INDIAN STATISTICAL INSTITUTE

Mid-semestral Exam

BStat 2nd year

Maximum marks: 20 Time: 120 minutes

September 10, 2025

Answer Question 5 and any three from the rest.

1. (a) Let R be a commutative ring with 1 and I, J be two ideals of R such that the ring R/IJ does not have any nonzero nilpotent element. Prove that $IJ = I \cap J$. [3]

   (b) Consider the ring $C[0,1] := \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$. Let $f_1, f_2, \cdots, f_n \in C[0,1]$ be such that they have no common zeros. Prove that the ideal $\langle f_1, f_2, \cdots, f_n \rangle$ is the whole ring $C[0,1]$. [2]

2. Let R be a commutative ring with 1.

   (a) Let $a, b, \lambda, \delta \in R$ be such that $b = \lambda a + \delta$. Prove that the ideals $\langle a, b \rangle$ and $\langle a, \delta \rangle$ are equal. [2]

   (b) Prove that the following ideals of $R[X]$ are equal: $\langle f(X), X - c \rangle$ and $\langle f(c), X - c \rangle$, where $f(X) \in R[X]$ and $c \in R$. [3]

3. Let p be a prime number. Prove that the ideal $\langle p, X \rangle$ is a proper ideal of $\mathbb{Z}[X]$ and that it is not a principal ideal of $\mathbb{Z}[X]$. [5]

4. Let R be a commutative ring with 1 and let M be a proper ideal of R. Prove that M is a maximal ideal of R if and only if R/M is a field. [5]

5. Consider the multiplicative group of nonzero complex numbers $\mathbb{C}^*$. Let H be the subgroup $\{1, -1, i, -i\}$. Prove that $\mathbb{C}^*/H \simeq \mathbb{C}^*$. [5]