INDIAN STATISTICAL INSTITUTE, KOLKATA
ASSIGNMENT: FIRST SEMESTER 2024 -'25
M. MATH I

Subject: **Linear Algebra** Date: **November 28, 2025**
Duration: **3 hrs** Maximum score: **60**

Attempt all the problems. Justify every step in order to get full credit of your answers. All arguments should be clearly mentioned on the answerscript. Points will be deducted for missing or incomplete arguments.

(1) Let V be a finite dimensional complex inner product space and $T \in \text{End}(V)$. If $\mathbb{R}\text{e}(\langle Tv, v\rangle) = 0$ for all $v \in V$, then show that (a) T is normal and (b) $\sigma(T)$ is a subset of the purely imaginary line.

[8 marks]

(2) (i) Let $T \in \text{End}(V)$ be normal where V is a finite dimensional complex inner product space. If $\sigma(T)$ is a subset of (a) $\mathbb{R}$, (b) $\{0, 1\}$, (c) S^1, then show that T is (a) self-adjoint, (b) orthogonal projection, (c) unitary.
(ii) Find an example of $T \in \text{End}(\mathbb{C}^2)$ (with the standard inner product on $\mathbb{C}^2$) such that $\sigma(T) = \{\pm 1\}$ but T is not a unitary.

[11 marks]

(3) Let V be an inner product space. Consider the norm defined by $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$ for $u \in V$. If $v_1, v_2 \in V$ such that $\left\|\frac{1}{2}(v_1 + v_2)\right\| = 1 \geq \|v_j\|$ for $j = 1, 2$, then show that $v_1 = v_2$.
Hint: Parallelogram law.

[5 marks]

(4) Let V be finite dimensional complex inner product space, $P \in \text{End}(V)$ is an orthogonal projection and $W := \text{Range}P$. Show that $\text{dist}(v, W) := \inf\{\|v - w\| : w \in W\}$ is equal to $\|v - P(v)\|$ for all $v \in V$. Further show that $P(v)$ is the unique element of W at which the infimum is attained.
Hints: Note that $u = P(u) + (u - P(u))$, $Pu \in W$ and $(u - P(u)) \in W^\perp$ for all $u \in V$. Apply norm on both sides and proceed.

[5 marks]

(5) If $A, B \in M_n(\mathbb{F})$ such that $AB = BA$ and χ_B has n distinct roots in $\mathbb{F}$, then show that A is diagonalizable.

[5 marks]

(6) Provide an example $A \in M_n(\mathbb{F})$ satisfying the following conditions one at a time:
(i) A is cyclic but not simple
(ii) m_A is irreducible but A is not cyclic
(iii) A is semisimple but not diagonalizable
(iv) m_A splits in $\mathbb{F}$ but A is not semisimple

[8 marks]

(7) Let V be a finite dimensional vector space and $T \in \text{End}(V)$.

(a) Prove that $\deg(m_T) = \dim(V) \Leftrightarrow m_T = \chi_T \Leftrightarrow T$ is cyclic.

(b) If $V = U \underset{\text{int}}{\oplus} W$ such that $T(U) \subset U$, $T(W) \subset W$ and $\gcd\left(m_{T|_U}, m_{T|_W}\right) \neq 1$, then show that T is not cyclic.

[9 marks]

(8) Suppose $J_{\lambda,m} \in M_m(\mathbb{F})$ denote the Jordan block matrix with $\lambda \in \mathbb{F}$ in the diagonal. Let A be a square matrix over $\mathbb{F}$ and $\alpha, \beta, \gamma \in \mathbb{F}$ such that

$$m_A(x) = (x-\alpha)^2(x-\beta)(x-\gamma)^3 \quad \text{and} \quad \chi_A(x) = (x-\alpha)^5(x-\beta)^3(x-\gamma)^6 .$$

List all possible Jordan canonical forms (that is, direct sum of Jordan blocks) of A which are distinct up to similarity.

[6 marks]

(9) Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \in M_2(\mathbb{R})$.

(a) Find $\sigma(A)$ and eigen vector corresponding to each eigen value.

(b) Construct $P \in M_2(\mathbb{R})$ such that $P^{-1}AP$ is a diagonal matrix.

(c) **Use part (b)** to simplify A^{100}.

[9 marks]