}$. Clearly, $S = S' \cdot (V^*|_{\mathcal{O}})$ is a continuous map. Moreover, for every $x \in \mathcal{O}$, $S(x)$ is invertible with $S(x)^{-1} = V(x)S'(x)^{-1}$, so that $S \in C\big(\mathcal{O}; GL_n(\mathbb{C})\big)$.

Since for each partition π, the range of $S'_\pi B|_{O_\pi} {S'_\pi}^{-1}$ lies in $GUT(\pi)$, it is clear that $S(x)A(x)S(x)^{-1} = S'(x)V^*(x)A(x)V(x)S'(x)^{-1} = S'(x)B(x)S'(x)^{-1}$, is a good upper-triangular matrix for every point x in $\mathcal{O}$. □

Theorem 5.3.11. *Let $\mathcal{X}$ be a Stonean space, and A be a matrix in $M_n\big(\mathcal{N}(\mathcal{X})\big)$. Then there is a unique commuting pair of matrices $D, N \in M_n\big(\mathcal{N}(\mathcal{X})\big)$ such that D is diagonalizable in $M_n\big(\mathcal{N}(\mathcal{X})\big)$, N is nilpotent, and $A = D + N$. In other words, every matrix in $M_n\big(\mathcal{N}(\mathcal{X})\big)$ admits a unique Jordan-Chevalley decomposition in $M_n\big(\mathcal{N}(\mathcal{X})\big)$. (We refer to D, N, respectively, as the diagonalizable part, nilpotent part, respectively, of A.)*

Proof. In view of Remark 5.2.9, from Proposition 5.3.10, there is an open dense subset $\mathcal{O}$ of $\mathcal{X}$ which is contained in $\mathrm{dom}(A)$, and an $S \in C\big(\mathcal{O}; GL_n(\mathbb{C})\big)$ such that $S(x)A(x)S(x)^{-1}$ is a good upper-triangular matrix for every $x \in \mathcal{O}$.

Viewing S as a matrix in $GL_n\big(C(\mathcal{O})\big)$, and $A|_{\mathcal{O}}$ as a matrix in $M_n\big(C(\mathcal{O})\big)$, let $T := S(A|_{\mathcal{O}})S^{-1} \in UT_n\big(C(\mathcal{O})\big)$. Let D_T be the diagonal matrix in $M_n\big(C(\mathcal{O})\big)$, whose diagonal coincides with the diagonal of T. Since $T(x)$ is a good upper-triangular matrix for every $x \in \mathcal{O}$, it follows from Lemma 5.3.5 that $D_T(x)$ is the diagonalizable part and $T(x) - D_T(x)$ is the nilpotent part in the Jordan-Chevalley decomposition of $T(x)$.

For $S^{-1}D_T S$, $S^{-1}(T - D_T)S$, viewed as elements in $C\big(\mathcal{O}; M_n(\mathbb{C})\big)$, we denote their normal extensions by D, N, respectively, and view them as elements of $M_n\big(\mathcal{N}(\mathcal{X})\big)$ (via Remark 5.2.9). Note that D is diagonalizable in $M_n\big(\mathcal{N}(\mathcal{X})\big)$, N is a nilpotent, and $\mathrm{dom}(D) = \mathrm{dom}(N)$. Clearly, $A = D + N$ and $DN = ND$ follow from Remark 5.2.8; and $A(x) = D(x) + N(x)$ for every $x \in \mathrm{dom}(D)$.

The uniqueness of the pair D, N, follows from the uniqueness of the Jordan-Chevalley decomposition of complex matrices, $A(x)$ for $x \in \mathrm{dom}(D)$, and the uniqueness of normal extensions. □

Proposition 5.3.12. *Let $\beta\mathbb{N}$ denote the Stone-Čech compactification of the set of natural numbers endowed with the discrete topology. Then there is a matrix $A \in M_3\big(C(\beta\mathbb{N})\big) \subseteq M_3\big(\mathcal{N}(\beta\mathbb{N})\big)$ such that the diagonalizable and nilpotent parts of A do not lie in $M_3\big(C(\beta\mathbb{N})\big)$.*

Proof. Let $A : \mathbb{N} \to M_3(\mathbb{C})$ be the bounded mapping given by,

$$A(n) = \begin{bmatrix} \frac{1}{n} & 1 & 0 \\ 0 & \frac{1}{n} & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 1.4.6, there is a unique continuous extension of A to $\beta\mathbb{N}$ which we again denote by A by abuse of notation. As discussed in Remark 5.2.9, below we view the $*$-algebras $\mathcal{N}\big(\beta\mathbb{N}; M_3(\mathbb{C})\big)$ and $M_3\big(\mathcal{N}(\beta\mathbb{N})\big)$ interchangeably. Let $D(A) \in M_3\big(\mathcal{N}(\beta\mathbb{N})\big)$ denote the diagonalizable part in the Jordan-Chevalley decomposition of A, as obtained in Theorem 5.3.11. From the uniqueness of the Jordan-Chevalley decomposition for matrices in $M_3(\mathbb{C})$ and Theorem 2.6.4, it is clear that $D(A)(x)$ is the diagonalizable part of the complex 3×3 matrix $A(x)$, for every $x \in \mathrm{dom}\big(D(A)\big)$. It follows from equation (2.6.1) that $\{D(A(n))\}_{n\in\mathbb{N}}$ is an unbounded sequence, which implies that $\mathrm{dom}\big(D(A)\big)$ is a proper subset of $\beta\mathbb{N}$. Thus, neither $D(A)$ nor $N(A)$ belongs to $M_3\big(C(\beta\mathbb{N})\big)$. ☐

Corollary 5.3.13. *Let $\mathcal{X}$ be an infinite Stonean space. Then there is a matrix A in $M_3\big(C(\mathcal{X})\big) \subseteq M_3\big(\mathcal{N}(\mathcal{X})\big)$ such that the diagonalizable and nilpotent parts of A do not lie in $M_3\big(C(\mathcal{X})\big)$.*

Proof. By Proposition 1.4.7, there is a closed subset $\mathbb{S}$ of $\mathcal{X}$ which is homeomorphic to $\beta\mathbb{N}$. By Proposition 5.3.12, there is a continuous function $A : \mathbb{S} \to M_3(\mathbb{C})$. By the Tietze extension theorem (cf. [Mun00, Theorem 35.1]), there is a continuous extension of A to the whole of $\mathcal{X}$. Using the argument by contradiction, as in the proof of Proposition 5.3.12, we arrive at the desired conclusion. ☐

5.4 Affiliated operators and Murray-von Neumann algebras

In this section, we review foundational concepts related to affiliated operators and Murray–von Neumann algebras, with particular emphasis on their behaviour under direct sums, in preparation for the results developed in §5.5. Our main references for the theory of affiliated operators and Murray von Neumann algebras are [GN24] and [Nay21], respectively; for basic definitions we refer to §1.6.

In [GN24], Ghosh and Nayak have investigated algebraic aspects of affiliated operators in the setting of general von Neumann algebras, which are, in particular, applicable to finite von Neumann algebras. Below we record two remarks pertinent to the finite setting which facilitates our study.

***Remark* 5.4.1.** For a finite von Neumann algebra $\mathscr{N}$, and $A \in \mathrm{Aff}(\mathscr{N})$, we note two types of quotient representation for A.

(i) There are bounded operators $P, Q \in \mathscr{N}$ with Q one-to-one such that

$$\text{dom}(A) = \text{ran}(Q), \ \text{ran}(A) = \text{ran}(P), \ \text{and } A = PQ^{-1}.$$

In other words, A is given by the mapping, $Qx \mapsto Px$, for $x \in \mathscr{H}$ (see [GN24, Theorem 5.4-(i)] and the discussion in [GN24, §4.2]).
(Note that Q automatically has dense range as $\mathscr{N}$ is finite, and thus $Q^{-1} \in \text{Aff}(\mathscr{N})$.)

(ii) The operator A is of the form $A = Q^{-1}P$, where $P, Q \in \mathscr{N}$ with Q being one-to-one (see [GN24, Theorem 5.4-(ii)]).

***Remark* 5.4.2.** Let $\mathscr{N}, \mathscr{N}'$ be finite von Neumann algebras acting on the Hilbert spaces $\mathscr{H}, \mathscr{H}'$, respectively, and $\Phi : \mathscr{N} \to \mathscr{N}'$ be a unital normal $*$-homomorphism. Let $A \in \text{Aff}(\mathscr{N})$, and P be an operator in $\mathscr{N}$ and Q be a one-to-one operator in $\mathscr{N}$ such that $A = Q^{-1}P$. By [GN24, Lemma 2.2], $\Phi(Q)$ is a one-to-one operator in $\mathscr{N}'$ and from [GN24, Theorem 5.5, Theorem 4.14], we have

$$\Phi_{\text{aff}}(A) = \Phi(Q)^{-1}\Phi(P). \tag{5.4.1}$$

In fact, it may be deduced from [GN24, Theorem 5.8] that Φ_{aff} is the unique extension of Φ (see Definition 1.6.6) which maps operators of the form $Q^{-1}P$ to $\Phi(Q)^{-1}\Phi(P)$.

Lemma 5.4.3. *Let $\mathscr{N}$ be a finite von Neumann algebra, $\mathscr{N}'$ be a von Neumann algebra, and $\Phi : \mathscr{N} \to \mathscr{N}'$ be a surjective unital normal $*$-homomorphism. Then we have the following.*

(i) The kernel of Φ is a WOT-closed two-sided ideal of $\mathscr{N}$ and there is a unique central projection E of $\mathscr{N}$ such that $\ker(\Phi) = \mathscr{N}(I_{\mathscr{N}} - E)$. For every $A' \in \mathscr{N}'$, there is a unique $A \in \mathscr{N}$ in the preimage of A' under Φ that satisfies $A = EAE$.

(ii) The von Neumann algebra $\mathscr{N}'$ is finite.

(iii) The map $\Phi_{\text{aff}} : \text{Aff}(\mathscr{N}) \to \text{Aff}(\mathscr{N}')$, as defined in Definition 1.6.6, is surjective. For every $A' \in \text{Aff}(\mathscr{N}')$, there is a unique $A \in \text{Aff}(\mathscr{N})$ in the preimage of A' under Φ_{aff} that satisfies $A = EAE$.

Proof. (i) Since Φ is WOT-WOT continuous, it is clear that the kernel of Φ is a WOT-closed two-sided ideal of $\mathscr{N}$. By [KR97, Theorem 6.8.8], there is a unique central projection E of $\mathscr{N}$ such that $\ker(\Phi) = \mathscr{N}(I_{\mathscr{N}} - E)$.

For $A' \in \mathscr{N}'$, let $B \in \mathscr{N}$ be such that $\Phi(B) = A'$. Then, $A := EBE \in \mathscr{N}$ is such that $\Phi(A) = \Phi(EBE) = \Phi(B) = A'$ and $EAE = E(EBE)E = EBE = A$. If $A_1, A_2 \in \mathscr{N}$ are such that $\Phi(A_1) = A' = \Phi(A_2)$, and $EA_1E = A_1, EA_2E = A_2$, then $(A_1 - A_2)(I_{\mathscr{N}} - E) = A_1(I_{\mathscr{N}} - E) - A_2(I_{\mathscr{N}} - E) = 0_{\mathscr{N}}$, and since $A_1 - A_2 \in \ker(\Phi) = \mathscr{N}(I_{\mathscr{N}} - E)$, we have $A_1 - A_2 = (A_1 - A_2)(I_{\mathscr{N}} - E) = 0_{\mathscr{N}}$.

(ii) From part (i), it is clear that $\Phi : E\mathscr{N}E \to \mathscr{N}'$ is a unital normal $*$-isomorphism. Clearly $\mathscr{N}'$ is finite as $E\mathscr{N}E$ is finite (see [KR83, Exercise 6.9.16]).

(iii) Let $A' \in \text{Aff}(\mathscr{N}')$. From Remark 5.4.1-(ii), $A' = Q'^{-1}P'$ where $P', Q' \in \mathscr{N}'$ with Q' one-to-one. Since Φ is surjective, there are operators $P, Q \in \mathscr{N}$ such that $\Phi(P) = P', \Phi(Q) = Q'$. Replacing Q with $EQE + (I_{\mathscr{N}} - E)$ if necessary, we may assume that Q is one-to-one. Using equation (5.4.1), $A = Q^{-1}P$ is in the preimage of A'. Thus, Φ_{aff} is surjective.

For $A' \in \text{Aff}(\mathscr{N}')$, let $A \in \text{Aff}(\mathscr{N})$ be such that $\Phi_{\text{aff}}(A) = A'$, then $EAE \in \text{Aff}(\mathscr{N})$ is such that $\Phi_{\text{aff}}(EAE) = A'$, and $E(EAE)E = EAE$. For the uniqueness part, it suffices to show that if $A \in \text{Aff}(\mathscr{N})$ is in the preimage of $0_{\mathscr{N}'}$ and satisfies $A = EAE$, then $A = 0_{\mathscr{N}}$. Let A be of the form $Q^{-1}P$ with $P, Q \in \mathscr{N}$, and Q one-to-one, so that $Q^{-1} \in \text{Aff}(\mathscr{N})$. Since $\Phi_{\text{aff}}(A) = 0_{\mathscr{N}'}$, from equation (5.4.1), we must have $\Phi(P) = 0_{\mathscr{N}'}$. That is, $P \in \ker(\Phi)$, and hence $PE = 0_{\mathscr{N}}$. Thus, $A = EAE = ER^{-1}PE = 0_{\mathscr{N}}$. □

Lemma 5.4.4. *Let $\mathscr{N}, \mathscr{N}'$ be finite von Neumann algebras, $\Phi : \mathscr{N} \to \mathscr{N}'$ be a unital normal $*$-homomorphism, and $\Phi_{\text{aff}} : \text{Aff}(\mathscr{N}) \to \text{Aff}(\mathscr{N}')$ be the extension of Φ as defined in Definition 1.6.6. Let $A \in \text{Aff}(\mathscr{N})$.*

(i) For every $k \in \mathbb{N}$, we have $\Phi_{\text{aff}}(|A^k|^{\frac{1}{k}}) = |\Phi_{\text{aff}}(A)^k|^{\frac{1}{k}}$.

(ii) If $\mathfrak{m}\text{-}\lim_{k\to\infty} |A^k|^{\frac{1}{k}} = B$, then $\mathfrak{m}\text{-}\lim_{k\to\infty} |\Phi_{\text{aff}}(A)^k|^{\frac{1}{k}} = \Phi_{\text{aff}}(B)$.

Proof. (i) Since Φ_{aff} is a $*$-homomorphism, $\Phi_{\text{aff}}\left((A^k)^* A^k\right) = \left(\Phi_{\text{aff}}(A)^k\right)^* \Phi_{\text{aff}}(A)^k$ for all $k \in \mathbb{N}$. From the functorial nature of the Borel function calculus (see [Nay21, Corollary 6.11]) in the context of the continuous function $t \mapsto t^{\frac{1}{2k}}$ on $\mathbb{R}_{\geq 0}$, we have,

$$\Phi_{\text{aff}}(|A^k|^{\frac{1}{k}}) = |\Phi_{\text{aff}}(A)^k|^{\frac{1}{k}} \text{ for every } k \in \mathbb{N}.$$

(ii) The assertion follows from part (i) and the $\mathfrak{m}$-continuity of Φ_{aff}. □

Definition 5.4.5 ($\mathfrak{u}$-scalar-type and $\mathfrak{m}$-quasinilpotent operators)**.** Let $\mathscr{N}$ be a finite von Neumann algebra acting on the Hilbert space $\mathscr{H}$. We say that an operator $D \in \text{Aff}(\mathscr{N})$ is a $\mathfrak{u}$-*scalar-type* operator, if there is an invertible operator S in $\text{Aff}(\mathscr{N})$ such that $S \hat{\cdot} D \hat{\cdot} S^{-1}$ is a normal operator in $\text{Aff}(\mathscr{N})$.

We say that an operator $N \in \text{Aff}(\mathscr{N})$ is $\mathfrak{m}$-*quasinilpotent* if the normalized power sequence of N, $\{|N^k|^{\frac{1}{k}}\}_{k\in\mathbb{N}}$, converges to $0_{\mathscr{N}}$ in the $\mathfrak{m}$-topology (see Definition 1.6.3).

Since we allow unbounded similarity, in the definition of $\mathfrak{u}$-scalar-type operators, it is not clear whether bounded $\mathfrak{u}$-scalar-type operators are necessarily scalar-type operators (see Theorem 3.2.2), in the sense of Dunford; the interested reader may take upon themselves to address this curiosity.

5.4.1 Restrictions and direct sums of Murray-von Neumann algebras

In this subsection, we essentially define and work with restrictions and direct sums of operators in Murray-von Neumann algebras, but only to the extent necessary for our problem of interest (see Theorem 5.5.3). The definitions for restrictions and direct sums are recorded in Proposition 5.4.8 and 5.4.10 respectively.

Lemma 5.4.6. *Let $\mathscr{N}$ be a finite von Neumann algebra, and $(E_\gamma)_{\gamma\in\Gamma}$ be an increasing net of central projections in $\mathscr{N}$ with least upper bound $I_{\mathscr{N}}$. Let $(A_i)_{i\in\Lambda}$ be a net of operators in Aff($\mathscr{N}$) such that for every $\gamma \in \Gamma$, the net $(A_i E_\gamma)_{i\in\Lambda}$ converges in the $\mathfrak{m}$-topology. Then $\{A_i\}_{i\in\Lambda}$ converges in the $\mathfrak{m}$-topology.*

Proof. For $\gamma \in \Gamma$, let $B_\gamma := \mathfrak{m}\text{-}\lim_{i\in\Lambda}(A_i E_\gamma)$. Since $\{E_\gamma\}_{\gamma\in\Gamma}$ is an increasing net, for indices $\eta' \geq \eta$ in Γ, we have $E_{\eta'}E_\eta = E_\eta$, and hence $(A_i E_{\eta'})E_\eta = A_i E_\eta$, for all $i \in \Lambda$. Using Theorem 1.6.4, and passing to $\mathfrak{m}$-limits, we have

$$B_{\eta'} E_\eta = B_\eta \text{ whenever } \eta' \geq \eta. \tag{5.4.2}$$

Let τ be a normal tracial state on $\mathscr{N}$, and $\varepsilon, \delta > 0$. Since $\{E_\gamma\}_{\gamma\in\Gamma} \uparrow I_{\mathscr{N}}$, the normality of τ tells us that $\{\tau(I_{\mathscr{N}} - E_\gamma)\}_{\gamma\in\Gamma} \downarrow 0$; thus, there is an index $\eta \in \Gamma$ such that $\tau(I_{\mathscr{N}} - E_\eta) \leq \delta$. Then

$$\|(B_\gamma - B_{\gamma'})E_\eta\| = \|B_\eta - B_\eta\| = 0 \leq \varepsilon \text{ for all } \gamma, \gamma' \geq \eta,$$

shows that $B_\gamma - B_{\gamma'}$ lies in the fundamental neighborhood $O(\tau, \varepsilon, \delta)$ for all $\gamma, \gamma' \geq \eta$. Thus, $\{B_\gamma\}_{\gamma\in\Gamma}$ is Cauchy, and hence convergent to an operator $B \in$ Aff($\mathscr{N}$) in the $\mathfrak{m}$-topology.

Next we show that $(A_i)_{i\in\Lambda}$ converges to B in the $\mathfrak{m}$-topology. From equation (5.4.2), and Theorem 1.6.4 (joint $\mathfrak{m}$-continuity of multiplication), note that

$$BE_\eta = \mathfrak{m}\text{-}\lim_{\gamma\in\Gamma} B_\gamma E_\eta = B_\eta \text{ for all } \eta \in \Gamma. \tag{5.4.3}$$

Fix $\alpha \in \Gamma$ such that $\tau(I_{\mathscr{N}} - E_\alpha) \leq \frac{\delta}{2}$. Since $\mathfrak{m}\text{-}\lim_{i\in\Lambda} A_i E_\alpha = B_\alpha$, there is an index $j(\varepsilon, \delta)$ such that for every for every $i \geq j(\varepsilon, \delta)$, there is a projection $F_i \in \mathscr{N}$ satisfying $\tau(I_{\mathscr{N}} - F_i) \leq \frac{\delta}{2}$ and $\|(A_i E_\alpha - B_\alpha)F_i\| \leq \varepsilon$. Since E_α is a central projection, $E_\alpha F_i$ and $(I_{\mathscr{N}} - E_\alpha)(I_{\mathscr{N}} - F_i)$ are projections in $\mathscr{N}$. Since $(I_{\mathscr{N}} - E_\alpha)(I_{\mathscr{N}} - F_i) \geq 0_{\mathscr{N}}$, we see that $I_{\mathscr{N}} - E_\alpha F_i \leq (I_{\mathscr{N}} - E_\alpha) + (I_{\mathscr{N}} - F_i)$, whence $\tau(I_{\mathscr{N}} - E_\alpha F_i) \leq \tau(I_{\mathscr{N}} - E_\alpha) + \tau(I_{\mathscr{N}} - F_i)$. Then, $\tau(I_{\mathscr{N}} - E_\alpha F_i) \leq \delta$ for all $i \geq j(\varepsilon, \delta)$. Moreover, using equation (5.4.3), we have

$$\|(A_i - B)E_\alpha F_i\| = \|(A_i E_\alpha - B_\alpha)F_i\| \leq \varepsilon \quad \forall\ i \geq j(\varepsilon, \delta).$$

Thus, $A_i - B \in O(\tau, \varepsilon, \delta)$ for all $i \geq j(\varepsilon, \delta)$, and we conclude that $\{A_i\}_{i\in\Lambda}$ converges to B in the $\mathfrak{m}$-topology. ☐

Corollary 5.4.7. *Let $\mathscr{N}$ be a finite von Neumann algebra, and $\{E_\gamma : \gamma \in \Gamma\}$ be a collection of mutually orthogonal central projections in $\mathscr{N}$ such that $\sum_{\gamma\in\Gamma} E_\gamma = I_{\mathscr{N}}$. Let $(A_i)_{i\in\Lambda}$ be a net of operators in Aff$(\mathscr{N})$ such that for every $\gamma \in \Gamma$, the net $(A_i E_\gamma)_{i\in\Lambda}$ converges in the $\mathfrak{m}$-topology. Then $(A_i)_{i\in\Lambda}$ converges in the $\mathfrak{m}$-topology.*

Proof. Let $\mathcal{F}(\Lambda)$ be the set of all finite subsets of Λ with the partial order given by set inclusion. For $\mathbb{F} \in \mathcal{F}(\Lambda)$, let $E_{\mathbb{F}} := \sum_{\gamma\in\mathbb{F}} E_\gamma$. Clearly $(E_{\mathbb{F}})_{\mathbb{F}\in\mathcal{F}(\Lambda)}$ is an increasing net of central projections in $\mathscr{N}$, directed by inclusion with $\{E_{\mathbb{F}}\}_{\mathbb{F}\in\mathcal{F}(\Lambda)} \uparrow I_{\mathscr{N}}$. From Theorem 1.6.4 ($\mathfrak{m}$-continuity of addition), for $\mathbb{F} \in \mathcal{F}(\Lambda)$ note that

$$\mathfrak{m}\text{-}\lim_{i\in\Lambda}(A_i E_{\mathbb{F}}) = \sum_{\gamma\in\mathbb{F}} \mathfrak{m}\text{-}\lim_{i\in\Lambda}(A_i E_\gamma).$$

Thus, the net $(A_i E_{\mathbb{F}})_{i\in\Lambda}$ converges in the $\mathfrak{m}$-topology for every $\mathbb{F} \in \mathcal{F}(\Lambda)$. From Lemma 5.4.6, the net $(A_i)_{i\in\Lambda}$ converges in the $\mathfrak{m}$-topology. □

Proposition 5.4.8. *Let $\mathscr{N}$ be a finite von Neumann algebra acting on the Hilbert space $\mathscr{H}$, and E be a central projection in $\mathscr{N}$. $\big($Note that for $A \in \mathscr{N}$, $AE = EAE$ implies that the range of $A|_{E(\mathscr{H})}$ lies in $E(\mathscr{H})$, and allows us to view $A|_{E(\mathscr{H})}$ as an operator in $\mathscr{B}\big(E(\mathscr{H})\big)\big)$. Then, we have the following:*

(i) The restriction mapping $\Pi|_E : \mathscr{N} \to \mathscr{B}\big(E(\mathscr{H})\big)$ given by, $A \mapsto A|_{E(\mathscr{H})}$, is a unital normal $*$-homomorphism. Moreover, its image, which we denote by $\mathscr{N}|_E$, is a finite von Neumann algebra acting on $E(\mathscr{H})$.

(ii) The map $(\Pi|_E)_{\mathrm{aff}} :$ Aff$(\mathscr{N}) \to$ Aff$(\mathscr{N}|_E)$, which is the extension of $\Pi|_E$ as described in Definition 1.6.6, is the restriction mapping, $A \mapsto A|_{E(\mathscr{H})}$, where $A|_{E(\mathscr{H})}$ denotes the restriction of A to $\mathrm{dom}(A)\cap E(\mathscr{H})$. For $A \in$ Aff$(\mathscr{N})$, we define $A|_{E(\mathscr{H})} := (\Pi|_E)_{\mathrm{aff}}(A)$.

(iii) For an operator $A' \in$ Aff$(\mathscr{N}|_E)$, there is a unique operator $A \in$ Aff$(\mathscr{N})$ such that $A = EAE$, and $A|_{E(\mathscr{H})} = A'$. In fact, $A = A'E$.

Proof. It is straightforward to verify that $\Pi|_E$ is a unital $*$-homomorphism. Let $(A_i)_{i\in\Lambda}$ be a net in $\mathscr{N}$, SOT-convergent to $A \in \mathscr{N}$. Clearly, $A_i|_{E(\mathscr{H})} \to A|_{E(\mathscr{H})}$ in SOT. It follows that $\Pi|_E$ is SOT-SOT continuous, whence it is normal using [KR97, Theorem 7.1.12]. Thus, $\Pi|_E : \mathscr{N} \to \mathscr{N}|_E$ is a surjective unital normal $*$-homomorphism.

(i) Follows immediately from Lemma 5.4.3-(i)-(ii).

(ii) Let $A \in$ Aff$(\mathscr{N})$. As noted in Remark 5.4.1-(i), $A = PQ^{-1}$, for some $P, Q \in \mathscr{N}$, with Q one-to-one, and $\mathrm{dom}(A) = \mathrm{ran}(Q)$. From equation (5.4.1) in Remark 5.4.2, we see that,

$$(\Pi|_E)_{\mathrm{aff}}(A) = (\Pi|_E)_{\mathrm{aff}}(PQ^{-1}) = \Pi|_E(P)\Pi_E(Q)^{-1} = P|_{E(\mathscr{H})}(Q|_{E(\mathscr{H})})^{-1}.$$

Thus, in particular,

$$\operatorname{dom}\left((\Pi|_E)_{\text{aff}}(A)\right) = \operatorname{ran}(Q|_{E(\mathscr{H})}) = \operatorname{ran}(QE) = \operatorname{ran}(Q) \cap E(\mathscr{H})$$
$$= \operatorname{dom}(A) \cap E(\mathscr{H}).$$

Since $\operatorname{dom}(A)$ is dense in $\mathscr{H}$, by [GN24, Theorem 5.5-(i)], we note that $\operatorname{dom}(A) \cap E(\mathscr{H})$ is a dense linear subspace of $E(\mathscr{H})$. It is easily verified that $(\Pi|_E)_{\text{aff}}(A) = P|_{E(\mathscr{H})}(Q|_{E(\mathscr{H})})^{-1}$ is the restriction of $A = PQ^{-1}$ to $E(\mathscr{H})$.

(iii) Follows immediately from Lemma 5.4.3-(iii) □

Lemma 5.4.9. *Let $\mathscr{N}$ be a finite von Neumann algebra acting on the Hilbert space $\mathscr{H}$, and E be a central projection in $\mathscr{N}$. Then, we have the following:*

(i) If $S \in \text{Aff}(\mathscr{N})$ is invertible, then so is $S|_{E(\mathscr{H})} \in \text{Aff}(\mathscr{N}|_E)$;

(ii) If $M \in \text{Aff}(\mathscr{N})$ is normal, then so is $M|_{E(\mathscr{H})} \in \text{Aff}(\mathscr{N}|_E)$;

(iii) If $D \in \text{Aff}(\mathscr{N})$ is $\mathfrak{u}$-scalar-type, then so is $D|_{E(\mathscr{H})} \in \text{Aff}(\mathscr{N}|_E)$;

(iv) If $N \in \text{Aff}(\mathscr{N})$ is $\mathfrak{m}$-quasinilpotent, then so is $N|_{E(\mathscr{H})} \in \text{Aff}(\mathscr{N}|_E)$.

Proof. Let $(\Pi|_E)_{\text{aff}} : \text{Aff}(\mathscr{N}) \to \text{Aff}(\mathscr{N}|_E)$ be the unital normal $*$-homomorphism as described in Proposition 5.4.8.

(i) Follows from Theorem 5.5 and Theorem 4.14-(iii) of [GN24] in the context of $(\Pi|_E)_{\text{aff}}$.

(ii) Follows from Theorem 5.5 and Theorem 6.6-(vi) of [GN24] in the context of $(\Pi|_E)_{\text{aff}}$.

(iii) Since $D \in \text{Aff}(\mathscr{N})$ is $\mathfrak{u}$-scalar-type, there exists an invertible operator S, and a normal operator M, in $\text{Aff}(\mathscr{N})$ such that $D = S^{-1} \hat{\cdot}\, M \,\hat{\cdot}\, S$. Using [GN24, Theorem 4.14-(iii)], note that $(\Pi|_E)_{\text{aff}}(S^{-1}) = (\Pi|_E)_{\text{aff}}(S)^{-1}$. Thus,

$$D|_{E(\mathscr{H})} = (\Pi|_E)_{\text{aff}}(D) = (\Pi|_E)_{\text{aff}}(S)^{-1} \,\hat{\cdot}\, (\Pi|_E)_{\text{aff}}(T) \,\hat{\cdot}\, (\Pi|_E)_{\text{aff}}(S),$$

is $\mathfrak{u}$-scalar-type.

(iv) Follows from Lemma 5.4.4-(ii). □

Proposition 5.4.10. *Let $\mathscr{N}$ be a finite von Neumann algebra, acting on the Hilbert space $\mathscr{H}$, and $\{E_i : i \in \Lambda\}$ be a collection of mutually orthogonal central projections in $\mathscr{N}$ partitioning the identity operator, that is, $\sum_{i\in\Lambda} E_i = I_{\mathscr{N}}$. If $\{A_i : i \in \Lambda\}$ is a collection of operators such that $A_i \in \text{Aff}(\mathscr{N}|_{E_i})$, then there is a unique operator $A \in \text{Aff}(\mathscr{N})$ such that $(\Pi|_{E_i})_{\text{aff}}(A) = A_i$ for all $i \in \Lambda$. We define $\bigoplus_{i\in\Lambda} A_i := A$.*

Proof. For $i \in \Lambda$, note that $A_i E_i \in \text{Aff}(\mathscr{N})$, as described in Proposition 5.4.8-(iii). Then $(A_i E_i)_{i\in\Lambda}$ is a net of operators in $\text{Aff}(\mathscr{N})$. Since for each $j \in \Lambda$, the net $\big((A_i E_i)E_j\big)_{i\in\Lambda}$ has all but one term, corresponding to j, equal to $0_{\mathscr{N}}$, it is convergent in the $\mathfrak{m}$-topology. From Corollary 5.4.7, the net $(A_iE_i)_{i\in\Lambda}$ converges in $\mathfrak{m}$-topology to an operator $A \in \mathrm{Aff}(\mathscr{N})$, given by

$$A = \mathfrak{m}\text{-}\lim_{\mathbb{F}\in\mathcal{F}(\Lambda)} \Big(\mathfrak{m}\text{-}\lim_{i\in\Lambda} \big(A_iE_i\textstyle\sum_{j\in\mathbb{F}} E_j\big)\Big) = \mathfrak{m}\text{-}\displaystyle\lim_{\mathbb{F}\in\mathcal{F}(\Lambda)} \Big(\sum_{j\in\mathbb{F}} A_jE_j\Big) = \mathfrak{m}\text{-}\sum_{i\in\Lambda} A_iE_i.$$

From the $\mathfrak{m}$-continuity of $(\Pi|_{E_i})_{\mathrm{aff}}$ (see Proposition 5.4.8-(ii)),

$$(\Pi|_{E_i})_{\mathrm{aff}}(A) = (\Pi|_{E_i})_{\mathrm{aff}}\Big(\mathfrak{m}\text{-}\sum_{j\in\Lambda} A_jE_j\Big) = A_i.$$

Let $A' \in \mathrm{Aff}(\mathscr{N})$ be another operator such that $(\Pi|_{E_i})_{\mathrm{aff}}(A') = A_i$ for all $i \in \Lambda$. Then, $A'E_i = A_iE_i = AE_i$, and hence $A' = \sum_{i\in\Lambda} A'E_i = \sum_{i\in\Lambda} AE_i = A$. □

Lemma 5.4.11. *Let $\mathscr{N}$ be a finite von Neumann algebra acting on the Hilbert space $\mathscr{H}$, and $\{E_i\}_{i\in\Lambda}$ be a family of mutually orthogonal central projections in $\mathscr{N}$ such that $\sum_{i\in\Lambda} E_i = I_{\mathscr{N}}$. For $i \in \Lambda$, let $R_i, S_i \in \mathrm{Aff}(\mathscr{N}|_{E_i})$. Then,*

$$\textstyle\bigoplus_{i\in\Lambda}(R_i \hat{\cdot} S_i) = \Big(\bigoplus_{i\in\Lambda} R_i\Big) \hat{\cdot} \Big(\bigoplus_{i\in\Lambda} S_i\Big), \tag{5.4.4}$$

$$\textstyle\big(\bigoplus_{i\in\Lambda} R_i\big)^* = \bigoplus_{i\in\Lambda} R_i^*. \tag{5.4.5}$$

Proof. Since $(\Pi|_{E_i})_{\mathrm{aff}}$ is a $*$-homomorphism (by Proposition 5.4.8-(ii)), for $i \in \Lambda$, we have,

$$\begin{aligned}(\Pi|_{E_i})_{\mathrm{aff}}\Big(\big(\textstyle\bigoplus_{i\in\Lambda} R_i\big) \hat{\cdot} \big(\bigoplus_{i\in\Lambda} S_i\big)\Big) &= \Big((\Pi|_{E_i})_{\mathrm{aff}}\big(\textstyle\bigoplus_{i\in\Lambda} R_i\big)\Big) \hat{\cdot} \Big((\Pi|_{E_i})_{\mathrm{aff}}\big(\bigoplus_{i\in\Lambda} S_i\big)\Big) \\ &= R_i \hat{\cdot} S_i.\end{aligned}$$

Similarly, for $i \in \Lambda$,

$$(\Pi|_{E_i})_{\mathrm{aff}}\Big(\big(\textstyle\bigoplus_{i\in\Lambda} R_i\big)^*\Big) = (\Pi|_{E_i})_{\mathrm{aff}}\big(\bigoplus_{i\in\Lambda} R_i\big)^* = R_i^*.$$

The result immediately follows from the uniqueness clause in Proposition 5.4.10. □

Lemma 5.4.12. *Let $\mathscr{N}$ be a finite von Neumann algebra acting on the Hilbert space $\mathscr{H}$, and $\{E_i\}_{i\in\Lambda}$ be a family of mutually orthogonal central projections in $\mathscr{N}$ such that $\sum_{i\in\Lambda} E_i = I_{\mathscr{N}}$. By Proposition 5.4.8-(i), note that $\mathscr{N}|_{E_i}$ is a finite von Neumann algebra.*

(i) For $i \in \Lambda$, let S_i be an invertible operator in $\mathrm{Aff}(\mathscr{N}|_{E_i})$. Then $\bigoplus_{i\in\Lambda} S_i$ is an invertible operator in $\mathrm{Aff}(\mathscr{N})$.

(ii) For $i \in \Lambda$, let M_i be a normal operator in $\mathrm{Aff}(\mathscr{N}|_{E_i})$. Then $\bigoplus_{i\in\Lambda} M_i$ is a normal operator in $\mathrm{Aff}(\mathscr{N})$.

(iii) For $i \in \Lambda$, let H_i be a self-adjoint operator (positive operator, respectively) in $\mathrm{Aff}(\mathscr{N}|_{E_i})$. Then $\bigoplus_{i\in\Lambda} H_i$ is a self-adjoint operator (positive operator, respectively) in $\mathrm{Aff}(\mathscr{N})$.

(iv) For $i \in \Lambda$, let D_i be a $\mathfrak{u}$-scalar-type operator in $\mathrm{Aff}(\mathscr{N}|_{E_i})$. Then $\oplus_{i\in\Lambda} D_i$ is a $\mathfrak{u}$-scalar-type operator of $\mathrm{Aff}(\mathscr{N})$.

(v) For $i \in \Lambda$, let A_i be an operator in $\mathrm{Aff}(\mathscr{N}|_{E_i})$ whose normalized power sequence converges in the $\mathfrak{m}$-topology to a positive operator H_i in $\mathrm{Aff}(\mathscr{N}|_{E_i})$. Then the normalized power sequence of the operator $\oplus_{i\in\Lambda} A_i$ in $\mathrm{Aff}(\mathscr{N})$, converges in the $\mathfrak{m}$-topology to the positive operator $\oplus_{i\in\Lambda} H_i$ in $\mathrm{Aff}(\mathscr{N})$.

(vi) For $i \in \Lambda$, let N_i be an $\mathfrak{m}$-quasinilpotent operator in $\mathrm{Aff}(\mathscr{N}|_{E_i})$. Then $\oplus_{i\in\Lambda} N_i$ is an $\mathfrak{m}$-quasinilpotent operator in $\mathrm{Aff}(\mathscr{N})$.

Proof. (i) Let $S_i^{-1} \in \mathrm{Aff}(\mathscr{N}|_{E_i})$ be the inverse of S_i. Using Lemma 5.4.11, we have

$$\left(\bigoplus_{i\in\Lambda} S_i\right) \hat{\cdot} \left(\bigoplus_{i\in\Lambda} S_i^{-1}\right) = \bigoplus_{i\in\Lambda} (S_i \hat{\cdot} S_i^{-1}) = \bigoplus_{i\in\Lambda} E_i = I_{\mathscr{N}}.$$

Similarly, $\left(\oplus_{i\in\Lambda} S_i^{-1}\right) \hat{\cdot} \left(\oplus_{i\in\Lambda} S_i\right) = I_{\mathscr{N}}$, which proves the result in this part.

(ii) Since M_i is a normal operator in $\mathrm{Aff}(\mathscr{N}|_{E_i})$, we have $M_i \hat{\cdot} M_i^* = M_i^* \hat{\cdot} M_i$. Using equations (5.4.4) and (5.4.5), we have,

$$\begin{aligned}\left(\bigoplus_{i\in\Lambda} M_i\right) \hat{\cdot} \left(\bigoplus_{i\in\Lambda} M_i\right)^* &= \left(\bigoplus_{i\in\Lambda} M_i\right) \hat{\cdot} \left(\bigoplus_{i\in\Lambda} M_i^*\right) = \bigoplus_{i\in\Lambda} (M_i \hat{\cdot} M_i^*) \\ &= \bigoplus_{i\in\Lambda} (M_i^* \hat{\cdot} M_i) = \left(\bigoplus_{i\in\Lambda} M_i^*\right) \hat{\cdot} \left(\bigoplus_{i\in\Lambda} M_i\right) \\ &= \left(\bigoplus_{i\in\Lambda} M_i\right)^* \hat{\cdot} \left(\bigoplus_{i\in\Lambda} M_i\right).\end{aligned}$$

(iii) If H_i is self adjoint in $\mathrm{Aff}(\mathscr{N}|_{E_i})$, we have $H_i^* = H_i$ for all $i \in \Lambda$. Using equation (5.4.5), we get

$$\left(\bigoplus_{i\in\Lambda} H_i\right)^* = \left(\bigoplus_{i\in\Lambda} H_i^*\right) = \left(\bigoplus_{i\in\Lambda} H_i\right).$$

For $i \in \Lambda$, let H_i be a positive operator in $\mathrm{Aff}(\mathscr{N}|_{E_i})$, then $H_i = A_i^* A_i$ for some operator A_i in $\mathrm{Aff}(\mathscr{N}|_{E_i})$ (see [Nay21, Proposition 6.14]). Thus from equation (5.4.4),

$$\bigoplus_{i\in\Lambda} H_i = \bigoplus_{i\in\Lambda} (A_i^* \hat{\cdot} A_i) = \left(\bigoplus_{i\in\Lambda} A_i\right)^* \hat{\cdot} \left(\bigoplus_{i\in\Lambda} A_i\right),$$

is a positive operator in $\mathrm{Aff}(\mathscr{N})$.

(iv) Since D_i is a $\mathfrak{u}$-scalar-type operator in $\mathrm{Aff}(\mathscr{N}|_{E_i})$, there is an invertible operator S_i, and a normal operator M_i, in $\mathrm{Aff}(\mathscr{N}|_{E_i})$, such that $D_i = S_i^{-1} \hat{\cdot} M_i \hat{\cdot} S_i$. From part (i), $\left(\oplus_{i\in\Lambda} S_i\right)$ is an invertible operator in $\mathrm{Aff}(\mathscr{N})$ with $\left(\oplus_{i\in\Lambda} S_i\right)^{-1} = \left(\oplus_{i\in\Lambda} S_i^{-1}\right)$. From part (ii), $\left(\oplus_{i\in\Lambda} M_i\right)$ is a normal operator in $\mathrm{Aff}(\mathscr{N})$. Using equation (5.4.4) and part (i), we have

$$\begin{aligned}\bigoplus_{i\in\Lambda} D_i = \bigoplus_{i\in\Lambda} (S_i \hat{\cdot} M_i \hat{\cdot} S_i^{-1}) &= \left(\bigoplus_{i\in\Lambda} S_i\right) \hat{\cdot} \left(\bigoplus_{i\in\Lambda} M_i\right) \hat{\cdot} \left(\bigoplus_{i\in\Lambda} S_i^{-1}\right) \\ &= \left(\bigoplus_{i\in\Lambda} S_i\right) \hat{\cdot} \left(\bigoplus_{i\in\Lambda} M_i\right) \hat{\cdot} \left(\bigoplus_{i\in\Lambda} S_i\right)^{-1}.\end{aligned}$$

Thus, $\oplus_{i\in\Lambda} D_i$ is a $\mathfrak{u}$-scalar-type operator in $\mathrm{Aff}(\mathscr{N})$.

(v) By repeated application of equation (5.4.4), for every $k \in \mathbb{N}$, we have $(\bigoplus_{i\in\Lambda} A_i)^k = \bigoplus_{i\in\Lambda} A_i^k$. Using Lemma 5.4.4-(i) in the context of the $\Pi|_{E_i}$'s and equations (5.4.4) and (5.4.5), for every $k \in \mathbb{N}$ we see that

$$\bigoplus_{i\in\Lambda} |A_i^k|^{\frac{1}{k}} = \left|(\bigoplus_{i\in\Lambda} A_i)^k\right|^{\frac{1}{k}}. \tag{5.4.6}$$

For $i \in \Lambda$, since $\mathfrak{m}\text{-}\lim_{k\to\infty} |A_i^k|^{\frac{1}{k}} = H_i$, by Lemma 5.4.4-(ii) in the context of the $\Pi|_{E_i}$'s, equation (5.4.6), and the uniqueness clause in Proposition 5.4.10, we have,

$$\mathfrak{m}\text{-}\lim_{k\to\infty} \left|(\bigoplus_{i\in\Lambda} A_i)^k\right|^{\frac{1}{k}} = \bigoplus_{i\in\Lambda} H_i.$$

(vi) Since each $N_i \in \text{Aff}(\mathscr{N}|_{E_i})$ is an $\mathfrak{m}$-quasinilpotent operator, the normalized power sequence, $\{|N_i^k|^{\frac{1}{k}}\}_{k\in\mathbb{N}}$, of N_i, converges to $0_{\mathscr{N}|_{E_i}}$ in the $\mathfrak{m}$-topology for all $i \in \Lambda$. The result immediately follows from part (v). □

5.5 The Jordan-Chevalley-Dunford decomposition in type *I* Murray-von Neumann algebras

In this section, using the groundwork laid in §5.3 and §5.4, we establish the existence and uniqueness of Jordan-Chevalley-Dunford decomposition of operators in type *I* Murray-von Neumann algebras. Furthermore, we discuss ramifications of these results for any meaningful version of the Jordan-Chevalley-Dunford decomposition in the context of type II_1 Murray-von Neumann algebras.

Proposition 5.5.1. *For $n \in \mathbb{N}$, let $\mathscr{M}_n$ be a finite von Neumann algebra of type I_n, acting on the Hilbert space $\mathscr{H}$, and let $A \in \text{Aff}(\mathscr{M}_n)$. Then we have the following:*

(i) There is a unique pair of commuting operators D, N in $\text{Aff}(\mathscr{M}_n)$ such that D is $\mathfrak{u}$-scalar-type, N is nilpotent, and $A = D \hat{+} N$.

(ii) The normalized power sequence of A converges in the $\mathfrak{m}$-topology to a positive operator in $\text{Aff}(\mathscr{M}_n)$.

(iii) The operator A is $\mathfrak{m}$-quasinilpotent if and only if it is nilpotent.

Proof. Using Theorem 1.5.2, the discussion in [Kad86, §3-§4], and [Nay21, Theorem 4.15], we note that there is a Stonean space $\mathcal{X}$ such that $\mathscr{M}_n$ is $*$-isomorphic to $M_n\big(C(\mathcal{X})\big)$, and $\text{Aff}(\mathscr{M}_n)$ is $*$-isomorphic to $M_n\big(\mathcal{N}(\mathcal{X})\big)$. Throughout this proof, we move back and forth between the operator-theoretic and topological viewpoints depending on which is more apt for the situation.

(i) The assertion is simply a rephrasing of Theorem 5.3.11 in the context of $\text{Aff}(\mathscr{M}_n) \cong M_n\big(\mathcal{N}(\mathcal{X})\big)$.

(ii) Let $A = D+N$ be the Jordan-Chevalley decomposition of the matrix A in $M_n\big(\mathcal{N}(\mathcal{X})\big)$ as given by Theorem 5.3.11. Let $S \in M_n\big(\mathcal{N}(\mathcal{X})\big)$ be an invertible matrix such that $SDS^{-1} = M$ is a diagonal matrix in $M_n\big(\mathcal{N}(\mathcal{X})\big)$.

Let $\mathcal{O} := \mathrm{dom}(A) \cap \mathrm{dom}(D) \cap \mathrm{dom}(S)$; note that $\mathcal{O}$ is an open dense subset of $\mathcal{X}$. We define an increasing sequence of open subsets of $\mathcal{X}$ as follows,

$$O_m := \big\{x \in X : \max\{\|A(x)\|, \|D(x)\|, \|S(x)\|, \|S(x)^{-1}\|\} < m\big\} \quad ; \quad m \in \mathbb{N}.$$

Since the clopen set $\overline{O_m}$ is contained in O_{m+1} and contains O_m, clearly $\mathcal{O} = \bigcup_{m\in\mathbb{N}} O_m = \bigcup_{m\in\mathbb{N}} \overline{O_m}$; moreover, $\|M(x)\| = \|S(x)D(x)S(x)^{-1}\| < m^3$ for all $x \in O_m$. The indicator function for the clopen set $\overline{O_m}$,

$$E_m(x) = \begin{cases} I_n, & \text{if } x \in \overline{O_m} \\ \mathbf{0}_n, & \text{if } x \in \mathcal{X}\backslash\overline{O_m} \end{cases}$$

corresponds to a central projection in $\mathscr{M}_n$, which we also denote by E_m. As noted above, $\{\overline{O_m}\}_{m\in\mathbb{N}}$ is an increasing sequence of clopen sets and $\mathcal{O} = \bigcup_{m\in\mathbb{N}} \overline{O_m}$ is a dense open subset of $\mathcal{X}$, whence $E_m \uparrow I_{\mathscr{M}_n}$ as $m \to \infty$.

Note that $AE_m, SE_m, DE_m, S^{-1}E_m$ are bounded operators in $\mathscr{M}_n$ as their norm is less than or equal to m, and ME_m is a bounded normal operator in $\mathscr{M}_n$ as its norm is less than or equal to m^3. Thus $SE_m + (I_{\mathscr{M}_n} - E_m)$ has bounded inverse $S^{-1}E_m + (I_{\mathscr{M}_n} - E_m)$, and

$$ME_m = (SE_m + (I_{\mathscr{M}_n} - E_m))\,(DE_m)\left(S^{-1}E_m + (I_{\mathscr{M}_n} - E_m)\right).$$

Thus, DE_m is a scalar-type operator in $\mathscr{B}(\mathscr{H})$ (see Theorem 3.2.2), whence AE_m is a spectral operator in $\mathscr{B}(\mathscr{H})$ (see Theorem 3.2.1) with the Dunford decomposition $AE_m = DE_m + NE_m$. By Theorem 3.4.2-(i), the normalized power sequence of AE_m converges in norm to a positive operator in $\mathscr{M}_n$, and hence also converges in the $\mathfrak{m}$-topology to the same operator, as the $\mathfrak{m}$-topology is coarser than the norm topology.

For every $m, k \in \mathbb{N}$, clearly $|(AE_m)^k|^{\frac{1}{k}} = |A^k|^{\frac{1}{k}} E_m$ as E_m is a central projection in $\mathscr{M}_n$. Using Lemma 5.4.6, we conclude that the normalized power sequence of A converges in the $\mathfrak{m}$-topology. Since the normalized power sequence of A comprises of positive operators, by [Nay21, Proposition 4.10-(ii)], the $\mathfrak{m}$-limit must be a positive operator in $\mathrm{Aff}(\mathscr{M}_n)$.

(iii) If A is nilpotent, then the normalized power sequence of A is eventually $0_{\mathscr{M}_n}$, and thus clearly $\mathfrak{m}$-quasinilpotent. Conversely, assume that A is $\mathfrak{m}$-quasinilpotent, that is, $\mathfrak{m}\text{-}\lim_{k\to\infty} |A^k|^{\frac{1}{k}} = 0_{\mathscr{M}_n}$. For $\ell \in \mathbb{N}$, we have $\mathfrak{m}\text{-}\lim_{k\to\infty} |A^k|^{\frac{1}{k}} E_\ell = 0_{\mathscr{M}_n}$, and from part (ii), we know that norm-$\lim_{k\to\infty} |(AE_\ell)^k|^{\frac{1}{k}}$ exists; since its $\mathfrak{m}$-limit is $0_{\mathscr{M}_n}$, so must be its norm-limit. Thus for every $\ell \in \mathbb{N}$ and $x \in \overline{O_\ell}$, using the spectral radius formula, $\mathrm{sp}(A(x)) = \{0\}$, that is, $A(x)$ is a nilpotent matrix in $M_n(\mathbb{C})$. We conclude that A^n is $\mathbf{0}_n$ on $\mathcal{O}$ which is an open dense subset of $\mathcal{X}$. Thus, $A \in M_n\big(\mathcal{N}(\mathcal{X})\big)$ is nilpotent. □

***Remark* 5.5.2.** For $m \in \mathbb{N}$, let $J_m(0)$ be the $m \times m$ Jordan matrix in $M_m(\mathbb{C})$ with 0's on the diagonal. As noted in Lemma 5.4.12-(v), $\mathfrak{m}$-quasinilpotence is preserved under arbitrary direct sums, whereas nilpotence is not, as demonstrated by the direct sum, $\bigoplus_{m\in\mathbb{N}} J_m(0)$, which is not nilpotent in the type I finite von Neumann algebra $\bigoplus_{m\in\mathbb{N}} M_m(\mathbb{C})$. The theorem below shows that substituting nilpotence with the weaker condition of $\mathfrak{m}$-quasinilpotence allows for the desired generalization of Proposition 5.5.1 to the setting of type I finite von Neumann algebras.

Theorem 5.5.3. *Let $\mathscr{M}$ be a type I finite von Neumann algebra acting on the Hilbert space $\mathscr{H}$, and let $A \in \mathrm{Aff}(\mathscr{M})$. Then we have the following.*

(i) There is a unique pair of commuting operators D, N in $\mathrm{Aff}(\mathscr{M})$ such that D is $\mathfrak{u}$-scalar-type, N is $\mathfrak{m}$-quasinilpotent, and $A = D \hat{+} N$.

(ii) The normalized power sequence of A converges in the $\mathfrak{m}$-topology to a positive operator in $\mathrm{Aff}(\mathscr{M})$.

Proof. By the type decomposition of type I von Neumann algebras (see [KR97, Theorem 6.5.2]), there is a subset Λ of $\mathbb{N}$ and a collection of mutually orthogonal non-trivial central projections $\{E_m : m \in \Lambda\}$ such that $\sum_{m\in\Lambda} E_m = I_{\mathscr{M}}$ and for every $m \in \Lambda$, the von Neumann algebra $\mathscr{M}|_{E_m}$ acting on the Hilbert space $E_m(\mathscr{H})$ is of type I_m.

(i) By Proposition 5.5.1, there are operators D_m, N_m in $\mathrm{Aff}(\mathscr{M}|_{E_m})$ such that

$$A|_{E_m(\mathscr{H})} = D_m \hat{+} N_m, \quad D_m \hat{\cdot} N_m = N_m \hat{\cdot} D_m, \quad \text{and } N_m^m = 0_{\mathscr{M}|_{E_m}}.$$

From Lemma 5.4.12-(iv), $D := \oplus_{m\in\Lambda} D_m$ is $\mathfrak{u}$-scalar-type, and by Lemma 5.4.12-(vi), $N := \oplus_{m\in\Lambda} N_m$ is $\mathfrak{m}$-quasinilpotent. From Proposition 5.4.10, $A = \oplus_{m\in\Lambda} A|_{E_m(\mathscr{H})} = D \hat{+} N$, and it is clear from Lemma 5.4.11 that $D \hat{\cdot} N = N \hat{\cdot} D$.

Let $A = D' \hat{+} N'$ also be another such decomposition. Since each $(\Pi|_{E_m})_{\mathrm{aff}}$ is a homomorphism, note that $A|_{E_m(\mathscr{H})} = D|_{E_m(\mathscr{H})} \hat{+} N|_{E_m(\mathscr{H})}$, and $D|_{E_m(\mathscr{H})}$ and $N|_{E_m(\mathscr{H})}$ commute with each other. From Lemma 5.4.9, $D|_{E_m(\mathscr{H})}$ is of $\mathfrak{u}$-scalar-type, $N|_{E_m(\mathscr{H})}$ is $\mathfrak{m}$-quasinilpotent. By Proposition 5.5.1-(iii), $N|_{E_m(\mathscr{H})}$ is nilpotent. Then from the uniqueness clause in Proposition 5.5.1-(i), $D|_{E_m(\mathscr{H})} = D'|_{E_m(\mathscr{H})}, N|_{E_m(\mathscr{H})} = N'|_{E_m(\mathscr{H})}$ for every $m \in \Lambda$. Using Proposition 5.4.10, we conclude that $D = D'$ and $N = N'$, which proves the uniqueness of the decomposition.

(ii) Note that $A = \bigoplus_{m\in\Lambda} A_m$ where $A_m := A|_{E_m(\mathscr{H})}$ is an operator in $\mathrm{Aff}(\mathscr{M}|_{E_m(\mathscr{H})})$. Since $\mathscr{M}|_{E_m(\mathscr{H})}$ is a type I_m von Neumann algebra, by Proposition 5.5.1-(ii), the normalized power sequence of A_m converges in the $\mathfrak{m}$-topology to a positive operator in $\mathrm{Aff}(\mathscr{M}|_{E_m(\mathscr{H})})$. By Lemma 5.4.12-(v), the normalized power sequence of A converges in the $\mathfrak{m}$-topology to a positive operator in $\mathrm{Aff}(\mathscr{M})$. □

Definition 5.5.4. For a type I finite von Neumann algebra $\mathscr{M}$, we call the decomposition of an operator A in $\text{Aff}(\mathscr{M})$ as described in Theorem 5.5.3 as the *Jordan-Chevalley-Dunford decomposition* of A.

Corollary 5.5.5. *Let $\mathscr{M}, \mathscr{M}'$ be type I finite von Neumann algebras, $\Phi : \mathscr{M} \to \mathscr{M}'$ be a unital normal $*$-homomorphism, and $\Phi_{\text{aff}} : \text{Aff}(\mathscr{M}) \to \text{Aff}(\mathscr{M}')$ be the extension of Φ, as given in Definition 1.6.6. For $A \in \text{Aff}(\mathscr{M})$, let $A = D \hat{+} N$ be the Jordan-Chevalley-Dunford decomposition of A. Then $\Phi_{\text{aff}}(A) = \Phi_{\text{aff}}(D) \hat{+} \Phi_{\text{aff}}(N)$, is the Jordan-Chevalley-Dunford decomposition of $\Phi_{\text{aff}}(A)$.*

Proof. Note that D is $\mathfrak{u}$-scalar-type, N is $\mathfrak{m}$-quasinilpotent, and $D \hat{\cdot} N = N \hat{\cdot} D$. Since Φ_{aff} is an $\mathfrak{m}$-continuous $*$-homomorphism, clearly we have,

$$\Phi_{\text{aff}}(A) = \Phi_{\text{aff}}(D \hat{+} N) = \Phi_{\text{aff}}(D) \hat{+} \Phi_{\text{aff}}(N).$$

From the uniqueness of the Jordan-Chevalley-Dunford decomposition of $\Phi_{\text{aff}}(A)$ in $\text{Aff}(\mathscr{M}')$, it suffices to show that $\Phi_{\text{aff}}(D)$ is a $\mathfrak{u}$-scalar-type operator, $\Phi_{\text{aff}}(N)$ is an $\mathfrak{m}$-quasinilpotent operator, and that $\Phi_{\text{aff}}(D)$ and $\Phi_{\text{aff}}(N)$ commute with each other.

Since D is $\mathfrak{u}$-scalar-type, there is a normal operator M and an invertible operator S in $\text{Aff}(\mathscr{M})$ such that $D = S \hat{\cdot} M \hat{\cdot} S^{-1}$. From [GN24, Theorem 4.14-(iii)] and [GN24, Theorem 6.6-(vi)], it follows that $\Phi_{\text{aff}}(S)$ is an invertible operator in $\text{Aff}(\mathscr{M}')$ with $\Phi_{\text{aff}}(S)^{-1} = \Phi_{\text{aff}}(S^{-1})$, and $\Phi_{\text{aff}}(M)$ is a normal operator in $\text{Aff}(\mathscr{M}')$, respectively. Thus, $\Phi_{\text{aff}}(D) = \Phi_{\text{aff}}(S) \hat{\cdot} \Phi_{\text{aff}}(M) \hat{\cdot} \Phi_{\text{aff}}(S)^{-1}$ is $\mathfrak{u}$-scalar-type.

Since $|N^k|^{\frac{1}{k}} \to 0_{\mathscr{M}}$ in the $\mathfrak{m}$-topology, from Lemma 5.4.4, we have $|\Phi_{\text{aff}}(N)^k|^{\frac{1}{k}} \to 0_{\mathscr{M}'}$ in the $\mathfrak{m}$-topology. Thus, $\Phi_{\text{aff}}(N)$ is $\mathfrak{m}$-quasinilpotent. Since D and N commute, we have $\Phi_{\text{aff}}(D) \hat{\cdot} \Phi_{\text{aff}}(N) = \Phi_{\text{aff}}(N) \hat{\cdot} \Phi_{\text{aff}}(D)$. □

***Remark* 5.5.6.** A standard faithful normal representation of the von Neumann algebra $\ell^\infty(\mathbb{N})$ is on the Hilbert space $\ell^2(\mathbb{N})$ via the multiplier action. Let $\mathscr{M}_3$ denote the type I_3 von Neumann algebra, $M_3\big(\ell^\infty(\mathbb{N})\big)$, acting on the Hilbert space $\mathscr{K} := \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Since $\ell^\infty(\mathbb{N})$ is $*$-isomorphic to $C(\beta\mathbb{N})$, it follows from Proposition 5.3.12 that there is an element of $\mathscr{M}_3$ for which its $\mathfrak{u}$-scalar-type part and nilpotent part both lie in $\text{Aff}(\mathscr{M}_3)\backslash\mathscr{M}_3$, that is, they are densely-defined closed operators on $\mathscr{K}$ that are not bounded.

Proposition 5.5.7. *For $n \in \mathbb{N}$, let $\mathscr{M}_n$ denote the type I_n von Neumann algebra, $M_n\big(\ell^\infty(\mathbb{N})\big)$, and $\mathscr{L}$ be a type II_1 von Neumann algebra. Then $\mathscr{M}_n$ is $*$-isomorphic to a von Neumann subalgebra of $\mathscr{L}$.*

Proof. Without loss of generality, we may assume that $\mathscr{L}$ is a represented von Neumann algebra acting on the Hilbert space $\mathscr{H}$. We first prove the assertion for $n = 1$. Using [KR97, Lemma 6.5.6], we may inductively choose a sequence $\{E_m\}_{m\in\mathbb{N}}$ of mutually orthogonal projections in $\mathscr{L}$ whose sum converges to $I_{\mathscr{L}}$ in SOT. Define the map,

$$\phi : \ell^\infty(\mathbb{N}) \to \mathscr{L} \text{ given by } \alpha \mapsto \sum_{m\in\mathbb{N}} \alpha(m)E_m.$$

It is straightforward to verify that ϕ is a unital injective $*$-homomorphism. In what follows, we show that ϕ is normal. Let $\{\alpha_i\}_{i\in I}$ be an increasing net in $\ell^\infty(\mathbb{N})^+$ with least upper bound $\alpha \in \ell^\infty(\mathbb{N})^+$. Clearly, for every $m \in \mathbb{N}$, $\alpha_i(m) \uparrow \alpha(m)$, and as E_m is a projection, we have $\alpha_i(m)E_m \uparrow \alpha(m)E_m$ in SOT. Since the E_m's are mutually orthogonal and $\sup_i \|\alpha_i\|_\infty \leq \|\alpha\|_\infty < \infty$, it is easy to verify using Parseval's identity ($\|x\|^2 = \sum_{m\in\mathbb{N}} \|E_m x\|^2$ for $x \in \mathscr{H}$), that $\sum_{m\in\mathbb{N}} \alpha_i(m)E_m \uparrow \sum_{m\in\mathbb{N}} \alpha(m)E_m$ in SOT. In other words, $\phi(\alpha_i) \uparrow \phi(\alpha)$ in SOT. This shows that ϕ is a unital normal embedding of $\mathscr{M}_1 = \ell^\infty(\mathbb{N})$ into $\mathscr{L}$.

We next prove the result for general $n \in \mathbb{N}$. By [KR97, Lemma 6.5.6], there are mutually orthogonal Murray-von Neumann equivalent projections $E_1, E_2, \ldots, E_n \in \mathscr{L}$ such that

$$E_1 + E_2 + \cdots + E_n = I_{\mathscr{L}}.$$

Using [KR97, Lemma 6.6.3-6.6.4], we observe that $\mathscr{L}$ is $*$-isomorphic to $M_n(E_1\mathscr{L}E_1)$. Note from [KR97, Exercise 6.9.16] that $\mathscr{L}' := E_1\mathscr{L}E_1$ is a type II_1 von Neumann algebra acting on the Hilbert space $E_1(\mathscr{H})$. From the case of $n = 1$, we have a unital normal embedding $\phi : \ell^\infty(\mathbb{N}) \to \mathscr{L}'$. It follows from [KR97, Theorem 11.2.9-11.2.10] that

$$\phi \otimes \mathrm{id}_n : M_n\big(\ell^\infty(\mathbb{N})\big) \to M_n(\mathscr{L}') \cong \mathscr{L},$$

is a unital normal embedding of $M_n\big(\ell^\infty(\mathbb{N})\big)$ into $\mathscr{L}$. □

***Remark* 5.5.8.** Let $\mathscr{L}$ be a type II_1 von Neumann algebra. In light of Corollary 5.5.5, Proposition 5.5.7, and the example in Remark 5.5.6, it appears that any meaningful notion of Jordan-Chevalley-Dunford decomposition for operators in $\mathscr{L}$ would involve operators in $\mathrm{Aff}(\mathscr{L})$, that do not necessarily lie in $\mathscr{L}$.

5.6 Concluding remarks

Inspired by Theorem 5.5.3, we conclude our discussion with two conjectures about operators affiliated with a type II_1 von Neumann algebra $\mathscr{L}$.

Conjecture 1:. Every operator A in $\mathrm{Aff}(\mathscr{L})$ can be uniquely decomposed as $A = D \hat{+} N$, where D is a $\mathfrak{u}$-scalar-type operator in $\mathrm{Aff}(\mathscr{L})$ and N is a $\mathfrak{m}$-quasinilpotent operator in $\mathrm{Aff}(\mathscr{L})$ such that D and N commute.

Conjecture 2: The normalized power sequence of every operator A in $\mathrm{Aff}(\mathscr{L})$ converges in the $\mathfrak{m}$-topology.

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List of Publications

1. S. Nayak, and R. Shekhawat
   On the Convergence of the Normalized Power Sequence of Spectral Operators on Hilbert space (accepted for publication in the Journal of Operator Theory)
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