Similarity Based Approximate Reasoning

This thesis is submitted to Indian Statistical Institute Calcutta, in partial fulfilment of the requirements for the award of Ph.D. Degree

Swapan Raha
Indian Statistical Institute, Calcutta, I N D I A
1999

Acknowledgement

To my supervisor, Prof. Kumar Sankar Ray, I owe my greatest debt. His allowances for all my difficulties and appreciation of my little efforts have been an enduring source of encouragement. This has enabled me to finish my present thesis.

Prof. Mihir Kumar Chakraborty, Department of Pure Mathematics, Calcutta Universiy, has helped me acquire a better understanding of the logical approach to approximate reasoning.

The faculty and staff of ECS Unit, at ISI Calcutta have been ungrudging in their support. Tapan Kumar Dinda and Susanta Mukherjee have helped me till the end.

I am grateful to the Directors, Prof. B.L.S.P.Rao and Prof. S.B.Rao, of Indian Statistical Institute, for allowing me to complete my stay as INSA and SERC visiting fellow and perform collaborative research at ECS Unit.

I express my sincere thanks to Prof. Nikhil Ranjan Pal, ECS Unit, ISI, Calcutta for having many useful discussions on approximate reasoning.

I take this opportunity to thank my colleagues in the Department of Mathematics, Visva-Bharati, for all their help and support without which it would not have been possible to finish this thesis. They have frequently relieved me of much of my departmental duties to enable me to concentrate on my research. Special thanks are due to Prof. S.K.Samanta, Head of the Department of Mathematics, for his useful suggestions on the definition of similarity between arbitrary fuzzy sets. Dr. Arani Chakraborty and Sabyasachi Kar, Visva-Bharati, have helped me in preparing the figures presented in this thesis.

My debt to my family is too great to be enumerated. My mother has been a constant source of strength. It will be my eternal regret that my father did not live to see the completion of my thesis. My sons, Arka and Aritra, have frequently reminded me that there is more to life than fuzzy reasoning. My debt to my wife is too private to be made public.

Contents

List of Tables

List of Figures

Chapter 1

Introduction - scope of work

Many years of research in Artificial Intelligence, Cognitive Science and allied areas reveal that the cognitive process of human reasoning deals with imprecise premises. As cognitive process of human reasoning is mainly concerned with the individual's perception, it is liable to be imprecise in nature. Precise traditional two-valued logic and/or multi-valued logics are not effective in handling such reasoning processes. This motivated Zadeh [109] to investigate how these impreciseness in human reasoning could be modeled through some computable entities. In this regard, Zadeh has shown how such imprecise linguistic terms could be expressed through fuzzy sets over universes of discourse, how linguistic variables could be handled and how deductive processes could be modeled for reasoning with imprecise concepts. These researches together with research on the calculus of linguistic variables [111] resulted in a general theory of approximate reasoning.

The main motivation of the theory of approximate reasoning is apparently the desire to build up a quantitative framework that will allow us to derive an approximate conclusion from imprecise knowledge. Approximate reasoning is somewhat structured yet flexible and more importantly, reliable. The basic principles of the theory of approximate reasoning have been formulated by Zadeh [110, 113, 114]. Since then, different forms of approximate reasoning have been studied by many researchers [35, 37, 38, 39].

A collection of imprecise knowledge given by human experts gives a fuzzy system. The task of a fuzzy system is to exploit the expert's knowledge and model the world with it. A fuzzy system reasons with its knowledge. Reasoning with imprecise knowledge is one of the main problems in the design of fuzzy systems. It has been extremely difficult for traditional mathematical models to capture such inexact reasoning in all its power and flexibility. Reasoning with imprecise data also shows that there is much flexibility in the reasoning of a cognitive agent. Different patterns of approximate reasoning in human beings indicate a need for similarity matching, in situations where there is no directly applicable knowledge, to come up with a plausible conclusion. In such cases, the confidence in a conclusion may be determined, based on a degree of similarity. In order to capture this, we also need corresponding flexibility in our model. Specifically, we need means to handle graded information on the one hand and the concept of similarity on the other hand. Conventional approximate reasoning does not consider the concept of similarity measure in deriving a consequence. Existing similarity based reasoning methods modify the consequence of a rule, based on a measure of similarity and therefore, the consequence becomes independent of the conditionals. To satisfy both the requirements simultaneously, we need to integrate conventional approximate reasoning and similarity based reasoning for an adequate theory of similarity based approximate reasoning.

In the present thesis, we attempt to use the concept of similarity between prototypes of fuzzy sets in approximate reasoning methodology. This is why the title of the thesis is 'Similarity based approximate reasoning'. New similarity measures are proposed and existing phenomenon of similarity based reasoning and approximate reasoning methods are combined. We attempt to account for different patterns of human reasoning through the proposed similarity based approximate reasoning mechanism.

1 A brief outline

By the term imprecise we mean only 'vague' linguistic entities, which may be due to lack of perception about any situation at hand. Let us consider the degree of human maturity - 'infancy', 'childhood', 'adolescence' and 'adulthood'. They are mutually inconsistent yet, lack sharp boundaries with respect to their neighbours in the scale of human maturation. More exactly, if we take a sufficiently small interval of time and suppose that someone matures in a typical fashion, then at no stage will (s)he effect within such an interval of time a transition from one stage of maturity to the next. The sentence 'the patient is adult' is true to some degree. Truth of such a proposition is a matter of degree. This is exactly what *vagueness* is. Whereas, the statement 'the patient will survive next week', may be either absolutely true or absolutely false.

In what follows, we shall use 'vague' in two senses - in the sense of being *indefinite*, i.e., lacking definite boundaries e.g., between 'infant' and 'child', as well as in the sense of being *indeterminate* as applies to 'mountain' and 'hill'.

A simple statement from a natural language may be broadly decomposed into two components, viz., a subject and a predicate. In the present thesis, we are mainly concerned with the vagueness of predicate. Vague predicates are rampant in everyday life. As a case in point, let's quote a verse from the 'Aranya' chapter of the '*Ramayana*',

> When the rains are heavy the grass grows so tall that it is difficult to find the right path.

Consider the two underlined predicates viz., heavy and tall in this connection. Suppose, that the meaning of the two predicates are given by the following clauses:

1. a) height is tall, if height > 25 cms.
   b) height is not tall, if height < 18 cms.
2. a) rainfall is heavy, if quantity/hour > 2 cms.
   b) rainfall is not heavy, if quantity/hour < 0.8 cms.

In clauses (1) and (2), both predicates are vague as their meanings are underdetermined for the interval not covered by sub-clauses (a) and (b). Quantifiers may also be vague. Indeed, quantifier over a vague domain may be represented by an appropriately relativised (/approximated) quantifier over a more precise domain [24, 32].

Let us look into a set of examples for a better understanding of the patterns of approximate reasoning mostly used by cognitive agents.

- From two approximately equal numbers we conclude that if one of them is small then the other will be more or less small.
- The temperature in Jaisalmir is high, so the water holding capacity of air is high too. We conclude that Jaisalmir is moist.
- High interest rates will cause low money supply growth. We conclude that very high interest rates cause low inflation rates.

- Either the general budget will be people friendly or the Government will be in trouble. If now the Government falls then we may conclude that it was not a common people's budget.

- Each student is either intelligent or meritorious. Arnab tried hard but, failed to qualify. We conclude that Arnab is not intelligent.

In the above examples, we see that most of the key concepts involved are not all-or-nothing, but somehow graded, so the conclusions must be graded, in correspondence with the confidence values of known facts and conditions. A close look at them reveal that, they are vague in nature - either indefinite or indeterminate. We focus our study on only approximate reasoning with vague linguistic entities, descriptions that often appear in a subject-predicate formulation of natural language. For this, we consider a set theoretic approach for the representation of such vague concepts, i.e., we use a set to exemplify such a concept and then use laws of the underlying set theory to manipulate them. Here, only fuzzy sets are used for the representation and the theory of fuzzy sets are used for manipulation of such vague concepts. Another important aspect noticed here is that, when there is no matching existing knowledge, we perform similarity matching in manipulation.

Zadeh's form of inference in approximate reasoning is based on the Compositional Rule of Inference (CRI) :

from 'X is A' and '(X, Y) is R' infer 'Y is B'.

CRI scheme is such that for a large class of A, each different from the other, the concluded B remains the same. As an example, let us take R as in the following :

$$R = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1.00 & 0.75 & 0.50 & 0.25 \\ 0.75 & 1.00 & 0.75 & 0.50 \\ 0.50 & 0.75 & 1.00 & 0.75 \\ 0.25 & 0.50 & 0.75 & 1.00 \end{bmatrix} \end{array} . \qquad (1.1)$$

For all fuzzy sets, $A = \mu_1/u_1 + \mu_2/u_2 + 1.0/u_3 + \mu_3/u_4$, where $0 \le \mu_i \le 0.50$, i= 1, 2, 3; the output of CRI will be the same $B = 0.50/v_1 + 0.75/v_2 + 1.0/v_3 + 0.75/v_4$. There is no scope to incorporate such a change in the input, A.

Also, such relations may produce significant conclusions from an almost dissimilar pair $\{A, A^*\}$. In [84], Zadeh's inference mechanism was critically appreciated and an alternative proposal for similarity based analogical approximate reasoning has been made. Recently in [82], the authors proposed a similar scheme for similarity based reasoning. In similarity based reasoning, from a given fact the conclusion is derived based, on a measure of similarity between the fact and the antecedent.

As an illustration, consider the statements

$$p : \text{if } X \text{ is } A \text{ then } Y \text{ is } B,\ \tau \text{ and } \quad q : X \text{ is } A'.$$

Here A and A' are fuzzy sets defined over the same universe of discourse U and the fuzzy set B is defined over the universe of discourse V, τ is the firing strength of the rule. Let $S(A, A')$ denote any measure of similarity between the fuzzy sets A, A'. Now, if $S(A, A') > \tau$ then the rule may be fired and the consequent of the rule is modified to produce the desired conclusion B'.

These methods [82, 84] use the similarity measure for a direct computation of inference without forming the induced fuzzy relation. Consequently, these methods provide the same conclusion if A and A' are interchanged in the propositions concerned. This means that the outcome of

$$p1 : \text{if } X \text{ is } A' \text{ then } Y \text{ is } B,\ \tau \text{ and } \quad q1 : X \text{ is } A$$

will be the same as that given by p and q. This is not appealing. Moreover, such a method of derivation becomes unsuitable when the conditional knowledge used for the derivation contains a disjunctive operator, i.e., of the form

$$p2 : X \text{ is } A \text{ or } Y \text{ is } B,\ \tau \text{ and } \quad q2 : X \text{ is } A'.$$

Let us consider a typical inference from p and q. Let, $A = 1/u + 1/v + 1/w$, $A' = 1/v$ are fuzzy sets defined over $U = \{u, v, w\}$ and $B = 1/a + 0.1/b$ be a fuzzy set defined over the universe of discourse $V = \{a, b\}$. By this conditional statement, we mean that, if X is A then Y is B, i.e., more explicitly, if X is u or X is v or X is w then we may conclude that Y is B. Here, we know that X is v. Therefore, the consequence should be Y is B.

In existing similarity based reasoning, we compute the similarity between A and A' and if found to be greater than τ, the rule consequent is modified for a conclusion.

certainly different from B. Let us suppose, instead of p and q, we have $p1$ and $q1$, where A and A' are as before. In this case, we are saying that Y is B if the value of X is v. We note that, here, X may be either u or v or w. We cannot, therefore, conclude anything about Y. Interestingly, the existing similarity based reasoning method would conclude the same in such a situation. It is not difficult to see that the consequences of the application of Compositional rule of inference, in these cases, are different.

Keeping all these in mind, we propose that a reasoning system should consider every change in A' and A (i.e., the fact and a prototype of the same appearing in the conditional statement) so that the inference is influenced by the said change and at the same time the more the change (in the linguistic descriptions), the less specific the conclusion. It is also necessary that nothing better than what the condition reveals should be allowed as a valid consequence. It should also consider some form of matching in the process of derivation of a consequence when the fact is different from the cause of the derivation.

As a solution to the problem, we show that existing similarity based reasoning schema need some simple modification for the said purpose and that the conventional compositional principle, due to Zadeh, are still useful in derivation. Till recently, similarity based reasoning was considered as an application of fuzzy production rule [24]. With such a modification in similarity based reasoning we show that many fuzzy systems may be designed effectively.

Our method of inference is based on a similarity measure. In practice, the conditional statement is first expressed as a fuzzy relation. We interpret it as a conditional fuzzy relation. Then, the similarity between the fact and a prototype of the same appearing in the conditional statement is used to modify the conditional relation. Such a modification of relation may be performed in many ways. We interpret the modified relation as a fuzzy relation induced by the fact. Then, we project the induced relation on the domain of the linguistic variable defining the consequence. We use two different forms of conditional statement - 'if a then b' and 'either a or b'.

To verify the effectiveness of the proposed method of similarity based inference we use it in designing two rule-based fuzzy systems --- a pattern classifier and a fuzzy controller. Simulation is done on real data and interesting results are presented. In developing similarity based approximate reasoning, we have been able to establish certain relevant concepts, like a generalized rule-based model for inference in an incomplete environment, the resolution-based model and the concept of generalized disjunctive syllogism in fuzzy inference.

Some other useful models are also extensively discussed. The mechanism of avoidance of the use of rule firing strength and its automatic introduction in the inference mechanism is an effective one. The use of specificity of fuzzy sets in decision making in an uncertain environment, in the framework of similarity based approximate reasoning is another important issue. The utility of the specificity based approach has been demonstrated by the results.

As another approach, Zadeh has shown that the theory of approximate reasoning is an application of the theory of possibility [114]. In the theory of possibility, such vague propositions may be conveniently represented by an appropriate possibility distribution. Each possibility distribution is assigned a fuzzy set/relation defined over an appropriate domain via some possibility assignment function. Therefore, reasoning with vague propositions may also be performed through manipulation of possibility distributions.

Next, we cultivate the potentiality of similarity based approximate reasoning in different other areas of reasoning, viz., reasoning with truth-qualified proposition, default reasoning and temporal reasoning. Here, we show that the theory of possibility may be effectively used in the representation of such knowledge and then reasoning may be made, using the rules for manipulation of possibility distribution.

We know that, fuzzy equivalence relations are a useful models to describe the indistinguishability inherent to prototypes of fuzzy sets. In such similarity based reasoning, it is possible to avoid the application of input fuzzy sets in rule firing, based on a measure of indistinguishability because, a higher accuracy than the indistinguishability inherent to the fuzzy set or fuzzy equivalence relation does not influence the resulting output of a fuzzy system. It is also possible to perform approximate reasoning based on a similarity relation. These methods are rule-based and as in the conventional cases, make a class of fuzzy sets indistinguishable with respect to a similarity relation.

If we view the membership grade of the elements of a fuzzy set as the resemblance degree between the element and prototypes of the fuzzy set then, one of the possible semantics of a fuzzy set may be given in terms of similarity [17]. For this we only need to equip the referential set with a similarity relation. Eventually, it is possible to find another fuzzy reasoning method based on similarity, more in a logical sense.

This is useful in considering interpolation in a logical setting. Work in detail may be found in [17]. The applicability of these methods are limited in scope.

2 Organization

The material of the present thesis is divided into two parts each consisting of a number of chapters. Chapters 2 to 5 constitute the first part of the thesis, in which existing similarity based reasoning methodology and approximate reasoning methodology are integrated in developing an unified similarity based approximate reasoning method. Different models of approximate reasoning based on similarity are formulated. They are applied in designing fuzzy systems and their performance in solving real life problems are studied and results are analysed. The second part consists of the remaining chapters 6,7 and 8. In this part, we apply the said methodology in reasoning with truth-qualified vague propositions, in default reasoning and in temporal reasoning.

A thorough exposition of inferring in approximate reasoning paradigm begins in chapter 2 with an adequate review of a collection of well used terms needed for a comprehensive understanding of the methodology of approximate reasoning. This chapter is a collection of results, most of which are already available in the literature. The focus, in this chapter, is on fuzzy set theoretic approach to inference in approximate reasoning which is crucial to the understanding of the contents of the following chapters. A possibilistic approach to approximate reasoning is also discussed. Different models of reasoning —- rule-based and resolution-based, ordinary and extended form —- are discussed. A new model of rule-based approximate reasoning is also discussed. In order to provide a better understanding of the semantics of approximate reasoning methodology, we present a pictorial description of the different operations underlying conventional approximate reasoning, by considering a typical problem.

Chapter 3 is on the concept of similarity index for measuring the nearness of fuzzy sets over a given universe of discourse. Different approaches for a solution to the problem may be found in the literature (, such as cardinality consideration, subsethood). Since, our main intention is to use such an index in modifying a fuzzy set/fuzzy relation, we consider a point wise comparison of the membership values of elements underlying the sets. First of all, a brief review of the existing measures is presented. Then new methods are proposed, based on some basic properties of similarity. Results are extensively discussed. Based on these properties we propose a set of axioms for defining a similarity index between two fuzzy sets. These works, certainly, make the task of choosing a particular measure for use, easier. A methodological note is in order here : in describing different similarity measures, we assume the universe of discourse to be finite. Similarity measures for infinite fuzzy sets may also be considered. In the end, we present some measures of similarity for arbitrary sets. They are found to be good for mathematical purposes.

In chapter 4, we introduce the similarity measure into approximate reasoning methodology. Different existing approaches to similarity based reasoning are discussed and then a new mechanism of approximate reasoning based on similarity measures is developed. Existing similarity based approach as well as ordinary approximate reasoning mechanism for deriving a consequence are combined together in order to develop an effective mechanism. First, a fuzzy relation is constructed from the translation of the conditional statement. The mechanism is such that every change in the concept as appears in the conditional(,general) statement and that in the fact, is incorporated in the fuzzy relation between the variables defining the condition. This procedure ensures the inference as a function of the concerned change. The more the change, the less specific the conclusion. For that, we formulate two schema for the modification of the conditional fuzzy relation. Another characteristic of the scheme to be noted here is that we may avoid the use of certainty factor concept for rule-misfiring. To do that, we need to modify the inference scheme in such a way that a significant change will make the conclusion less specific. This is done if an expansion type of inference scheme is chosen. Here, the 'UNKNOWN' case, is taken as the threshold. Explicitly, when the similarity value becomes low, i.e, when the change is significant, the reasoning process is such that the inference becomes unknown. At the same time, when there is no change, i.e., a perfectly matching case, we show that it is possible to derive the expected consequence (, the consequence of the condition in a rule-based system). In other cases, the consequence is found to be no better than what the condition allows. Different interesting results in this direction are extensively discussed in this section. Here, we show that the concept of specificity measure of fuzzy sets is inherent in such similarity based approximate reasoning methodology. Examples are considered to demonstrate the computations under the procedure. In order to generate some 'feeling' about the proposed similarity based approximate reasoning, we present a pictorial representation of the same. We compare the result of our method with the results of other existing approaches to similarity based reasoning and Zadeh's compositional rule of inference. In the process, we formulate different models for similarity based approximate reasoning - rule-based as well as resolution-based, ordinary and extended forms.

In chapter 5, we test the effect of the proposed method of similarity based approximate reasoning, by considering the application of the said method in the design of rule-based fuzzy systems. In fact, we consider the design of a rule-based pattern classifier as well as a rule-based fuzzy control system. Defuzzification, a basic operation, used in the development of fuzzy systems is extensively discussed in the light of new similarity based approximate reasoning mechanisms. New schema for defuzzzification, suitable for similarity based approximate reasoning, are defined. These defuzzification methods are then used in pattern classification problems as well as in fuzzy control problems. Simulations are performed with some real data for Telugu vowel classification problem and inverted pendulum control problems. Results are then tabulated along with their pictorial representations. This completes the first part of the present thesis.

We then explore further possibilities of the use of similarity based approximate reasoning in other emerging areas of research. The second part consists of the application of similarity concept in two important fields of recent research in reasoning — default reasoning and temporal reasoning. Here, we also consider the representation and manipulation of truth-qualified propositions.

So far we consider only those vague statements which are always taken for granted. In chapter 6, we consider the representation and manipulation of truth-qualified vague statements which cannot be taken for granted. Such statements are represented as a pair of predictions about two objects, of which one is about the linguistic truth value of the prediction. Rules for the composition of two or more such formulae are proposed. Reasoning with such partially true premises are considered using the proposed method of similarity based approximate reasoning.

In chapter 7, we utilize the concept of graded certainty in representing default values and consider similarity based approximate reasoning. This gives a new dimension to the study of default reasoning. We show that the technique maintains the nonmonotonicity property and blocks undesirable transitivity. The representation and manipulation of default and default values are illustrated with simple and concrete problems.

The need to make time-dependent assumptions is frequently encountered in many systems, especially in dynamical systems. In the final chapter 8, we attempt a theoretical study of the representation and manipulation of some vague and incomplete knowledge in the temporally changing world. Some simple yet concrete examples have been used to illustrate the usefulness of the proposed models.

The present thesis is concluded in the last section with a list of areas recommended for future work in this direction. In the end, a comprehensive list of relevant works, either referred to in this thesis or consulted during the preparation of the thesis, is presented.

Chapter 2

Approximate Reasoning - a brief review

1 Introduction

First of all, it may be recalled that reasoning is a mental activity — actually a combination of intellectual activities — which allow the passage from a set of propositions, stated as premises, to a further proposition, taken as the conclusion(/consequence), by virtue of a logical connection, with some degree of strength, which link the conclusion to the premises. Taking a decision amounts to giving reason for or against it and hence reasoning is an essential activity in decision-making. Reasoning is approximate when some of the propositions are imprecise and the rules for derivation are inexact in nature.

Approximate reasoning is considered to be a powerful tool to study the remarkable human ability to understand real-world activities in terms of computational entities with immense confidence. As for instance, crossing the railway track safely even after seeing a train approaching with high speed or avoidance of a particular road and use of by-lanes to beat the office-hour rush are typical examples of approximate reasoning.

Approximate reasoning is defined as the process or processes by which an approximate conclusion may be deduced from a set of possibly imprecise information using some inexact rule for the derivation. It was first formally introduced by Zadeh

[109]. Since its inception in 1973 [109], significant theoretical advances have established approximate reasoning as an important field of research. Different mechanisms of approximate reasoning have been proposed and discussed in the literature [37, 38, 66, 70, 73]. We have witnessed among many other things the birth of fuzzy logic controller.

Zadeh's concept of approximate reasoning is based on fuzzy logic and the theory of fuzzy sets. In order to have an adequate understanding of the theory of approximate reasoning, some basic concepts are studied in the following.

2 Fuzzy Set - basic operations

In representing human understanding of various kinds of real world activities, Zadeh formally introduced the concept of a fuzzy set [106]. A fuzzy set is the theoretical primitive of fuzzy mathematics just as a classical set is the theoretical primitive of classical mathematics. In fuzzy set theory, classical sets are called crisp sets, in order to distinguish them from fuzzy sets.

A (classical)set is characterized by objects which are called members of the set. Like (classical)set, a fuzzy set is also characterized by its members. But unlike (classical)set, members of the universal set U may or may not belong to the fuzzy subset to some degree.

Definition 2.1 : 'A fuzzy set(class) A in(a given set) U is characterized by a membership(characteristic) function $f_A(u)$ which associates with each point in U, a real number in the interval [0,1] (or a suitable, partially ordered set P) with the value $f_A(u)$ at u representing the grade of membership of u in A'.

In order to have a better understanding of a fuzzy set, let us elaborate some basic concepts, associated with the *Definition 2.1* of fuzzy set. By a fuzzy set, we mean:
i) A set of elements, e.g., m (man) $\in M$ (men);
ii) A label, e.g., X (height of men), for an attribute of elements $m \in M$;
iii) An adjective / adverb, e.g., A (tall man) associated with height of men;
iv) A referential set, e.g., R (the interval [0,250] cms.) associated with height of men and;
v) A purely subjective assignment function

$$\mu : U \rightarrow [0, 1]$$

known as the gradual set-membership function, to denote a concept of grade/degree with which an element $u \in U$ may belong to the fuzzy set A. Throughout this thesis, a fuzzy set A in U is represented by the collection of pairs $(u, f_A(u))$ as :

$$A = [(u, f_A(u)) \mid u \in U, f_A(u) > 0]. \tag{2.1}$$

When U is finite, we may use

$$A = \sum_{u \in U} f_A(u)/u. \tag{2.2}$$

Definition 2.2 : A fuzzy set, A, in which there exists, at least one element with membership grade '1', i.e.,

$$\sup_{u \in U} f_A(u) = 1. \tag{2.3}$$

is called a normal fuzzy set.

Definition 2.3 : The support of a fuzzy set A is a crisp set that contains all elements of A with non-zero membership degree. This is denoted by $S(A)$ and is formally defined by

$$S(A) = \{u \in U \mid \mu_A(u) > 0\}. \tag{2.4}$$

Example 2.1 : Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the set of number of members a family may have. Then the fuzzy set A = 'sensible number of members in a family' may be described as follows :

$$A = \{(0, 0.1), (1, 0.4), (2, 0.7), (3, 1.0), (4, 1.0), (5, 0.7), (6, 0.4), (7, 0.1)\}.$$

The membership grades of this fuzzy set are obviously subjective measures. Since $f_A(u)$ may take values other than 0 and 1, it is possible to represent vague/imprecise concepts through fuzzy sets.

2.1 Set operators

Let A and B be two fuzzy sets. The simplest complementation operator is :

$$C(x) = 1 - x, \quad \forall x \in [0, 1] \tag{2.5}$$

and the membership function of $\bar{A}$ is defined by :

$$\mu_{\bar{A}}(x) = C(\mu_A(x)). \tag{2.6}$$

A discussion about more general complementation operators may be found in [81].

Mainly three operators are used for fuzzy set intersection :

$$I(x, y) = \min(x, y); I(x, y) = x.y; I(x, y) = \max(0, x + y - 1). \qquad (2.7)$$

The intersection operator is denoted as $\cap$ and $A \cap B$ is defined by :

$$\mu_{A\cap B}(x) = I(\mu_A(x), \mu_B(x)). \qquad (2.8)$$

Conjointly, three associated union operators may be deduced from the above operators by preserving De Morgan's laws :

$$U(x, y) = \max(x, y) \ : \ U(x, y) = x + y - x.y \ ; \ U(x, y) = \min(1, x + y). \qquad (2.9)$$

The union operator is denoted as $\cup$ and $A \cup B$ is defined by :

$$\mu_{A\cup B}(x) = U(\mu_A(x), \mu_B(x)). \qquad (2.10)$$

A more detailed discussion on such operators may be found in [18].

Definition 2.4 : Two fuzzy sets A and B are equal ($A = B$) if and only if

$$\mu_A(u) = \mu_B(u) \ ; \ \forall u \in U. \qquad (2.11)$$

Definition 2.5 : Fuzzy set A is said to be a subset of fuzzy set B if and only if

$$\mu_A(u) \leq \mu_B(u) \ ; \ \forall u \in U. \qquad (2.12)$$

2.2 T-operators

The triangular norm (T-norm) and the triangular conorm (T-conorm) are frequently used in fuzzy set theory in defining intersection and union of fuzzy sets. Zadeh's conventional T-operators 'min' and 'max' have been extensively used in many design of fuzzy systems. Let us define T-operators in the following.

Definition 2.6 : Let $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$. T is a T-norm, if and only if, for all $x,y,z \in [0, 1]$;

2.6.1.) $T(x, y) = T(y, x)$,

2.6.2.) $T(x, y) \leq T(x, z)$, if $y \leq z$,

2.6.3.) $T(x, T(y, z)) = T(T(x, y), z)$,

2.6.4.) $T(x, 1) = x$.
A T-norm is Archimedean, if and only if,
2.6.5.) $T(x, y)$ is continuous.
2.6.6.) $T(x, x) < x$, for all $x \in (0, 1)$.
An Archimedean T-norm is strict, if and only if,
2.6.7.) $T(x', y') < T(x, y)$, if $x' < x$, $y' < y$, for all $x', y', x, y \in (0, 1)$.

Definition 2.7 : Let $T^* : [0, 1] \times [0, 1] \rightarrow [0, 1]$. T^* is a T-conorm, if and only if, for all $x,y,z \in [0, 1]$;
2.7.1.) $T^*(x, y) = T^*(y, x)$.
2.7.2.) $T^*(x, y) \leq T^*(x, z)$, if $y < z$.
2.7.3.) $T^*(x, T^*(y, z)) = T^*(T^*(x, y), z)$.
2.7.4.) $T^*(x, 0) = x$.
A T-conorm is Archimedean, if and only if,
2.7.5.) $T^*(x, y)$ is continuous.
2.7.6.) $T^*(x, x) > x$, for all $x \in (0, 1)$.
An Archimedean T-conorm is strict, if and only if,
2.7.7.) $T^*(x', y') < T^*(x, y)$, if $x' < x$, $y' < y$, for all $x', y', x, y \in (0, 1)$.
Let us now present some T-operators in the following. A detailed study on the same may be found in [36].
1.) $T_1(x, y) = \min(x, y), T_1^*(x, y) = \max(x, y)$.
2.) $T_2(x, y) = x.y, T_2^*(x, y) = x + y - x.y$.
3.) $T_3(x, y) = \max(x + y - 1, 0), T_3^*(x, y) = \min(x + y, 1)$.
Now we list some important properties of T and T^* in the following :
1.) $T(x, y) = T(y, x)$,$T^*(x, y) = T^*(y, x)$ (commutativity),
2.) $T(x, T(y, z)) = T(T(x, y), z)$,$T^*(x, T^*(y, z)) = T^*(T^*(x, y), z)$ (associativity),
3.) $T(x, T^*(y, z)) = T^*(T(x, y), T(x, z))$,$T^*(x, T(y, z)) = T(T^*(x, y), T^*(x, z))$ (distributivity),
4.) $T(T^*(x, y), x) = x$,$T^*(T(x, y), x) = x$ (absorption),
5.) $T(x, x) = x$, $T^*(x, x) = x$ (idempotency).

2.3 Extension principle

A very important notion in fuzzy set theory is the extension principle. It provides a method for combining non-fuzzy and fuzzy concepts of different kinds including the operation of a mathematical function on fuzzy sets.

Let $A_1, A_2, \cdots, A_n$ be fuzzy sets, defined on $U_1, U_2, \cdots, U_n$ respectively, and let f be a non-fuzzy function

$$f : U_1 \times U_2 \times \cdots \times U_n \rightarrow V.$$

Let F be the extension of f on V.

Definition 2.8 : The extension of f, operating on $A_1, A_2, \cdots, A_n$ results in the following membership function for F

$$\mu_F(v) = \sup_{\{u_1,\cdots,u_n \mid f(u_1,\cdots,u_n)=v\}} \{\min(\mu_{A_1}(u_1), \cdots, \mu_{A_n}(u_n))\} \tag{2.13}$$

when $f^{-1}(v)$ exists. Otherwise, $\mu_F(v) = 0$.

2.4 Specificity measure of fuzzy sets

In this section, we discuss specificity measure of a fuzzy set, which is used in measuring the preciseness of a fuzzy information. A fuzzy set with maximum specificity value corresponds to a precise assessment of the values. In trying to capture the form of a specificity index, a number of properties are required or desirable. According to Dubois and Prade, a specificity measure $Sp(A)$ [19] should satisfy the following properties.

Let X be a linguistic variable defined on a universe of discourse U and A, B be normalized fuzzy subsets of U.

P1. For all $A \subseteq U$, $Sp(A) \in [0, 1]$.

P2. $Sp(A) = 1$, if and only if A is a singleton of U.

P3. $A \subseteq B \rightarrow Sp(A) \geq Sp(B)$.

Yager [96] has introduced one such measure of specificity which satisfy the above properties. When U is finite, Yager has proposed the following expression for defining the specificity.

Definition 2.9 : Let us assume that A be a fuzzy set defined over the universal set U. The specificity associated with A is denoted as Sp(A) and is defined as

$$\text{Sp(A)} = \int_0^{\alpha_{max}} \frac{1}{CardA_\alpha} d\alpha. \tag{2.14}$$

Here, $A_\alpha = \{u \in U \mid \mu_A(u) \geq \alpha\}$ and Card A denotes the cardinality of the fuzzy set A, i.e., Card $A = \sum_{\alpha \in (0,1]} \alpha . A_\alpha$.

Example 2.2 : Let $U = \{u_1, u_2, u_3, u_4\}$, and $A = \{1.0/u_1 + 0.75/u_2 + 0.5/u_3 + 0.25/u_4\}$. In this case we find that $\alpha_{max} = 1.0$. The level sets are

$$\begin{aligned}
A_\alpha &= \{u_1, u_2, u_3, u_4\} && \text{for} && \alpha \leq 0.25 \\
A_\alpha &= \{u_1, u_2, u_3\} && \text{for} && 0.25 < \alpha \leq 0.50 \\
A_\alpha &= \{u_1, u_2\} && \text{for} && 0.50 < \alpha \leq 0.75 \\
A_\alpha &= \{u_1\} && \text{for} && 0.75 < \alpha \leq 1.00
\end{aligned}$$

and from these we may get the cardinalities as

$$CardA_\alpha = \begin{cases} 4 & \text{for} \quad \alpha \leq 0.25 \\ 3 & \text{for} \quad 0.25 < \alpha \leq 0.50 \\ 2 & \text{for} \quad 0.50 < \alpha \leq 0.75 \\ 1 & \text{for} \quad 0.75 < \alpha \leq 1.00 \end{cases} .$$

Therefore, we find from *Definition 2.9* that Sp(A) = 0.52.

Let us now list some important properties [96] associated with the above *Definition 2.9*.

Theorem 1 For all A, Sp(A) assumes its maximum value 1, when $A = \{1/u\}$ for some particular $u \in U$.

Theorem 2 For all A, Sp(A) $\in [0, 1]$ and it assumes its minimum value 0, when $A = \Phi$.

Theorem 3 If for all A, $\mu_A(u) = k$ for all $u \in U$ then Sp(A) $= \frac{k}{n}$ where n is the cardinality of the ordinary set U.

3 Fuzzy Logic

Looking at the number of papers dealing with fuzzy logic and its successful applications in engineering, science and technology hopefully, we may conclude that it is the logic for future generation Intelligent systems. In short, fuzzy logic is a logic for inexact reasoning. The underlying set theory being the theory of fuzzy sets. Classical theory of sets and formal logic are dual representations of the same information. Often the statement 'something is a member of a set' is used to mean that 'it is true that the object possesses the defining property of the set'. But this is not the case with fuzzy sets and fuzzy logic. For the degree of ease of being an element of a set and the degree of truth about a particular statement cannot be considered to be the same. In fuzzy logic, the truth-values are qualitative and possibly imprecise. Such truth assignments may be best represented by a fuzzy set over the unit interval [0,1].

In [55], Zadeh has pointed out that the term *fuzzy logic* has two different meanings - wide and narrow.

> In a narrow sense, *fuzzy logic*, FLn, is a logical system which aims at a formalization of approximate reasoning. In this sense, FLn is an extension of multivalued logic. However, the agenda of FLn is quite different from that of traditional multivalued logics. In particular, such key concepts in FLn as the concept of a linguistic variable, canonical form, fuzzy if-then rule, fuzzy quantification and defuzzification, predicate modification, truth-qualification, the extension principle, the compositional rule of inference and interpolative reasoning, among others, are not addressed in traditional systems. This is the reason why FLn has a much wider range of applications than traditional systems. In its wide sense, *fuzzy logic*, FLw, is fuzzily synonymous with fuzzy set theory, FST, which is the theory of classes with unsharp boundaries. FST is much broader than FLn and includes the latter as one of its branches.

Recently, Petr Hajěk concluded that fuzzy logic is a logic in its wide sense, anything dealing with fuzziness may be called fuzzy logic. Whereas, in its narrow sense, it is a beautiful logic and is important for applications. There are various systems of fuzzy logic, not just one. Formal calculi of non-traditional multivalued logic is the kernel or base of fuzzy logic. Formally, fuzzy logic may be defined as an algebraic system $\langle L, \wedge, \vee \rangle$ where L, the truth set, is a complete residuated lattice. $\wedge$ is completely distributive with respect to $\vee$, $\wedge$ and $\vee$ are the well-known 'min' and 'max' operators. For convenience, let us take L to be the unit interval [0,1] where 0 is the bottom element and 1 is the top element of the lattice.

3.1 Linguistic Variable

As imprecision is intrinsic in natural language, therefore, only an approximate characterization of the values of the variables is possible. The concept of linguistic variable plays an essential role in the theory of approximate reasoning. It is a tool for approximate characterization of the values of the variables and their interrelations [111]. For example, *height* of a person may be short, *volume* of a container may be huge, *code section* of some programme may be tiny, *pressure* in a boiler may be low, *time* of occurrence of an event may be mid-night, two *numbers* may be approximately equal, and so on. Zadeh called such variables - linguistic variables [111].

Such a variable takes as value, linguistic terms that are possibly imprecise. For example, *age* of a person may be described by the terms infant, teenage, youth, middle-age, old etc.. So a linguistic variable *age* may take any value such as infant, teenage, young, etc.. These linguistic values of a linguistic variable may be conveniently represented as computational entities by means of fuzzy sets. The set {infant, teenage, young, middle-aged, old } may constitute a primary term set and the term set will be the totality of the primary terms together with terms generated from the primary terms in conjunction with modifiers as permissible, e.g., very young, more or less old etc..

3.2 Inference rule

Now, we present some basic inference rules in fuzzy logic which are commonly used in approximate reasoning. Among them, two rules are of major importance, viz., the compositional rule of inference and generalized modus ponens. In [70], we develop a new rule of inference in fuzzy logic and call it generalised disjunctive syllogism. We also show the importance of the same in reasoning with vague concepts.

- **Entailment** : From 'X is A' we infer 'X is A^*', if $\mu_A(x) \leq \mu_A^*(x)$.
- **Cylindrical extension** : From 'X is A' we infer '(X,Y) is A^+', where $(\forall y)$ $\mu_{A^+}(x,y) = \mu_A^*(x)$.
- **min-combination** : From 'X is A_1' and 'X is A_2' infer 'X is $A_1 \cap A_2$', where $\mu_{A_1 \cap A_2}(x) = \min(\mu_{A_1}(x), \mu_{A_2}(x))$.
- **sup-projection** : From '(X,Y) is R' infer 'Y is B', where $(\forall y)$ $\mu_B(y) = \sup_x \mu_R(x,y)$.
- **max-min composition** : From 'X is A' and '(X,Y) is R' infer 'Y is B', where $\mu_B(y) = \max_x \min(\mu_A(x), \mu_R(x,y))$.

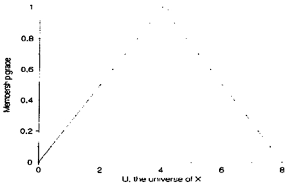

Figure 2.1: The fuzzy set A

- **Generalized modus ponens :** From 'X is A^*' and 'if X is A then Y is B' infer 'Y is B^*', where $\mu_{B^*}(y) = \sup_x(\mu_{A^*}(x) \circ (\mu_A(x) \to \mu_B(y)))$, $\circ$ and $\to$ may have different interpretation. If we choose $\to$ to be any T-norm operator, we call it sup $-t$ combination.

- **Generalized disjunctive syllogism :** From 'X is A^*' and 'either X is A or Y is B' infer 'Y is B^*', where $\mu_{B^*}(y) = \inf_x(\mu_{A^*}(x) \bullet (\mu_A(x) \vee \mu_B(y)))$, $\bullet$ and $\vee$ may have different interpretation. If we choose $\vee$ to be any T-conorm operator, we call it inf $-s$ combination.

In a recent work [58], different reasoning methods based on point-valued and interval-valued fuzzy sets have been discussed and results analyzed.

4 Possibility distribution

The theory of approximate reasoning, due to Zadeh [109, 113, 114], may be viewed alternatively as a suitable application of the theory of possibility. The theory of possibility is a special branch of fuzzy measure theory. A lower semi-continuous fuzzy measure is called a possibility measure, Pos, iff it satisfies the requirement

$$\mathrm{Pos}(\cup_{i\in I} A_i) = \sup_{i\in I} \mathrm{Pos}(A_i)$$

for any family $\{A_i \mid A_i \in \mathcal{P}(X), i \in I\}$ of subsets of X, where I is an arbitrary index set and X is a given universal set.

Given a possibility measure Pos, a function f on X defined by

$$f(x) = \mathrm{Pos}(\{x\}) \tag{2.15}$$

for all $x \in X$ is called a ***possibility distribution function.*** The function uniquely characterizes the fuzzy measure via the formula

$$\mathrm{Pos}(A) = \sup_{x\in A} f(x) \tag{2.16}$$

for all $A \in \mathcal{P}(X)$.

The basic assumption underlying the theory of approximate reasoning is that the imprecision in natural language is possibilistic in nature. Possibility theory plays an essential role in translating natural language expression into possibility distributions to which, the rules of inference may be applied for a possible inference. In the following, we discuss the basic principles of the theory of possibility. A thorough exposition of the same may be found in Zadeh [113, 114].

Approximate reasoning is concerned with knowledge expressed in a formal atomic primitive, which are actually given in a natural language form, e.g., *Pressure is low.* The translation of this statement into atomic forms proceeds as follows:

- a symbol is used to denote the physical variable *Pressure* (P),
- another symbol is chosen to denote the particular linguistic value of the variable *low* (B),
- the natural language statement is rearranged according to the form *Pressure has the property of being low,*
- the atomic primitive is given by P is B, where 'is' stands for the phrase 'has the property of being'.

In this thesis, such an expression is called an atomic fuzzy proposition. Composite fuzzy propositions may be generated in terms of atomic fuzzy propositions using fuzzy connective. The meaning of the atomic fuzzy proposition is defined by the membership function of elements in the fuzzy set B. The meaning helps us to decide the degree with which the symbolic expression is satisfied, given a specific physical value of the physical variable ,say Pressure.

Although possibility theory is well formulated in terms of fuzzy measures [93], as already pointed out, it is important to examine whether this formulation may also capture the fundamental connection between possibility measure and fuzzy set, which is facilitated by the associated possibility distribution function. To explain this function, let X be a linguistic variable that takes values in a universal set U and let information about the actual value of the variable be expressed as a vague proposition

$$p : X \text{ is } A.$$

To express the above information in measure-theoretic terms, it is natural to interpret the membership degrees $\mu_A(u)$ for each $u \in U$ as the degree of possibility that $X = u$. This interpretation induces a possibility distribution π_X which associates to each generic element u of the universe U of the linguistic variable X, the possibility that u could be a value of X, which is exactly what the fuzzy set A is in this case. Thus the vague proposition p induces a possibility distribution

$$\Pi_X = A$$

such that

$$\pi_X(u) = \mu_A(u) \ ; \ u \in U. \tag{2.17}$$

Example 2.3 : Let $U = \{n \in N \mid 1 \leq n \leq 6\}$ be the universe of a linguistic variable X. The proposition

$$p : X \text{ is close to } 4$$

indicates that any natural number n, $1 \leq n \leq 6$ could possibly be a value of the variable X with varying degrees of possibility as is given below :

Poss $[X < 1] = 0$,

Poss $[X = 1] = 0.25$,

$$\text{Poss}[X=2] = \text{Poss}[X=6] = 0.5,$$
$$\text{Poss}[X=3] = \text{Poss}[X=5] = 0.75,$$
$$\text{Poss}[X=4] = 1,$$
$$\text{Poss}[X>6] = 0.$$

This idea generates the concept of a possibility assignment equation given by

$$\Pi_X = \text{'close to 4'}$$
$$= \{(1,.25),(2,.5),(3,.75),(4,1.0),(5,.75),(6,.5),\}.$$

The simplest and most usual expressions in fuzzy logic are of the form X is A, with the intended meaning : the linguistic variable X takes the linguistic value A, represented by a fuzzy set $\{u, \mu_A(u)\}$ on a certain universal set U. Equivalently, it is said that the fuzzy set A restricts, in a flexible manner, the possible values the variable X may take. In terms of possibility distributions, this is expressed as by saying that the assertion of the above fuzzy expression induces the following inequality :

$$\pi_X(u) \leq \mu_A(u)$$

where π_X stands for the possibility distribution corresponding to the variable X. If there be no known constraint else, on π_X, the principle of minimum specificity would assign to π_X the least specific (,the biggest) possibility distribution satisfying the above constraint, i.e.,

$$\pi_X(u) = \mu_A(u), \forall u \in U.$$

This is the so called possibility assignment principle.

In Fuzzy logic, elementary expressions may involve a multidimensional linguistic variable Z, for instance, '$Z$is R', where Z denotes the vector of variables (X,Y) and R is a fuzzy relation on the product space $U \times V$. V being the universal set for Y. Thus, such an expression induces the constraint

$$\pi_{(X,Y)}(u,v) \leq \mu_R(u,v)$$

on the joint possibility distribution of the variables X and Y.

Within this possibilistic view of fuzzy inference, constraint propagation is carried out almost exclusively by three kinds of operations : extension, combination and projection. The operation of extension allows us to extend a constraint on a marginal possibility distribution to a constraint on a joint possibility distribution.

The operation of combination allows us to combine two distributions of a particular variable to a new distribution on the same variable. The projection operation allows us to get constraints on respective marginal distributions from their joint distributions.

4.1 Principle of minimum specificity

If Π_X and $\Pi_X{}'$ are such that $\Pi_X < \Pi_X{}'$ then Π_X is said to be more specific than $\Pi_X{}'$ in the sense that $\Pi_X{}'$ gives no better information than Π_X and hence in such cases, the distribution $\Pi_X{}'$ may be discarded [24]. The principle of minimum specificity says that if a piece of information comes from several sources, the possibility distribution is the least specific possibility distribution that satisfies the constraint induced by the pieces of information given by different sources. Thus if Π_X^1, Π_X^2, $\cdots$ be the available possibility distributions for X then the principle of minimum specificity asserts that

$$\Pi_X = \min_i \Pi_X^i. \tag{2.18}$$

4.2 Conjunction principle

Let p and q be two typical propositions whose translations are expressed through possibility assignment equations as

$$p \to \Pi_{X_1,X_2,\cdots,X_n} = S \quad \text{and}$$
$$q \to \Pi_{X_1,X_2,\cdots,X_m,Y_1,Y_2,\cdots,Y_k} = T$$
$$(m \leq n; m, n, k < \infty)$$

Here S and T are fuzzy relations. Let U_i $(1 \leq i \leq n)$ and V_j $(1 \leq j \leq k)$ be the universe of discourse of the respective linguistic variables. Set

$$X = (X_1, X_2, \cdots, X_n);$$
$$X' = (X_1, X_2, \cdots, X_m);$$
$$Y = (Y_1, Y_2, \cdots, Y_k).$$

Then the conjunction principle asserts that [113] r may be inferred from p and q according to the following scheme :

$$p \to \Pi_X = S(, say)$$
$$q \to \Pi_{(X',Y)} = T(, say)$$
$$r \leftarrow \Pi_{(X,Y)} = \overline{S} \cap \overline{T}$$

where $\bar{S}$ and $\bar{T}$ respectively denote the cylindrical extension of S and T over $U_1 \times U_2 \times$

$U_n \times V_1 \times V_2 \times \cdots \times V_k$ which actually produces the greatest possibility distribution on the said domain, in accordance with the principle of minimum specificity.

4.3 Projection principle

Let p be a vague proposition whose translation is expressed as the possibility assignment equation

$$p \rightarrow \Pi_{(X_1, X_2, \cdots, X_n)} = F \tag{2.19}$$

where F is a fuzzy relation and let $X' = (X_1, X_2, \cdots, X_m)$; denote a sub-variable $(m < n)$. To preserve the order of appearance in case $m < n$ it may require renaming and rearranging. The projection principle asserts that r may be inferred from p according to [113]

$$p \rightarrow \Pi_{(X_1, X_2, \cdots, X_n)} = F$$
$$r \leftarrow \Pi_{(X_{m+1}, \cdots, X_n)} = Proj_{U_1 \times \cdots \times U_m} F$$

such that $\pi_{X'}(u_{m+1}, \cdots, u_n) = \sup_{u_1, \cdots, u_m} \mu_F(u_1, \cdots, u_n)$.

4.4 Cylindrical extension principle

The cylindrical extension operation is more or less the inverse operation of the projection operation. It extends fuzzy sets to fuzzy relations. Let R be a fuzzy relation on $U = U_1 \times \cdots \times U_m$. Let for $n > m$, $V = V_1 \times \cdots \times V_n$.

<u>*Definition 2.10*</u> **:** The cylindrical extension of R into V will be a fuzzy relation S defined by

$$\mu_S(v_1, \cdots, v_n) = \mu_R(v_1, \cdots, v_m). \tag{2.20}$$

Thus the cylindrical extension clearly produces the largest fuzzy relation that is compatible with the given projection. Such a relation is the least specific of all relation compatible with the projection.

5 Different models of approximate reasoning

Since the first formal description of fuzzy reasoning by Zadeh, many researchers have discussed different forms of fuzzy reasoning. Mizumoto presented a possible extension of fuzzy reasoning in [57]. In ordinary fuzzy reasoning, as suggested

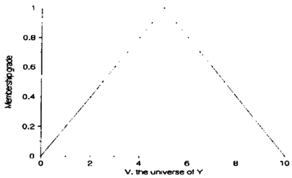

Figure 2.2: The fuzzy set B

by Zadeh [114], we make inferences of the form presented in Table 2.1. Here the variables X, Y take values in universes of discourse U, V respectively and A, A', B, B' are (possibly vague) descriptions of the linguistic variables X and Y, which are approximated by fuzzy sets over their respective universal sets. We may deduce the conclusion B' from premises p and q according to the following scheme. First we translate the conditional statement as a fuzzy implication relation and then use max-min composition($\circ$) of the fuzzy set A' with the fuzzy relation $A \to B$. Thus, we have

$$B' = A' \circ (A \to B) \tag{2.21}$$

where

$$\mu_{B'}(v) = \sup_{u \in U}\{\mu_{A'}(u) \wedge \mu_{A \to B}(u,v)\}; \quad \wedge \quad \text{is a min-operator.} \tag{2.22}$$

A pictorial description of the above may be given as in the following [69]. Let us

p :	X is A	then	Y is B
q :	X is A'		
r :			Y is B'.

Table 2.1: Model 1 : Rule-based ordinary approximate reasoning

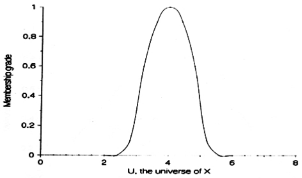

Figure 2.3: The fuzzy set A'

take A, B and A' to be normalized fuzzy sets and define them as:

$$\mu_A(x) = \begin{cases} 1 - \frac{1}{4} \mid x - 4 \mid, & \text{if } 0 \leq \mid x - 4 \mid \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_B(y) = \begin{cases} 1 - \frac{1}{5} \mid y - 5 \mid, & \text{if } 0 \leq \mid y - 5 \mid \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_{A'}(x) = \begin{cases} 1 - \frac{1}{16}(x - 4)^2, & \text{if } 0 \leq \mid x - 4 \mid \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

The support of each fuzzy set is uniformly quantized into 20 levels. Figure 2.1 depicts the representation of the fuzzy sets A, the antecedent of the conditional rule. Figure 2.2 corresponds to the representation of the fuzzy set B, the consequent part of the conditional statement p. The fuzzy set A' is shown in Figure 2.3. The conclusion using Zadeh's compositional rule of inference has been presented in Figure 2.4. Figure 2.5 gives a pictorial representation of the relational matrix $R(A, B)$ using the min-rule for the translation of the conditional statement. The cylindrical extension of the fuzzy set A' over the entire space $U \times V$ is shown in Figure 2.6. In Figure 2.7, a pictorial representation of the conjunction of the two relations have been shown.

In extended fuzzy reasoning, as suggested by Mizumoto, we make inferences of the form presented in Table 2.2. where X_i's are variables taking values in universes of

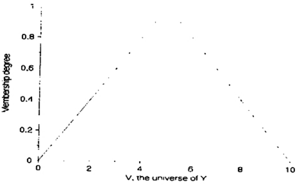

Figure 2.4: The inferred fuzzy set B'

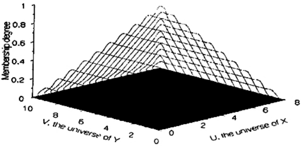

Figure 2.5: The fuzzy conditional relation S

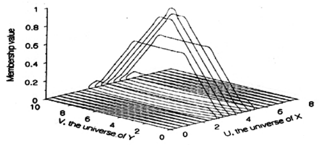

Figure 2.6: The cylindrical relation R, induced by A'

discourse U_i, $i = 1, 2, \cdots, n$. B and B' are fuzzy subsets of V. We use this second form of reasoning in fuzzy control problems. One possible way to obtain the conclusion B' is the application of max-min compositional rule. Here, we first translate the premise p into a fuzzy relation between the inputs(,rule-antecedents) and the output(,the consequent). There are different ways to obtain such a relation. Then we translate the second premise q into another fuzzy relation between the input variables and cylindrically extend the same over the product space of the input and output variables. We then compose these two relations using the (min)conjunction principle and (max)project it over the universe(,V) of the output variable, for the desired conclusion.

Here, in order to obtain a conclusion, we need prior information about all the variables, appearing in the body of the rule. But in controlling a real plant/system, it has often been found that all information regarding the firing of the rule may

p	:	if	X_1 is A_1	and	X_2 is A_2	and	$\cdots$	X_n is A_n	then	Y is B
q	:		X_1 is A'_1	and	X_2 is A'_2	and	$\cdots$	X_n is A'_n		
r	$\leftarrow$							Y is B'		

Table 2.2: Model 2 : Rule-based extended approximate reasoning

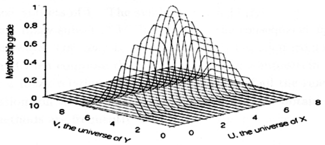

Figure 2.7: The compositional relation between S and R

not be explicitly available. For instance, all state parameters of a rule-based output feedback design may not be field measurable(observable). Hence a generalization of the said form of reasoning have been proposed in [71].

In [71], the reasoning process is carried out as in the following. The rule p is first translated as an implication relation R, between the antecedent linguistic variables X_i; $i = 1, 2, \cdots, n$ and the consequent linguistic variable Y, in the form $R = A \rightarrow B$ where A is defined by

$$A = \bigcap_{i=1}^{n} A_i$$

which is the point-valued fuzzy set determined as a combination of the fuzzy sets $A_i (i = 1, 2, \cdots, n)$. Then we translate the premise q into another fuzzy relation A', defined over the product space of the input variables and then apply compositional rule of inference according to Zadeh's sense to obtain

$$B' = A' \circ R$$

where $\circ$ is a compositional operator and A' is defined by

$$A' = \bigcap_{i=1}^{m} A_i'$$

in a similar manner as above.

As an applicable form of approximate reasoning, let us consider a generalized form, mostly used in rule-based systems as in Table 2.3. There are m rules in n linguistic variables, where X_i's are variables taking values in universes of discourse U_i, $i = \{1, 2, \cdots, n\}$. For each value of $j = \{1, 2, \cdots, m\}$, A^i_j is a fuzzy subset of U^i and B^i's are fuzzy subsets of V. The symbols $A_i, i = \{1, 2, \cdots, n\}$ are n fuzzy subsets of U_i and B is a fuzzy subset of V. In this case, the consequence may be obtained in two different ways. One way is to compute the conclusion from each rule with the premise q and then compose them by some rule of composition. The other way is to compute a fuzzy relation from the combination of all the rules (each of them is a fuzzy relation) on the product space and then use q to obtain the desired result. Both the methods are frequently used in rule-based systems.

Let us now consider two different models for reasoning with vague propositions. The two models are presented in Table 2.4 and Table 2.5. Detail work on these models may be found in [70]. The motivation for this form of approximate reasoning is to handle disjunctive form of vague knowledge in the framework of approximate reasoning methodology. This may be used in deriving resolvents in fuzzy logic. Here, the consequence r may be taken as a fuzzy resolvent of the two formulae p and q. For a resolvent of the form r we first translate premise p into a fuzzy binary relation(,say R_1), possibly using a T-conorm operator. Then we perform cylindrical extension of A' over $U \times V$. Let it be denoted as R_2. We then compose R_1 and R_2 to generate one binary fuzzy relation (R) on $U \times V$. Since, in generating the fuzzy relation we use T-conorm operator, it would reasonable here to choose some conjunction operation for a meaningful consequence. Thus we perform inf-s combination operation on R to find the desired fuzzy set B' according to (2.23)

$$\mu_{B'}(v) = \inf_{u \in U} \mu_R(u, v). \tag{2.23}$$

In Table 2.5, we find a generalised model for approximate reasoning based on dis-

p1 :	if	X_1 is A^1_1	and	X_2 is A^2_1	and	$\cdots$	X_n is A^n_1	then	Y is B^1
p2 :	else if	X_1 is A^1_2	and	X_2 is A^2_2	and	$\cdots$	X_n is A^n_2	then	Y is B^2
	$\vdots$		$\vdots$		$\vdots$				
pm :	else if	X_1 is A^1_m	and	X_2 is A^2_m	and	$\cdots$	X_n is A^n_m	then	Y is B^m
q :		X_1 is A_1	and	X_2 is A_2	and	$\cdots$	X_n is A_n		
r ←									Y is B

Table 2.3: Model 3 : Applicable form of approximate reasoning

p :	X is A	or	Y is B	
q :	X is A'			
r ←			Y is B'.	

Table 2.4: Model 4 : Resolution-based ordinary approximate reasoning

p :	X_1 is A_1	or	X_2 is A_2	or	$\cdots$	X_n is A_n
q :	X_1 is A'_1					
r :			X_2 is A'_2	or	$\cdots$	X_k is A'_k

Table 2.5: Model 5 : Resolution-based extended approximate reasoning

junctive knowledge. Here X_i's are n linguistic variables taking values in universes of discourse U_i, $i = 1, 2, \cdots, n$. For a resolvent of the form r as given in Table 2.5 we first translate premise p into a fuzzy binary relation (,say R_1), possibly using a T-conorm operator. Then we perform cylindrical extension of A'_1 over $U_1 \times U_2 \times \cdots \times U_n$. Let it be denoted as R_2. Then we compose R_1 and R_2 to generate one binary fuzzy relation (R) on $U_1 \times U_2 \times \cdots \times U_n$. Then we perform inf-s combination on R separately on $U_i; i = 2, 3, \cdots, n$ to find the desired conclusion according to (2.24)

$$\mu_{A'_i}(u) = \inf_{u_j \in U_j : j \neq i} \mu_R(u_1, u_2 \cdots, u_n). \tag{2.24}$$

Chapter 3

Similarity Indices

1 Introduction

The similarity of two objects suggest the degree to which properties of one may be inferred from those of the other. The notion of similarity plays a fundamental role in theories of knowledge and behaviour and has been dealt with broadly in the psychology literature [34]. There are different ways to define the similarity between two fuzzy sets. Accordingly, they are classified. The most important class of distance function is the Minkowski's r-metric, defined as follows :

$$d_r(x,y) = \left[\sum_{i=1}^{n} | x_i - y_i |^r\right]^{\frac{1}{r}}, \quad r \geq 1 \tag{3.1}$$

where x and y are two points in an n-dimensional space with components (x_i, y_i), $i = 1, 2, \cdots, n$. The well known Euclidean metric corresponds to the case of $r = 2$. The three distance functions, corresponding to $r = 1, 2$ and ∞, are mostly used in psychological theory [121].

Another important class of distance function is the Hausdorff metric. It is a generalisation of the distance between two points in a metric space to two compact non-empty subsets of the space. If, U and V are such compact non-empty sets of real numbers then Hausdorff distance is defined as :

$$d'(U,V) = \max\left[\sup_{v\in V} \inf_{u\in U} d_2(u,v), \sup_{u\in U} \inf_{v\in V} d_2(u,v)\right] \tag{3.2}$$

where d_2 is defined as in (3.1).

Different generalisation of the Hausdorff metric have been proposed by many researchers. A similarity matching degree may be defined from the distance functions according to the following :

$$s(\bullet, \bullet) = 1 - d(\bullet, \bullet). \tag{3.3}$$

A set theoretic approach to a family of similarity functions may also be given by

$$s(A, B) = \theta f(A \cap B) - \alpha f(A - B) - \beta f(B - A) \tag{3.4}$$

for some function f and parameters $\theta, \alpha, \beta \geq 0$ [87]. Typically, the function f is taken to be the cardinality function.

Similarity concept is not new in fuzzy set theory. Several attempts have been made to generalise the classical concepts of similarity, to fuzzy sets. In this chapter, we investigate different lines of research on this aspect and introduce some notion of similarity of fuzzy sets, a measure of 'indistinguishability'(,'sameness'), defined over the same universe of discourse.

Approximation is inherent in fuzzy set theory. It has been noted that small deviations from what might be considered as 'precise membership values' should normally be considered of no practical significance [107]. On the other hand, approximation is implied when considering, for example, the multitude of solutions of the inverse problem :

Given a fuzzy relation R on $U \times V$ and a fuzzy subset B of V, to find all fuzzy subsets A of U such that $A \circ R = B$, $\circ$ being the maximin composition.

A solution to the above problem clearly indicate the need for similarity matching and some form of analogy in reasoning. In such cases, we may find similar concepts within the current context and come up with some conclusion based on their similarity. The confidence of the conclusion may be determined based on the degree of similarity.

The concepts of similarity and proximity of fuzzy sets plays a fundamental role in reasoning with vague knowledge [74, 75, 121]. The authors in [84] commented and showed that the notion of a similarity measure between two fuzzy sets may be successfully applied in fuzzy reasoning. Recently, similarity based approximate reasoning mechanisms are being applied to pattern recognition, process control and many other areas [75].

Here, we assume that the universe of discourse is a finite set. In the process we also discuss similarity measure between two countable infinite fuzzy sets. Let

$$A = \sum_{u \in U}\{\mu_A(u)/u\} \text{ and } B = \sum_{u \in U}\{\mu_B(u)/u\}.$$

be two fuzzy sets defined over the universe of discourse U. A similarity index between the pair $\{A, B\}$ is denoted as $S(A, B; U)$ or simply $S(A, B)$. In the following, a number of existing similarity measures are listed from the literature. The properties of several measures of similarity are studied.

2 A brief review

Similarity indices in a fuzzy set theoretic framework may be found in the work of Dubois and Prade [22]. There, it is expected that, besides being symmetric, $S(A, B)$ should also satisfy the following properties :

$$S(A, B) = 1 \text{ iff } A\nabla B = \Phi \text{ where } \mu_{A\nabla B}(x) = D(\mu_A(x), \mu_B(x)) \tag{3.5}$$

and

$$D(A, B) = \min[\max(a, b), \max(1-a, 1-b)] \tag{3.6}$$
$$= \max[\min(1-a, b), \min(a, 1-b)] \quad 0 \le a, b \le 1. \tag{3.7}$$

If A and B have disjoint support then $S(A, B) = 0$. Again, $S(A, B)$ depends on a scalar evaluation of $\overline{A\nabla B}$. With such indices, it was claimed that, fuzzy pattern (mis)matching problem may be handled efficiently. The following indexes have been proposed in [18] as dissimilarity measures between fuzzy sets :

$$d_1(A, B) = 1 - \frac{|A \cap B|}{|A \cup B|}; \tag{3.8}$$

$$d_2(A, B) = |A \circ B|; \tag{3.9}$$

$$d_3(A, B) = \sup_{u \in U} \mu_{A \circ B}(u); \tag{3.10}$$

where

$$\forall u \in U, \mu_{A\circ B}(u) = \max[\min(\mu_A(u), 1-\mu_B(u)), \min(1-\mu_A(u), \mu_B(u)] \tag{3.11}$$

and finally a disconsistency index as

$$d_4(A, B) = 1 - \sup_{u \in U} \mu_{A \cap B}(u). \tag{3.12}$$

In [22], out of S they have also defined a measure of non-dissimilarity as $d(\bar{A}, \bar{B})$ and concluded that 'usually with fuzzy sets $S(A, B)$' and $d(\bar{A}, \bar{B})$ do not convey the same information and accordingly, similarity and non-dissimilarity are not synonymous' [22].

The measure as given in (3.12) may be used for countable infinite fuzzy sets also.

A measure of similarity of fuzzy sets may also be given by the formula of neighbourhoodness of two points as

$$S(A,B) = \frac{\sum_{u \in U} \{\mu_A(u).\mu_B(u)\}}{\max\{\sum_{u \in U} \mu_A^2(u), \sum_{u \in U} \mu_B^2(u)\}}. \tag{3.13}$$

Work on this measure may be found in [9]. It was claimed that such a measure of likeness of fuzzy sets may be useful in fuzzy decision making problem.

A family of measures of equality of fuzzy sets having a strong logical background may be given as in the following :

$$S(A,B) = \frac{1}{2}[(A \leftrightarrow B) + (\bar{A} \leftrightarrow \bar{B})] \tag{3.14}$$

where $(A \leftrightarrow B) = (A \to B) \wedge (B \to A)$, $\wedge$ is a conjunction operator and $\to$ is an implication operator. Different interpretation of the operators will result in different measures of similarity between fuzzy sets. More work on this measures may be found in [62]. These definitions may also be used for countable infinite fuzzy sets. By a proper choice of implication and conjunction operation we may also make them compuationally simple.

A simple modification of the said measure result in another measure of similarity of fuzzy sets with finite support as follows :

$$S(A,B) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}[(\mu_A(u) \leftrightarrow \mu_B(u)) + (\mu_{\bar{A}}(u) \leftrightarrow \mu_{\bar{B}}(u))] \tag{3.15}$$

where

$$(\mu_A(u) \leftrightarrow \mu_B(u)) = (\mu_A(u) \to \mu_B(u)) \wedge (\mu_B(u) \to \mu_A(u)), \tag{3.16}$$

n being the cardinality of the universal set U, $\wedge$ is a conjunction operator and $\to$ is an implication operator. A thorough exposition on the same may be found in [6]. The authors used this measure to model bi-directional approximate reasoning through an inference network.

Let us now consider a measure of similarity of fuzzy sets based on maximum difference of corresponding grades of membership values as in the following :

$$S(A, B) = 1 - \max_{u \in U}(\mid \mu_A(u) - \mu_B(u) \mid). \tag{3.17}$$

A measure of similarity of fuzzy sets based on the difference and sum of corresponding grades of membership values may be given as in the following :

$$S(A, B) = 1 - \frac{\sum_{u \in U} \mid (\mu_A(u) - \mu_B(u)) \mid}{\sum_{u \in U} (\mu_A(u) + \mu_B(u))}. \tag{3.18}$$

A measure of similarity of fuzzy sets based on the operations of union and intersection of corresponding grades of membership values may be given as :

$$S(A, B) = \frac{\sum_{u \in U} \min(\mu_A(u), \mu_B(u))}{\sum_{u \in U} \max(\mu_A(u), \mu_B(u))}. \tag{3.19}$$

Here, 'min' and 'max' are used for the intersection and union operations respectively. All these three measures may be used for countable infinite fuzzy sets. A detailed work on these measures may be found in [61].

A modification of the measures given in (3.15) and (3.17) may also produce two other measures of similarity between fuzzy sets having finite support, based on comparison of membership values of corresponding elements in the fuzzy sets as

$$S(A, B) = \frac{1}{n} \sum_{i=1}^{n} \frac{\min(\mu_A(u), \mu_B(u))}{\max(\mu_A(u), \mu_B(u))} \tag{3.20}$$

here n, a finite number, is supposed to be the cardinality of the universal set U and

$$S(A, B) = \frac{1}{n} \sum_{i=1}^{n} [1 - \mid \mu_A(u) - \mu_B(u) \mid] \tag{3.21}$$

In [94], the authors claim that these measures give more reliable results.

A good working measure of similarity between two countable infinite fuzzy sets may be given as in the following :

$$S(A, B) = 1 - \sup_{u \in U} \mid \mu_A(u) - \mu_B(u) \mid . \tag{3.22}$$

It is easy to see that this may be an effective measure if we consider an infinite fuzzy set. But, the problem in working with such a measure is that, it gives importance to the 'sup'-operation only. Work in this direction may be found in [40].

Another similarity measure, between two finite fuzzy sets, may be defined as follows:

$$S(A, B) = \max_{u \in U}\{\min(\mu_A(u), \mu_B(u))\}. \quad (3.23)$$

Properties like reflexivity, symmetry, normalization, boundedness and dissimilarity between fuzzy sets based on this measure have been studied in [49]. The authors also claimed that their measure may be found to be useful in behaviour analysis.

Several other measures of similarity of fuzzy sets have been suggested by different researchers at different instances of time and for different purposes and have been claimed to be useful.

In this regard, in [94], a definition of correction coefficient has been provided. The correction coefficient of two fuzzy sets A and B is denoted as $k(A, B)$ and is defined as :

$$k(A, B) = \frac{C(A, B)}{\sqrt{T(A).T(B)}} \quad (3.24)$$

where

$$T(A) = \sum_{i=1}^{n}\left[(\mu_A^2(x_i).\nu_A^2(x_i)\right]; \nu_A(x_i) = 1 - \mu_A(x_i) \quad (3.25)$$

and

$$C(A, B) = \sum_{i=1}^{n}\left[(\mu_A(x_i).\mu_B(x_i) + (\nu_A(x_i).\nu_B(x_i)\right]. \quad (3.26)$$

Now, Once an index is defined -

How can we compare this with other existing indices?

How should we judge the goodness of such an index?

Questions of this nature carry immense importance for all practical purposes. In the following, we attempt to provide a comparative study on the same.

3 Comparative study

In this regard, the authors in [121] have reviewed different similarity measures, as suggested in the literature in the general case and as adapted to fuzzy sets. They have also presented an experimental design for linguistic approximation and discussed at length the suitability of application of different measures of similarity.

In [94], the authors presented a comparative study on the basis of a set of axioms. There, it has been found that the performance of the similarity index given by

(3.23) is not at all satisfactory. They have also investigated some similarities and dissimilarities in performance.

First of all, we see that all the similarity measures listed in the previous section satisfy the reflexivity, symmetry and boundedness property. These three properties are a must for any similarity measures. In this regard, all measures are equally well. Besides these three properties, similarity measures should also satisfy properties like computational simplicity, monotonicity and non-dissimilarity. These are some desirable properties.

Similarity measures based on the computation of overall sup-operation as well as max between elements are such that they give more importance to a particular value and ignore the presence of others. Thus, two fuzzy sets are often found to be similar when they have the same sup and/or max.

Of course, we may define two fuzzy sets to be similar as and when they have the same cardinality or they have the same support. This may work for mathematical theory construction. But in order assist the decision-maker in the real life situation, the practical meaning of similarity concept is of vital importance. We are considering those indices that play a crucial role in the theory of fuzzy reasoning. This demands similarity measures based on a comparison of membership degrees of each concept.

Thus the measure defined in eqn (3.13) is workable in practical situations. A major drawback underlying the said measure is that $S(A, B)$ does not, in general, satisfy any monotonicity criterion. Again, here, measure of similarity does not indicate the measure of non-dissimilarity.

Next, let us consider the similarity measure defined as in (3.22). In order to illustrate the drawback underlying it, let us consider a simple example as in the following :

$$\mu_A(u) = 1 \ \forall u \in U \text{ and} \tag{3.27}$$

$$\begin{aligned} \mu_B(u_0) &= 0 \text{ for a particular } u_0 \in U \text{ and} \\ &= 1 \text{ otherwise.} \end{aligned} \tag{3.28}$$

Even in such an almost similar pair of fuzzy sets it is found that the similarity index is 0, showing thereby that they are completely dissimilar.

The measure proposed in (3.24) is an appropriate measure. Only problem in working with this measure arises from the fact that $S(A, B) = 0$ does not imply that A and B are disjoint fuzzy sets. In this case, we find that $A = 1 - B$.

Now, we may safely conclude that, it is practically impossible to single out one possible similarity measure that works well for all purpose. It is intended in the present thesis to provide the user with certain measures of similarity each of which, satisfies certain basic needs of being a measure of similarity.

In the following section, we present some definition of similarity between fuzzy sets, defined over the same universe of discourse. Some important deductions are also provided to illustrate the proposed measure's soundness.

4 Proposed measure - definition and properties

In order to provide a definition for similarity index [74], a number of factors must be considered. A primary consideration is that, whatever way we choose to define such an index, it must satisfy the properties as already mentioned. Similarity measures when expressed through pure numbers are found to be non-transitive.

Keeping all these in mind, we expect that, a similarity measure $S(A, B)$ should satisfy the following properties.

For all fuzzy sets A , B

P1. $S(B, A) \;=\; S(A, B)$.

P2. $S(A^c, B^c) = S(A, B)$, A^c being some negation of A.

P3. $0 \leq S(A, B) \leq 1$.

P4. $A = B$ iff $S(A, B) \;=\; 1$.

P5. if $S(A, B) = 0$ then either $A \cap B = \Phi$ (null); or $A^c \cap B^c = \Phi$; or $B = 1 - A$.

For $0 \leq \epsilon \leq 1$, if $S(A, B) \geq \epsilon$, we say that the two fuzzy sets A and B are ϵ-similar. Thus, the case for $\epsilon = 1$ correspond to equality of fuzzy sets. There may be many functions satisfying properties **P1** through **P5**. One such measure of similarity satisfying properties **P1** through **P5** is given in *Definition 3.1*.

Definition 3.1 : Let A and B be two fuzzy sets defined over the universe of discourse U. The similarity index of the pair $\{A, B\}$ is defined by

$$S(A, B) \;=\; \min\; \{\alpha(A, B),\; \alpha(A^c, B^c)\} \qquad (3.29)$$

where

$$\alpha(A,B) = \left\{ \frac{\sum_{u \in U} \{\mu_A(u).\mu_B(u)\}}{\sum_{u \in U} \{\max(\mu_A(u), \mu_B(u))\}^2} \right\}^{\frac{1}{2}} \tag{3.30}$$

$$\text{and } \mu_{A^c}(u) = 1 - \mu_A(u).$$

In case, $\sum_{u \in U} \{\max(\mu_A(u), \mu_B(u))\}^2 = 0$ we find that, A and B are null fuzzy sets, and we set

$$\alpha(A,B) = 1 = S(A,B).$$

When A, B are binary fuzzy relations the summations in (3.30) may be extended to all possible elements of the associated composite universe. Thus if,

$$A = \sum_i \sum_j \{\mu_A(u_{ij})/u_{ij}\}$$

$$\text{and } B = \sum_i \sum_j \{\mu_B(u_{ij})/u_{ij}\} \; ; \; u_{ij} = (u_i, v_j) \in U \times U$$

are two fuzzy binary relations, then we define

$$S(A,B) = \min \{\alpha(A,B), \; \alpha(A^c, B^c)\}$$

where

$$\alpha(A,B) = \left\{ \frac{\sum_i \sum_j \{\mu_A(u_{ij}).\mu_B(u_{ij})\}}{\sum_i \sum_j \{\max(\mu_A(u_{ij}), \mu_B(u_{ij}))\}^2} \right\}^{\frac{1}{2}}$$

$$\text{and } \mu_{A^c}(u_{ij}) = 1 - \mu_A(u_{ij}).$$

It is easy to see that properties **P1** and **P2** are satisfied by *Definition 3.1.* The other properties may be verified as follows:

Verification of P3.

Let A and B be two fuzzy sets defined on the same universe of discourse U. It is known that, $0 \le \mu_A(u), 0 \le \mu_B(u)$. If

$$\sum_{u \in U} [\max\{\mu_A(u), \mu_B(u)\}]^2 = 0,$$

then $\forall u \in U$, $\mu_A(u) = \mu_B(u) = 0$ and hence $\alpha(A,B) = 1$. Also in this case, $1 - \mu_A(u) = 1 - \mu_B(u) = 1$. So that $\alpha(A^c, B^c) = 1$. Therefore, $S(A,B) = 1$.

Otherwise, $\exists u \in U$, such that $\max\{\mu_A(u), \mu_B(u)\} > 0$, making $\alpha(A,B) \ge 0$. Similarly, $\alpha(A^c, B^c) \ge 0$. Therefore, $S(A,B) \ge 0$.

Again $\forall u \in U$,

$$\mu_A(u) \leq \max\{\mu_A(u), \mu_B(u)\}$$
$$\text{and} \;\; \mu_B(u) \leq \max\{\mu_A(u), \mu_B(u)\}$$

will imply at once that

$$\sum_{u \in U} \{\mu_A(u).\mu_B(u)\} \leq \sum_{u \in U} [\max\{\mu_A(u), \mu_B(u)\}^2],$$

i.e., $\alpha(A, B) \leq 1$. Similarly, $\alpha(A^c, B^c) \leq 1$ and hence $S(A, B) \leq 1$. Thus the proposition

$$0 \leq S(A, B) \leq 1$$

is valid.

Verification of P4.

Let us suppose that $S(A, B) = 1$ i.e.,

$$\min\{\alpha(A, B), \alpha(A^c, B^c)\} = 1. \tag{3.31}$$

By P3, since

$$\alpha(A, B) \leq 1 \;\text{ and }\; \alpha(A^c, B^c) \leq 1,$$

it follows from (3.31) that,

$$\alpha(A, B) = \alpha(A^c, B^c) = 1.$$

Now, $\alpha(A, B) = 1$ will imply that,

$$\sum_{u \in U} \{\mu_A(u).\mu_B(u)\} = \sum_{u \in U} [\max\{\mu_A(u), \mu_B(u)\}]^2. \tag{3.32}$$

Again,

$$\mu_A(u) \leq \max\{\mu_A(u), \mu_B(u)\}$$
$$\text{and } \mu_B(u) \leq \max\{\mu_A(u), \mu_B(u)\}$$

i.e.,

$$\mu_A(u).\mu_B(u) \leq [\max\{\mu_A(u), \mu_B(u)\}]^2; \;\; \forall u \in U. \tag{3.33}$$

Each term on either side of the inequality (3.33) is non-negative. For inequation (3.33) to be true, we must have

$$\mu_A(u).\mu_B(u) = [\max\{\mu_A(u), \mu_B(u)\}]^2. \tag{3.34}$$

Otherwise, if for a particular $u \in U$

$$\mu_A(u).\mu_B(u) < [\max\{\mu_A(u), \mu_B(u)\}]^2$$

then $\alpha(A, B)$ would become less than unity, which is not true. Hence,

$$\mu_A(u).\mu_B(u) = [\max\{\mu_A(u), \mu_B(u)\}]^2; \ \forall u \in U. \quad (3.35)$$

If now $\mu_A(u) \geq \mu_B(u)$ then we find that

$$\mu_A(u).\mu_B(u) = \{\mu_A(u)\}^2. \quad (3.36)$$

If $\mu_A(u) = 0$ then from the given condition we find that $\mu_B(u) = \mu_A(u) = 0$; $\forall u \in U$.

Otherwise, cancelling $\mu_A(u)$ from both sides of (3.36), we find that $\mu_B(u) = \mu_A(u)$.

$$\Rightarrow \mu_B(u) = \mu_A(u) \neq 0$$

$$\text{or } \mu_B(u) = \mu_A(u) = 0.$$

So that, in any case, it is true that $B = A$.

If, on the other hand, $\mu_B(u) \geq \mu_A(u)$ then from (3.35) we find that

$$\mu_A(u).\mu_B(u) = \{\mu_B(u)\}^2 \quad (3.37)$$

In this case also,

$$\text{either } \mu_A(u) = \mu_B(u) \neq 0$$

$$\text{or } \mu_A(u) = \mu_B(u) = 0.$$

Hence the proposition $B = A$ is valid.

If, we suppose that two fuzzy sets A and B are such that $A = B$ then from the *Definition 3.1* it follows at once that $S(A, B) = 1$.

Verification of P5.

Here it is given that $S(A, B) = 0$. By *Definition 3.1*, it follows that,

$$\min\{\alpha(A, B), \alpha(A^c, B^c)\} = 0,$$

i.e., either or both of $\alpha(A, B) = 0$ or $\alpha(A^c, B^c) = 0$ or both. Let us suppose that $\alpha(A, B) = 0$, i.e.,

$$\sum_{u \in U} (\mu_A(u).\mu_B(u)) = 0. \quad (3.38)$$

$$\text{Since, } \mu_A(u).\mu_B(u) \geq 0 \; : \; \forall u \in U$$

$$\text{therefore, (3.38)} \quad \Rightarrow \quad \mu_A(u).\mu_B(u) = 0 \; : \; \forall u \in U,$$

$$\text{i.e., } \min\{\mu_A(u), \mu_B(u)\} = 0 \; : \; \forall u \in U,$$

$$\text{i.e., } A \cap B = \Phi.$$

On the other hand, if $\alpha(A^c, B^c) = 0$, then

$$\sum_{u \subset U} \{(1 - \mu_A(u)).(1 - \mu_B(u))\} = 0. \tag{3.39}$$

Since each term under the summation is non-negative

$$\text{(3.39)} \quad \Rightarrow (1 - \mu_A(u)).(1 - \mu_B(u)) = 0 \; : \; \forall u \in U,$$

$$\text{i.e., } \min\{(1 - \mu_A(u)), (1 - \mu_B(u))\} = 0 \; ; \; \forall u \in U,$$

$$\text{i.e., } A^c \cap B^c = \Phi, \text{ the NULL set.}$$

And when both are zero then

$$\mu_A(u).\mu_B(u) = 0 \; ; \; \forall u \in U,$$

$$\text{as well as } (1 - \mu_A(u)).(1 - \mu_B(u)) = 0 \; ; \; \forall u \in U.$$

$$\text{Now if } \mu_A(u) = 0 \text{ then } \mu_B(u) = 1$$

$$\text{else if } \mu_B(u) = 0 \text{ then } \mu_A(u) = 1,$$

$$\text{i.e., } B = 1 - A.$$

Hence the above proposition is valid.

Example 3.1 : Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and

$$\begin{aligned} A &= .1/u_1 + .3/u_2 + .5/u_3 + .75/u_4 + 1/u_5; \\ B &= .01/u_1 + .09/u_2 + .25/u_3 + .5625/u_4 + 1/u_5; \\ C &= .316/u_1 + .55/u_2 + .707/u_3 + .866/u_4 + 1/u_5 \end{aligned}$$

be defined as above. Then,

$$\alpha(A, B) \;=\; \underline{0.82346405} \; (approximately).$$

Similarly,

$$\alpha(A^c, B^c) \;=\; \alpha(1 - A, 1 - B) \;=\; \underline{0.78543776} \; (approximately).$$

Therefore,

$$S(A,B) = \underline{0.78543776}\ (\textit{approximately}).$$

Again if C is considered, instead of B, then

$$\alpha(A,C) = \underline{0.82936142}\ (\textit{approximately}).$$

$$\text{Similarly, } \alpha(A^c,C^c) = \alpha(1-A,1-C) = \underline{0.68874419}(\textit{approximately}).$$

$$\text{Therefore, } S(A,C) = \underline{0.68874419}(\textit{approximately}).$$

Although the last property **P5** is a plausible and an intuitively appealing one, it is possible to argue in favour of a stricter condition for which $S(A,B)$ should be zero. Two crisp sets A and B are completely dissimilar only when $A \cap B = \Phi$. If $A \cap B \neq \Phi$, then they have some similarity as A and B have some elements in common, in terms of grade. The similarity between the two increases as the number of elements by which the two sets differ decreases. The similarity becomes maximum (the maximum value may be thought of as 1) when the two sets are identical, i.e.,

$$|A \cap B| = |A| = |B|. \tag{3.40}$$

Here, we consider a direct extension of this concept (3.40) in defining the similarity between fuzzy sets. For two fuzzy sets, it is reasonable to assume, that the similarity should be zero if and only if $A \cap B = \Phi$. Property **P5** may now be reformulated as

P5′. For all fuzzy sets A, B ; $S(A,B) = 0$ iff $A \cap B = \Phi$.

The need, thus, arises to find measures of similarity satisfying properties **P1** *through* **P4** and **P5′**. There could be several such measures, a family of such simple measures may be given by the following *Definition 3.2*.

Definition 3.2 : Let

$$A = \sum_{u \in U} \mu_A(u)/u \quad \text{and} \quad B = \sum_{u \in U} \mu_B(u)/u$$

be two fuzzy sets defined over the same universe of discourse U. The similarity index of the pair $\{A, B\}$ is denoted by $S(A,B)$ and is defined by

$$S(A,B) = 1 - \left(\frac{\sum_u |\mu_A(u) - \mu_B(u)|^q}{n} \right)^{\frac{1}{q}} \tag{3.41}$$

where n is the cardinality of the universe of discourse and q is the family parameter.

From the above *Definition 3.2* it is clear that **P1**, i.e., $S(A, B) = S(B, A)$, is satisfied.

$$\text{Next, } S(A,B) = 1 - \left(\frac{\sum_u | \mu_A(u) - \mu_B(u) |^q}{n} \right)^{\frac{1}{q}}$$

$$= 1 - \left(\frac{\sum_u | (1 - \mu_A(u)) - (1 - \mu_B(u)) |^q}{n} \right)^{\frac{1}{q}}$$

$= S(A^c, B^c)$. Hence property **P2**.

In order to verify property **P3** we proceed as follows :

$$0 \leq | \mu_A(u) - \mu_B(u) | \leq 1$$

i.e., $$0 \leq | \mu_A(u) - \mu_B(u) |^q \leq 1$$

i.e., $$0 \leq \sum_{u=1}^{n} | \mu_A(u) - \mu_B(u) |^q \leq n$$

i.e., $$0 \leq \left(\frac{\sum_{u=1}^{n} | \mu_A(u) - \mu_B(u) |^q}{n} \right) \leq 1$$

i.e., $$0 \leq \left(\frac{\sum_{u=1}^{n} | \mu_A(u) - \mu_B(u) |^q}{n} \right)^{\frac{1}{q}} \leq 1.$$

Therefore, $0 \leq S(A, B) \leq 1$, which is property **P3**.

For **P4**, it is easy to see that

$$S(A,B) = 1 \quad \Leftrightarrow \left(\frac{\sum_u | \mu_A(u) - \mu_B(u) |^q}{n} \right)^{\frac{1}{q}} = 0$$

i.e., $\Leftrightarrow \mu_A(u) - \mu_B(u) = 0 \;\; \forall u$

i.e., $\Leftrightarrow \mu_A(u) = \mu_B(u) \;\; \forall u$

i.e., $\Leftrightarrow A = B$. Hence property **P4** is established.

Note that this also ensures $S(\Phi, \Phi) = 1$. Next, property **P5′** is verified. For that, let us suppose.

$$S(A,B) = 1 - \left(\frac{\sum_u \mid \mu_A(u) - \mu_B(u) \mid^q}{n} \right)^{\frac{1}{q}} = 0.$$

$$\Leftrightarrow \left(\frac{\sum_u \mid \mu_A(u) - \mu_B(u) \mid^q}{n} \right)^{\frac{1}{q}} = 1$$

$$\Leftrightarrow \sum_u \mid \mu_A(u) - \mu_B(u) \mid^q = n$$

$$\Leftrightarrow \mid \mu_A(u) - \mu_B(u) \mid^q = 1 \ \ \forall u$$

$$\Leftrightarrow \mid \mu_A(u) - \mu_B(u) \mid = 1 \ \ \forall u.$$

If $\mu_A(u) \geq \mu_B(u)$ then $\mu_A(u) = 1$ and $\mu_B(u) = 0$; while if $\mu_A(u) \leq \mu_B(u)$ then $\mu_A(u) = 0$, $\mu_B(u) = 1$ and this is true for all u. Thus $S(A,B) = 1$ iff $A \cap B = \Phi$: i.e., when A and B are crisp and $A = B^c$.

Example 3.2 : Let $U = \{u_1, u_2, u_3, u_4, u_5\}$. And

$$\begin{aligned} A &= .1/u_1 + .3/u_2 + .5/u_3 + .75/u_4 + 1/u_5; \\ B &= .01/u_1 + .09/u_2 + .25/u_3 + .5625/u_4 + 1/u_5; \\ C &= .316/u_1 + .55/u_2 + .707/u_3 + .866/u_4 + 1/u_5. \end{aligned}$$

With $q = 2$ it is found that

$$S(A,B) = \underline{0.92257746} \ (approximately).$$

Similarly, if C is considered in place of B then

$$S(A,C) = \underline{0.91864620} \ (approximately).$$

$S(A,B) \geq S(A,C)$ will imply that 'B is at least as close to A as C is close to A'. $S(A,B)$, as given in *Definition 3.2*, is quite sensitive — every change in A or B will be reflected in $S(A,B)$. Next some more property of S is discussed in the following.

Theorem 1. If $S(A,B) = 1$ and $S(B,C) = 1$ then $S(A,C) = 1$.

Proof : $S(A,B) = 1 \Rightarrow A = B$, $S(B,C) = 1 \Rightarrow B = C$
$\Rightarrow A = B = C \Rightarrow S(A,C) = 1$.

Of course, in general, for all fuzzy sets A, B and C the numbers $S(A,B)$ and $S(B,C)$ cannot always determine $S(A,C)$. If some structural arrangement between them may be prescribed then, we may find an estimate for the same as in Theorem 2.

Theorem 2. For all fuzzy sets A, B, C

$$\text{if either } A \subseteq B \subseteq C \text{ or } A \supseteq B \supseteq C$$

$$\text{then } S(A,C) \leq \min\{S(A,B), S(B,C)\}.$$

Proof. : Suppose $A \subseteq B \subseteq C$,

i.e., $\mu_A(u) \leq \mu_B(u) \leq \mu_C(u)$; $\forall\, u \in U$

i.e., $\mu_B(u) - \mu_A(u) \leq \mu_C(u) - \mu_A(u)$; $\forall\, u \in U$

i.e., $|\, \mu_B(u) - \mu_A(u) \,| \leq |\, \mu_C(u) - \mu_A(u) \,|$; $\forall\, u \in U$

i.e., $\frac{1}{n} \sum_{u \in U} |\, \mu_B(u) - \mu_A(u) \,|^q \leq \frac{1}{n} \sum_{u \in U} |\, \mu_C(u) - \mu_A(u) \,|^q$;

i.e., $\left(\frac{1}{n} \sum_{u \in U} |\, \mu_B(u) - \mu_A(u) \,|^q\right)^{\frac{1}{q}} \leq \left(\frac{1}{n} \sum_{u \in U} |\, \mu_C(u) - \mu_A(u) \,|^q\right)^{\frac{1}{q}}$;

i.e., $1 - \left(\frac{1}{n} \sum_{u \in U} |\, \mu_B(u) - \mu_A(u) \,|^q\right)^{\frac{1}{q}} \geq 1 - \left(\frac{1}{n} \sum_{u \in U} |\, \mu_C(u) - \mu_A(u) \,|^q\right)^{\frac{1}{q}}$;

i.e., $S(A,B) \geq S(A,C)$.

Similarly, $S(B,C) \geq S(A,C)$. Hence

$$S(A,C) \leq \min\{S(A,B), S(B,C)\}.$$

On the other hand, Suppose that $A \supseteq B \supseteq C$

i.e., $\mu_A(u) \geq \mu_B(u) \geq \mu_C(u)$; $\forall\, u \in U$

i.e., $\mu_B(u) - \mu_C(u) \leq \mu_A(u) - \mu_C(u)$; $\forall\, u \in U$

i.e., $|\, \mu_B(u) - \mu_C(u) \,| \leq |\, \mu_A(u) - \mu_C(u) \,|$; $\forall\, u \in U$

i.e., $\frac{1}{n} \sum_{u \in U} |\, \mu_B(u) - \mu_C(u) \,|^q \leq \frac{1}{n} \sum_{u \in U} |\, \mu_A(u) - \mu_C(u) \,|^q$;

i.e., $\left(\frac{1}{n} \sum_{u \in U} |\, \mu_B(u) - \mu_C(u) \,|^q\right)^{\frac{1}{q}} \leq \left(\frac{1}{n} \sum_{u \in U} |\, \mu_A(u) - \mu_C(u) \,|^q\right)^{\frac{1}{q}}$;

i.e., $1 - \left(\frac{1}{n} \sum_{u \in U} |\, \mu_B(u) - \mu_C(u) \,|^q\right)^{\frac{1}{q}} \geq 1 - \left(\frac{1}{n} \sum_{u \in U} |\, \mu_A(u) - \mu_C(u) \,|^q\right)^{\frac{1}{q}}$;

i.e., $S(B,C) \geq S(A,C)$.

Similarly, $S(A,B) \geq S(A,C)$. Hence

$$S(A,C) \leq \min\{S(A,B), S(B,C)\}.$$

Hence the theorem.

Theorem 2 motivates us to consider the property of monotonicity of similarity between fuzzy sets to satisfy another axiom for some kind of monotonicity. So, we are now in a position to state the axioms for similarity measure as follows :

For all fuzzy sets A , B

A1. $S(B, A) = S(A, B)$.

A2. $S(A^c, B^c) = S(A, B)$, A^c being some negation of A.

A3. $0 \leq S(A, B) \leq 1$.

A4. $A = B$ iff $S(A, B) = 1$.

A5. $S(A, B) = 0$ if and only if $A \cap B = \Phi$.

A6. If $A \supseteq B \supseteq C$, then $S(A, B) \geq S(A, C)$.

Here, we note that A^c, the complement of a fuzzy set A is to be defined first. Throughout the thesis, we used the idea of '$1-$' as the complementation. A general characterisation of similarity index satisfying the set of axioms is not in the present scope of the thesis. On the basis of the above axioms, it is easy to see that the family of similarity measures defined in *Definition 3.2* is a valid choice.

5 More on similarity measure

Our interest is on machine-oriented measures for solving practical problems. For that, so far we have considered similarity measures for fuzzy sets having finite support set. In this section, we study some measures of similarity between fuzzy sets having arbitrary support set. Already we have listed some of them in the review section. For example, the measures defined in (3.14) and (3.22) may be directly extended to infinite sets.

Let U be any arbitrary set and let $\mathcal{F}(U)$ be the collection of all fuzzy subsets of U. To specify a degree of similarity between elements of $\mathcal{F}(U)$, we first define a distance function

$$d(A, B) = \sup_{u \in U} \mid \mu_A(u) - \mu_B(u) \mid, \quad A, B \in \mathcal{F}(U). \tag{3.42}$$

The corresponding similarity index $S(A, B)$ may be obtained as

$$S(A, B) = 1 - d(A, B). \tag{3.43}$$

Thus, given any $\epsilon \in [0, 1]$, two members of $\mathcal{F}(U)$ are said to be ϵ-similar, if and only if $S(A, B) \geq \epsilon$.

If we go back to the axioms of the previous section we find that (3.43) satisfies all except axiom **A5.**. Nevertheless, it is a reasonable definition. Work on this measure may be found in [40]. Let us now propose a family of similarity indices for countably infinite fuzzy sets as follows :

$$d(A, B) = \left[\sum_{n=1}^{\infty} \frac{\mid \mu_A(u_n) - \mu_B(u_n) \mid^q}{2^n} \right]^{\frac{1}{q}}, \quad A, B \in \mathcal{F}(U); q \geq 1. \tag{3.44}$$

The corresponding similarity index $S(A, B)$ may be obtained as

$$S(A, B) = 1 - d(A, B). \tag{3.45}$$

Now, since

$$0 \leq \mid \mu_A(u_n) - \mu_B(u_n) \mid \leq 1, \tag{3.46}$$

it is clear that, the infinite series is convergent and converges to 1.

In the above case, we see that the similarity index depends not only on the corresponding membership values but also on the order of their appearance because of the presence of the weight factor. This may be avoided, if we consider the supremum over all possible combinations. The above concept is sound for mathematical theory construction but found to be inappropriate in terms of computational aspect.

Thus, we see that, similarity or indistinguishability between fuzzy sets may be captured by aggregating the dissimilarity or distinguishability between membership values of elements in the corresponding fuzzy sets. Such an index, being a pure number, does not give any information about the ordering. This is why these measures are not transitive. Similarity matching is inherent in reasoning with imprecise concepts. In the next chapter, we show how similarity measure may be made to work in reasoning with vague concepts.

Chapter 4

Similarity based approximate reasoning

1 Introduction

In the previous chapter, we developed the concept of similarity index for measuring the likeness of fuzzy sets over a given universe of discourse and proposed two new measures for the same. We discussed some basic properties and results in connection with these measures. Here we restrict ourselves to the said two similarity measures for fuzzy sets having finite support (Definition 3.1, Definition 3.2). We will consider the introduction of the concept of similarity in approximate reasoning methodology.

To begin with, in this chapter, we take a close look at the different methods of inference based on a similarity measure. In [84], the authors proposed a similarity based method called Approximate analogical reasoning schema. It was shown that the method is applicable to both point-valued and interval valued-fuzzy sets. In [9], the author proposed two similar methods for medical diagnosis problems. Two other methods based on different modification procedures have been proposed in [104]. In the framework of existing approaches to similarity based inference methodology, recently, in [105], the authors proposed another two similarity based methods for reasoning and made a comparative study of the above similarity based fuzzy reasoning methods.

In all these works, the authors considered that similarity based fuzzy reasoning methods do not require the construction of a fuzzy relation. Accordingly, they are based on the computation of the degree of similarity between the fact and the antecedent of a rule, in a rule-based system. Then, based on the similarity value between the membership values of the elements of the fuzzy set representation of the fact and the corresponding fuzzy set in the antecedent of the rule, the membership value of each element of the consequent fuzzy set of the rule is modified to obtain a conclusion. This is the same for all existing similarity based reasoning schema. The modification procedure is different for different schema.

We propose two new similarity based approximate reasoning methods. Our first method is a modification of the method presented in [84]. The second method is an integration of similarity based reasoning and Zadeh's compositional rule of inference. In the process, we consider both similarity basedd models and resolution based models, for similarity based approximate reasoning. With different results we show that the proposed similarity based approximate reasoning methods are reasonable. In the proposed methods, for inference in a rule-based system, the conditional rule and for inference in a resolution-based system, the disjunction are first expressed as a fuzzy binary relation. In translation, we prefer to use triangular norms in the first case and triangular conorms in the other case, for a better understanding. Other interpretations are also possible. Then, new facts are used to compute the similarity between the fact and the antecedent of the rule, in a rule-based system and one of the disjunct defined over the same universe in a resolution-based system, to modify the above fuzzy binary relation and not the consequence of the rule as done in the existing similarity based reasoning mechanisms. The modification is based on a measure of similarity following some scheme to be presented. The result is interpreted as the induced fuzzy binary relation. Then the inference is computed from the induced fuzzy binary relation using the well known sup-operation, in a rule-based system and inf-operation in a resolution-based system.

In the following, we present a brief review on the existing similarity based fuzzy reasoning mechanisms. Then, we formulate two new schema of reasoning based on similarity measures. The above schema are used in formulating different models (, rule-based and resolution-based). We provide some simple examples for a better understanding of the proposed schema.

p :	X is A then	Y is B
q :	X is A'	
r :		Y is B'.

2 A brief review

Many fuzzy systems are based on Zadeh's compositional rule of inference [114]. Despite their success in various systems, researchers have indicated certain drawbacks [84] in the mechanism. This motivates the introduction of similarity based reasoning mechanisms as proposed in [9, 10, 82, 84, 104, 105].

A brief review on the similarity based reasoning, in general, may be found in [24]. A detailed description on some similarity related reasoning may be found in [84].

In such similarity based reasoning schema, we see that, from a given fact, the desired conclusion is derived using only a measure of similarity between the fact and the antecedent, in a rule-based system. In some cases, a threshold value τ is associated with a rule. If the degree of similarity, between the antecedent of the rule and the given fact, exceeds the real value of τ, associated with the rule under consideration, then only that rule is assumed to be fired. The conclusion is derived using some modification procedure.

As an illustration, let us consider the two premises as in Table 2. Here A and A' are fuzzy sets defined over the same universe of discourse $U = \{u_1, u_2, \cdots, u_m\}$ and B, B' are defined over the universe of discourse $V = \{v_1, v_2, \cdots, v_n\}$. Let $S(A, B)$ denote some measure of similarity between two fuzzy sets A, B. In [84], the authors used the Euclidean distance

$$D_2(A, A') = \left[\frac{\sum_{i=1}^{m}[\mu_{A'}(u_i) - \mu_A(u_i)]^2}{m}\right]^{1/2}. \quad (4.1)$$

The similarity is then defined as

$$S(A, B) = [1 + D_2(A, A')]^{-1}. \quad (4.2)$$

In the existing mechanisms, if $S(A, A') > \tau$ then the rule will be fired and the consequent of the rule is modified to produce the desired conclusion. Based on the change of membership grade of the consequent, two types of modification procedures may be proposed as in [84, 86] - expansion type inference and reduction type inference.

Let $B' = \sum_{i=1}^{n}\{\mu_{B'}(v_i)/v_i\}$ and $s = s(S(A,B),\tau)$.

$$\text{Expansion form : } \mu_{B'}(v_i) = \min(1, \mu_B(v_i)/s). \tag{4.3}$$

$$\text{Reduction form : } \mu_{B'}(v_i) = (\mu_B(v_i).s). \tag{4.4}$$

The authors have also extended this method to handle interval-valued fuzzy sets. Both the methods proposed in [9, 10] use the threshold value, a confidence factor and the reduction form of inference without providing any argument as to the choice of modification procedure. In one of them [9], each fuzzy set is first conceived as an m-component vector and then use the concept of vector dot product for finding the similarity, called the matching function as :

$$S(A,B) = \frac{|A||A'|\cos(\theta)}{(\max(|A|^2|A'|^2))} \tag{4.5}$$

where $|A|$ is the length of the vector A and $cos(\theta)$ is the cosine of the angle between the two vectors. If $S(A,B) \geq \tau$, the predefined threshold value, then the rule will be fired and strength of confirmation is calculated by $S(A,B) \star \mu$, where μ is the confidence factor associated with the rule. In the other method [10], the author used weights with each propositions for the calculation of similarity. In this case, the similarity between fuzzy sets is computed as :

$$S(A,B) = \sum_{i=1}^{m}\left[T(\mu_{A'}(u_i), \mu_A(u_i)).\frac{w_i}{\sum_{k=1}^{m} w_k}\right] \tag{4.6}$$

where $T(\mu_{A'}(u_i), \mu_A(u_i)) = 1 - |\mu_{A'}(u_i) - \mu_A(u_i))|$. The procedure for the computation of the conclusion remains the same.

In [104] the authors used the value of certainty factor associated with the rules in the modification procedure. The inference is based on the number of propositions in the antecedent of the rule(s) as well as the operator(s) connecting them. In each case, the inference is one of expansion type. In [105] they have also presented two more modification procedures and claimed two new fuzzy reasoning methods. One modification is based on Zadeh's inclusion and cardinality measure and the other one is based on equality and cardinality measure. Other operations remain almost identical.

3 Proposed method

In this section, we show how conclusions may be obtained from given premises with the help of such a similarity measure. In the process we consider both rule-based and resolution-based models. Let X, Y be two linguistic variables and let $\mathcal{U}$, $\mathcal{V}$ respectively denote the universes of discourse. Two typical propositions p and q are given and we derive a conclusion according to similarity based inference. The scheme may be best described in Table 4.1. Let

$$\mathcal{U} = \{u_1, u_2, \cdots, u_l\},$$

$$\mathcal{V} = \{v_1, v_2, \cdots, v_m\}$$

denote the respective universe of discourse of the linguistic variables X and Y. Let fuzzy sets A, A' and B in Table 4.1 be defined as :

$$A = \sum_{i=1}^{l}\{\mu_A(u_i)/u_i\} ; \quad (4.7)$$

$$A' = \sum_{i=1}^{l}\{\mu_{A'}(u_i)/u_i\} ; \quad (4.8)$$

$$B = \sum_{i=1}^{m}\{\mu_B(v_i)/v_i\} ; \quad (4.9)$$

$$B' = \sum_{i=1}^{m}\{\mu_{B'}(v_i)/v_i\}. \quad (4.10)$$

3.1 Outline

It is easy to see that all the existing methods [82, 84, 104, 105] use the similarity measure for a direct computation of inference without considering the induced

p :	X is A	then	Y is B
q :	X is A'		
r :			Y is B'.

Table 4.1: Ordinary approximate reasoning

relation, i.e., how the underlying relation(, a condition) is modified in presence of the given fact. This is important in deriving a consequence of the fact from the rule. Consequently, those methods provide the same conclusion, if A and A' are interchanged in the propositions concerned. Thus, if p, q and p', q' be defined as in the following:

$$(i) \quad p: \text{if } X \text{ is } A \text{ then } Y \text{ is } B, \tau \;\text{ and }\; q: X \text{ is } A'$$

$$(ii) \quad p: \text{if } X \text{ is } A' \text{ then } Y \text{ is } B, \tau \text{ and }\; q: X \text{ is } A$$

then both (i) and (ii) will produce the same conclusion. This is not appealing. This happens because the conclusion is derived just by a modification of the consequent of the rule. It should be noted here that, this is not the case with Zadeh's compositional rule of inference. Another notable fact is that we need to consider the threshold or certainty factor in order to tackle the problem of rule misfiring.

The first drawback may at once be eliminated if we consider the interpretation of the relational operator present in the conditional premise, as is done in executing compositional rule of inference. It is easy to verify that for a class of nested fuzzy sets, each different from the other, the consequence of a rule using compositional rule of inference(CRI), becomes the same. We seek a reasoning system, where every change in the concept(s) as appear in the conditional statement and that in the fact, be incorporated in the induced relation between the variables defining the condition, in this case, X and Y. Only then the inference will be influenced by the change.

In order to avoid the use of certainty factor for rule-misfiring, we modify the inference scheme in such a way that significant change will make the conclusion less specific. This is done if an expansion type of inference scheme be chosen. Here, the 'UNKNOWN' case, i.e., the fuzzy set $B' = V$, may be taken as the limit. Explicitly, when the similarity value becomes low, i.e, when A and A' differ significantly, the reasoning process should be such that the only inference be $B' = V$. As $A' = A$, we expect that $B' = B$. This, in turn, implies that nothing better than what the rule says should be allowed as a valid conclusion.

3.2 Schema

In view of the above observations, we propose a similarity based inference method for deriving the consequence r. We first generate the fuzzy relation between the antecedent variable(s) and the consequent variable as done in executing CRI. Then we compute the absolute change in linguistic labels, represented as fuzzy sets, and systematically propagate the same into the conditional relation in order to obtain the induced modified conditional relation. From this induced modified relation, a possible conclusion may be drawn using the sup-operation. The scheme for computation may be presented in the following algorithm.

ALGORITHM SAR : Similarity based ordinary approximate reasoning

Step 1. Translate premise p and compute $R(A,B)$ using any suitable translating rule possibly, a T-norm operator.
Step 2. Compute $S(A,A')$ according to either *Definition 3.1* or *Definition 3.2* or by some other similar definitions.
Step 3. Modify $R(A,B)$ with $S(A,A')$ to obtain the modified conditional relation $R(A \mid A',B)$ according to some scheme C.
Step 4. Use sup-projection operation on $R(A \mid A',B)$ to obtain B' as

$$\mu_{B'}(v) = \sup_u \mu_{R(A'|A,B)}(u,v). \tag{4.11}$$

Now, for a given fact $q : XisA'$ and from the condition $p : ifXisAthenYisB$, we propose two schema $C1$ and $C2$ for computation of the modified conditional relation $R(A \mid A',B)$ as given in Step 3.

Scheme C1

The first scheme C1 is based on a concept similar (, but NOT identical) to the method proposed in [84]. We may recall here that, the authors computed the conclusion $B' = \min(1, B/s)$, where s is the measure of similarity between fuzzy sets A and A' without considering the information suggested by the conditional rule. Here, we propose to modify the conditional relation acording to the (4.12).

$$R(A \mid A',B) = [r'_{u,v}]_{l\times m} = \begin{bmatrix} r'_{u,v} & = \min(1, r_{u,v}/s) & \text{if } s > 0 \\ & = 1 & \text{otherwise.} \end{bmatrix} \tag{4.12}$$

The difference between the proposed scheme and the one presented in [84] may be easily noted. It is clear that, the proposed scheme, unlike the schema in [84, 82], does not produce the same conclusion when A and A' are interchanged. It is not difficult to see that in (4.12), if $s \leq r_{u,v}$ for some $v \in V$ then $r'_{u,v}$ becomes equal to one. Thus, making the membership of that v in the resultant fuzzy set equal to one.

Example 4.1 : Let us consider a problem as posed schematically in Table 4.1, where $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3, v_4\}$. Also let

$$A = 1.00/u_1 + 0.75/u_2 + 0.50/u_3 + 0.25/u_4 \ ;$$

$$A' = 1.00/u_1 + 0.80/u_2 + 0.40/u_3 + 0.10/u_4 \ ;$$

$$B = 0.25/v_1 + 0.50/v_2 + 0.75/v_3 + 1.00/v_4 \ .$$

Using Mamdani's min-rule for translation we first construct the fuzzy binary relation on $U \times V$ as,

$$R(A,B) \quad = \quad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0.25 & 0.50 & 0.75 & 1.00 \\ 0.25 & 0.50 & 0.75 & 0.75 \\ 0.25 & 0.50 & 0.50 & 0.50 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \end{array} . \tag{4.13}$$

Then, using *Definition 3.2*, we compute $S(A, A') = 0.95322928$(approx.). The modified relation, $R(A, B)$ i.e. $R(A \mid A', B)$ using <u>scheme C1</u> may be found to be :

$$R(A' \mid A,B) \quad = \quad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0.2623 & 0.5245 & 0.7868 & 1.00 \\ 0.2623 & 0.5245 & 0.7868 & 0.7868 \\ 0.2623 & 0.5245 & 0.5245 & 0.5245 \\ 0.2623 & 0.2623 & 0.2623 & 0.2623 \end{bmatrix} \end{array} . \tag{4.14}$$

Therefore, using (4.11), we find the consequence as,

$$B' = \sup_{u \in U}[R(A' \mid A, B)] = 0.2623/v_1 + 0.5245/v_2 + 0.7868/v_3 + 1/v_4. \tag{4.15}$$

Note here that the conclusion is the same as it would have been using the scheme proposed in [84]. Instead of using Mamdani's min operator if we use $\max(0, a+b-1)$ for the translation of the conditional statement then we find the conditional relation as :

$$R(A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0.2500 & 0.5000 & 0.7500 & 1.0 \\ 0.2623 & 0.5245 & 0.7868 & 1.0 \\ 0.2751 & 0.5503 & 0.8254 & 1.0 \\ 0.2886 & 0.5773 & 0.8659 & 1.0 \end{bmatrix} \end{array}. \tag{4.16}$$

The modified relation $R(A, B)$ i.e. $R(A' \mid A, B)$ will be given by

$$R(A' \mid A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0.2623 & 0.5245 & 0.7868 & 1.0 \\ 0.2751 & 0.5503 & 0.8254 & 1.0 \\ 0.2886 & 0.5773 & 0.8659 & 1.0 \\ 0.3028 & 0.6056 & 0.9084 & 1.0 \end{bmatrix} \end{array} \tag{4.17}$$

Therefore, using (4.11), the consequence become,

$$B' = \sup_{u \in U}[R(A' \mid A, B)] = 0.3028/v_1 + 0.6056/v_2 + 0.9084/v_3 + 1/v_4. \tag{4.18}$$

This scheme, although a heuristic one, is intuitively a plausible scheme. Our next <u>scheme C2</u> for computation of $R(A' \mid A, B)$ is based on a set of axioms.

<u>Scheme C2</u>

We believe that in a similairty based reasoning methodology, a scheme for computation of the induced relation, when a fact and a conditional statement is given, should satisfy the following axioms :

A4.1. If $S(A, A') = 1$, i.e., if $A' = A$, then

$$\mu_{R(A'|A,B)}(u, v) = \mu_{R(A,B)}(u, v); (\forall(u, v)) \in U \times V. \tag{4.19}$$

A4.2. If $S(A, A') = 0$, i.e., if $A' \cap A = \Phi$, then

$$\mu_{R(A'|A,B)} = 1 \quad \forall(u, v) \in U \times V. \tag{4.20}$$

A4.3. As $S(A, A')$ increase from 0 to 1 $\mu_{R(A'|A,B)}(u, v)$ decreases uniformly from 1 to $\mu_{R(A,B)}(u, v); \forall(u, v) \in U \times V$.

Axiom **A1** asserts that we should not modify the conditional relation as and when A' and A remain equal. Axiom **A2.** asserts that when A' is completely dissimilar

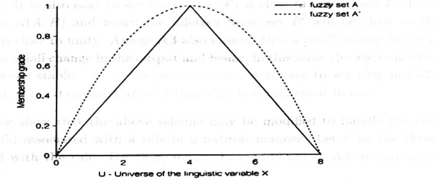

Figure 4.1: Simultaneous representation of the fuzzy sets A and A'

to A, i.e., A' and A have disjoint support, we should not conclude specifically. In such a situation, anything is possible. A3 says that as the fact A' changes from the most dissimilar case (, similarity value zero) to the most similar one (,similarity value one), the inferred conclusion should change from the most non-specific case i.e, the UNKNOWN case($B' = V$) to the most specific case, i.e., $B' = B$. This, in turn, means that whatever A' be, $R(A' \mid A, B) \supseteq R(A, B)$,i.e., the induced relation should not be more specific than what is given as a condition. For notational simplicity, let us denote $S(A, A')$ by s and $R_{A'|A,B}$ by r'. Now, axiom A3 *uniquely* suggest a function of the form

$$\frac{dr'}{ds} = k(, \textit{a constant})$$

$$\Rightarrow r' = k\ s + c, \quad c \text{ is a constant.}$$

These two constants may be determined from the conditions already prescribed in axiomA1 and axiom A2. More explicitly, when $s = 1$ we know that $r' = r$ (from axiom A1) and when $s = 0$ we know that $r' = 1$ (from axiom A2). This gives,

$$r' = 1 - (1 - r).s \tag{4.21}$$

as our new scheme for the modification of the conditional relational.

Therefore, axiom A1 through axiom A3 *uniquely* suggest the scheme C2 as

$$\mu_{R(A'|A,B)}(u, v) = 1 - (1 - \mu_{R(A,B)}(u, v)).S(A, A'). \tag{4.22}$$

From the (4.11) and (4.22) it is easy to see that when $S(A, A') = 0$ we have $B' = V$, in other words, it is impossible to conclude anything when $\{A, A'\}$ are completely dissimilar. It is also easy to see when $S(A, A')$ is close to unity, then $R(A' \mid A, B)$ is close to $R(A, B)$ and hence the inferred fuzzy set B' will be close to B, i.e., $S(B, B')$ is close to unity. Axiom **A3** also ensures that a small change in the input produces a small change in the output and hence, in this sense the above mechanism of inference is stable. As in the previous case, it is easy to see that in (4.22), if either $S(A, A') = 0$ or $\mu_R(A, B) = 1$ then $r'_{u,v}$ becomes equal to one.

Let us now show that the above scheme may be modified to handle the concept of threshold associated with a rule in a natural manner. Let τ be the threshold associated with the rule. For that, we are to modify axiom **A2** according to the following :

A4. If $S(A, A') \leq \tau < 1$ then

$$\mu_{R(A'|A,B)} = 1 \quad \forall (u, v) \in U \times V. \tag{4.23}$$

Accordingly, simple calculations, as before, resulted in the following

$$\mu_{R(A'|A,B)} = \min[1, (1 - (1 - \mu_{R(A,B)}) . \frac{s - \tau}{1 - \tau})] \tag{4.24}$$

as the general scheme for relation membership modification. It is easy to see that the case $\tau = 0$ correspond to the scheme presented by (4.22). This scheme ensures that with all fuzzy sets A' having similarity value $S(A, A')$ less or equal to the threshold value τ, the inference B' using (4.11) will be 'UNKNOWN'.

Example 4.2 : Let us consider the problem as posed in Table 4.1. For simplicity, let $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3, v_4\}$. Also let the fuzzy sets be given as follows :

$$A = 1.00/u_1 + 0.75/u_2 + 0.50/u_3 + 0.25/u_4 ;$$
$$A' = 1.00/u_1 + 0.80/u_2 + 0.40/u_3 + 0.10/u_4 ;$$
$$B = 0.25/v_1 + 0.50/v_2 + 0.75/v_3 + 1.00/v_4 .$$

It is required to find a fuzzy set B' using the mechanism described in algorithm SAR. Using min-rule for translation, we first compute the conditional relation $R(A, B)$ as follows :

$$R(A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0.25 & 0.50 & 0.75 & 1.00 \\ 0.25 & 0.50 & 0.75 & 0.75 \\ 0.25 & 0.50 & 0.50 & 0.50 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{array} \right] \end{array}. \tag{4.25}$$

Then, using **Definition 3.2**, we compute the similarity between A and A' as $S(A, A') = 0.95322928$(approx.). Next, using (4.22), the modified relation $R(A' \mid A, B)$ may be computed as

$$R(A' \mid A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0.2851 & 0.5234 & 0.7617 & 1.00 \\ 0.2851 & 0.5234 & 0.7617 & 0.7617 \\ 0.2851 & 0.5234 & 0.5234 & 0.5234 \\ 0.2851 & 0.2851 & 0.2851 & 0.2851 \end{array} \right] \end{array}. \tag{4.26}$$

Therefore, using (4.11), we find the consequence as

$$B' = \sup_{u \in U}[R(A' \mid A, B)] = 0.2851/v_1 + 0.5234/v_2 + 0.7617/v_3 + 1/v_4. \tag{4.27}$$

Instead of using similarity based approximate reasoning methodology in deriving a consequence, if we consider the existing max-min compositional rule of inference then the result would be

$$B_1' = 0.25/v_1 + 0.5/v_2 + 0.75/v_3 + 1/v_4. \tag{4.28}$$

From (4.28), it is easy to see that, there is no change in the output although the inputs differ significantly. Also, it may be shown that the same happens for a large class of fuzzy sets each different from the other. This is supposed to be a drawback in executing max-min compositional rule of inference in its present form. In order to generate some 'feeling' about the proposed inference mechanism, in the following, we present a pictorial description. For that, let us consider the two propositions p and q as in the following. where A, B and A' are normalized fuzzy sets defined as follows :

$$\mu_A(x) = \begin{cases} 1 - \frac{1}{4} \mid x - 4 \mid, & \text{if } 0 \leq \mid x - 4 \mid \leq 4 \\ 0 & \text{otherwise.} \end{cases} \tag{4.29}$$

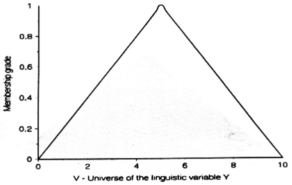

Figure 4.2: Representation of the fuzzy set B

$$\mu_B(y) = \begin{cases} 1 - \frac{1}{5} \mid y - 5 \mid, & \text{if } \; 0 \leq \mid y - 5 \mid \leq 5 \\ 0 & \text{otherwise.} \end{cases} \tag{4.30}$$

$$\mu_{A'}(x) = \begin{cases} 1 - \frac{1}{16}(x - 4)^2, & \text{if } \; 0 \leq \mid x - 4 \mid \leq 4 \\ 0 & \text{otherwise.} \end{cases} \tag{4.31}$$

From p and q, we would like to conclude a proposition r : Y is B' using algorithm SAR. The support of each fuzzy set is uniformly quantized into 50 levels. Figure 4.1 depicts the simultaneous representation of the fuzzy sets A and A'. Figure 4.2 corresponds to the representation of the fuzzy set B, the consequent part of the conditional statement p. Figure 4.3 gives a pictorial representation of the relational matrix $R(A, B)$ using the min-rule for the translation of the conditional statement. Using **Scheme C2**, the modified relational matrix $R(A' \mid A, B)$ has been shown in Figure 4.4. In Figure 4.5, a pictorial representation of the conclusion B', has been shown. For comparison, the conclusion using Zadeh's compositional rule of inference has been presented in Figure 4.6. It is interesting to see that the concluded B' as shown in Figure 4.6 and the fuzzy set B as shown in Figure 4.2 are the same,

p : X is A then Y is B
q : X is A'

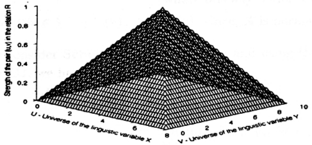

Figure 4.3: Representation of the conditional relation

although A and A' are significantly different. Another important feature of the proposed similarity based approach that B is more specific B', may be seen after the comparison of Figure 4.2 and Figure 4.5. Thus, the conclusion is not better than B, the consequent of the rule.

Remarks : If Mamdani's min-rule is used for the translation of the implication statement and only normal fuzzy sets are considered in the manipulation then $A' = A$ will imply that $B' = B$. This is simply because, in this case, $S(A, A') = 1$ and hence $R(A' \mid A, B)$ will be equal to $R(A, B)$.

Let A be a normal fuzzy set. If we assume that the translating rule used in generating the conditional relation is one of T-norm type then, as is already proposed, a basic and desirable result of the inferred proposition, *nothing better than what the rule says may be concluded* may be established as in the following. For that, let us consider the model as in Table 4.1. For all A, A', the following proposition is valid.

Theorem 1. $B' \supseteq B$.

Proof. : Let us first consider **Scheme C2**. From (4.11) and the result of application of **Scheme C2**, we have,

$$\mu_{B'}(v) \quad = \quad \sup_{u \in U} \mu_{R(A'|A,B)}(u, v)$$

$$
\begin{aligned}
&= \sup_{u \in U}\{1 - (1 - \mu_{R(A,B)}(u,v)).S(A,A')\} \\
&\geq \sup_{u \in U}\{\mu_{R(A,B)}(u,v)\}; \text{(since, } 0 \leq S(A,A') \leq 1) \\
\text{i.e., } \mu_{B'}(v) &\geq \sup_{u \in U}\{\mu_A(u) o \mu_B(v)\} \\
&\quad \text{where } o \text{ is any T-norm operator.} \\
\text{Therefore, } (\forall v \in V),\ \mu_{B'}(v) &\geq \mu_B(v) \ \text{ since, } A \text{ is normal.}
\end{aligned}
$$

Let us now consider **Scheme C1**. From (4.11) and using the result of application of **Scheme C1**, we have,

$$
\begin{aligned}
\mu_{B'}(v) &= \sup_{u \in U} \mu_{R(A'|A,B)}(u,v) \\
&= \sup_{u \in U} \min\{1, \mu_{R(A,B)}(u,v)/S(A,A')\} \\
&\geq \sup_{u \in U}\{\mu_{R(A,B)}(u,v)/S(A,A')\}; \\
&\geq \sup_{u \in U} \mu_{R(A,B)}(u,v); \text{(since, } 0 \leq S(A,A') \leq 1) \\
\text{i.e., } \mu_{B'}(v) &\geq \sup_{u \in U}\{\mu_A(u) o \mu_B(v)\} \\
&\quad \text{where } o \text{ is any T-norm operator.} \\
\text{Therefore, } (\forall v \in V),\ \mu_{B'}(v) &\geq \mu_B(v) \ \text{ since, } A \text{ is normal.}
\end{aligned}
$$

With the above understanding of similarity based reasoning methodology, let us now propose different models for reasoning.

4 Applications to different models

In this section, we consider the application of the proposed similarity based approximate reasoning mechanism to different models of approximate reasoning – rule-based and resolution-based.

4.1 Rule-based models

Let $X_1, X_2, \cdots, X_k, Y$ be $k+1$-linguistic variables defined respectively over universes of discourse $U_1, U_2, \cdots, U_k, V$ and let $U_i = \{u_i^j\}; j = 1, 2, \cdots, j_i$. Let us consider a pattern for approximate reasoning with vague knowledge, as presented in the following : A consequence r may be derived according to the following basic steps.

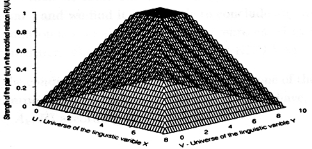

Figure 4.4: Representation of the induced relation

Let the representation of the imprecise concepts in the propositions through fuzzy sets be given by

$$\begin{aligned} A_i &= \sum_{j=1}^{j_i} \mu_A(u_i^j)/u_i^j \; ; i = 1, 2, \cdots, k; \\ A'_i &= \sum_{j=1}^{j_i} \mu_{A'_i}(u_i^j)/u_i^j \; ; i = 1, 2, \cdots, k; \\ B &= \sum_{i=1}^{m} \mu_B(v_i)/v_i \; . \end{aligned}$$

Here the conditional proposition p is first translated into a fuzzy relation R on the product space $U_1 \times \; U_2 \times \; \cdots \; U_k \; \times V$. Now R may be computed using any suitable translating rule, possibly a T-norm operator. Then, we compute $S(A_i, A'_i)$

p	:	if	X_1 is A_1	and	X_2 is A_2	and	$\cdots$	X_k is A_k	then	Y is B
q	:		X_1 is A'_1	and	X_2 is A'_2	and	$\cdots$	X_k is A'_k		
r	$\leftarrow$									Y is B'

Table 4.2: Extended approximate reasoning

for $i = 1, 2, \cdots, k$ and set

$$s = \min\{S(A_1, A_1'), S(A_2, A_2'), \cdots, S(A_k, A_k')\}.$$

If now $s = 0$ then at least one pair (A_i, A_i') are complementary (,disjoint support/dissimilar) and we find it impossible to conclude anything in particular, until further information is available. This may be represented by the fact that anything follows as conclusion. Hence, we set $B' = V$(UNKNOWN).

Otherwise, the conditional relation is modified using one of the two schema **Scheme C1** and **Scheme C2**. If **Scheme C1** is used then we have, $R' = R(A_1' \mid A_1, A_2' \mid A_2, \cdots, A_k' \mid A_k, B)$ according to

$$\mu_{R'}(u_1, u_2, \cdots, u_k, v) = \min\left\{1, \frac{1}{s}\mu_R(u_1, u_2, \cdots, u_k, v)\right\}.$$

If, instead, **Scheme C2** is used then we find

$$\mu_{R'}(u_1, u_2, \cdots, u_k, v) = 1 - (1 - \mu_R(u_1, u_2, \cdots, u_k, v)).s.$$

In both cases, the conclusion B' will be given by

$$\mu_B'(v) = \sup_{u_1, u_2, \cdots, u_k} \mu_{R(A_1'|A_1, A_2'|A_2, \cdots, A_k'|A_k, B)}(u_1, u_2, \cdots, u_k, v). \tag{4.32}$$

ALGORITHM A1 :

Step 1. Compute $S(A_i, A_i')$ for $i = 1, 2, \cdots, k$ and set

$$s = \min\{S(A_1, A_1'), S(A_2, A_2'), \cdots, S(A_k, A_k')\}.$$

Step 2. Translate premise p and compute $R(A_1, A_2, \cdots, A_k, B)$ using any suitable translating rule possibly, a T-norm operator.

Step 3. Modify $R(A, B)$ with s to obtain the modified conditional relation $R' = R(A_1' \mid A_1, A_2' \mid A_2, \cdots, A_k' \mid A_k, B)$ according to either (4.12) or (4.22).

Step 4. Use sup-projection operation on R' to obtain B' as given in (4.32).

Example 4.3 : Let us consider the model presented in Table 4.2. Let $k = 2$ and

$$\begin{aligned}
\mathcal{U}_1 &= \{u_1^1, u_1^2, u_1^3, u_1^4, u_1^5\}; \\
\mathcal{U}_2 &= \{u_2^1, u_2^2, u_2^3, u_2^4, u_2^5\}; \\
\mathcal{V} &= \{v_1, v_2, v_3, v_4, v_5\}; \\
A_1 &= 0.3/u_1^1 + 0.55/u_1^2 + 0.7/u_1^3 + 0.85/u_1^4 + 1.0/u_1^5;
\end{aligned}$$

if	X_1 is A_{11}	and	X_2 is A_{12}	$\cdots$	X_n is A_{1n}	then	Y is B_1
else if	X_1 is A_{21}	and	X_2 is A_{22}	$\cdots$	X_n is A_{2n}	then	Y is B_2
			$\vdots$		$\vdots$		
else if	X_1 is A_{m1}	and	X_2 is A_{m2}	$\cdots$	X_n is A_{mn}	then	Y is B_m
	X_1 is A_1	and	X_2 is A_2	$\cdots$	X_n is A_n		
Conclusion							Y is B

Table 4.3: Applicable form of approximate reasoning

$$\begin{aligned}
A_1' &= 0.09/u_1^1 + 0.3025/u_1^2 + 0.49/u_1^3 + 0.7225/u_1^4 + 1.0/u_1^5; \\
A_2 &= 0.9/u_2^1 + 1.0/u_2^2 + 0.85/u_2^3 + 0.7/u_2^4 + 0.55/u_2^5; \\
A_2' &= 0.9486/u_2^1 + 1.0/u_2^2 + 0.922/u_2^3 + 0.8367/u_2^4 + 0.7416/u_2^5; \\
B &= 1.0/v_1 + 0.75/v_2 + 0.5/v_3 + 0.25/v_4 + 0.05/v_5.
\end{aligned}$$

Using *Definition 3.2* with $q = 2$ it is found that $S(A_1, A_1') = 0.81794918$(approx.) and $S(A_2, A_2') = 0.8878007$(*approx.*). So, we set $s = 0.81794918$(*approx.*). Let us use min-rule for translation of premises p and q. On the basis of scheme C2, and (4.32), it is found that

$$B' = 1.0/v_1 + 0.795513/v_2 + 0.591025/v_3 + 0.386538/v_4 + 0.222948/v_5.$$

If instead, scheme C1 is used, then the inference would become

$$B' = 1.0/v_1 + 0.916927/v_2 + 0.611285/v_3 + 0.305642/v_4 + 0.061128/v_5.$$

The difference in the last two results show that the change of membership values in the second case is more than the other one. In the first case (using Scheme C2), the expansion in fuzzy set membership is gradual. In the second case (using Scheme C1), the set membership becomes all one i.e., the inference becomes 'UNKNOWN' even for non-zero and sometimes high similarity value.

Next, let us consider a generalized model as presented in Table 4.3. This form of reasoning is used in many rule-based fuzzy systems. In particular, it is used in pattern classification and fuzzy control. Let there be n-linguistic variables associated with another linguistic variable Y according to the following m-fuzzy rules. The problem is to find the linguistic value of the variable Y as suggested by the rules, when the values of the n-variables are given. Under the conventional mechanism, for each rule, the consequent fuzzy set is calculated according to existing method of inference as already described and then the union of all consequent fuzzy sets is taken

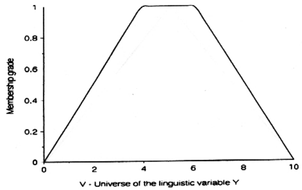

Figure 4.5: The inferred fuzzy set B'

as the conclusion which is then defuzzified, if necessary, using some defuzzification scheme. In the present case of similarity based reasoning we cannot do this, as the membership values computed from the modified induced relation becomes less and less specific as the similarity between the facts and antecedent of a rule decreases. In conventional paradigm also, the membership values of various elements becomes equal to the maximum, making it an ambiguous one (more alternatives with similar membership values at the positive level) with the reduction of the firing strength (used in deriving a conclusion), but the membership values at which the ambiguity occurs becomes less than one. For example, in case of Mamdani-type of reasoning, if the firing strength of a rule is, say 0.3, then all alternatives which have membership values greater than or equal to 0.3 take membership values of 0.3. On the other hand, in the present case, if the similarity value is 0.3, then the membership values of elements in the inferred fuzzy set will be at least 0.3. Moreover, the elements having membership value greater than or equal to 0.3 in the consequent of the rule will be equal to '1' in the consequent fuzzy set. This means that, with decrease in similarity the computed membership values increase and ultimately moves close to the least specific case (with membership values of 1 for all alternatives). The above discussion is illustrated with the help of a diagram. In Figure 4.7, let us suppose, the symmetric triangular fuzzy set represents the consequent of a rule. When the firing strength of the rule is 0.3, the derived conclusion from the rule is given by the trapezoid with height 0.3. Clearly, every value of [a,b] in Figure 4.7

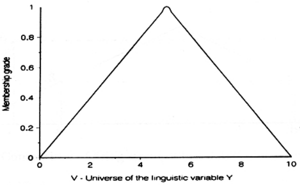

Figure 4.6: The inferred fuzzy set B' using Zadeh's CRI

has the same membership grade 0.3 i.e., the possibility of becoming the solution. On the other hand, if the similarity between the fact and the antecedent of the rule is 0.3, then the conclusion derived using similarity based mechanisms is given by the trapezoidal fuzzy set with height $\alpha \geq 0.3$ (shown in dotted line in the same Figure 4.7). In this case, every alternative of [c,d] in Figure 4.7 could be a solution with membership value of α. Here, not only more alternatives have been offered (since [a,b] $\subseteq$ [c,d]) with the same membership value than the previous case but also the conclusion becomes more close to the least specific case. For this reason, we propose a new scheme, for computing the final conclusion, based on a measure of similarity. A detailed discussion on the same is presented in the next section. Our method is based on rule-selection and then rule-execution. In both cases, we use the concept of similarity between fuzzy sets as a basis of the task. For that, first of all we compute $S(A_ij, A_i); i = 1, 2, \cdots, m$. Then we perform the same operation for different $j = 1, 2, \cdots, n$. Let s_{ij} denote the different similarity values. Next, we compute the overall rule matching index from the above data as

$$s^i = \min_j s_{ij} \tag{4.33}$$

From among the m distinct rules we choose those rules for which $s^i > \epsilon$. This ϵ may be interpreted as a threshold in our case. Then we apply algorithm **A1** to generate a conclusion from each rule conformal for firing. The output may be generated using the intersection of fuzzy sets. It is important to note that the intersection

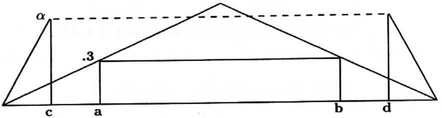

Figure 4.7: Comparison of firing strength based and similarity based reasoning schema

operation is chosen in order to justify the rule-selection procedure. Here, fewer rules are fired and the output of each rule is significant.

ALGORITHM A2 :

Step 1. Compute s_{ij} for $i = 1, 2, \cdots, m; j = 1, 2, \cdots, n$ and then s^i according to (4.33).
Step 2. Define ϵ and find the rules conformal for firing. **Step 3.** Translate the i^{th}-rule, provided $s^i > \epsilon$ and compute the relation R_i using any suitable translating rule possibly, a T-norm operator.
Step 4. Modify R_i with s^i to obtain the modified conditional relation R_i' according to either (4.12) or (4.22).
Step 5. Use sup-projection operation on R_i' to obtain B_i' as given in (4.34).

$$\mu_{B_i'}(v) = \sup_{u_1, u_2, \cdots, u_k} \mu_{R(A_1|A_{i1}, A_2|A_{i2}, \cdots, A_k'|A_{ik}, B)}(u_1, u_2, \cdots, u_k, v). \qquad (4.34)$$

Step 6. Compute the output $B = \bigcap_i B_i'$.

Examples and relevant issues are considered in the next chapter.

4.2 Resolution-based models

Now, let us consider resolution-based models for reasoning with vague concepts. Let there be two typical premises p and q and we derive a conclusion r using simialrity based reasoning mechanisms. The entire scheme may be put in a Table 4.4. Let

p :	X is A	or	Y is B
q :	X is A'		
r ←			Y is B'.

Table 4.4: Resolution-based approximate reasoning

$$\mathcal{U} = \{u_1, u_2, \cdots, u_l\},\ \mathcal{V} = \{v_1, v_2, \cdots, v_m\}$$

denote the respective universe of discourse of the linguistic variables X and Y. First we translate the imprecise concepts in the premises p and q into appropriate fuzzy sets and obtain

$$A = \sum_{i=1}^{l} \mu_A(u_i)/u_i\ ;$$
$$A' = \sum_{i=1}^{l} \mu_{A'}(u_i)/u_i\ ;$$
$$B = \sum_{i=1}^{m} \mu_B(v_i)/v_i\ ;$$

Next, we translate the conditional premise p into some binary fuzzy relation R on $\mathcal{U} \times \mathcal{V}$ using an appropriate operator for the fuzzy connector 'or'. We use some T-conorm operator for the translation. Thus, we have,

$$\mu_R(u, v) = \mu_A(u) \circ \mu_B(v); \quad \circ \text{ is some T-conorm operator.} \tag{4.35}$$

Then using the fuzzy set A', we modify the fuzzy binary relation $R(A, B)$ with $S(A, A')$. Here, in modification, we use the fact that for a resolution-based model the dissimilarity between the fact and the antecedent of the rule is considered for a meaningful resolvent. The dissimilarity index is defined as in (3.3). Then we generate a modified fuzzy binary relation $R(A' \mid A, B)$ on $\mathcal{U} \times \mathcal{V}$. Here, we use either

$$\mu'_R(u, v) = 1 - (1 - \mu_R(u, v)) * d \tag{4.36}$$

or

$$\mu'_R(u, v) = \mu_R(u, v)/d, \text{ d is the dissimilarity index} \tag{4.37}$$

for modification of the conditional relation. Since we use a T-conorm function in translation, and we choose expansion type modification procedure, therefore, for the best solution from among a class of solutions we perform inf-operation on R' in order to obtain the desired fuzzy value for the linguistic variable Y.

Here, d is taken as a measure of dissimilarity between the two fuzzy sets A and A'. In order to formulate the disjunctive syllogism in fuzzy logic, we need to introduce d in the inference procedure. When $d = 0$, we find that $B' = V$. So that, nothing, in particular, may be concluded about Y. Again when $d = 1$ i.e. $S(A, A') = 0$ i.e., A and A' are disjoint sets, we need not modify the relation and see that $B' = B$ may be obtained. This is, in accordance with, the law of generalized disjunctive syllogism [70].

ALGORITHM A3 :

Step 1. Compute $S(A, A')$ according to either *Definition 3.1* or *Definition 3.2* or by some other similar definitions. Set $d = 1 - S(A, A')$.
Step 1. Translate premise p and compute $R(A, B)$ using any suitable translating rule possibly, a T-conorm operator.
Step 3. Modify $R(A, B)$ with $S(A, A')$ to obtain the modified conditional relation $R(A \mid A', B)$ according to either (4.36) or (4.37).
Step 4. Use inf-operation on $R(A \mid A', B)$ to obtain B' as

$$\mu_{B'}(v) = \inf_{u \in U} \mu_{R(A'|A,B)}(u, v). \tag{4.38}$$

Example 4.4 : Let us consider the model presented in Table 4.4. Let

$$\begin{aligned} U &= \{u_1, u_2, u_3, u_4, u_5\}; \\ V &= \{v_1, v_2, v_3, v_4\}; \\ A &= 1.0/u_1 + 0.5625/u_2 + 0.25/u_3 + 0.0625/u_4; \\ A' &= 0.4375/u_2 + 0.75/u_3 + 0.9375/u_4 + 1.0/u_5; \\ B &= 0.25/v_1 + 0.5/v_2 + 0.75/v_3 + 1.0/v_4; \end{aligned}$$

For a conclusion of the form r, let us first compute $S(A, A')$ according to Definition 3.2. Here, we find that $S(A, A') = \underline{0.3475}$. We set $d = 1 - S(A, A') = \underline{0.6525}$. Now, let us translate premise p as a relation $R(A, B)$ using max operator, where

$$\mu_{R(A,B)}(u, v) = \max\left[\mu_A(u), \mu_B(v)\right]. \tag{4.39}$$

More explicitly, we have,

p :	X_1 is A_1 or X_2 is A_2 or $\cdots$ X_k is A_k
q :	X_1 is A'_1
r :	X_2 is A'_2 or $\cdots$ X_k is A'_k

Table 4.5: Resolution-based generalised approx. reasoning

$$R(A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \end{array} \left[\begin{array}{cccc} 1.0 & 1.0 & 1.0 & 1.0 \\ 0.5625 & 0.5625 & 0.75 & 1.0 \\ 0.25 & 0.50 & 0.75 & 1.0 \\ 0.25 & 0.50 & 0.75 & 1.0 \end{array} \right]. \tag{4.40}$$

Next, using (4.36) with d, the modified relation $R(A' \mid A, B)$ may be given by

$$R(A'|A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \end{array} \left[\begin{array}{cccc} 1.0 & 1.0 & 1.0 & 1.0 \\ 0.7145 & 0.7145 & 0.8368 & 1.0 \\ 0.5106 & 0.6737 & 0.8368 & 1.0 \\ 0.5106 & 0.6737 & 0.8368 & 1.0 \end{array} \right]. \tag{4.41}$$

Therefore, $B' = 0.5106/v_1 + .6737/v_2 + .8368/v_3 + 1.0/v_4$.

On simple calculation it is found that $S(B, B') = \underline{0.9157}$. This is simply because A, A' are almost complementary pair, having $S(A, A') = \underline{0.3475}$.

In the end, let us consider a generalized resolution-based model as presented in Table 4.5. Let $X_1, X_2, \cdots, X_k$ be k-linguistic variables defined respectively over universe of discourses $U_1, U_2, \cdots, U_k$ and let $U_i = \{u_i^j\}$, $j = 1, 2, \cdots, j_i$. We consider a generalized model, where from premises p and q we derive a conclusion r. As usual, we first compute $S(A_1, A'_1)$ and set $d = 1 - S(A_1, A'_1)$. Then with d we follow the following steps :

<u>ALGORITHM A4</u> :

Step 1. Translate premise p into a fuzzy relation and compute $R(A_1, A_2, \cdots, A_k)$ using any suitable translating operator for the 'or' in the rule possibly, a T-conorm operator.
Step 2. Compute $S(A_1, A'_1)$ and set $d = 1 - S(A_1, A'_1)$.
Step 3. Modify $R(A_1, A_2, \cdots, A_k)$ with d to obtain the modified conditional relation $R' = R(A'_1 \mid A_1, A'_2 \mid A_2, \cdots, A'_k \mid A_k)$ according to either (4.36) or (4.37).
Step 4. Perform inf-operation separately to obtain

$$\mu_{A'_i}(u_i^j) = \inf_{j:j \neq i} \mu_{R'}(u_1, \cdots, u_k); j = 1, \cdots, j_i; i = 2, \cdots, k. \tag{4.42}$$

In the process, we find a conceptual change in similarity based inference mechanism. A closer look at the connection between the proposed schema and the existing schema allow us to conclude that our schema may be thought of as an integration of Zadeh's compositional rule of inference and similarity based inference schema. Such a scheme is expected to produce efficiency in inference mechanisms. In the next chapter, the above procedure is applied in the design of important rule-based fuzzy systems.

Chapter 5

Applications of similarity based approximate reasoning

1 Introduction

The object of this chapter is to design rule-based fuzzy systems using the concept of similarity between fuzzy sets and similarity based approximate reasoning instead of conventional approximate reasoning. Since the invention of fuzzy logic, the variety of applications have been growing. Approximate reasoning, the inference mechanism, plays a leading role in many areas of application. Many scientific problems in real-world setting may be solved using approximate reasoning methodology.

In the previous chapter, we applied similarity concept in reasoning mechanism. In an attempt to eventually investigate how the similarity concept in reasoning affect the design of rule-based fuzzy systems, we present *rule-based solution of two* important and useful problems - pattern classification and fuzzy control.

In pattern classification we are interested in *object data classification*. In object data, each pattern or object (say, a human being, tank, animal etc.) is characterized by a set of attributes (features) like height, weight etc.. A classifier partitions the feature space in such a way that any unlabeled data point may be assigned an appropriate class (crisp or fuzzy) label. Usually, a classifier is designed using some training data, for which the actual class labels are known. For example, in order to design an analysis system for remotely sensed images, one or more images may be given for which the class labels of the pixels are known. In other words, for every given pixel, it is known that, whether it corresponds to water, land or vegetation etc. Based on these (training) images, possible schema are devised in such a way that as and when new images come, class labels are assigned to every pixel considering 'similarity' of the pixels in the new image with that in the training image.

In the real world, the problem of pattern classification, connected with human cognitive activities, is faced with fuzziness. Humans possess a remarkable innate ability to exploit the tolerance for fuzziness(,imprecision) to achieve tractability, robustness and low solution cost whereas the traditional techniques fail to do so. The problem of classification is the assignment of an object to one of a number of predetermined groups or classes. It is an easy task as long as the object concerned is well-defined and the boundaries of the groups or classes. In most practical instances, the classes are found to be overlapping. In such situations, although humans may perform the recognition task efficiently, it is not an easy task for other mechanical systems. A crisp model is often found to be inadequate. Similarity or indistinguishability is another important concept in pattern classification.

In such cases, fuzzy set theory may be used in pattern classification. The application of fuzzy sets in pattern classification may be found in [12, 13]. Approximate reasoning methodology based pattern classification may be found in [77]. Here, we present a classification algorithm based on similarity between features using a similarity based inference mechanism. For illustration, the proposed method is applied to classify Telugu (an Indian language) vowels.

Next we consider the design of a second class of problems, which has so far received possibly the maximum attention in the application of fuzzy logic, the rule-based fuzzy control of complex systems. Zadeh outlined the basic ideas underlying fuzzy control in [109, 111]. Among them, the concept of linguistic variables, fuzzy if-then rules, fuzzy algorithms and the compositional rule of inference are some essential ideas in the design of a fuzzy controller. However, it was the seminal work of Mamdani and Assilian [54] that showed how these ideas could be translated into a working control system. The contributions of Mamdani and his associates at Queen Mary College, along with the contribution of Van Naute Lemke and others at Delft Institute of Technology and those of Sugeno and others at the Tokyo Institute of Technology played a key role in the development of fuzzy control systems.

It was for the first time in 1974, that the application of fuzzy logic in process control was proposed at the 4th IFAC/IFIP International conference on digital computer applications to process control, held in Zurich, Switzerland [2]. Since then fuzzy logic control has attracted great attention from both the academic and industrial communities. Many people have devoted a great deal of time and effort to both theoretical research and implementation techniques for the design of fuzzy controllers. Considerable progress has been made in successfully applying fuzzy logic control to industries. In 1982, a fuzzy controller for a cement kiln was developed by Smidth and Co., at Denmark. Then an automatic-drive fuzzy control system for subway trains in Sendai, Japan was opened in 1987. A turning point in 1987 was the first consumer product- a shower head conceived and produced by Matshuhita. Other consumer products - washing machines, vacuum cleaners, camcorders, cameras and air conditioners - followed in quick succession. Fuzzy logic control technology has drastically reduced the development time and deployment cost for the synthesis of non-linear controllers for dynamic systems. All these gave fuzzy logic a much higher visibility in control.

Rule-based fuzzy control, as introduced by Zadeh [109] and implemented by Mamdani [52], is a useful tool for systems where the exact model is either not known or is too complex to be tractable in real time. The design of a rule-based fuzzy controller consists of a sequence of activities such as knowledge acquisition, controller structure definition, selection of rules, tuning a variety of gains and other controller parameters, modification of rules to improve performance and defuzzification.

Based on this understanding, in the following sections, we briefly address the two problems, mentioned earlier, and describe a typical similarity based solution for both application types (pattern classification and fuzzy control). We also present computer simulation results, to establish the success.

2 Application to pattern classification

Different approaches to pattern classification [12, 63] have been reported in the literature. Application of approximate reasoning methodology in pattern classification may also be found in [67, 77]. In this section, we present an illustrative application of similarity based reasoning using a rule-base for classification of patterns.

We represent a pattern or object as a vector of feature values. In approximate reasoning approach to pattern classification, each component of the feature vector is represented by a linguistic variable, instead of a real variable. Let there be p features in the feature vector, selected for the classification of a typical pattern. Each of which may be represented by the real-valued variables $\{F_1, F_2, \cdots, F_p\}$. Let there be a training data set $X = \{\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}\}$, where each $\mathbf{x_i} \in R^p$ and let there be c distinct classes. Let us assume that, for each point in the training data set, the actual class it has come from, is known. The problem is to design a rule-based system, using the similarity based reasoning scheme introduced in the previous chapter, so that unknown points may be classified.

The classifier may be designed, based on a set of rules of the form

$$\textit{If } X_1 \textit{ is } A_1^i \textit{ and } X_2 \textit{ is } A_2^i \textit{ and} \cdots X_p \textit{ is } A_p^i \textit{ then } C \textit{is} B$$

where, $X_j; j = 1, 2, \cdots, p$ are p linguistic variables corresponding to the p feature values $\{F_1, F_2, \cdots, F_p\}$, $A_j^i; j = 1, 2, \cdots, p$ are the linguistic values of the respective variables in the $i-th$ rule and B is a fuzzy set defined on the set of classes. It may be mentioned here that A_j^i, the value of the linguistic variable X_j in the i–th rule are not all distinct for each i and j. Each such rule is interpreted as a condition for the classification of patterns. Rules are generated either using expert operator's knowledge or by exploratory data analysis, keeping in consideration all practical possibilities.

At the learning stage, we first partition the feature space (the product of the domain of definitions of each feature) into representative classes. The partition is based on the definition of the linguistic term sets used for each linguistic variable representing the feature values. The training data set lie entirely on the feature space. The selected features are then fuzzified using some parametric membership definition function. Other membership function definitions may also be used. Next, we generate ***if-then*** rules as precondition for classification. The membership grade of a particular class in the fuzzy set B may be computed by the ratio of individual population over total population in a particular zone, determined by the linguistic terms. These rules are tested against the classification of the training data. If found satisfactory, we proceed further; otherwise, the rules may be modified either by trial-and-error method or by some systematic methods like Genetic algorithms, gradient search [12, 13].

At the classification stage, the selected features are fuzzified using triangle shaped fuzzy sets as the triangular-shaped membership function is the most economic one. For a particular feature, we compute the similarity between the fuzzified feature value and the different linguistic values of that feature, as used in the rule-base.

From them, we are interested in finding the maximum similarity value and the corresponding fuzzy set for which the similarity value is maximum. Next we choose those rules with the said feature having the corresponding fuzzy value. This process is continued for all features, defining a pattern, and for all rules for which the previous features have fuzzy values with maximum similarity. Ultimately, a single rule will be found satisfying the maximum similarity value criterion. For, otherwise, the system would have rules with identical rule-antecedent. Here lies the novelty in similarity based approach to pattern classification. The best rule has been chosen from a class of rules for possible firing. Then the similarity based approximate reasoning methodology is applied (with similarity value equal to the minimum of all the maximum similarity values computed earlier for each features) to obtain a fuzzy set representation of different classes, using the algorithm **SAR** for model 6. At the time of making a non-fuzzy decision, the class with maximum membership value is selected. Ties (for patterns lying in the overlapped zone), if arises, may be broken arbitrarily (use first-of-maxima or last-of-maxima). Multiple classification is a product of fuzzy algorithm.

Let us now present the entire discussion in the following algorithm :

ALGORITHM 5.1 : Pattern Classification

Step 1 : Take an input vector $\mathbf{x} \in R^p$ whose i^{th} component x_i is the value of F_i for $i = 1, 2, \cdots, p$.

Step 2 : Fuzzify each real values x_i using triangular membership function. Let these fuzzified values be denoted as $\{A_1', A_2', \cdots, A_p'\}$.

Step 3 : Compute $S(A_1', A_1^i)$ where the index i ranges over all fuzzy sets for the first feature F_1. Set $A_1^{k_1} = \max_i S(A_1', A_1^i)$. Then compute $A_j^{k_j} = \max_i S(A_j', A_j^i)$ where i varies over those rules for which F_1 is $A_1^{k_1}$ and F_2 is $A_2^{k_2}$ and $\cdots$ and F_{j-1} is $A_{j-1}^{k_{j-1}}$ $j = 2, 3, \cdots, p$. Let the rule to be fired be if F_1 is $A_1^{k_1}$ and F_2 is $A_2^{k_2}$ and $\cdots$ and F_p is $A_p^{k_p}$ then B^k.

Step 4 : Apply SAR algorithm for model 6, with fuzzified $\{A_1', A_2', \cdots, A_p'\}$, for firing the rule as found in Step 3, using the minimum of all the maximum similarity values as obtained in Step 3 and obtain a consequence B' on the class of patterns.

Step 5 : The fuzzy set B', as obtained in Step 4, is used to find the class with maximum membership value as the class represented by the input vector $\mathbf{x}$. Ties, if arises, may be broken arbitrarily.

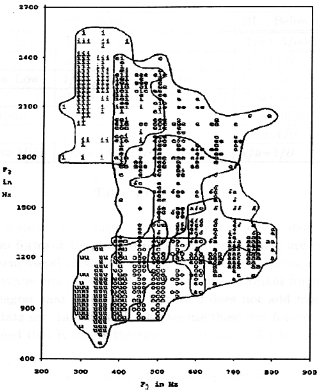

Figure 5.1: Telugu vowel recognition data

2.1 An application to Telugu vowel classification

Let us now apply Algorithm **Pattern classification** to classify Telugu vowels. Telugu is one of the Indian languages spoken in the southern part of the country. The data set consists of 710 discrete phonetically balanced speech samples for the Telugu vowels in consonant-vowel nucleus-consonant (CNC) form. These samples are generated from three male informants(in the age group of 28 to 30 years) on an AKAI-type recorder. The spectrographic analysis has been done on a Kay Sonograph Model 7029-A. The total bandwidth of the system is 80 Hz to 8 kHz with a resolution of 300 Hz.

∧	ZE : Zero	BL : Below Low
BZ : Below Zero	.5/o+1/u	.1/e+.5/o+1/u
ZE : Zero	.5/e+1/i	.5/e+1/i
BL : Below Low	1/e+.1/o+.1/u+.5/∂	1/e+.5/i+.5/∂
LO : Low	.5/a+.1/e+1/o+.1/u+.1/∂	.5/a+.5/e+1/o+.1/u+1/∂
ME : Medium	1/e+.5/i	.1/a+.5/o
HI : High	1/e+1/∂	1/e+.1/∂
AH : Above High	1/a+.1/o+1/∂	1/a+1/∂

Table 5.1: The Rule-base

The data set is shown in Figure 5.1). Actually, Figure 5.1 shows the scatter plot of the first two formant frequencies of the data set. There are significant overlap between different vowel classes. Let X_1 and X_2 be two linguistic variables used to represent the vague description about the two given formant frequencies. Reported results [78] suggest that inclusion of feature 3 does not add much discriminating power to the data set. In this study also we use these two features. The initial rule set is so designed that it covers the entire input space. Each rule is of the form

$$\textit{If } X_1 \textit{ is } A_1^i \textit{ and } X_2 \textit{ is } A_2^i \textit{ then } B^i.$$

Here, A_j^i are the linguistic values of the linguistic variable $X_j; j = 1, 2$; that appear in the body of the i^{th} rule. The fuzzy set on the consequent part of the rule may be generated using the technique mentioned earlier [78]. They may also be assigned by experts or may be learnt. Since our intention is to show the efficiency of the proposed similarity based reasoning methodology, we use a rule set similar to the one used in [78].

For X_1 and X_2 only 5 and 7 primary linguistic values are allowed which spanned the entire input space. Therefore, we consider thirty five rules as shown in Table 2.1 and Table 2.1. The exact definitions of the fuzzy sets describing the antecedents of the rules are given in Table 5.3 and Table 5.4. In order to classify an unknown pattern, the given feature values are first fuzzified using appropriate triangular fuzzy sets. These are compared with the antecedents of each rule, in order to find out the most suitable rule for firing, on the basis of similarity index. This rule is fired using the concept of similarity based reasoning, for the determination of a particular

∧	LO : Low	ME : Medium	HI : High
BZ	.5/e+.5/o+1/u	.5/e+.1/o+.1/u	.5/e+.5/i
ZE	1/o+1/u	.1/e+1/o+1/u+.1/∂	1/e+1/o+.5/u+.5/∂
BL	1/e+1/i+.1/∂	1/e+1/i	.1/a+1/o+.1/u
LO	.5/a+1/e+.1/o+1/∂	1/e+.1/i+1/∂	1/e+.5/i+.1/∂
ME	1/a+1/o+.1/∂	1/a+.1/e+.5/o+1/∂	1/a+0.5/e+.1/o+1/∂
HI	.5/e	.1/a+.1/o	1/a+.1/o+.1/∂
AH	.5/e+.5/∂	.5/e+.1/∂	.1/e

Table 5.2: The Rule-base(contd.)

ZE	0.1 0.5 1.0 0.5 0.1 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
BL	0.0 0.0 0.1 0.5 1.0 0.5 0.1 0.0 0.0 0.0 0.0 0.0 0.0
LO	0.0 0.0 0.0 0.0 0.1 0.5 1.0 0.5 0.1 0.0 0.0 0.0 0.0
ME	0.0 0.0 0.0 0.0 0.0 0.0 0.1 0.5 1.0 0.5 0.1 0.0 0.0
HI	0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.1 0.5 1.0 0.5 0.1

Table 5.3: Fuzzy set representation of the linguistic values for Feature F_1

BZ	0.25 0.5 1.0 0.5 0.25 0.10 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
ZO	0.00 0.1 0.25 0.5 1.0 0.5 0.25 0.1 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
BL	0.00 0.0 0.0 0.1 0.25 0.5 1.0 0.5 0.25 0.1 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
LO	0.00 0.0 0.0 0.0 0.0 0.1 0.25 0.5 1.0 0.5 0.25 0.1 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0
ME	0.00 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.1 0.25 0.5 1.0 0.5 0.25 0.1 0.0 0.0 0.0 0.0 0.0
HI	0.00 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.1 0.25 0.5 1.0 0.5 0.25 0.1 0.0 0.0
AH	0.00 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.1 0.25 0.5 1.0 0.5 0.25 0.1

Table 5.4: Fuzzy set representation of the linguistic values for Feature F_2

class, which the input data correspond to. At the time of making a hard decision for a single class, the classes with maximum possibility value have been considered. Let there be $n(> 0)$ classes having maximum possibility value. When $n = 0$, we are done and when n exceeds 1, we may use two strategies as our hardening scheme. One strategy is to choose the class with maximum membership value, that appears in a fixed order (first, middle, last) from the list of classes just obtained. The other one is to generate uniformly distributed random number in the unit interval. Then partition the unit interval into n disjoint sub-intervals such that the union of the sub-intervals is the unit interval. The generated random number falls in exactly one sub-interval. Let it be the $p - th$ sub-interval $(1 \leq p \leq n)$. The $p - th$ candidate from the list of classes with maximum membership value is chosen as the class representing the input frequencies.

2.2 Results and discussion

The results of Telugu vowel recognition may be explained by a confusion matrix as presented in Table 5.5. The number in a cell of the confusion matrix represents the number of instances in which the same decision was made by the proposed algorithm driven by similarity based reasoning. The $(i, j) - th$ entry in the confusion matrix shows the number of instances in which the actual class is known to be i and the proposed similarity based algorithm driven classifier suggested j. Thus, the diagonal values indicate the number of utterances correctly identified. The performance of the proposed classifier shows that the confusion in classification is limited only to neighbouring classes with which it has some overlap as may be seen from Figure 5.1. While generating Table 5.5, as mentioned earlier, ties were broken by choosing the first class with maximum grade in the resultant fuzzy set. From Table 5.5, it is found that the average recognition score is 70%. If, instead, we choose the last class with maximum grade, to break a tie situation, then the entries in the corresponding confusion matrix would be as shown in Table 5.6. Here, we find the average recognition score is 58.17%. Table 5.7 presents the same result as shown in Table 5.5, when a middle class is chosen, in a tie situation and the recognition score is found to be 70.00%. The middle is a unique class if the number of contestants are odd and is the first from the two middle classes when the number of candidates are even.

Here, we are using similarity based reasoning mechanisms and we are using triangular possibility distributions. Considering the discussions around Figure 4.7 we

	a	e	i	o	u	ə	Total	Recognition Score(%)
a	60	4	0	19	0	0	83	72.29
e	1	190	9	0	0	0	200	95.00
i	0	54	79	0	0	0	133	59.40
o	8	4	0	104	0	0	116	89.66
u	0	2	0	46	64	0	112	57.14
ə	27	25	0	14	0	0	66	00.00
Correct choice	60	190	79	104	64	0	avg	70.00

Table 5.5: Recognition score for Telugu vowels using first-of-maxima

	a	e	i	o	u	ə	Total	Recognition Score(%)
a	56	3	0	10	0	14	83	67.47
e	5	112	28	10	2	43	200	56.00
i	0	28	105	0	0	0	133	78.95
o	3	8	0	63	38	4	116	54.31
u	0	4	0	27	80	1	112	71.43
ə	2	7	0	10	0	47	66	71.21
Correct choice	56	112	105	63	80	47	avg	66.56

Table 5.6: Recognition score for Telugu vowels using last-of-maxima

may conclude that this is an useful technique for breaking a tie situation. These results are shown in order to make a comparative analysis among different hardening schema. The result of a random choice to break the tie situation is provided in Table 5.8. First of all, uniformly distributed random numbers are generated in the unit interval $[0, 1]$. Next, the interval is partitioned into n sub-intervals, where the number of candidates waiting for the selection is exactly n. The generated random number lies in one of the sub-intervals. The corresponding serial number of the sub-interval in the unit interval is used in our selection. Here, the result of average recognition score is found to be 64.37%. Considering Figure 5.1, the performance of the classifier is quite satisfactory because in the 2-space there are significant overlappings between different pairs of classes. Some improvement in performance may be realized through tuning of the membership functions as well as through change of the rule-base.

	a	e	i	o	u	∂	Total	Recognition Score(%)
a	60	4	0	19	0	0	83	72.29
e	1	190	9	0	0	0	200	95.00
i	0	54	79	0	0	0	133	59.40
o	8	4	0	104	0	0	116	89.66
u	0	2	0	46	64	0	112	57.14
∂	27	25	0	14	0	0	66	00.00
Correct choice	60	190	79	104	64	0	avg	70.00

Table 5.7: Recognition score for Telugu vowels using middle-of-maxima

	a	e	i	o	u	∂	Total	Rec. Score(%)
a	41	0	0	7	0	35	83	49.40
e	1	131	20	10	0	38	200	65.50
i	0	33	100	0	0	0	133	75.19
o	3	7	0	71	17	18	116	61.21
u	0	8	0	23	79	2	112	70.54
∂	12	7	0	12	0	35	66	53.03
Correct choice	57	186	120	123	96	128	avg	64.37

Table 5.8: Recognition score for Telugu vowels using random selection

	a	e	i	o	u	∂	Total	Recognition Score(%)
a	72	0	0	8	0	3	83	86.75
e	0	184	0	5	0	11	200	92.00
i	0	8	125	0	0	0	133	93.98
o	1	0	0	112	0	3	116	96.55
u	0	8	0	11	92	1	112	82.14
∂	0	2	0	0	0	64	66	96.97
Correct choice	72	184	125	112	92	64		91.41

Table 5.9: Recognition score of Telugu vowels assuming correct classification when the suggested choices include the actual class

Different types of classifiers	Recognition Score(%) data(, Figure 5.1)
First-of-maxima	70.00
middle-of-maxima	70.00
last-of-maxima	66.56
Random-of-maxima	64.37

Table 5.10: A comparative study

Figure 5.1 suggests that, for any classifier which uses only two features, some misclassifications are bound to result. If the proposed scheme is a consistent one, then any fuzzy logic based classifier is likely to suggest more than one choice with the highest membership value for points lying in the overlapping regions. In order to establish that it is indeed the case, let us assume that the system output is correct if the alternatives suggested by the rule-based system include the correct class. Table 5.9 is generated keeping this in mind. To make it more clear, the scores are computed and presented in Table 5.9.

If the rule-base suggests only one class then the recognition score for that class is increased or else, if the system suggests more than one class containing the correct class then the recognition score for the correct class is increased. This results in a significant improvement in the recognition score(91.41%) as depicted in Table 5.9. On scrutiny, we find that, when more than one alternative satisfies our criterion for selecting a single class, then the input frequencies correspond to (comes from) some overlapped region containing the correct class. In order to compare, all the results are tabled in Table 5.10.

3 Application to fuzzy control

The field of fuzzy logic control(FLC) is one of the most active and fruitful areas of research in which fuzzy set theory is applied. Fuzzy controllers may perform at least as well, and in some cases better, than conventional model-based controllers, especially, when applied to processes difficult to model; and when there is a significant heuristic knowledge from human operators. The important properties of fuzzy controllers are their high flexibility and low sensitivity to parametric variations,

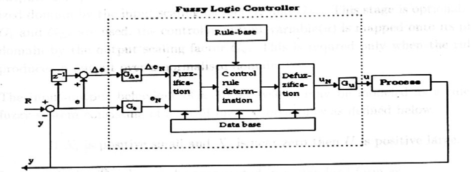

Figure 5.2: Block diagram of a fuzzy logic controller

which enables their application to varying problems.

Mamdani and Assilian reported the first implementation of a fuzzy controller in 1974 [53], which demonstrated the viability of a fuzzy controller for a small model steam engine. The purpose was to control engine speed and boiler steam pressure by adjusting the heat applied to the boiler and the throttle setting on the engine.

The action of a fuzzy controller is based on a collection of *if-then* rules whose antecedent(s) and consequent are fuzzy values. The process state parameters are first fuzzified and then combined with these rules for a fuzzy action. The fuzzy action is defuzzified for a precise action for the said process. Here, we use similarity concept in deriving the fuzzy control action. In order to demonstartte the effectiveness of similarity based approach to inference, we first propose and then use a new specificity based defuzzification procedure. For that, let us begin with a discussion on some basic concepts underlying conventional fuzzy control.

3.1 Conventional fuzzy control

For simplicity sake, in this investigation, we consider a multi-input single output(MISO) fuzzy system. A block diagram of such a fuzzy logic controller is shown in Figure 5.2. In Figure 5.2, the physical values of the current process state variables i.e., the error and the change of error are respectively denoted as e and Δe. The two symbols y and u respectively denote the system output and the controller information for the proper functioning of the fuzzification module, the rule-base and the defuzzification module. Whereas, the function of the rule-base is to represent, in a structured way, the control policy of an experienced process operator and/or control engineer given in the form of a set of production rules as described earlier.

Inference - This is a decision making stage under the control rule determination block, which generates fuzzy output corresponding to fuzzified inputs using fuzzy rules from the rule-base. A typical fuzzy rule actually enumerate the process input-output relation on a portion of the input space. The method of approximate reasoning is applied in the derivation of the fuzzy output. The inputs to this block are the fuzzified state variables, which are first used in rule selection and then used in output determination through rule firing. Two types of inference are commonly used in the design of fuzzy logic controllers with multiple rules. In the first case, all rules in the rule-base are combined to produce a single relation to represent the input and the output behaviour of the system over the entire input space. Then the fuzzy input is used to infer according to compositional rule of inference. The resultant is a fuzzy set. In the second case, each rule is fired separately with the same fuzzy input, using approximate reasoning mechanism, to generate the fuzzy output of the rule and then the resulting fuzzy outputs are combined to produce a single fuzzy set as the possible fuzzy action.

Defuzzification - The result of rule firing, using any of the above mentioned approaches to inference, is a fuzzy set. This is interpreted at the semantic level as the possible values of the desired output. We need to determine a precise action for the process to be controlled. The purpose of defuzzification is to obtain a scaler value $u \in U$, from the said output fuzzy set, as the control action. Then, if necessary, de-normalisation is performed on the output so as to obtain the corresponding action on its physical domain.

Different methods of defuzzification are in use [16]. Let A be any fuzzy set defined over the universe of discourse $U = \{u_1, u_2, \cdots, u_n\}$ and let a defuzzified value of A be denoted as u^*. In the following, some of the well known methods of defuzzification for the computation of u^* are presented.

Center-of-gravity :

This is the most commonly used defuzzification method. Here, the defuzzified value is given by

$$u^* = \frac{\sum_{i=1}^{n} u_i.\mu_A(u_i)}{\sum_{i=1}^{n} \mu_A(u_i)}. \tag{5.1}$$

First-of-maxima :

This is the simplest of all the defuzzification schema, in use. As the name suggest, first member with the maximum membership value in the output fuzzy set is taken as the corresponding defuzzified value. Thus, the defuzzified value will be given by

$$u^* = \inf_{u \in U}\{u \in U \mid \mu_U(u) = \max_{v \in U} \mu_U(v)\}. \tag{5.2}$$

We may use an alternative version of the above scheme, known as the last-of-maxima.

So far, we are concerned with defuzzification of a fuzzy set. In rule-based fuzzy control, when we use the firing of individual rules for inference, we find a collection of clipped fuzzy sets as possible outputs. Let us now consider some widely used methods of defuzzification procedure in such cases. For that, let there be m fuzzy sets $\{A^{(k)}; k = 1, 2, \cdots, m\}$.

Center-of-sums :

In this method, the overlapping areas are considered more than once. The computational part is simple as compared with Center-of-gravity, and works fast. Here, the defuzzified output value is given by

$$u^* = \frac{\sum_{k=1}^{m}\sum_{i=1}^{n} u_i.\mu_A^{(k)}(u_i)}{\sum_{k=1}^{m}\sum_{i=1}^{n} \mu_A^{(k)}(u_i)}. \tag{5.3}$$

Note here that the Center-of-gravity method of defuzzification for clipped fuzzy sets is exactly the same as Center-of-sums method of defuzzification, except that the latter uses the overlapped areas only once.

Height :

This method of defuzzification demands strictly convex fuzzy sets. The individual peak values of consequent fuzzy sets of the fired rules are used to generate the weighted average of these peak values. It is a simple method and works faster than the Center-of-sums method. Let $p^{(k)}$ be the peak value of $A^{(k)}$ and $h^{(k)}$ be the corresponding height of the clipped version of $A^{(k)}$ or the firing strength of the $k-th$ rule. Then the defuzzified value will be given by

$$u^* = \frac{\sum_{k=1}^{m} p^{(k)}.h^{(k)}}{\sum_{k=1}^{m} h^{(k)}}. \tag{5.4}$$

A detailed study on the conventional approach to fuzzy control may be found in [50].

3.2 Similarity based fuzzy control

Now, let us propose a different strategy for fuzzy control based on the concept of similarity. The concept of similarity between fuzzy sets are used in selecting rules from the rule-base, to be fired for the particular input specification and then in deriving the control action from a set of rules and input values. In this regard, a new scheme for defuzzification based on a measure of specificity of fuzzy sets is proposed. Let us consider a process controlled by p inputs.

Fuzzification - Since we are considering similarity based reasoning, in this module, we propose to consider only parametric, functional definitions of certain uniform geometric shaped fuzzy sets. The most popular choices include, triangular-, trapezoidal- and bell-shaped functions. Among them, the parametric definition of a triangular shaped function is the most economic one. We use a triangular membership function both for fuzzification and rule-base generation. Since, we are considering similarity concept in choosing a rule from the rule-base for possible firing, the real values of each process state variables are fuzzified using similar triangles, i.e., of the same width and height as is used in the rule-base. As for example, let e_t denote the observed value of the state variable error e at time t, then e_t is fuzzified by a symmetric triangular membership function with peak at e_t(, membership grade 1) and base-width $b_t = b$, where b is the base-width of every fuzzy set defined for the linguistic variable corresponding to error, for the purpose of rule-base generation. The values of b_t may be different also. Note here that, in conventional fuzzy controller design, normally fuzzy singletons are used.

Knowledge base - This module remains the same as described in the previous section.

Inference - Here we shall consider only rule-based inference using similarity concept. In this approach, we first select the rules to be fired for a given input combination. In selection, we use the similarity between the given and the antecedent fuzzy sets as a basis. In order to select the rules for firing, we proceed as folows :

Let A_i' be the fuzzified values of $x_i; i = 1, 2, \cdots, p$. x_i is the i^{th}-component of the state vector $\mathbf{x} = (x_1, x_2, \cdots, x_p) \in R^p$. Let the $k-th$ rule be

$$R^k : \text{ if } x_1 \text{ is } A_1^k \text{ and } x_2 \text{ is } A_2^k \text{ and } \cdots \text{ and } x_p \text{ is } A_p^k \text{ then } u \text{ is } U^k.$$

Let us set

$$\alpha_k = \min_{i=1,2,\cdots,p} \{S(A_i', A_i^k)\}. \tag{5.5}$$

Now, R^k is fired if $\alpha_k > 0$. This value of α_k may be interpreted at the semantic level as a measure of agreement between the fact and the antecedent of the $k-th$ rule. This completes the first part of reasoning.

Next, in the rule execution stage of inference, we first compute the input-output relation from the translation of the rule in consideration. Then we modify the said relation using the rule firing strength (α_k) and compute the output fuzzy action.

Defuzzification - The result of rule firing is a class of clipped fuzzy sets defined over the same universe of discourse. We are required to determine a single real value from those fuzzy outputs. Earlier we discussed several methods of defuzzification like Center-of-gravity, Center-of-sums, height, etc. For these schema, the basic idea is as follows :

If the membership grade of a particular element, in the output fuzzy set, is high then this contributes more to the defuzzified output.

Such concepts cannot be used in the present case of similarity based reasoning paradigm. Here, the lower the similarity value between the rule-antecedent and the fact, the closer the output to the least specific case (i.e., unknown) with the membership grade of elements in the output fuzzy set close to '1'. In such cases, a natural choice would be to use specificity information of the output fuzzy sets in defuzzification. In our scheme, the basic idea will be : the element with high membership value should come from the most specific output fuzzy set.

Our first defuzzification scheme is based on this concept. The most specific among the output fuzzy sets has the maximum impact on the resultant choice. For that we compute specificity value of each output fuzzy sets separately.

Specificity based defuzzification :

Let there be m clipped fuzzy sets $\{A^{(k)}; k = 1, 2, \cdots, m\}$ and let $\{s_{(k)}, p^{(k)}; k = 1, 2, \cdots, m\}$ be the specificity associated with $A^{(k)}$ as well as the height of the consequent of the k^{th}-rule. Then the defuzzified value u^* will be given by

$$u^* = \frac{\sum_{k=1}^{m} p^{(k)}.s_{(k)}}{\sum_{k=1}^{m} s_{(k)}}. \tag{5.6}$$

Other methods of defuzzification may also be defined. We next suggest another scheme which looks similar to the center-of-gravity method as described earlier. Notice here that among the set of rules fired, the rule corresponding to best matching between the fact and the antecedent will give the most specific output as compared to the outputs from other rules having lower similarity value. Thus, unlike conventional defuzzification techniques, the conjunction of all output fuzzy sets may be taken as the output of the rule-based system, which then may be defuzzified using center-of-gravity method of defuzzification as already defined.

Modified center-of-gravity :

Let there be m clipped fuzzy sets $\{A^{(k)}; k = 1, 2, \cdots, m\}$ and let $A = \bigcap_{i=1}^{m} A^{(k)}$. Then the defuzzified value u^* will be given by

$$u^* = \frac{\sum_{i=1}^{n} u_i.\mu_A(u_i)}{\sum_{i=1}^{n} \mu_A(u_i)}. \tag{5.7}$$

Based on above discussions, let us write an algorithm in the following.

ALGORITHM 5.2 : Fuzzy Control

Step 1: Let $\mathbf{x}^t = (x_1^t, x_2^t, \cdots, x_p^t)$ be the process state vector at time t.

Step 2: Fuzzify each x_i^t, the real value for the process state variable, using triangular membership function.

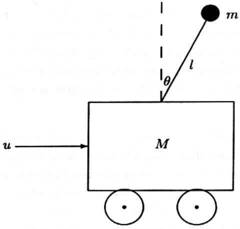

Figure 5.3: The Inverted Pendulum on a moving cart

Step 3: For each rule R^k, compute similarity index between the input fuzzy set and the antecedent fuzzy set, for all variables. Obtain α^k, the minimum of all the similarity indices. Take this as a matching grade of the rule.

Step 4: Perform similarity based approximate reasoning, taking one rule at a time for which $\alpha^k \geq \epsilon$; $0 \leq \epsilon \leq 1$.

Step 5: Defuzzify the fuzzy sets, as obtained in Step 4, by using either the specificity based defuzzification scheme or the modified center-of-gravity method.

Step 6: Use the defuzzified result, as found in Step 5, as the process input for the next time interval. Set $t \leftarrow t + 1$. Go to Step 2.

3.3 Inverted Pendulum problem

In this section, we consider an illustrative application of the proposed algorithm **Fuzzy Control** for controlling an inverted pendulum mounted on a cart as may be seen in Figure 3.3. The problem is to balance the pole in a vertical position by applying an appropriate horizontal force u on the cart. Here, we shall consider only the motion of the angular position of the pole on the cart, and not the position of the cart or the velocity of the cart. Let us assume that the cart travels in one direction only along a frictionless track, i.e., the motion to be purely two-dimensional. It is also assumed that the pendulum mass is concentrated at the end of the rod and the rod is massless. The inverted pendulum is unstable in that it may fall over anytime in any direction unless a suitable control force is applied. Therefore, the problem is to design a control system in a way that, given any initial conditions (may be caused by disturbances), the pendulum may be brought back in the vertical position. The motion of the car will not be considered. A detailed description may be found in [60].

For a solution to the above problem, let us first define the process parameters. Let θ be the angular position of the pole with respect to the vertical. Since we expect to keep the inverted pendulum vertical, the angle θ is assumed to be small. Let u be the driving force on the cart, $2l$ be the length of the pole, g be the gravitational acceleration(, a constant), M be the mass of the cart and m be that of the pole. Let x denote the distance traversed at time t, measured from some fixed point in the plane of the motion.

Let x_1,x_2,x_3 and x_4 denote the four state variables used to represent the angle dynamics of the pole as under

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \\ x_3 &= x \\ x_4 &= \dot{x} \end{aligned}$$

where x denotes the location of the cart from a fixed point, about which the pendulum moves. Then it is easy to describe the dynamics of the motion of the pendulum [60] in terms of state equations as below:

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= \frac{M+m}{Ml} g x_1 - \frac{1}{Ml} u \\ \dot{x_3} &= x_4 \\ \dot{x_4} &= -\frac{m}{M} g x_1 + \frac{1}{M} u \end{aligned}$$

where g is taken to be $9.8m/s^2$. Let us set $m = 0.1kg$, $M = 2kg$ and $l = 0.5m$ as exactly in [60]. These numerical values when substituted in (5.8), we obtain

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= 20.601 x_1 - u \end{aligned}$$

	∧	NB	NM	NS	ZE	PS	PM	PB
Neg. Big :	NB	NB	NB	NB	NM	NS	NS	ZE
Neg. Medium :	NM	NB	NM	NM	NM	NS	ZE	PS
Neg. Small :	NS	NB	NM	NS	NS	ZE	PS	PS
Zero :	ZE	NM	NM	NS	ZE	PS	PM	PM
Pos. Small :	PS	NS	NS	ZE	PS	PS	PM	PB
Pos. Medium :	PM	NS	ZE	PS	PM	PM	PM	PB
Pos. Big :	PB	ZE	PS	PS	PM	PB	PB	PB

Table 5.11: The rule-base

$$\dot{x_3} = x_4$$
$$\dot{x_4} = -0.4905x_1 + 0.5u.$$

In this example, the operating ranges for θ(,the angle) is set to be $[-0.1, 0.1]$, measured in *radian* and that for $\dot{\theta}$ (the angular velocity) is set to be $[-0.15, 0.15]$, measured in *radian/sec*. Over these ranges, seven equi-spaced (similar) as well as half-overlapping (crosses at the membership value 0.5) isosceles triangular fuzzy sets have been generated off-line, in order to define seven linguistic labels for each of the linguistic variables. Using trial and error method, the operating range for the applied force is found to be $[-6.0, 6.0]$, measured in Newton unit. Over this ranges, seven similar such fuzzy sets are generated off-line. According to human expert's description the following fuzzy conditional statements may be taken to represent the relation between the input (the angle and the angular velocity) and the output (the controlling force to be applied on the cart) variables. The rule-base is shown in Table 5.7. There are seven rows and an equal number of columns in the table, indicating the presence of exactly forty-nine distinct rules in the rule-base. Considering the symmetry of the motion of the pendulum about the vertical, the relational Table 5.7 is found to be symmetric about the diagonal. Each entry in the Table 5.11 is a fuzzy set, denoting the control action to be taken when the cause of action will be determined by the position of that particular entry in the table. The normalized triangular fuzzy sets as given in Table 5.11 for the three different categories θ, $\dot{\theta}$ and the applied controlling force u are defined using the following function with adjustable parameters:

$$\mu(x) = \begin{cases} 1 - \frac{1}{c} \mid x - l \mid, & \text{if } l \leq x \leq l + 2c \\ 0 & \text{otherwise.} \end{cases} \quad (5.8)$$

	l-value	θ	$\dot{\theta}$	action
Neg. Big :	NB	-0.100	-0.1500	-2.4
Neg. Medium :	NM	-0.075	-0.1125	-1.8
Neg. Small :	NS	-0.050	-0.0750	-1.2
Zero :	ZE	-0.025	-0.0375	-0.6
Pos. Small :	PS	0.000	0.0000	0.0
Pos. Medium :	PM	0.025	0.0375	0.6
Pos. Big :	PB	0.050	0.0750	1.2

Table 5.12: Left-end points of the triangular distribution for different fuzzy sets under a single category

The parameters of the membership function of the elements in the different category of fuzzy sets, as defined by (5.8) are given in the following Table 5.8 and Table 5.9. There are seven rows, indicating seven linguistic labels and three columns, representing three different linguistic variables, used in the process. There are, in all, twenty-one entries in this Table 5.8, representing the left-end point of the parametric triangular membership function. The three values in Table 5.9 represent the respective half-length of the base of each triangle formed under three different classes. In each case, the fuzzy sets are defined by membership values at fifty equi-spaced points, generated using the step-lengths listed in Table 5.11, from the respective universe of discourse, as stated.

3.4 Results and discussion

The results of four simulations, following algorithm **Fuzzy Control** and using the said two (specificity based, centroid based) defuzzification methods are presented in Table 5.15 and Table 5.16. The variation of θ, the angular position of the pole, over time are also provided in the Figures 5.4, 5.5, 5.6, 5.7, 5.8), for specificity based defuzzification method. The centroid based results are presented in Figures 5.9, 5.10, 5.11, 5.12, 5.13. In all cases, initially the pole is given a small angular displacement in the range of $\pm 5°$ together with a small angular velocity $\pm 10°$/per second and let it go. The results in Table 5.15 and Table 5.16 show that the behaviour of the pendulum is consistently good.

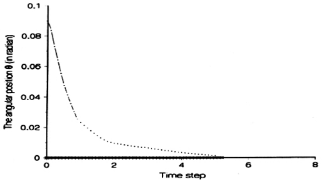

Figure 5.4: Specificity based controller characteristic : $\dot{\theta} = 0.0$

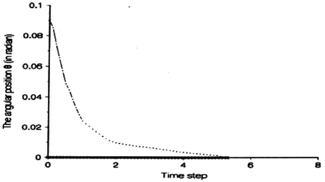

Figure 5.5: Specificity based controller characteristic : $\dot{\theta} = 0.075$

c-value	θ	$\dot{\theta}$	action
	0.025	0.0375	0.6

Table 5.13: Width of triangle for different categories of fuzzy sets

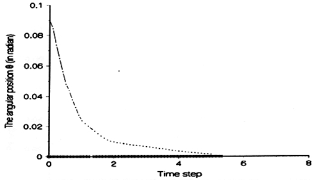

Figure 5.6: Specificity based controller characteristic : $\dot{\theta} = -0.075$

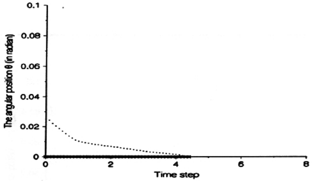

Figure 5.7: Specificity based controller characteristic : $\dot{\theta} = 0.0$

h-value	θ	$\dot{\theta}$	action
	0.004	0.006	0.096

Table 5.14: Step length for the generation of three categories of fuzzy sets

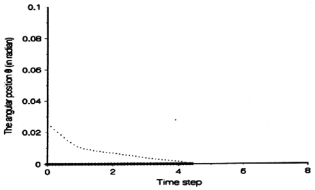

Figure 5.8: Specificity based controller characteristic : $\dot{\theta} = -0.05$

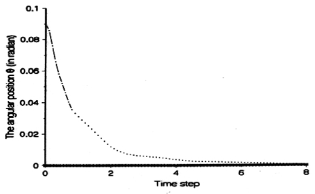

Figure 5.9: Centroid based controller characteristic : $\dot{\theta} = 0.0$

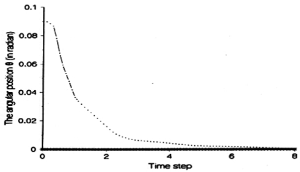

Figure 5.10: Centroid based controller characteristic : $\dot{\theta} = 0.075$

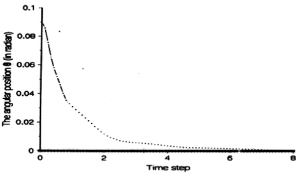

Figure 5.11: Centroid based controller characteristic : $\dot{\theta} = -0.1$

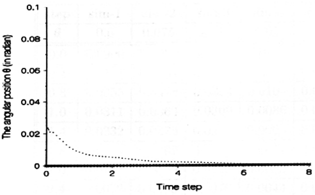

Figure 5.12: Centroid based controller characteristic : $\dot{\theta} = 0.05$

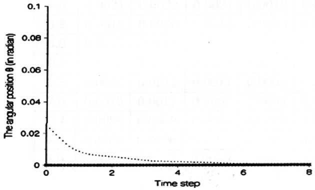

Figure 5.13: Centroid based controller characteristic : $\dot{\theta} = -0.05$

step	sim-1	sim-2	sim-3	sim-4	sim-5
$\dot{\theta}$	0.0	0.075	-0.1	0.05	-0.05
0.0	0.0900	0.0900	0.0900	0.0250	0.0250
0.4	0.0581	0.0781	0.0579	0.0170	0.0173
0.8	0.0355	0.0478	0.0354	0.0104	0.0105
1.0	0.0311	0.0364	0.0309	0.0086	0.0087
1.4	0.0232	0.0275	0.0231	0.0065	0.0065
1.8	0.0154	0.0197	0.0154	0.0057	0.0057
2.0	0.0118	0.0159	0.0118	0.0053	0.0053
2.4	0.0079	0.0096	0.0079	0.0044	0.0045
2.8	0.0062	0.0069	0.0062	0.0034	0.0035
3.0	0.0059	0.0063	0.0058	0.0030	0.0030
3.4	0.0052	0.0055	0.0051	0.0024	0.0024
3.8	0.0042	0.0048	0.0042	0.0022	0.0022
4.0	0.0037	0.0043	0.0037	0.0021	0.0021
4.4	0.0028	0.0033	0.0028	0.0019	0.0019
4.8	0.0024	0.0025	0.0024	0.0016	0.0016
5.0	0.0023	0.0024	0.0023	0.0015	0.0015
5.4	0.0021	0.0022	0.0021	0.0012	0.0012
5.8	0.0018	0.0020	0.0018	0.0009	0.0009
6.0	0.0017	0.0018	0.0017	0.0008	0.0008
6.4	0.0014	0.0016	0.0014	0.0007	0.0007
6.8	0.0011	0.0013	0.0011	0.0005	0.0005
7.0	0.0010	0.0012	0.0009	0.0004	0.0004
7.4	0.0008	0.0009	0.0007	0.0003	0.0003
7.8	0.0006	0.0008	0.0006	0.0002	0.0002
8.0	0.0005	0.0007	0.0005	0.0002	0.0002

Table 5.15: Simulation results - centroid based defuzzification

step	sim-1	sim-2	sim-3	sim-4	sim-5
$\dot{\theta}$	0.0	0.075	-0.075	0.0	-0.05
0.0	0.0900	0.0900	0.0900	0.0250	0.0250
0.2	0.0730	0.0745	0.0736	0.0221	0.0219
0.4	0.0554	0.0570	0.0562	0.0187	0.0184
0.6	0.0414	0.0446	0.0438	0.0153	0.0149
0.8	0.0304	0.0331	0.0327	0.0124	0.0122
1.0	0.0232	0.0242	0.0238	0.0103	0.0103
1.2	0.0196	0.0206	0.0202	0.0093	0.0093
1.4	0.0162	0.0171	0.0152	0.0086	0.0086
1.6	0.0131	0.0139	0.0122	0.0079	0.0079
1.8	0.0108	0.0112	0.0102	0.0073	0.0073
2.0	0.0096	0.0098	0.0093	0.0069	0.0069
2.2	0.0089	0.0090	0.0086	0.0063	0.0063
2.4	0.0081	0.0083	0.0078	0.0056	0.0056
2.6	0.0075	0.0076	0.0073	0.0051	0.0051
2.8	0.0070	0.0071	0.0068	0.0044	0.0044
3.0	0.0065	0.0066	0.0063	0.0038	0.0038
3.2	0.0058	0.0059	0.0056	0.0032	0.0033
3.4	0.0052	0.0053	0.0051	0.0028	0.0028
3.6	0.0046	0.0048	0.0044	0.0024	0.0024
3.8	0.0040	0.0041	0.0038	0.0020	0.0020
4.0	0.0034	0.0035	0.0033	0.0016	0.0016
4.2	0.0029	0.0030	0.0028	0.0007	0.0009
4.4	0.0025	0.0026	0.0024	0.0000	0.0000
4.6	0.0021	0.0022	0.0020	0.0000	0.0000
4.8	0.0017	0.0018	0.0016	0.0000	0.0000
5.0	0.0013	0.0014	0.0006	0.0000	0.0000
5.2	0.0000	0.0002	0.0000	0.0000	0.0000

Table 5.16: Simulation results - specificity based defuzzification

Chapter 6

Truth-qualified propositions

1 Introduction

In the previous chapters we considered only those fuzzy statements which are taken *to be always true, i.e., held for granted. In this chapter, we consider the represen*tation and manipulation of such truth-qualified knowledge based on the theory of fuzzy sets, due to Zadeh. A possibilistic approach is also presented. First of all, we represent a truth-qualified statement in a standard form. Then we prescribe a list of equivalent formulae for their composition. The concept of similarity between prototypes of fuzzy sets is used to formulate methods for reasoning with truth-qualified formulae in the framework of similarity based approximate reasoning.

Truth-qualified vague propositions are often used in human reasoning. Simple such statements are of the form:

- Payel is tall is quite-true;
- Pressure is low is absolutely-false;
- Temperature is moderate is fairly-true;
- It is mostly-true that if wages rise then there is inflation.

It is easy to see that propositions of the above nature consists of two different yet interactive concepts and, therefore, it may be logically split into two parts, one of which is the truth-value assigned to the statement. As a case in point, when it is said that *Arka is intelligent* we mean that *Arka is intelligent is true.* Therefore, any true proposition may be conceived as a truth-qualified proposition. On the other hand, given any truth-qualified proposition as, *Aritra is handsome is true*, we may always construct an unqualified proposition *Aritra is handsome*, which is always true. This concept of conversion also holds for partially truth-qualified propositions, with which we are mainly concerned.

Bellman and Zadeh [3] have discussed the notion of partial truth and suggested a method for constructing a true statement underlying the incomplete statement. Tsukamoto [83] has taken advantage of the said concept and used it in reasoning with partially truth-qualified statements. We show that truth-qualified statements may be manipulated in the framework of similarity based approximate reasoning. Work in this regard may be found in [72, 73].

Bellman and Zadeh [3] assumed that this partial truth-qualification is local rather than absolute. In [3], the authors suggested that a partially true statement is equivalent to some modified fuzzy statement, which may be taken to be true as and when the partially true statement is given. They reconstructed the true statement, by some truth function modification process, from the partially true statement and obtained the corresponding representation of that partially true statement. Thus, from a given statement

$$X \text{ is } A \text{ is } \tau\text{\_true}$$

we may construct a fuzzy statement

$$X \text{ is } B \text{ is true}$$

according to the following

$$\mu_B(u) = \mu_\tau(\mu_A(u)). \tag{6.1}$$

The above result at once yields the following :

X is A is true is equivalent to X is A,

X is A is false is equivalent to X is not A.

In representing a truth-qualified statement we show that the above method of construction of a true statement out of a partially true statement is not necessary for similarity based reasoning purpose. Truth values may be manipulated at the time of deriving a consequence from truth-qualified propositions. To do that, we use a set of rules for the manipulation of truth value. Now, the partial truth-values are vague and are represented by fuzzy sets over the universe of discourse $[0, 1]$. Therefore, manipulation of such truth-values is performed according to the existing rules for manipulation of fuzzy sets. Then similarity based method for reasoning with vague concepts is used to infer a statement that may also be partially true. In the following section, let us first consider the issues concerning the representation of truth-qualified statements.

2 Representation of partially true knowledge

In this section, we consider issues concerning the representation of truth-qualified knowledge, in the framework of approximate reasoning. For this, let us consider the following simple statements and their symbolisation.

Payel is healthy is quite_true may be symbolised as :

Constitution(Payel) = *healthy* and Truth = *quite_true*;

Paulami is tall is absolutely_false, may be translated as :

Height(Paulami) = *tall* and Truth = *absolutely_false*;

Icy roads are slippery is fairly_true may take the form as :

Condition(Icy roads) = *slippery* and Truth = *fairly_true*.

Here, each of the attributes, Constitution, Height, Condition and Truth may be represented by linguistic variables and the terms *healthy*, *tall*, *slippery*, *quite_true* etc., the values of the attributes may be conveniently represented by fuzzy sets over the universe of discourse of the linguistic variables.

Thus, symbolically, we may take a simple truth-qualified statement in the form of

$$X \text{ is } A \text{ is } \tau\text{--true}$$

where τ−true is a fuzzy truth-value such as almost-true, mostly-true, etc. and X is A, a fuzzy proposition. Each such partially true statement may be logically split into two components a prediction about an object type component and an associated truth-value component. We consider fuzzy sets and the theory of fuzzy sets for the representation of such partially true statements. We also consider a possibilistic approach to the representation and manipulation of partially truth-qualified statements.

Throughout the thesis, the unit interval $[0,1]$ is taken as the set of possible truth-values. Any vague description about truth(/falsity) of a proposition may be represented as a fuzzy set over the unit interval, $[0,1]$. Typically, a simple prescription about truth(/falsity) may be as follows :

Truth-value is mostly_true.

The concept may be represented by a fuzzy set of possible truth values as :

$$\begin{aligned}\mu_{Truth}(w) &= \mu_{\text{mostly\_true}}(w)\\ &= 0.25/0.65 + 0.50/0.70 + 0.75/0.75 + 1/0.80 +\\ &\quad 0.75/0.85 + 0.50/0.90 + 0.25/0.95\\ &= \sum_{w\in[0,1]} \mu_{\text{mostly\_true}}(w)/w.\end{aligned}$$

In possibility theory, it implies that

$$\pi_{\text{Truth}}(w) = \text{Poss}\{\text{Truth=w}\} = \mu_{\text{mostly\_true}}(w).$$

When $\pi(w) = 0$, it is certain that the truth-value is different from w and when $\pi(w) = 1$ it is certain that the truth-value is equal to w. Further, when $\pi(w) = 1 - w$; $\forall\, w \in [0,1]$ we may say that the truth-value is false or that the statement is false. When $\pi(w) = w$; $\forall w \in [0,1]$ it is said that the particular statement is true. To represent a vague truth, we consider only unimodal possibility distribution.

In the natural language, truth-value of a statement may be explicitly referred, e.g.,

Temperature is moderate is absolutely-true,

or it may be implicitly referred, e.g.,

Swedes are tall.

If the fuzzily truth-qualified statements like X is A is τ−true be interpreted as 'it is (fuzzily) τ−true that $X = A$' then at the semantic level, the latter statement may be represented as 'it is true that Swedes are tall' or that *Swedes are tall is true.* Let us now present a formal approach to the representation of such truth-qualified propositions.

We see that a simple statement may be of the general/standard form

$$X \text{ is } F;\ T \text{ is } C$$

where X and T are two linguistic variables of which T explicitly denote the truth-value of the fuzzy proposition X is F, F and C respectively denote the vague descriptions about the object X and the truth of the proposition X is F. In the sequel, the following two formulae

$$X \text{ is } F;\ T \text{ is } C$$

$$\text{and } X \text{ is not } F;\ T \text{ is } ant\_C$$

are taken to be logically equivalent(, in terms of possibility assignment functions this would lead to the same possibility distribution whereas in terms of fuzzy set representation this would lead to the same representation). This, in turn, means that, we may replace any one of the formula with the other wherever they occur. As a case in point,

$$X \text{ is small };\ T \text{ is fairly\_true}$$

$$\text{and } X \text{ is ant\_small };\ T \text{ is fairly\_false}$$

are logically equivalent [24] where fuzzy truth-value ant_C is denoted by the following function

$$\chi_{\text{ant\_}C} : [0,1] \rightarrow [0,1],$$

and is defined by

$$\chi_{\text{ant\_}C}(w) = 1 - \chi_C(w)\ ;\ \forall w \in [0,1].$$

Also $\tau-$true and ant $\tau-$false are taken as logically equivalent, where

$$\mu_{\text{ant}(\tau-\text{false})} = 1 - \mu_{\tau-\text{true}}(t).$$

Thus, without any loss of generality, we consider truth-qualified statements as a collection of simple formula such as

$$X \text{ is } F;\ T \text{ is } C\_\text{true} = F(X); C\_mboxtrue(T).$$

Composite formulae may be generated using the following equivalent formulae :

$$(F(X); C\_\text{true}(T)) \rightarrow (G(Y); D\_\text{true}(T)) = (F(X) \rightarrow G(Y); C(T));$$
$$(F(X); C\_\text{true}(T)) \wedge (G(Y); D\_\text{true}(T)) = (F(X) \wedge G(Y); C(T));$$
$$(F(X); C\_\text{true}(T)) \vee (G(Y); D\_\text{true}(T)) = (F(X) \vee G(Y); C(T));$$
$$(F(X); C\_\text{true}(T)) \leftrightarrow (G(Y); D\_\text{true}(T)) = (F(X) \leftrightarrow G(Y); C(T));$$
$$\sim (F(X); C\_\text{true}(T)) = (\sim F(X); C\_\text{true}(T))$$

Now let us consider a typical proposition $p : X$ is F; T is C_true. Let G be any fuzzy set defined over the universe of discourse U of X such that $F \subset G$. Let q be another truth-qualified proposition of the form

$$q : X \text{ is } G; \; T \text{ is undefined}$$

i.e., in standard notation we may restate q as

$$q : X \text{ is } G : \; T \text{ is } W\_\text{true} \; ; \; where \; \mu_{W\_\text{true}}(w) \; = \; 1 \; \forall w \in [0,1].$$

Combination of the two propositions p and q according to the above rules yields the following result:

$$\text{Now, } (X \text{ is } G; \; T \text{ is } Undefined) \wedge (X \text{ is } F; \; T \text{ is } C\_\text{true})$$

$$\begin{aligned} &= ((X \text{ is } F \wedge X \text{ is } G); \; T \text{ is } (C\_\text{true} \cap W\_\text{true})) \\ &= ((X \text{ is } F \cap G); \; T \text{ is } C\_\text{true}) \\ &= (X \text{ is } F; \; T \text{ is } C\_\text{true}). \end{aligned}$$

Thus, it is established that, from the given proposition

$$(X \text{ is } F; \; T \text{ is } C\_\text{true})$$

and the fact that X is G; T is $Undefined$, where $F \subset G$, it is possible to conclude (X is G; T is C_true).

With the above formulation, we now go into generating mechanisms for reasoning with truth-qualified propositions in the following section.

3 Reasoning with truth-qualified proposition

In this section, the issues concerning similarity based inference suggested by different models – rule-based and resolution-based – are extensively discussed. In the process, we consider both fuzzy set theoretic approach as well as possibilistic approach to reasoning with truth-qualified statements, separately.

Let us consider three linguistic variables X, Y and T of which T is to denote truth only and let $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ = $[0,1]$ respectively denote the universe of discourse. Let there be two typical propositions p and q as presented in Table 6.1. A natural conclusion is r and may be derived according to the following basic steps. Let

p :	if	X is A	then	Y is B ;	C_true
q :		X is A';			C'_true
r :				Y is B';	C"_true.

Table 6.1: Approximate reasoning with vague truth

$$\mathcal{U} = \{u_1, u_2, \cdots, u_l\},$$

$$\mathcal{V} = \{v_1, v_2, \cdots, v_m\},$$

$$\mathcal{W} = \{w_1, w_2, \cdots, w_n\}, 0 \leq w_i \leq 1.$$

First we translate the imprecise concepts in the propositions p and q using appropriate fuzzy sets. Let they be

$$\begin{aligned} A &= \sum_{i=1}^{l} \mu_A(u_i)/u_i \; ; \\ A' &= \sum_{i=1}^{l} \mu_{A'}(u_i)/u_i \; ; \\ B &= \sum_{i=1}^{m} \mu_B(v_i)/v_i \; ; \\ C\_\text{true} &= \sum_{i=1}^{n} \mu_{C\_\text{true}}(w_i)/w_i \; ; \\ C'\_\text{true} &= \sum_{i=1}^{n} \mu_{C'\_\text{true}}(w_i)/w_i \, . \end{aligned}$$

Now we represent proposition p as a fuzzy relation $R(A, B)$ whose interpretation would be the same as a conditional relation. $R(A, B)$ is either an implication relation or a conjunction relation. Next we compute $S(A, A')$ and if found zero, we set $B' = V$. Otherwise, we modify $R(A, B)$ with A' to obtain $R(A'|A, B)$, the induced fuzzy relation, according to any one of the following ways :

$$R(A' \mid A, B) = \frac{1}{S(A, A')} R(A, B)\}\text{where}$$

$$\mu_{R(A'|A,B)}(u, v) = \min\{1, \mu_{R(A,B)}(u, v)/S(A, A')\} \qquad (6.2)$$

$$\text{and } R(A' \mid A, B) = 1 - (1 - R(A, B)).S(A, A')\text{where}$$

$$\mu_{R(A'|A,B)}(u,v) = 1 - (1 - \mu_{R(A,B)}(u,v)).S(A,A'). \tag{6.3}$$

From the modified relation we derive a conclusion B' by the sup-operation as

$$\mu_{B'}(v) = \sup_{u \in U}\{\mu_{R(A'|A,B)}(u,v)\}; \forall v \in V. \tag{6.4}$$

Next, we compute $S(C,C')$ and set $t = S(C,C')$. The number t may be interpreted as the truth matching index. Using t we compute C''_true according to either (6.6) or (6.8).

$$C''\_true = 1 - (1 - C\_true).t \tag{6.5}$$

where

$$\mu_{C''\_true}(w) = 1 - (1 - \mu_{C\_true}(w)).t \tag{6.6}$$

and

$$C''\_true = \frac{1}{t} C\_true \tag{6.7}$$

where

$$\mu_{C''\_true}(w) = \min\{1, \mu_{C\_true}(w)/t\}. \tag{6.8}$$

Certainly, C''_true $\supseteq C$_true. The truth value of the inferred proposition can never be more specific than the truth of the condition. Note here that, when C'_true $= C$_true and $A' = A$ it may be found that $S(A,A') = 1$ and in that case, we find that $B' = B$ and C''_true $= C$_true. This is, in accordance with, the concept of modus ponens.

Example 6.1 : For a better understanding of the mechanism, let us consider a simple problem as posed in Table 6.1. With usual notation, let,

$$\begin{aligned}
\mathcal{U} &= \{u_1, u_2, u_3, u_4\};\\
\mathcal{V} &= \{v_1, v_2, v_3, v_4\};\\
\mathcal{W} &= \{w_1 = 0.25, w_2 = 0.40, w_3 = 0.65, w_4 = 0.85, w_5 = 1.0\};\\
\{A, A'\} &= \{1.0, 0.75\}/u_1 + \{0.75, 1.0\}/u_2 + \{0.5, 0.75\}/u_3 + \{0.25.0.5\}/u_4;\\
B &= 0.5/v_1 + 0.75/v_2 + 1.0/v_3 + 0.75/v_4;\\
C_\text{true} &= 0.1/w_1 + 0.25/w_2 + 0.75/w_3 + 0.875/w_4 + 1.0/w_5;\\
C'_\text{true} &= 0.01/w_1 + 0.06/w_2 + 0.56/w_3 + 0.76/w_4 + 1.0/w_5.
\end{aligned}$$

For a consequence of the form r, we first compute the fuzzy binary relation $R(A,B)$ from the translation of the statement form $A \longrightarrow B$ using the most frequently used min-rule. Let it be

$$R(A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0.50 & 0.75 & 1 & 0.75 \\ 0.50 & 0.75 & 0.75 & 0.75 \\ 0.50 & 0.50 & 0.50 & 0.50 \\ 0.25 & 0.25 & 0.25 & 0.25. \end{array}\right] \end{array}. \tag{6.9}$$

Then we compute the similarity between A and A'. Here, using Definition 3.2, we find that $S(A, A') = \underline{0.75}$(approximately). Now, with the proposition q we modify the relation $R(A, B)$ according to the scheme presented in (6.2) and obtain the modified relation $R(A'|A, B)$ as

$$R(A'|A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0.6666 & 1.0 & 1.0 & 1.0 \\ 0.6666 & 1.0 & 1.0 & 1.0 \\ 0.6666 & 0.6666 & 0.6666 & 0.6666 \\ 0.3333 & 0.3333 & 0.3333 & 0.3333. \end{array}\right] \end{array}. \tag{6.10}$$

Using sup-operation as given in (6.4), a consequence of the form r is given by $B' = 0.6666/v_1 + 1/v_2 + 1/v_3 + 1/v_4$. In order to determine the truth of the said consequence we compute the similarity between fuzzy sets C_true and C'_true. Here, we find that $S(C, C') = \underline{0.8208}$ (approximately). Then we modify C_true with $S(C.C')$ according to (6.8) and obtain

$$C''\_\text{true} = 0.1218/w_1 + 0.3046/w_2 + 0.9137/w_3 + 1/w_4 + 1/w_5.$$

Now the statement that $R(A, B)$ is C_true according to Bellman and Zadeh [3] yields a fuzzy relation $R'(A, B)$ given by (6.1) as

$$R'(A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0.45 & 0.8125 & 1.0 & 0.8125 \\ 0.45 & 0.8125 & 1.0 & 0.8125 \\ 0.45 & 0.45 & 0.45 & 0.45 \\ 0.01 & 0.01 & 0.01 & 0.01. \end{array}\right] \end{array} \tag{6.11}$$

Similarly, the statement X is A' ; T is C'_true when translated according to the same procedure yield a fuzzy set A'' as the true value for the variable X, where

$$A'' = 0.8125/u_1 + 1.0/u_2 + 0.8125/u_3 + 0.45/u_4. \quad (6.12)$$

Using compositional rule of inference, this would lead to

$$B'' = A'' \circ R'(A, B) = 0.45/v_1 + 0.8125/v_2 + 1.0/v_3 + 0.8125/v_4 \quad (6.13)$$

as a consequence of the above problem. It is interesting to see that the truth values are close to fuzzily true and although A and A' differ significantly, B and B'' are close to each other. In other words, $S(B, B'')$ is close to 1. Our inference, as obtained through the application of the proposed scheme, B' is C'''_true, when interpreted according to the same (6.1) yield a fuzzy set $B^* = 0.91/v_1 + 1.0/v_2 + 1.0/v_3 + 1.0/v_4 = V$. Thus the result is that nothing in particular may be concluded.

If the other scheme is used in modification of relation, then the result becomes

$$R(A'|A, B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \end{array} \left[\begin{array}{cccc} 0.6250 & 0.8125 & 1.0 & 0.8125 \\ 0.6250 & 0.8125 & 0.8125 & 0.8125 \\ 0.6250 & 0.6250 & 0.6250 & 0.6250 \\ 0.4375 & 0.4375 & 0.4375 & 0.4375. \end{array} \right]. \quad (6.14)$$

Using sup-operation as given in (6.4), a consequence of the form r will given as

$$B' = 0.6250/v_1 + 0.8125/v_2 + 1/v_3 + 0.8125/v_4 \quad (6.15)$$

and truth value

$$C''\text{\_true} = 0.2612/w_1 + 0.3844/w_2 + 0.7948/w_3 + 0.8974/w_4 + 1/w_5. \quad (6.16)$$

This result when interpreted according to (6.1) yield a fuzzy set

$$B^* = 0.7537/v_1 + 0.882/v_2 + 1.0/v_3 + 0.882/v_4. \quad (6.17)$$

The specificity of which is very low, close to zero.

If, instead, we use $a \to b = a.b$ in the translation of the conditional premise p we find that

$$R(A,B) = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline u_1 & 0.5 & 0.75 & 1 & 0.75 \\ u_2 & 0.375 & 0.5625 & 0.75 & 0.5625 \\ u_3 & 0.25 & 0.375 & 0.5 & 0.375 \\ u_4 & 0.125 & 0.1875 & 0.25 & 0.1875. \end{array} \quad (6.18)$$

Then using our second scheme for modification as presented in (6.3), the modified relation may be found to be

$$R(A'|A,B) = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline u_1 & 0.625 & 0.8125 & 1.0 & 0.8125 \\ u_2 & 0.5312 & 0.6718 & 0.8125 & 0.6718 \\ u_3 & 0.4375 & 0.5312 & 0.625 & 0.5312 \\ u_4 & 0.3875 & 0.3906 & 0.4375 & 0.3906. \end{array} \quad (6.19)$$

Using sup-operation as given in (6.4), a consequence of the form r is given by $B' = 0.625/v_1 + 0.8125/v_2 + 1/v_3 + 0.8125/v_4$, the same as in (6.15), and the same truth value

$$C''\_\text{true} = 0.2612/w_1 + 0.3844/w_2 + 0.7948/w_3 + 0.8974/w_4 + 1/w_5.$$

Example 6.2 : In this example, we seek a conclusion of the form **It is almost fairly_true that people will feel not so uncomfortable(Uc) when it is true that humidity is moderate** from the general knowledge that

It is fairly_true that when humidity is high people feels uncomfortable(Uc)

using the model just described. Here,

$$\begin{aligned} A &= \text{(humidity is) high;} \\ B &= \text{(tolerance) uncomfortable;} \\ C\_\text{true} &= \text{fairly-true;} \\ A' &= \text{(humidity is) moderate;} \\ C'\_\text{true} &= \text{true.} \end{aligned}$$

p : if	X_1 is A_1	and	X_2 is A_2	$\cdots$	X_k is A_k	then	Y is B;	C_true(T)
q :	X_1 is A'_1	and	X_2 is A'_2	$\cdots$	X_k is A'_k;			C'_true(T)
r :							Y is B';	C''_true(T)

Table 6.2: Extended approximate reasoning with vague truth

Now, the task is to represent the imprecise concepts in the propositions into fuzzy sets over some appropriate universe of discourse. Let the universe of discourses be denoted as follows :

$$\begin{aligned} \text{Percentile humidity} &= [0,1]; \\ \text{Percentile tolerance index} &= [0,1]; \\ \text{Truth-value} &= [0,1]. \end{aligned}$$

Thus, at any particular moment of time, if the humidity of air will be 97.5% it would represent, in this choice of universe, as a point 0.975. Any vague description about the same would then be represented as a fuzzy set over the said universe of discourse. Similarly, tolerance index '1' will mean that **feeling comfortable** and anything less than '1' would be partially comfortable so that '0' becomes **absolutely uncomfortable**.

$$\begin{aligned} \text{high} &= .25/0.25 + .5/0.5 + .75/0.75 + 1.0/1.0; \\ \text{Uc} &= .8/0.0 + .65/0.125 + .55/0.25 + .35/0.5 + 0.2/0.75 + .05/1.0; \\ \text{fairly\_true} &= .25/0.75 + .5/0.8 + .75/0.85 + 1.0/0.9 + .75/0.95 + .5/1.0; \\ \text{moderate} &= .25/0.0 + .5/0.25 + .75/0.5 + 1.0/0.75 + .75/1.0; \\ \text{true} &= .75/0.75 + .80/0.80 + .85/0.85 + .9/0.9 + .95/0.95 + 1.0/1.0. \end{aligned}$$

Using the above scheme for a conclusion of the form B' is C''_true, we find that

$$B' = 1.0/0.0 + 1/0.125 + .775/0.25 + 0.543/0.5 + 0.31/0.75 + 0.07/1.0$$

$$\text{and C''\_true} = .25/.75 + .5/.8 + .75/.85 + 1.0/.9 + .75/.95 + .5/1.0.$$

Next let us consider a generalised model as presented in Table 6.2.

where $X_1, X_2, \cdots, X_k$ are linguistic variables defined respectively over universe of discourses $U_1, U_2, \cdots, U_k$ and let $U_i = \{u_i^j\}$, $j = 1, 2, \cdots, j_i$. As usual, let the translation of the imprecise concepts in the propositions into appropriate fuzzy sets be given by

$$\begin{aligned}
A_i &= \sum_{j=1}^{j_i} \mu_A(u_i^j)/u_i^j \; ; i = 1, 2, \cdots, k; \\
A_i' &= \sum_{j=1}^{j_i} \mu_{A_i'}(u_i^j)/u_i^j \; ; i = 1, 2, \cdots, k; \\
B &= \sum_{i=1}^{m} \mu_B(v_i)/v_i \; ; \\
C_\text{true} &= \sum_{i=1}^{n} \mu_{C_\text{true}}(w_i)/w_i \; ; \\
C'_\text{true} &= \sum_{i=1}^{n} \mu_{C'_\text{true}}(w_i)/w_i \; .
\end{aligned}$$

With these data, we first compute $R(A_1, A_2, \cdots, A_k, B)$ from the translation of premise p using a suitable translating rule and then we compute $S(A_i, A_i')$ for $i = 1, 2, \cdots, k$ and set

$$s = \min\{S(A_1, A_1'), S(A_2, A_2'), \cdots, S(A_k, A_k')\}.$$

The number s may be taken to be the strength of matching. If $s = 0$ then we set $B' = V$, i.e., nothing may be concluded. Otherwise, we compute $R(A_1' \mid A_1, A_2' \mid A_2, \cdots, A_k' \mid A_k, B)$ according to any one of the procedure for modification of fuzzy realtion similar to the one as given in (6.2) and (6.3).

$$R(A_1' \mid A_1, A_2' \mid A_2, \cdots, A_k' \mid A_k, B) = \frac{1}{s} R(A_1, A_2, \cdots, A_k, B)$$

where the domain D is given by $U_1 \times U_2 \times \cdots \times U_k \times V$ and

$$\mu_{R(A_1'|A_1, A_2'|A_2, \cdots, A_k'|A_k, B)}(u_1, u_2, \cdots, u_k, v) = \min\{1, \frac{1}{s}\mu_{R(A_1, \cdots, A_k, B)}(u_1, \cdots, u_k, v)\}.$$

A consequence B' may be given from the above relation by the sup-operation as

$$\mu_B'(v) = \sup_{u_1 \in U_1, u_2 \in U_2, \cdots, u_k \in U_k} \mu_{R(A_1'|A_1, A_2'|A_2, \cdots, A_k'|A_k, B)}(u_1, u_2, \cdots, u_k, v)$$

whose truth value is computed in a similar way as presented in (6.6) and (6.8).

Example 6.3 : As an illustration of the above model, let $k = 2$ and as usual, let

$$\begin{aligned}
\mathcal{U}_1 &= \{u_1^1, u_1^2, u_1^3, u_1^4\};\\
\mathcal{U}_2 &= \{u_2^1, u_2^2, u_2^3, u_2^4, u_2^5\};\\
\mathcal{V} &= \{v_1, v_2, v_3, v_4\};\\
\mathcal{W} &= \{w_1 = 0.1, w_2 = 0.25, w_3 = 0.5, w_4 = 0.75, w_5 = 1.0\};\\
A_1 &= 0.55/u_1^1 + 0.7/u_1^2 + 0.85/u_1^3 + 1.0/u_1^4;\\
A_1' &= 0.5/u_1^1 + 0.75/u_1^2 + 1.0/u_1^3 + 0.75/u_1^4;\\
A_2 &= 0.9/u_2^1 + 1.0/u_2^2 + 0.85/u_2^3 + 0.7/u_2^4 + 0.55/u_2^5;\\
A_2' &= 0.8/u_2^1 + 0.9/u_2^2 + 0.75/u_2^3 + 0.6/u_2^4 + 0.5/u_2^5;\\
B &= 1.0/v_1 + 0.8/v_2 + 0.6/v_3 + 0.5/v_4;\\
C_\text{true} &= 0.1/w_1 + 0.1/w_2 + 0.1/w_3 + 1.0/w_4 + 0.9/w_5;\\
C'_\text{true} &= 0.05/w_1 + 0.3/w_2 + 0.55/w_3 + 0.8/w_4 + 1.0/w_5.
\end{aligned}$$

First of all, on computation, we find that $S(A_1, A_1') = 0.85$, $S(A_2, A_2') = 0.9078$ and $S(C, C') = 0.7571$. We set the overall matching index, $s = 0.7571$. Using Mamdani's min rule for translation of premises p into an appropriate fuzzy with the connector 'and' as min, it is found that

$$B_1' = 1.0/v_1 + 1.0/v_2 + 0.796/v_3 + 0.6634/v_4;$$

and

$$C''_{\text{true}} = 0.1326/w_1 + 0.1326/w_2 + 0.1326/w_3 + 1.0/w_4 + 1.0/w_5.$$

Here, in computing the above result, we use (6.2) and (6.8). On the other hand, if we use (6.3) and (6.6) for the same, then the result would become

$$B_2' = 1.0/v_1 + 0.8492/v_2 + 0.6985/v_3 + 0.6231/v_4;$$

and

$$C''_{\text{true}} = 0.3216/w_1 + 0.3216/w_2 + 0.3216/w_3 + 1.0/w_4 + 0.9246/w_5.$$

The above two results when interpreted in terms of (6.1) yield two fuzzy sets

$$B_1^* = 1.0/v_1 + 1.0/v_2 + 1.0/v_3 + 0.6995/v_4(\text{, the first result});$$

$$B_2^* = 0.9246/v_1 + 0.8928/v_2 + 0.8602/v_3 + 0.6556/v_4(\text{, the second result}).$$

From the above, it is clear that nothing in particular may be concluded. Both the results are consistent. The specificity measure of the latter is high as compared

p :	X is A or Y is B;		$C\_true(T)$
q :	X is A';		$C'\_true(T)$
r :		Y is B';	T is C''_true.

Table 6.3: Resolution-based reasoning with vague truth

to the former. Here although the pairs A_1, A'_1 as well as A_2, A'_2 are sufficiently close to each other, the truth values differ significantly and, therefore, the overall matching index s becomes low. Thus, making the consequence almost close to the UNKNOWN case. Here, as s approaches the maximum value 1, specific conclusion results.

So far, we considered only rule-based models for reasoning with truth-qualified statement. Now, let us consider resolution-based models for reasoning with vague truth-qualified knowledge as presented in Table 6.3. A consequence r may be derived according to a similar procedure as stated in Chapter 4. First, we compute $S(A, A')$ and set,

$$d = 1 - S(A, A'). \tag{6.20}$$

Here, d may be taken as a measure of dissimilarity between the two fuzzy sets A and A'. As has already been pointed out that for a consequence from models of this type it is necessary to compute the dissimilarity index instead of a similarity index. The inferential procedure differs from the rule-based model, Table 4.4, in three main issues. First of all, here, a T-conorm is used in place of a T-norm operator in translation of premise p. Secondly, every occurrence of s is replaced with d, for a conclusion. Lastly, the sup-operation performed on the modified relation, for a possible conclusion is replaced with an inf-operation. The premise p is first translated into a binary fuzzy relation. For safe, we use the max operator for the translation. Then d is used to modify the above fuzzy relation according to any one of the scheme for modification, (6.2) and (6.3) with d for s. Finally, we apply inf-operation to obtain a fuzzy set as our conclusion B'. In order to compute the truth of the consequence, we compute $S(C\_true, C'\_true)$ and modify fuzzy set C_true according to any one of (6.6) and (6.8). When $d = 0$, we find that $B' = V$. So that, nothing, in particular, may be concluded about Y. Again when $d = 1$ i.e. $S(A, A') = 0$ i.e., A and A' are disjoint sets, we need not modify the relation and if $C'\_true = C\_true$ it is possible to see that $B' = B$ together with $C''\_true = C\_true$. This is, in accordance with, the law of generalized disjunctive syllogism [70].

Example 6.4 : As an illustration, let us consider the following data :

$$\begin{aligned}
U &= \{u_1, u_2, u_3, u_4\};\\
V &= \{v_1, v_2, v_3, v_4\};\\
\mathcal{W} &= \{w_1 = 0.1, w_2 = 0.25, w_3 = 0.5, w_4 = 0.75, w_5 = 1.0\};\\
A &= 0.75/u_1 + 1.0/u_2 + 0.75/u_3 + 0.5/u_4;\\
A' &= 0.25/u_1 + 0.0/u_2 + 0.25/u_3 + 0.5/u_4;\\
B &= 0.25/v_1 + 0.50/v_2 + 0.75/v_3 + 1.0/v_4;\\
C_\text{true} &= 0.01/w_1 + 0.2/w_2 + 0.5/w_3 + 0.75/w_4 + 1.0/w_5;\\
C'_\text{true} &= 0.001/w_1 + 0.04/w_2 + 0.25/w_3 + 0.5625/w_4 + 1.0/w_5.
\end{aligned}$$

For a conclusion of the form r, let us first compute $S(A, A')$ and $S(C, C')$ according to Definition 3.2. Here, we find that $S(A, A') = \underline{0.2929}$ and $S(C, C') = 0.8244$. We set $d = \min\{1 - S(A, A'), S(C, C')\} = \underline{0.7071}$. Now, let us translate premise p: A or B as a relation $R(A, B)$ using max operator, where

$$\mu_{R(A,B)}(u, v) = \max[\mu_A(u), \mu_B(v)]\,. \tag{6.21}$$

More explicitly, we have,

$$R(A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0.75 & 0.75 & 0.75 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 0.75 & 0.75 & 0.75 & 1.0 \\ 0.50 & 0.50 & 0.75 & 1.0 \end{array}\right] \end{array}. \tag{6.22}$$

Next, using d, the modified relation $R(A' \mid A, B)$ may be given by

$$R(A'|A,B) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0.8232 & 0.8232 & 0.8232 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 0.8232 & 0.8232 & 0.8232 & 1.0 \\ 0.6464 & 0.6464 & 0.8232 & 1.0 \end{array}\right] \end{array}. \tag{6.23}$$

Therefore, $B' = 0.6464/v_1 + .6464/v_2 + .8232/v_3 + 1.0/v_4$

p :	X_1 is A_1	or	X_2 is A_2	or $\cdots$	X_k is A_k;	C_true(T)
q :	X_1 is A_1';					C'_true(T)
r :			X_2 is A_2'	or $\cdots$	X_k is A_k';	C''_true(T).

Table 6.4: Resolution-based generalised app. reas. with vague truth

and $C''\_\text{true} = 0.30/w_1 + 0.43/w_2 + 0.63/w_3 + 0.82/w_4 + 1.0/w_5$.

On scrutiny, we find that B' is close to B, with $S(B, B') = \underline{0.7765}$ and the truth value is close to linguistically true. This is simply because A, A' are almost complementary pair, having $S(A, A') = \underline{0.2929}$ and C, C' are close to linguistically true. As d approaches its maximum value, 1, $S(B, B')$ also does so. Hence the result is a reasonable one.

Next, let $X_1, X_2, \cdots, X_k$ be k-linguistic variables defined respectively over universe of discourse $U_1, U_2, \cdots, U_k$ and let $U_i = \{u_i^j\}$, $j = 1, 2, \cdots, j_i$. We now consider a generalized model for reasoning as given in Table 6.4. In this model, a significant conclusion r may be derived only when $S(C, C')$ is high and A, A' are almost complementary pair. For the conclusion r, we first translate premise p using a suitable translating rule to obtain $R(A_1, A_2, \cdots, A_k)$. Here also we may use max operator for the translation. Then, we compute $S(A_1, A_1')$ and $S(C, C')$ and ultimately set

$$d = \min\{1 - S(A_1, A_1'), S(C, C')\}.$$

If now $d = 0$ nothing can be concluded. Otherwise, using d we compute the modified relation $R(A_1'|A_1, A_2, \cdots, A_k)$ according to any one of the scheme for modification of fuzzy relation as stated already. Thus, we have, either

$$R(A_1'|A_1, A_2, \cdots, A_k) = \frac{1}{d} R(A_1, A_2, \cdots, A_k)$$

$$\text{or, } R(A_1'|A_1, A_2, \cdots, A_k) = 1 - (1 - R(A_1, A_2, \cdots, A_k)).d,$$

where the domain $D = U_1 \times U_2 \times \cdots \times U_k$. Then we perform the inf-operation on R' separately to obtain $A_2', A_3', \cdots, A_k'$. In order to compute the truth of the consequence we compute $S(C'\_\text{true}, C'\_\text{true})$ and modify C_true(T) according to any one of (6.6) and (6.8). Observe here that, when $d = 0$, nothing, in particular, can be concluded and in this case, we set $B' = V$. Again when $d = 1$ i.e. $S(A, A') = 0$ i.e. A and A' are disjoint and if, now, $C'\_\text{true} = C\_\text{true}$ it is possible to obtain

$A'_i = A_i; i = 2, 3, \cdots, k$ together with C''_true $= C$_true. This is, in accordance with, the law of generalized disjunctive syllogism [70].

We may call the inference r as a valid resolvent of the formulae p and q. The abve scheme may be made more general by allowing more than one disjunct (, a simple formula of the form X is A) in the premise q. In such cases, we may use the same mechanism described just now for a meaningful resolvent. The similarity index is to be calculated for all pairs $A_i, A'_i; i = 1, 2, \cdots, k$. The pair to be used in finding a resolvent is the one having minimum similarity value. The procedure to find the resolvent will be the same as above.

Thus we see that similarity based inference could be used for a meaningful deduction from truth-qualified vague statements.

Chapter 7

Default reasoning

1 Introduction

It may be argued that a significant portion of our beliefs about the world is uncertain and incomplete. Most of the decisions one makes everyday may hardly be taken with complete certainty. Anytime they may contraindicate any chosen course. This is simply because a majority of our reasoning is dictated by common sense rules whose conclusions, may be invalidated as further information comes in.

The above discussion may be illustrated by looking at a simple example. Suppose that there are two typical propositions p and q as follows :

p : ' Arnab is married '.

q : ' Arnab is over 25 years old '.

Based on only supporting evidence and the absence of contradictory evidence, there may be times when conclusions like 'A married person is over 25 years old' is forced to be drawn. Now, there is a person named Arpan who is married to a woman named Rupa. Therefore, the above rule when invoked will result in the belief that *Arpan is over 25 years old.* But, at a latter stage, more information about Arpan suggests that he has just recently crossed *23 years of age.* This would force us to revise our previous belief that *Arpan is over 25 years old.*

This is a typical instance of default reasoning. We may assert a fact not only when we may prove it but also when there are no information available to the contrary. So in the absence of further information, we may apply information on Arpan to a conditional statement of the form 'married persons are typically over 25 years of age'. We may revise the same as and when some contradictory information is available. Again another statement 'Snakes are poisonous', mean that for anything that is a snake, that thing are poisonous. Now this statement is not absolutely true in the real world — a python is non-poisonous and it is not the only exception. But certainly it is an exception, in a list of few exceptions. Nevertheless, such a general remark about characteristics of snakes is acceptable to many, if not to all. We may say that, 'typically snakes are poisonous'. The problem then reduces to the rejection of derivation of exceptional cases, as they come. A simple solution to this problem is to include the list of exceptions in the conditional statement. Since our knowledge about the world is necessarily incomplete an explicit list of such exceptions is not possible to make. So there should be some mechanism, which would allow us to jump to conclusion, in the face of incomplete information and the modification (revision) of such results, if necessary, at an appropriate time, in order to avoid the possibility of accepting the fact and its contradiction at the same time. Thus, default reasoning is a form of non-monotonic reasoning.

1.1 Review

Many assertions about the real world express default properties of individuals or class of individuals, rather than strict conditional relations. As for example, 'hard-workers are adults', 'intelligents prosper', 'teenagers are unmarried', 'snakes are highly-poisonous', 'birds fly faster than bees', 'Pathans are tall' statements express default properties. It is easy to see that they allow vague concepts in expression. Humans consistently reason approximately about such knowledge with reasonably good results (retract without much difficulty, as an when necessary). To cope with such form of reasoning through symbolic manipulation is a difficult task.

Researchers in computer representation of human knowledge have shown considerable interest, in recent years, in the representation of defaults and in the design of an artificial mechanism for reasoning with defaults. Unfortunately, most of them are not concerned with the fact that, imprecision is inherent in human knowledge expressed through natural language. The difficulty in translating human knowledge into a formal representation lies not so much in what is said as in what is left unsaid. Every such statement induces a knowledge base about the world which is never spelt out but only implied.

In this thesis, we make an attempt to represent and manipulate incomplete and imprecise default knowledge, based on the theory of fuzzy sets and the theory of possibility. We use similarity concepts in reasoning with defaults. We find that possibility theory plays an important role in default reasoning. Work in this direction is not numerous. In [100], Yager proposed to represent default knowledge through possibility-qualification and used the framework of Zadeh's concept of approximate reasoning for reasoning with defaults. Dubois and Prade [21] have considered a modified fuzzy set augmented, with a certainty factor for the representation of default values. Representation and manipulation are based on the theory of possibility in an approximate reasoning framework.

Yager's approach is based on the derivation of higher order sets in terms of elements of the power set of the universal set and from that, it is possible to deduce a possibility-qualified statement corresponding to a default rule. A statement 'X is A is possible' is used to indicate a possibility-qualification. It should be mentioned here that, a possibility-qualification is a less restrictive data than the unqualified simple statement 'X is A'. The possibility-qualified statement induces a statement X is A^+, where A^+ is the set of all subsets of the power set of U, the universal set of the linguistic variable. A^+ is the set of all subsets of U which intersect A. Reasoning with default was performed in the framework of approximate reasoning based on the theory of possibility. Thus, from

if X is A is posssible then Y is B and the data X is C

a consequence Y is D may be obtained as

$$\mu_D(v) = (1 - \text{Pos}[A \mid C]) \vee \mu_B(v).$$

Dubois and Prade have shown that,'the above treatment of default rules in an approximate reasoning framework fails to capture the uncertain nature of conclusions obtained via default rules' [21]. They proposed that, some uncertainty should be attached with each default conclusion, when relevant, in a rigorous manner. They have also observed that in Yager's approach, 'conclusions inferred from defaults and that inferred from rules without exceptions, have the same status, while clearly, they are not valid in the same sense : the latter are undisputable while the former may be interpreted as default values'. So, they proposed to modify the assignment for an attribute, with a special class of fuzzy sets augumented with a certainty factor, and deduced a technique for the propagation of uncertainty in default conclusions in the framework of the theory of possibility.

In [21], the authors propose to represent a default value for an attribute X and an object u_0 by means of a special kind of fuzzy set A^{λ} [65], where

$$(\forall v), \mu_{A^{\lambda}}(v) = \max(\mu_A(v), 1-\lambda), \lambda < 1. \tag{7.1}$$

The subset A gathers the a priori more plausible values among the possible ones; $1-\lambda$ estimates to what extent it is possible that the value lies outside A, exceptional ones. Then reasoning with default rules was performed in the framework of conventional approximate reasoning.

In [4], the authors propose to model default rules in the framework of possibility theory. They expressed that the exceptional situation is less possible than the normal state of affairs. However, human reasoning does not use only generic pieces of knowledge pervaded with exceptions; but also take advantage of independent assumptions. They made an attempt to describe the notion of independence in possibility theory [5] and applied it to default reasoning.

1.2 Outline

In order to capture the uncertainty associated with a default value, in the present work, a default value assigned to an attribute X corresponding to a typical individual u_0 from U is represented by means of a composite statement of the form $(X(u_0) = A)$ is λ_certain or simply, $(X(u_0) = A)$ is λ, where A and λ are linguistic labels and, therefore, may be vague [73]. The vague certainty value assigned to a statement is based on only the consistency of the consequence of a default rule. A '1' is assigned to a certain statement, a '0' to an uncertain statement and a positive proper fraction to a graded certain statement. The possible certainty-value of a vague statement constitutes the continuum [0,1]. An imprecise certainty-value may be represented by a fuzzy set over the universe of discourse [0,1]. In the sequel, the statement X is A, is certain may be taken as X is A and the statement X is A is uncertain may be taken as X is U, the universe of discourse of the linguistic variable.

Another important issue that needs to be considered in default reasoning is, a way of blocking undesirable transitivity which arises because of absence of an appropriate mechanism to incorporate explicit reference to the list of exceptional circumstances which would block its application. The certainty factor approach used by

Dubois and Prade [21] helps to block such undesirable transitivity. Our certainty-qualification technique is also found to be useful in this regard.

2 Representation of default knowledge

As is known, defaults are general rules subject to exceptions. Since, knowledge about the exceptions may not be complete, it is, therefore, impossible to represent the set of exceptions exclusively in a default rule. Again number of possible instances may be listed whose conclusions are, in some sense, reasonable in the light of what is known about the world. The set of exceptions, although not completely known, are fewer in number. Thus, it may be concluded, that defaults are rules whose conclusions are almost always certain.

In the following, the above discussion may be brought down to earth by looking at a few simple examples although, it is well known that, simple examples are always somewhat misleading. The actual test of any representation lies in whether it may cope with complexity. Nevertheless, these small and simple ones at least illustrate how the concept of certainty-qualification may be put to use in default reasoning.

Let us have a rule-base that contains within it the following rule, beside others :

Rule 1: If a person has high blood pressure and is a heavy smoker then that person has a very high risk of heart attack.

Such a complex rule not only consists of several simple propositions connected with well known logical connectives but also admit of concepts which are vague. In general, we have seen that, fuzzy logic provides a framework to represent such vague concepts in terms of fuzzy sets. In fuzzy logic, the rule may be interpreted as a fuzzy relation between the linguistic variables appearing in it. Explicitly, in a Predicate like notation, we may represent the rule with a general fuzzy formula as

$$\text{If Pressure(Blood)} = \underline{\text{High}} \text{ and Consumption(Smoke)} = \underline{\text{Heavy}}$$
$$\text{then Risk(Heart Attack)} = \underline{\text{Very High}}.$$

The underlined terms may be conveniently represented using fuzzy sets and the rule may be represented by a fuzzy relation.

In possibility theory, let Π_X denote the induced possibility distribution of a linguistic variable X, which equates the possibility of X taking a value u to the grade of membership of u in the fuzzy set, representing a value of the linguistic variable. Then we translate the above knowledge in a predicate-like notation as

If $\Pi_{Pressure(Blood)}$ = High and $\Pi_{Consumption(Smoke)}$ = Heavy
then $\Pi_{Risk(HeartAttack)}$ = Very High.

Let us now assume that the attribute named *Consumption* may receive default value. Then we have to interpret the rule in the following manner :

If Pressure(Blood) = High and it is consistent to assume that
Consumption(Smoke) = Heavy
then Risk(Heart Attack) = Very High.

Next let us consider another rule :

Rule 2: If the Horse is a colt and its height is very tall and constitution healthy then, it can run fast.

In fuzzy logic, the above rule may be represented as

If Age(Horse)=colt and Height(Horse)=very tall and Constitution(Horse)=healthy
then Speed(Horse)=fast.

In the theory of possibility we may have the following representation

If(Sex = Male;Π_{Age} = colt) and (Π_{Height} = very tall) and ($\Pi_{Constitution}$ = healthy) then (Π_{Speed} = fast).

Again, let us assume here that the attributes *height* and *constitution* may be allowed to receive default value. As before, the above rule then translates into

If Age(Horse) = colt and it is consistent to assume that Height(Horse) = very tall and Constitution(Horse) = healthy then Speed(Horse) = fast.

Thus, a default rule may be represented by an expression of the form

$$\frac{A(X) : B_1(X), B_2(X), \cdots, B_m(X)}{C(X)}$$

where $A(X), \{B_i(X) \mid i = 1, 2, \cdots, m\}, C(X)$ are all simple formulae, as used in this thesis. The concepts $A, \{B_i \mid i = 1, 2, \cdots, m\}, C$ are respectively called the prerequisite, justifications and consequent of the default and is interpreted as :

> if X is A and it is consistent to assume that X is B_1 and X is B_2 and $\cdots$ and X is B_m then infer that X is C.

The case where $m = 0$, default with no justifications behave like standard inference rule. If, also, the prerequisite is empty, it may be taken to be a tautology. The presence of justifications in default rules actually restricts the set of beliefs sanctioned by them about the world.

Now, inference sanctioned by rules without exceptions are well established and hence acceptable to all. Whereas those sanctioned by default may be found to be contradictory with more knowledge pouring in. Since, we allow vague formulas in a default rule, therefore, anything inferred by default, in absence of proper justification, may or may not be true in graded term. This suggest a more interesting representation of a vague default value as

$$C(x_0) \textit{ is } \lambda\text{\_certain} \tag{7.2}$$

where λ is a certainty value modifier and is usually vague, such as most, almost, fairly, absolutely, etc. In the theory of fuzzy set they may be represented as fuzzy subset of the unit interval. This qualified certainty about a proposition actually estimates a degree of certainty that the value of the attribute A for the object u_0 does not correspond to an exception. This observation demands the concept of similarity based framework for reasoning with vague default.

3 Reasoning with vague default

In the previous section we have seen that a general default rule may be represented in a Predicate like notation, as

$$\frac{A(X) : B_1(X),\ B_2(X), \cdots, B_m(X)}{C(X)}$$

which may be taken to mean

$$\text{if } A(X) \text{ and } B_1(X) \text{ and } B_2(X) \text{ and } \cdots \text{ and } B_m(X) \text{ then } C(X)$$

where the symbols have their meaning as stated earlier. Let us now discuss the issues pertaining to reasoning with such possibly vague default [74]. For that, let us consider the set of justifications appearing in the body of a default rule. They used to play an important role in default reasoning, the process of making conclusions in absence of complete knowledge about the world. Actually, rules with exceptions restrict the acceptable beliefs about the world, generated by the rule without justifications. So, a typical proposition from the acceptable set of beliefs, as sanctioned in absence of justifications, may or may not hold in the face of the justification criterion for a typical object. Therefore, in absence of contradictory evidence we may assume that *Arnab being married is over 25 years of age* to be partially certain. That is why, in this work, we propose to take the beliefs sanctioned by defaults to be partially certain. This certainty-qualification is expressed through imprecise knowledge and is, therefore, represented by a fuzzy set over [0,1], the possible certainty values, rather than a precise number from the said universe. The problem then reduces to provide a mechanism to manipulate vague certainty.

Let us assume that the whole set [0,1] corresponds to the uncertain case. In order to provide a unified environment for the manipulation of vague concepts let us assume that, when X is A is certain we take it as simply X is A. From a partially certain proposition we may construct a valid proposition as and when necessary for manipulation. The statement

$$X \text{ is } A \text{ is } \lambda\_\text{certain}$$

may be interpreted at the semantic level as 'it is certain that X is B' where,

$$\mu_B(u) = \mu_{\lambda\_\text{certain}}(\mu_A(u)).$$

Next, we consider a typical inference pattern in the scheme consisting of the following:

Given a default rule :

$$\textit{if } X \textit{ is } A \textit{ and } X \textit{ is } B_1, \cdots, X \textit{ is } B_m \textit{ then } X \textit{ is } C$$

and a fact that

$$X_0 \textit{ is } A',$$

we expect a reasonable conclusion as

$$X_0 \text{ is } C' \text{ is } \lambda\text{\_certain.}$$

In absence of any information about the justification $\{B_i : i = 1, 2, \cdots, m\}$ we take them to be partially certain, a fuzzy set defined over the unit interval [0,1] such as almost-certain, very-certain, certain, not very-certain, fairly-certain, etc, according to information available. For instance, we may take it as almost-certain, where

$$\mu_{\text{almost\_certain}}(w) = \{\frac{1}{1 + 10(w-1)^2}\}^{1/2} ; \ w \in [0,1].$$

So, we propose to mention explicitly all known information about $B_i's$ and when necessary we evaluate the rule by means of assignment of default values for the unknown $B_i's$. A default value for an attribute B_i and a particular object X_0 is represented by means of a certainty-qualified statement such as

$$B_i(X_0) \text{ is } \lambda_i\text{\_certain.}$$

This partial certainty assignment although subjective must be meaningful and may act as an estimate of the consistency of the conclusion.

Let us represent the concepts A, A' as fuzzy subsets of U, B_i as a fuzzy subset of V_i $(i = 1, 2, \cdots, m)$ and C, C' as fuzzy subsets of W. Then a consequence may be computed according to the following basic steps :

Algorithm 7.1 : Default reasoning

1. For each i, for which no specific knowledge about $B_i(X_0)$ are available, set $B_i(X_0)$ *is* λ_i_certain.
2. Compute λ_certain $= \cup_i^m \lambda_i$_certain.
3. Compute $s = S(A, A')$ using Definition 3.2.
4. Compute the conditional relation $R = R(A, C)$ from the translation of the premise 'if $A(X)$ then $C(X)$' and set $R' = 1 - (1 - R).s$, where $\mu_{R'}(u, v) = 1 - (1 - \mu_R(u, v)).s$.
5. Set, $\mu_{C'(X_0)}(w) = \sup_{u \in U} \mu_{R'}(u, w)$.

In the above scheme, when $A' = A$ we may choose the translating rule in such a way that $C' = C$ holds. In any case, they are found to be close to each other, when A' and A are close i.e., $S(C, C')$ is close to one when $S(A, A')$ are so. Also, if $\forall i$, λ_i_certain = certain then it may be found that λ_certain = certain. As A and A' becomes more and more dissimilar, the value of similarity index gets lowered.

In such cases, the inference about X becomes more or less unknown. If any one of $B_i(X_0)$ is found to be false then λ becomes undefined (. high membership value for points close to certainty value zero as well as close to certainty value one). We may reject such conclusion.

It is easy to see that this certainty value assignment to a default value *obtained* from the application of a default rule helps us to revise our belief about the world through manipulation of fuzzy sets only. In fact, if, at any instance of time, we find that $C'(X_0)$ is false, then the corresponding λ_certain becomes uncertain and as in the previous case, the conjunction of the certainty values becomes undefined and we reject all such consequences. Thus, the previous belief that $C'(X_0)$ is partially certain is rejected.

Next, let us consider an interesting case, where,

$$A'' \subseteq A' \subseteq A$$

holds. Here, X_0 *is* A'' is more informative than X_0 *is* A'. Thus, from the above facts, we may make an inference X_0 *is* A'', the justifications being the same. It has already been discussed that in such a situation

$$S(A'', A) \leq S(A', A).$$

Assume that

$$0 < S(A'', A) \leq S(A', A) \leq 1.$$

Then

$$1 - (1 - r).S(A'', A) \geq 1 - (1 - r).S(A', A); 0 \leq r \leq 1$$

and hence $C' \leq C''$. Thus more knowledge about the prerequisite does not always guarantee more knowledge about the inference, purely non-monotonic.

A simple example is considered in the following to illustrate how the above Algorithm 7.1, may be used to deal with real problems.

Example 7.1 : Let us assume that the knowledge base contain the following default rules and information.

- Prem 1. if X is A then X is B ;
- Prem 2. X_0 is A' is $fairly$_certain ;
- Prem 3. if X is C and X is D then X is B ;

- Prem 4. X_1 is C';
- Prem 5. if X is B then X is D ;
- Prem 6. X_1 is D.

Let us represent the imprecise concepts in the premises as fuzzy subsets of

$$U^i = \sum_{j=1}^{j_i} u^i_j \ ; \ i = 1, 2, 3, 4.$$

respectively. More explicitly, let they be given by

$$A = 0.85/u^1_1 + 1.0/u^1_2 + 0.75/u^1_3 + 0.5/u^1_4 + 0.25/u^1_5 \ ;$$

$$A' = 0.723/u^1_1 + 1.0/u^1_2 + 0.563/u^1_3 + 0.25/u^1_4 + 0.063/u^1_5 \ ;$$

$$B = 1.0/u^2_1 + 0.7/u^2_2 + 0.53/u^2_3 + 0.30/u^2_4 \ ;$$

$$C = 0.8/u^3_1 + 0.9/u^3_2 + 1.0/u^3_3 + 0.9/u^3_4 + 0.8/u^3_5 \ ;$$

$$C' = 0.89/u^3_1 + 0.95/u^3_2 + 1.0/u^3_3 + 0.95/u^3_4 + 0.89/u^3_5 \ ;$$

$$D = 0.25/u^4_1 + 0.50/u^4_2 + 0.75/u^4_3 + 1.0/u^4_4 \ ;$$

$$\mu_{\text{almost\_certain}}(w) = \{\frac{1}{(1 + 10(w-1)^2)}\}^{1/2} \ ;$$

$$\mu_{\text{fairly\_certain}}(w) = \{\frac{1}{(1 + 7.5(w-1)^2)}\}^{1/2} \ .$$

First of all, let us consider Prem 1 and Prem 2. Since, there are no justification in the rule under Prem 1 but, the fact under Prem 2 is partially certain, therefore, we have enough reason to conclude that

Concl.1 :(X_0 *is* B') *is* fairly_certain.

Now, on computation, we find that,

$$S(A, A') = 0.81560681 \ (approx.).$$

Using similarity based approximate reasoning, for B', we find that

$$B'(X_0) \ = 1.0/u^2_1 + 0.755/u^2_2 + 0.617/u^2_3 + 0.43/u^2_4.$$

Now, the above result may also be put as

$$\begin{aligned} B'_1(X_0) \ &= \ (\ X_0 \ is \ B'\ ) \ is \ \text{fairly\_certain} \\ &= \ 1/u^2_1 + 0.83/u^2_2 + 0.69/u^2_3 + 0.539/u^2_4. \end{aligned} \quad (7.3)$$

Next, we consider the remaining premises numbered 3 to 6. From Prem 3 and Prem 4, it is reasonable to conclude, by default,

$$(\ X_1\ is\ B'\ )\ is\ \lambda_2\text{-certain}.$$

Here, it is easy to see that we have prior information about X_1 as in Prem 6. Now, since we know that X_1 is D is certain, therefore, we conclude with certainty that,

$$\text{Concl.2}: X_1 \text{ is } B' \text{ is certain or that } X_1 \text{ is } B'.$$

For that, as usual, we first compute $S(C, C')$ and in this case, we find

$$S(C, C') = 0.93488472\ (approx.).$$

Using similarity based approximate reasoning we compute $B_1'(X_1)$ as

$$B_1'(X_1) = 1/u_1^2 + 0.72/u_2^2 + 0.561/u_3^2 + 0.346/u_4^2.$$

Again from Prem5 and Concl.2 we may deduce the following:

$$\text{Concl.3}: (X_1\ is\ D')\ is\ \text{certain}\ or\ that\ (X_1\ is\ D').$$

Here, we find that $S(B, B') = 0.97051695$ (approx.), almost close to '1'. where

$$D' = 0.272/u_1^4 + 0.515/u_2^4 + 0.757/u_3^4 + 1.0/u_4^4.$$

Now, it is given that, X_1 is D (Prem 6). It is easy to see that, here,

$$S(D, D') \sim 1.0.$$

According to the principle of minimum specificity, we find that

$$(X_1\ is\ D).$$

Obviously, this work represents a meagre beginning and suggests a few of the relevant issues involved in default reasoning. More research on the use of certainty-qualification and the underlying semantics of the same in the representation of default value is required to understand more precisely the effect of the same on the cognitive processes involved in default reasoning. However, from this initial study we see that certainty-qualified statements and similarity based reasoning are useful in default reasoning.

Chapter 8

Temporal reasoning

1 Introduction

Most of the study on fuzzy systems are made in a steady environment. The need to make time-dependent assumptions is frequently encountered in many systems, especially in dynamical systems. In order to build effective fuzzy systems in a dynamic environment, the temporal knowledge must be taken into consideration. For that, conventional reasoning mechanisms should be modified in order to derive a consequence at an appropriate moment of time. For reasoning purpose, we find that, an approximate matching of time is necessary.

Reasoning with time is considered as a key problem in many AI areas. Its relevance has been shown in a number of domains — in medical diagnosis there is a clear need for representation of time; programmes developed for planning require a way of representing possible future, schedule operations require a processing time to be considered and many more. Considerable interest has been shown by different researchers in the representation and manipulation of time-dependent knowledge [1, 56, 90]. However, they have considered properties of systems and their behavior over precise temporal concepts only.

Human being consistently reason approximately about time with reasonably good results. In such cases, it is found that human perception plays a major role. As natural language is often used in expressing temporal information, we often find imprecision in temporal knowledge. Therefore, our knowledge about time may be imprecise, as e.g.,

Payel was rich;
Paulami was leaving for school;
Foggy mornings are common during late-January;
Temperature used to be low early in the morning.

Each of the statement is temporally vague. We show that such vague temporal propositions may be conveniently represented using fuzzy sets. Until quite recently, such vague temporal considerations have been given very little attention. A common justification was that such considerations would lead to enormous complexities without much practical gain. This leads to the acceptance of concepts which are simply no longer true over all time, e.g., *Arka likes soft-drinks* may be valid only at present whereas *Payel arrives home around 5.30 in the evening* may be valid in future too(,habitual).

Now, time is a collection of moment. Any vague description about a moment of time may be represented by a fuzzy set over the real line. Therefore, complex temporal descriptions may be represented by composite fuzzy relations. In a possibilistic approach, we associate a possibility distribution with every vague temporal statements and then use the rules for composition of possibility distributions in reasoning with such temporal knowledge. A unimodal possibility distribution is used for the representation of a moment of time as in [23].

In this chapter, we show the use of similarity concept between fuzzy sets in vague temporal knowledge matching. Our approach to modeling is based on the theory of fuzzy sets and the concept of similarity based reasoning. In the process we show that possibility theory plays an important role in representing temporal knowledge.

We begin with a brief review followed by a detailed discussion on a representation of vague temporal knowledge using fuzzy sets and the theory of possibility. Then we consider the derivation of a consequence from different models with simple yet concrete examples.

2 Review

Nowakowska [59] was the first to consider subjectivity in the perception of time. In the year 1983, M.Vitek [92] made a formal attempt to describe fuzzy time through fuzzy sets of time intervals. T.C.Fall [31] tried to propagate uncertainty and imprecision along the temporal axis using belief functions. Then there was an attempt made by S.Dutta [27, 28] to capture the lack of knowledge about events by means of fuzzy sets of time-intervals. D.Dubois and H.Prade [23] proposed to manage imprecision and uncertainty in temporal knowledge through possibility distributions — a notion first introduced by Zadeh [113]. This is an interesting and informative work. The approach to representation and manipulation is more general. Time points are considered as primitive notion of time. A fuzzy set of time scale models an ill-known date. They used the theory of possibility and necessity in verifying temporal relations and showed that fuzzy comparators may be modeled by a fuzzy relation. The possibilistic approach ensures the representation of precedence relation between dates. In reasoning with temporal knowledge they used the conventional approach to approximate reasoning, due to Zadeh. This is based on a combination of relevant pieces of informations and then projection on appropriate domains for a possibility distribution. Da-Qun-Qian [66] proposed a method for handling imprecise temporal knowledge in dynamical systems. Then in the year 1995, Van Le propose a somewhat adhoc logic [89] and a scheme for fuzzy temporal reasoning based on a combination of fuzzy and temporal logics. He has also described a model-theoretic semantics for the scheme. His work is closely related to that of Qian's [66] work. Temporal knowledge is represented through a temporal descriptor operator. A temporal descriptor operator describes the time characteristics of a fuzzy proposition. It consists of a fuzzy proposition, a time interval descriptor and a time unit descriptor. In this case, the successful pattern matching of fuzzy rule not only requires that all the fuzzy propositions in the antecedent of a fuzzy rule should match the data, but also requires that the temporal relations among these fuzzy propositions should match the corresponding dynamic information in the data. Recently, Maeda et.al. [51] proposed a qualitative reasoning method that incorporates a vague time delay into fuzzy *if-then* rules and showed its importance in dynamical fuzzy reasoning in system dynamics that has a time delay between premise and consequent in fuzzy rule-base knowledge. There the authors regarded time as an instant and as an interval. Then they showed that a fuzzy time interval may be derived from the fuzzy instants which delimit the time span during which an event occur. Then Virant and Zimic [91] proposed to consider a time variable in the domain of fuzzy logic. In view of giving a new extension to fuzzy logic in terms of time, there they use a modified fuzzy set which may change its elements with time as a representation of a fuzzy time variable, a fuzzy time dependent rule, a fuzzy time operator and similar other items.

The scope of the present thesis is representation and manipulation of vague time in the framework of approximate reasoning. To do that, we employ a fuzzy set theoretic approach in the representation of vague temporal concepts. Then we introduce the concept of time in similarity based approximate reasoning. In the following section, let us first consider the representation issues.

3 Representation of fuzzy temporal knowledge

In representing fuzzy temporal knowledge, like the work of Dubois and Prade in [23], we consider time points as our primitive temporal component. In this section, we consider the representation of dates, time-intervals and precedence relation between dates and events based on the same. For that we assume one continuous linear scale $\mathcal{T}$ for modeling time points(, moments of time). Each member of $\mathcal{T}$ represent a precise time instant. In practice, we use the real-line and a discretization of $\mathcal{T}$ for computational simplicity.

Following Dubois and Prade, we call a *date* **a** as any fuzzy instant of time represented by a fuzzy subset **A** of $\mathcal{T}$. If $\mu_A(t) = 0$, we are certain that **a** is different from t and if $\mu_A(t) = 1$, we are certain that **a** is t. With these interpretation, the fact $t' \neq t$ and $\mu_A(t) = \mu_A(t') = 1$ may be taken to mean that other values of t may also be equal to date **a**. In representing common sense temporal knowledge we often encounter linguistic description about dates. As, for example, if it is known that

> Temperature becomes low in the evening ;
> Rimi arrives home in the afternoon ;

then, in each of the cases, the date **a** is to denote the starting time of the respective event. The time period, in some cases, have been found to be explicitly referred, e.g., Arka is running slowly for *over an hour*; whereas in some other cases they are found to be implicitly referred, e.g., Roni likes snacks, which may be valid in the present.

In both cases, such knowledge may be split into two components — an atemporal component and an associated temporal component. Considering the fact that any definition of time is a collection of moments, we try to represent vague knowledge about moments, the temporal primitive of our work, by means of a fuzzy set over

T. Typically, a simple proposition about time, ***around 5.30*** defines a fuzzy set A as

$$\begin{aligned} A &= \sum_{t \in \mathcal{T}} \mu_{\text{around } 5.30}(t)/t \\ &= 0.12/5.05 + 0.25/5.10 + 0.5/5.15 + 0.75/5.20 + 1.0/5.25 + \\ &= 1.0/5.30 + 1.0/5.35 + 0.75/5.40 + 0.5/5.45 + 0.25/5.50 + 0.12/5.55. \end{aligned}$$

Here,

$$\mu_{\text{around } 5.30}(t) \;:\; \mathcal{T} \rightarrow [0,1]$$

such that, $\mu_{\text{around } 5.30}(t)$ is to represent the grade of membership of t in the said fuzzy set. Using the theory of possibility we may represent the above temporal information as in the following :

$$\begin{aligned} \Pi_{\mathcal{T}} &= \sum_{t \in \mathcal{T}} \mu_{\text{around } 5.30}(t)/t \\ &= 0.12/5.05 + 0.25/5.10 + 0.5/5.15 + 0.75/5.20 + 1.0/5.25 + \\ &= 1.0/5.30 + 1.0/5.35 + 0.75/5.40 + 0.5/5.45 + 0.25/5.50 + 0.12/5.55. \end{aligned}$$

implying that

$$\pi_{\text{around } 5.30}(t) = Poss\{\text{Time[around 5.30]} = t\} = \mu_{\text{around } 5.30}(t); \forall t \in \mathcal{T}.$$

When $\pi_{\text{around } 5.30}(t) = 0$, it is certain that the particular moment is different from t and when $\pi_{\text{around } 5.30}(t) = 1$, it is completely certain that the particular moment is equal to t. Other values may also exist which are completely possible for the moment. Although the assignment of membership values is subjective in nature, standard functions with adjustable parameters [114] are also available for the same. To represent a vague moment, only unimodal possibility distribution function is used as in [23] in order to take into consideration that the majority of possible values of the moment are clustered together [23].

Sometimes, it becomes necessary to represent composite time references using relational matrices, in the discrete case, e.g., the knowledge about any particular moment of some particular day may be represented by means of a fuzzy relation. The time component in the assertion that *the ice-cream seller comes in the evening during mid-April* may be translated into a fuzzy relation as in Table 8.1.

The above relational matrix may be obtained from the conjunction of two fuzzy sets τ_1 = *around 5.30* and τ_2 = *mid-April* in a usual way as in the following :

		Diff.	days	in	the	month	of	April
		12/4	13/4	14/4	15/4	16/4	17/4	18/4
	16.30	0.25	0.25	0.25	0.25	0.25	0.25	0.25
T	16.45	0.25	0.50	0.50	0.50	0.50	0.50	0.25
I	17.00	0.25	0.50	0.75	0.75	0.75	0.50	0.25
M	17.15	0.25	0.50	0.75	1.00	0.75	0.50	0.25
E	17.30	0.25	0.50	0.75	1.00	0.75	0.50	0.25
S	17.45	0.25	0.50	0.75	1.00	0.75	0.50	0.25
	18.00	0.25	0.50	0.75	0.75	0.75	0.50	0.25
	18.15	0.25	0.50	0.50	0.50	0.50	0.50	0.25
	18.30	0.25	0.25	0.25	0.25	0.25	0.25	0.25

Table 8.1: The fuzzy binary relation *around 5.30 in mid-April*

$$R(\tau_1, \tau_2) = R(\text{,say}); \text{ where}$$
$$\mu_R(t_1, t_2) = \min\{\tau_1(t_1), \tau_2(t_2)\}.$$

Such a relation may be used in determining the joint possibility distribution $\Pi_{(T_1,T_2)}$ from $\Pi_{T_1} = \tau_1$ and $\Pi_{T_2} = \tau_2$. Let,

$$\Pi_T = \Pi_{(T_1,T_2)} = \overline{\tau_1} \bigcap \overline{\tau_2} = \tau_1 \times \tau_2,$$

where $\overline{\tau}$ is to denote the cylindrical extensions of τ over the cartesian product of the domain of definitions.

Next, from the possibility distribution attached with a particular date a, we may define the fuzzy set of time points that are possibly after the given date as in the following :

$$\forall t \in \mathcal{T} \qquad \mu_{\text{after-a}}(t) = \sup_{t' \leq t} \pi_a(t')$$

and the fuzzy set of time points that are possibly before the given date **a** as

$$\forall t \in \mathcal{T} \qquad \mu_{\text{before-a}}(t) = \sup_{t' \geq t} \pi_a(t').$$

Furthermore, it is known that a time-interval is a period of time between two given dates and, therefore, they may be represented by an ordered pair of dates, denoting the end points of the interval. Thus, given two vague moments of time, m_1 and m_2, which induces two possibility distributions π_{m_1} and π_{m_2}, respectively and let

M_1 and M_2 denote the fuzzy sets of more or less possible values of m_1 and m_2 then it may be defined by the fuzzy interval of time-points $[M_1, M_2]$ of moments which are after-m_1 and before-m_2 according to the following

$$\forall t \in \mathcal{T} \qquad \mu_{[M_1, M_2]}(t) = \sup_{t' \leq t \leq t''} \{\min(\pi_{m_1}(t'), \pi_{m_2}(t''))\}.$$

Complex temporal statements are actually complex relations and in the discrete case they may be represented as higher dimensional relational matrices.

According to the above discussion and an understanding on the representation of temporal concepts we are in a position to consider simple vague temporal propositions. A vague temporal proposition may be taken in a standard form as X is A; T is B, where X and T are linguistic variables of which T always denote the time of occurrence of the event, A and B are possible vague description of X and T respectively. We take them as our atomic formula. In the sequel, the following two formulae

not (X is A ; T is B)
and (X is not A ; T is B)

are taken at the semantic level to be equivalent i.e., they may be used interchangebly. By this we mean that for all time reference B the two statements 'X is not A' and 'not X is A' are equivalent.

Since, the concepts A and B, values of the linguistic variables X and T, may be vague, therefore, each of them may be represented using appropriate fuzzy sets (, fuzzy relations). Each such proposition induces a possibility distribution through the possibility assignment function. Let they be denoted as Π_X and Π_T respectively and be given by $\Pi_X = A$ and $\Pi_T = B$. Hence a simple vague temporal knowledge may be represented as a pair of possibility distributions $\{\Pi_X, \Pi_T\}$.

Next, let symbolically, $F(A, B) = F[A(X); B(T)]$ is to denote the atomic formula (X is A ; T is B). In the present thesis we consider the following as formulas to be used in reasoning.

if F is a formula then not F is a formula,

if F and G are formulas then $F \to G$, $F \vee G$, $F \wedge G$ and $F \leftrightarrow G$ are formulae. The interpretation of a composite formula in terms of the constituent simple formulae may be given by

$$\begin{aligned} F \to G &= (X \text{ is } A;\ T \text{ is } B) \circ (Y \text{ is } C;\ T \text{ is } D) \\ &= ((X \text{ is } A \circ Y \text{ is } C);\ T \text{ is } \bar{B} \cap \bar{D})(, \circ \text{is to denote the operator}). \end{aligned}$$

After the introduction of formula for vague temporal knowledge, let us now consider their manipulation in reasoning with time in the following section.

4 Reasoning with time

First of all, given information about dates represented by a fuzzy subset of $\mathcal{T}$ and a relation between dates by some appropriate fuzzy relation on $\mathcal{T} \times \mathcal{T}'$, we may use algorithm **SAR** to compute a consequence from the pieces of temporal information as a fuzzy set of date on $\mathcal{T}'$. If, instead of a date-date relation, we are given a date-interval or interval-interval relationship, we try to cylindrically extend the information about date cylindrically over the appropriate domain of definition and the use the actual reasoning procedure.

In the temporally changing world many information may be processed in the framework of rule-based reasoning. For that, we consider the formulation of the same in the framework of similarity based approximate reasoning as in [75].

Let X, Y and T denote three linguistic variables of which T is to refer time only and let $\mathcal{U}, \mathcal{V}, \mathcal{T}$ respectively denote the universe of discourses. Let there be two typical premises

$$p : \text{if } X \text{ is } A \text{ then } Y \text{ is } B \text{ ; } T \text{ is } \tau$$

$$q : X \text{ is } A' \text{ ; } T \text{ is } \tau'.$$

From p and q we derive a conclusion r : Y is B' ; T is τ''.

First, we represent the time components τ and τ' in an appropriate frame of reference. For a reasonable conclusion, we check the two time references in p and q. If τ' is included in τ, entirely or in part, then we proceed as in the following. In this case, we set a time matching index t with a value of 1. Otherwise, we set $t = 0$. If $t = 1$, we translate the imprecise concepts in the premises p and q into appropriate fuzzy set representaion. We compute a similarity index between the fuzzy sets A and A' and set the overall matching index equal to the value just computed. Let it be s. Then we follow the algorithm SAR as proposed in Chapter 4.3, in order to obtain a fuzzy set representation of B'. The period at which the above result will hold may be computed as the intersection of τ with $after - \tau'$.

Next, let us consider a generalized model where from premises

$$p: \text{if } X_1 \text{ is } A_1 \text{ and } X_2 \text{ is } A_2 \text{ and } \cdots \text{ and } X_k \text{ is } A_k \text{ then } Y \text{ is } B \;;\; T \text{ is } \tau$$

$$q: X_1 \text{ is } A_1' \text{ and } X_2 \text{ is } A_2' \text{ and } \cdots \text{ and } X_k \text{ is } A_k';\; T \text{ is } \tau'$$

we would like to derive a conclusion

$$r: Y \text{ is } B' \;;\; T \text{ is } \tau''$$

For a consequence of the form r from the above model of approximate reasoning with time, the reasoning procedure will remain the same as already stated in the general case (, without the time component), in algorithm **A1** and the corresponding time may be given by the intersection of τ with $after - \tau'$.

In this way, we may derive consequences using other models for approximate reasoning involving temporal knowledge as well. Now, we illustrate the same considering some simple cases as in the following.

<u>*Example 8.1*</u> : Let

$$\begin{aligned} U &= \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}; \\ V &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}; \\ \mathcal{T} &= \{t_1, t_2, t_3, t_4, t_5, t_6, t_7\}. \end{aligned}$$

be the respective universe and the fuzzy subsets be as given below.

$$\begin{aligned} A &= 0.1/u_1 + 0.25/u_2 + 0.5/u_3 + 0.75/u_4 + 1.0/u_5 + 0.75/u_6 + 0.50/u_7; \\ A' &= 0.01/u_1 + 0.0625/u_2 + 0.25/u_3 + 0.5625/u_4 + 1.0/u_5 + 0.5625/u_6 + 0.25/u_7; \\ B &= 0.55/v_1 + 0.8/v_2 + 1.0/v_3 + 0.8/v_4 + 0.55/v_5 + 0.25/v_6 + 0.05/v_7; \\ \tau &= 0.15/t_1 + 0.3/t_2 + 0.45/t_3 + 0.6/t_4 + 0.75/t_5 + 0.9/t_6 + 1.0/t_7; \\ \tau' &= 0.06/t_1 + 0.16/t_2 + 0.3/t_3 + 0.46/t_4 + 0.65/t_5 + 0.85/t_6 + 1.0/t_7. \end{aligned}$$

Here, it is easy to see that $\tau \cap \tau' \neq \Phi$ (, null fuzzy set). In fact,

$$\tau'' = \tau' = 0.06/t_1 + 0.16/t_2 + 0.3/t_3 + 0.46/t_4 + 0.65/t_5 + 0.85/t_6 + 1.0/t_7.$$

Now let us compute $S(A, A')$. Using Definition 3.2 we find that $S(A, A') = \underline{0.8006}$ (approximately). Using algorithm **SAR** and min-rule for translation, obtain

$$B' = 0.6397/v_1 + 0.8399/v_2 + 1.0/v_3 + 0.8399/v_4 + 0.6397/v_5 + 0.3996/v_6 + 0.2394/v_7.$$

Thus we conclude that $B'(Y); \tau'(T)$ is a consequence of the above information.

Example 8.2 : In the following example, we derive a conclusion of the form *speed* may be *fairly_low around_nine* in the morning, assuming that *around_nine* in the morning *traffic congestion* remain *moderate* from the general knowledge that during *office_hours* if *traffic congestion* is *high* then *speed* is *low* using the model just described. Here

$$\begin{aligned} A &= \textit{traffic congestion} \text{ is } \textit{high}; \\ B &= \textit{speed} \text{ is } \textit{low}; \\ \tau &= \textit{office-hours}; \\ A' &= \textit{traffic congestion} \text{ is } \textit{moderate}; \\ \tau' &= \textit{around\_nine}. \end{aligned}$$

Now, the task is to represent the imprecise concepts in the propositions into fuzzy sets over some appropriate universe of discourse. Let the universe of discourses be denoted as follows :

$$\begin{aligned} \text{Percentile traffic congestion} &= [0, 1]; \\ \text{Speed} &= [0, 100]\text{miles/hour}; \\ \text{Time} &= [0, 24]\text{hours a day}. \end{aligned}$$

Thus at any particular moment of time, if the traffic congestion in some scale be 97.5% it would be represented, in the above choice of universe, as a point 0.975. Any vague description about the same would then be represented as a fuzzy set over the said universe of discourse. Now, let us translate the imprecise concepts in the propositions into fuzzy set representation as follows :
high traffic $= 0.1/0.5 + 0.25/0.55 + 0.40/0.60 + 0.55/0.65 + 0.70/0.70 + 0.85/0.75$
$+ 1.0/0.80 + 1.0/0.85 + 1.0/0.9 + 0.85/0.95 + 0.7/1.0$;
low speed $= 0.25/0 + 0.50/5 + 0.75/10 + 1.0/15 + 1.0/20$
$+ 1.0/25 + 1.0/30 + 0.75/35 + 0.50/40 + 0.25/45$;
office-hours $= 0.50/8:00 + 1.0/9:00 + 1.0/10:00 + 1.0/11:00 + 0.50/12:00$
$+ 0.5/16:00 + 1.0/17:00 + 1.0/18:00 + 1.0/19:00 + 0.5/20:00$;
moderate congestion $= 0.1/0.4 + 0.25/0.5 + 0.5/0.55 + 0.7/0.60 + 0.85/0.65$

$+ 1.0/0.7 + 1.0/0.75 + 1.0/0.8 + 0.85/0.85 + 0.70/0.90 + 0.55/0.95 + 0.40/1.0;$
$around\_nine = 0.1/7.00 + 0.5/8.00 + 1.0/9.00 + 0.5/10.00 + 0.1/11.00;$
Here, we first see that the two time definitions are actually overlapping, i.e., having some moments in common. Hence, we find that, in this case,

$$\tau'' = 0.50/8.00 + 1.0/9.00 + 0.5/10.00 + 0.1/11.00.$$

We are required to find the approximate expected speed of vehicles during that time in terms of a fuzzy set representation B'. For that, as usual, we compute, $S(A, A')$ and see that $S(A, A') = \underline{0.7514}$ (approximately). Using the algorithm **SAR**, we find that the fuzzy set representation of the expected speed of a vehicle around that time as

$$\begin{aligned} B' &= 0.4365/0 + 0.6243/5 + 0.8122/10 + 1.0/15 + 1.0/20 \\ &\quad + 1.0/25 + 1.0/30 + 0.8122/35 + 0.6243/40 + 0.4365/45. \end{aligned}$$

Obviously, this work on imprecise temporal knowledge processing represents a meagre beginning and suggests a few of the relevant issues pertaining to the representation and manipulation of temporal concepts. More study on the use of fuzzy relations, in defining temporal concepts are required to understand more precisely the effect of the same on the cognitive processes involved in temporal reasoning. However, from this initial study, we see that, similarity between fuzzy sets and similarity based approximate reasoning are useful in temporal reasoning.

Chapter 9

Conclusion - scope for future work

1 Summary

In this thesis, we made an attempt to represent and manipulate incomplete and vague knowledge in fuzzy logic that are based on manipulation of fuzzy sets and fuzzy relations. Moreover, we tried to present a possibilistic interpretation, wherever possible, during the development of the thesis.

In the first part of the thesis, we discussed the concept of similarity measure between fuzzy sets based, on a pairwise comparison of elements, and proposed a set of axioms for the choice of such a measure. A family of such measures are proposed, based on these axioms. The use of such measures in deriving a fuzzy consequence from given condition(s) and fact(s) have been extensively discussed. Then we developed a powerful mechanism, similarity based approximate reasoning, through the integration of existing similarity based reasoning mechanism and Zadeh's compositional rule of inference. We applied the said mechanism in designing different rule-based fuzzy systems. Two important problems, viz., pattern classification of Telugu vowels and fuzzy control of an inverted pendulum have been considered and solutions based on the said design have been proposed. In each case, pictorial descriptions have been presented for a better understanding of the solutions offered by the same. These examples serve to clarify the importance of similarity based approximate reasoning.

In the process, we have been able to establish certain relevant concepts, like a generalised rule-based model for inference in an incomplete environment, the resolution-based model, a deductive process —- generalised disjunctive syllogism in fuzzy inference, the automatic introduction of rule firing strength in inference mechanism and the use of specificity measure of fuzzy sets in decision making in an uncertain environment. The utility of the specificity based approach to defuzzification has been demonstrated with results.

We then extended the applicability of the proposed similarity based approximate reasoning to other challenging fields of current research in knowledge representation and manipulation. In the process, we have shown that similarity based approximate reasoning may be used in default reasoning as well as in temporal reasoning.

In the second part of the thesis, we have considered the representation and manipulation of truth-qualified statements. We have discussed the concept of partially truth-qualified vague statements (/formulae) and proposed means to handle them. Different models for reasoning with truth-qualified statements have been extensively discussed with supporting examples.

In this thesis, we tried to attach some uncertainty with a default value. We have shown that a default value may be considered as a partially certain statement and then reasoning with vague default may be performed efficiently, in the framework of similarity based approximate reasoning. In the process, we have deduced a technique to differentiate facts inferred from well established rules and that inferred from a default rule through certainty-qualification. This helps to rule out undesirable transitivity of default values which is needed for blocking the applicability of a default in certain circumstances.

Representation and manipulation of vague temporal knowledge in the development of intelligent systems is yet another challenging field of investigation. We have shown that the concept of similarity between fuzzy sets play an important role in temporal matching as well as in temporal reasoning. We have also shown that possibility theory plays an important role in temporal knowledge representation and manipulation.

2 Concluding remarks

Approximate reasoning is based on fuzzy logic. Admittedly, in the methodology of similarity based approximate reasoning, as reported in this thesis, there is as such not much of logic. Several aspects such as the concept of relation modification, fuzzification of a crisp input, use of similarity in rule selection from the rule-base and defuzzification are unrelated to any logic. Thus, we conclude that, similarity based approximate reasoning is *fuzzy logic more in a wide sense*. Certainly, in its wide sense, fuzzy logic has the expressive power to represent vagueness in a natural way and we may at least represent and manipulate concepts which are neither complete nor precise through such a system. Therefore, in absence of an agreed objective model of how human mind encompasses and deals with vague knowledge in the real world, it is inevitable that no single method of mechanizing the process of human reasoning will be suitable for all application types. Certain problems may be solved only by the conventional approximate reasoning mechanism, while some others may be solved by the existing similarity based reasoning mechanism alone. But, similarity based approximate reasoning mechanism may be applied to solve both kinds of problems.

We know that, if we view the membership grade of an element of a fuzzy set as the resemblance degree between the element and prototypes of the fuzzy set then, one of the possible semantics of a fuzzy set may be given in terms of similarity [17]. For this, we only need to equip the referential set with a similarity relation. Eventually, it is possible to find another fuzzy reasoning method based on similarity, more in a logical sense. This is useful in considering interpolation in a logical setting. Work in detail may be found in [17]. Another approach to similarity based reasoning is to find a relation between fuzzy equivalence relations and fuzzy sets. It is possible to show that fuzzy equivalence relations provide a useful description of the indistinguishability in fuzzy sets. In all these methods, several aspects related with the design of fuzzy systems could not be considered.

By the end, we hope to have established the following facts :

- Similarity based reasoning and conventional approximate reasoning may be integrated into a single framework for reasoning with vague concepts.
- Similarity based approximate reasoning is a reasonably deep theory.
- There could be not just one but, various systems of similarity based approximate reasoning schema.
- Similarity based approximate reasoning may be made popular because of the scope of its application in different wide and challenging fields of investigation.

- Similarity based approximate reasoning schema are applicable to both rule-based and resolution-based design of fuzzy systems.
- The concept of similarity between fuzzy sets is useful not only in deriving a consequence but also in selecting rules from the rule-base to be fired, depending on a particular input specification.
- In similarity based approximate reasoning, the concept of specificity measure of fuzzy sets plays an important role in decision making in an uncertain environment.

In addition, we claim the following :

- Further investigations of similarity based approximate reasoning are possible.
- Design of similarity based Fuzzy systems is possible.
- Development of resolution-based model and derivation of an unified resolution principle based on similarity concept are possible.

3 Scope for further research

In future, work is needed

- to characterize, in general, the structure of similarity indices satisfying the axioms presented in Chapter 3 of this thesis,
- to study the same for arbitrary fuzzy sets and to develop a general framework of similarity based approximate reasoning,
- to test the applicability of the mechanism in other day-to-day decision making activities,
- to define resolvent, based on similarity concept and then develop a resolution principle based on similarity,
- to include generalised quantifiers in the scope of similarity based approximate reasoning,

- to define some algebraic structures that will ultimately give a characterisation on the choice of operators and then to derive a logic from the same, in order to make similarity based approximate reasoning systems more powerful.

Bibliography

[1] Allen J.F., 'Towards a general theory of action and time', *Artificial Intelligence*, 1984, vol. 23, pp. 123-154.

[2] Assilian S., E.H.Mamdani, 'Learning control algorithms in real dynamic systems', in Proc. IFAC/IFIP Int. Conf. on Digital Computer Applications to Process Control, Zurich, Springer, Berlin, 1994.

[3] Bellman R.E., L.A.Zadeh, 'Local and fuzzy logics', in G.Epstein, ed., Modern uses of multiple-valued logic, 1977, pp. 103-165.

[4] Benferhat S., D.Dubois, H.Prade, 'Possibilistic independence and plausible reasoning', in Proc. FAPT'95, eds., G.Cooman, D.Ruan, E.E.Kerre, 1995, pp. 47-63.

[5] Benferhat S., D.Dubois, H.Prade, 'Representing default rules in possibilistic logic', in Proc. of the 3rd international conf. on *Principles of Knowledge representation and reasoning(KR'92)*, eds. B.Nebel, C.Rich, W.Swartout, 1992, pp. 673-684.

[6] Bien Z., M.G.Chun, 'An inference network for bidirectional approximate reasoning based on an equality measure', *IEEE Trans. on Fuzzy Systems*, 1994, vol. 2, no. 2, pp. 177-180.

[7] Brewka G., 'Non-monotonic reasoning : logical foundations of commonsense reasoning', Cambridge University Press, Cambridge, 1991.

[8] Cao Z., A.Kandel, 'A new model of fuzzy reasoning', *Fuzzy Sets and Systems*, 1990, vol. 36, pp. 311-325.

[9] Chen S.M., 'A new approach to handling fuzzy decision making problems', *IEEE Trans. on SMC*, 1988, vol. 18, pp. 1012-1016.

[10] Chen S.M., 'A weighted fuzzy reasoning algorithm for medical diagnosis', *Decision support systems*, 1994, vol.11, pp. 37-43.

[11] Chi Z., H.Yan, 'ID3-derived fuzzy rules and optimized defuzzification for handwritten numeral recognition', *IEEE Trans. Fuzzy Systems*, 1996, vol.4, no.1, pp. 24-31.

[12] Chiu S.L., 'Extracting fuzzy rules for pattern classification by cluster estimation', in Proc. sixth Int. Fuz. Syst. Assoc, World Congress (IFSA '95), 1995.

[13] Chiu S.L., 'Extracting fuzzy rules from data for function approximation and pattern classification', in D. Dubois, H. Prade and R.R. Yager, eds., *Fuzzy Information Engineering*, Wiley and Sons, NY, pp. 149-162.

[14] Crusca F., 'Special issue on tense and aspect', *Computational Linguistics*, 1988.

[15] Delgrande J.P., 'An approach to default reasoning based on a first order conditional logic : Revised report ', *Artificial Intelligence*, 1988, vol. 36, pp. 63-90.

[16] Driankov D., H.Hellendoorn and M.Reinfrank, 'An introduction to fuzzy control', Springer-Verlag, New York, 1993.

[17] Dubois D., F.Esteva, P.Garcia, L.Godo, H.Prade, 'Steps to a logic of similarity', in Proc. IFSA World Congress VI, 1995, vol. II, pp. 583-586.

[18] Dubois D., H.Prade, 'Fuzzy sets and systems : theory and applications', Academic press, New York, 1980.

[19] Dubois D., H.Prade, 'A note on measures of specificity for fuzzy sets',*Int. J. General Systems*, 1985, vol.10, pp. 279-283.

[20] Dubois D., H.Prade, 'Possibility theory', Plenum press, NY, 1988.

[21] Dubois D., H.Prade, 'Default reasoning and possibility theory', *Artificial Intelligence*, 1988, vol.35, pp. 243-257.

[22] Dubois D., H.Prade, 'A unifying view of comparison indices in a fuzzy set theoretic framework', in R.R.Yager ed., *Fuzzy set and possibility theory – Recent developments*, Part-I, Pergamon Press, 1988, pp. 3-13.

[23] Dubois D., H.Prade, 'Processing fuzzy temporal knowledge', *IEEE Trans. on SMC*, 1989, vol.19, no.4, pp. 729-744.

[24] Dubois D., H.Prade, 'Fuzzy sets in approximate reasoning, part I: inference with possibility distribution', ***Fuzzy sets and systems***, 1990, vol. 40, no. 1, pp. 143-202.

[25] Dubois D., H.Prade, 'Conditional objects, possibility theory and default rules', in G.Crocco, L.Farinas del Cerro, A.Herzig, eds., Conditionals : From Philosophy to Computer Sciences, Oxford University press, 1992, pp. 311-346.

[26] Dubois D., H.Prade, C.Testemale, 'Fuzzy pattern matching with extended capabilities: Proximity notions, importance assessment, random sets', in Proc. NAFIPS'86 - Recent developments in the theory and applications of fuzzy sets, 1986.

[27] Dutta S., 'An event-based fuzzy temporal logic', in Proc. 18^{th} IEEE Int. Symp. on Multiple-Valued logic, Palma de Mallorca, Spain, 1988, pp. 64-71.

[28] Dutta S., 'A temporal logic for uncertain events and an outline of a possible implementation in an extension of PROLOG', in Proc. 4^{th} AAAI workshop on uncertainty in Artificial Intelligence, Univ. of Minnesota, 1988, pp. 90-97.

[29] Etherington D.W., 'Formalizing non-monotonic reasoning system', *Artificial Intelligence*, 1987, vol. 31, pp. 41-85.

[30] Everitt B.S., 'Cluster analysis', Edward Arnold: A division of Hodder and Stoughton, London, Melbourne, Auckland, Third edition, 1993.

[31] Fall T.C., 'Evidential reasoning with temporal aspects', in Proc. 5^{th} AAAI Nat. Conf. Artificial Intelligence, 1988, pp. 891-895.

[32] Fine K., 'Vagueness, truth and logic', *Synthese*, 1975, vol.30, pp. 265-300.

[33] Gebhardt J., R.Kruse, 'The context model - an integrating view of vagueness and uncertainty', *Inter. Jour. approximate reasoning*, 1993, vol.9, pp. 283-314.

[34] Gregson R.M., 'Psychometrics of similarity', Academic press, New York, 1975.

[35] Gupta M.M., A.Kandel, W.Bandler, J.B.Kiszka, 'Approximate Reasoning in Expert Systems', North-Holland, Amsterdam, 1985.

[36] Gupta M.M., J.Qi, 'Theory of T-norms and fuzzy inference methods', *Fuzzy sets and systems*, 1991, vol.40, no.3, pp. 431-450.

[37] Gupta M.M., R.K.Ragade, R.R.Yager, 'Advances in Fuzzy set theory and Applications', North-Holland, Amsterdam, 1979.

[38] Gupta M.M., E.Sanchez, 'Approximate reasoning in Decision Analysis', North-Holland, Amsterdam, 1982.

[39] Gupta M.M., T.Yamakawa, 'Fuzzy Computing : Theory, Hardware and Applications', Elsevier Science Publisher, The Netherlands, 1988.

[40] Hong D.H., S.Y.Hwang, 'A note on the value similarity of fuzzy systems variables', *Fuzzy Sets and Systems*, 1994, vol. 66, pp. 383-386.

[41] Hornstein N., 'Towards a theory of tense', *Linguistic Inquiry*, 1977, vol. 8, pp. 521-557.

[42] Kahn K.M., 'Mechanization of Temporal Knowledge', *Artificial Intelligence*, 1977, vol. 9, pp. 87-108.

[43] Kandel A., 'Fuzzy hierarchical classification of dynamic patterns', in NATO ASI, Pattern recognition and classification, France, September 1975.

[44] Kandel A., 'Fuzzy techniques and pattern recognition', John Wiley and Sons., 1982.

[45] Kandel A., 'Fuzzy mathematical techniques with applications', Addison-Wesley, Reading MA, 1986.

[46] Kandel A., G.Langholz, 'Hybrid architectures for intelligent systems', CRC Press, 1992.

[47] Klir G.J., T.A.Folger, 'Fuzzy sets, Uncertainty and Information', Prentice-Hall, Englewood cliffs, NJ, 1988.

[48] Klir G.J., B.Yuan, 'Fuzzy sets and Fuzzy logic : Theory and Applications', Prentice-Hall PTR, Upper saddle river, NJ, 1995.

[49] Kwang H.L., Y.S.Song and K.M.Lee, 'Similarity measure between fuzzy sets and between elements', *Fuzzy Sets and Systems*, 1994, vol. 62, pp. 291-293.

[50] Lee C.C., 'Fuzzy logic in control systems: Fuzzy logic controller, Parts I and II', *IEEE Transaction on SMC*, 1990, vol.20, no.2, pp. 404-435.

[51] Maeda H., S.Asaoka, S.Murakami, 'Dynamical fuzzy reasoning and its application to system modeling', *Fuzzy Sets and Systems*, 1996, vol.80, no.2, pp. 101-109.

[52] Mamdani E.H., 'Application of fuzzy algorithms for control of simple dynamic plant', in Proc. IEEE, 121(12), 1974, pp. 1585-1588.

[53] Mamdani E.H., S.Assilian, 'A case study on the application of fuzzy set theory to automatic control', in Proc. IFAC Stochastic control symposium, Budapest, 1974.

[54] Mamdani E.H., S. Assilian, 'An experiment in linguistic synthesis with a fuzzy logic controller', *Int. J. of Man-Machine studies*, 1975, pp. 1-13.

[55] Marks II R.J., 'Fuzzy logic technology and applications', IEEE Technical Activities Board, 1994.

[56] Mcdermott D., 'A temporal logic for reasoning about processes and plans', *Cognitive Science*, 1982, vol.6, pp. 101-155.

[57] Mizumoto M., 'Extended fuzzy reasoning', in Gupta, Kandel, Bandler, Kiszka, eds., Approximate Reasoning in Expert Systems, North-Holland, Amsterdam, 1985, pp. 71-85.

[58] Nakanishi H., I.B.Turksen, M.Sugeno, 'A review and comparison of six reasoning methods', *Fuzzy Sets and Systems*, 1993, vol.57, no.3, pp. 257-294.

[59] Nowakowska M., 'A new theory of time : generation of time from fuzzy temporal relations', BUSEFAL, 1981, no.6., pp. 87-84.

[60] Ogata K., 'Modern Control Engineering', Englewood Cliffs, NJ: Prentice-Hall, 1970.

[61] Pappis C.P., N.I.Karacapilidis, 'A comparative assessment of measures of similarity of fuzzy values', *Fuzzy Sets and Systems*, 1993, vol.56, pp. 171-174.

[62] Pedrycz W., 'Neurocomputations in relational systems', *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 1991, vol.13, no.3, pp. 289-297.

[63] Pedrycz W., 'Fuzzy sets in pattern recognition : methodology and methods', *Pattern Recognition*, 1993, vol.23, no.1/2, pp. 121-146.

[64] Poole D., 'A logical framework for default reasoning', *Artificial Intelligence*, 1988, vol.36, pp. 27-47.

[65] Prade H., 'Reasoning with fuzzy default values', in Proceedings 15^{th} IEEE International Symposium on Multiple-valued logic, Kingston, Ont. , 1985, pp. 191-197.

[66] Qian D.Q., 'Representation and use of imprecise temporal knowledge in dynamic Systems', *Fuzzy Sets and Systems*, 1992, vol. 50, no.1, pp. 59-77.

[67] Raha S., 'Similarity based approximate reasoning', in methodologies for the conception, design and application of intelligent systems, Proc. IIZUKA'96, 4^{th} Int. Conf. on Soft Computing, IIZUKA, Japan, 1996, pp. 414-417.

[68] Raha S., K.S.Ray, 'A critical appreciation on fuzzy logic controller design', in Proc. IEEE-TENCON '91, New-Delhi, INDIA, 1991.

[69] Raha S., K.S.Ray, 'Analogy between approximate reasoning and the method of interpolation', *Fuzzy Sets and Systems*, 1992, vol.51, no.3, pp. 259-266.

[70] Raha S., K.S.Ray, 'Approximate reasoning based on generalized disjunctive syllogism', *Fuzzy Sets and Systems*, 1994, vol.61, no.2, pp. 143-151.

[71] Raha S., K.S.Ray, 'On extended fuzzy reasoning', *Fuzzy Sets and Systems*, 1994, vol.62, no.1, pp. 121-125.

[72] Raha S., K.S.Ray, 'Approximate reasoning using truth-qualified vague predicates', Tech. Report, INSA visiting fellowship, 1993 - 1994, no. BS/VF/2/482, 1994.

[73] Raha S., K.S.Ray, 'Reasoning with vague truth', *Fuzzy Sets and Systems*, 1998, to appear.

[74] Raha S., K.S.Ray, 'Reasoning with vague default', *Fuzzy Sets and Systems*, 1997, vol. 91, no. 3, pp. 327-338.

[75] Raha S., K.S.Ray, 'Approximate reasoning with time', *Fuzzy Sets and Systems*, 1998, to appear.

[76] Ray K.S., T.K.Dinda, M.K.Chakraborty, 'Pattern classification using fuzzy relation calculus', in Methodologies for the conception, design and application of intelligent systems, Proceedings of Iizuka '96, 1996, pp. 217-220.

[77] Ray K.S., J.Ghoshal, 'Approximate reasoning approach to pattern recognition', *Fuzzy Sets and Systems*, 1996, vol.77, pp. 125-150.

[78] Ray K.S., J.Ghoshal, 'Neuro fuzzy approach to pattern recognition', *Neural Networks*, 1977, vol.10, no. 1, pp. 161-182.

[79] Reiter R., 'A logic for default reasoning', *Artificial Intelligence*, 1980, vol.13, pp. 81-132.

[80] Reiter R., G.Criscuolo, 'Some representational issues in default reasoning', *Int. Jour. of comput. Math.*, 1983, vol. 9, pp. 1-13.

[81] Trillas E., 'Sobre funciones de negacion en la teoria de conjuntos difusos', *Stochastica*, 1979, vol.3, no.1, pp. 47-60.

[82] Tsang F.C.C, J.W.T.Lee, D.S.Yeung, 'Similarity based fuzzy reasoning methods for fuzzy production rules', in Proc. IFSA, World Congress, 1995, pp. 157-160.

[83] Tsukamoto Y., 'An approach to fuzzy reasoning method', in M.M.Gupta, R.K.Ragade, R.R.Yager, eds., *Advances in fuzzy set theory and applications*, North-Holland, Amsterdam, 1979, pp. 137-149.

[84] Turksen I.B., Z.Zhong, 'An approximate analogical reasoning approach based on similarity measures', *IEEE Trans. on SMC*, 1988, vol.18, no.6, pp. 1049-1056.

[85] Turksen I.B., Z.Zhong, 'An approximate analogical reasoning scheme based on similarity measures and interval valued fuzzy sets', *Fuzzy Sets and Systems*, 1990, vol.34, no.3, pp. 323-346.

[86] Turksen I.B., Y.Tian, 'Combination of rules or their consequences in fuzzy expert systems', *Fuzzy sets and systems*, 1993, vol.58, no.1, pp. 3-40.

[87] Tversky A., 'Features of similarity', *Psychol. Rev.*, 1977, vol.84, pp. 327-352.

[88] Tzafestas S.G., A.N.Venetsanopoulos, 'Fuzzy reasoning in Information, Decision and Control systems', Kluwer Academic Publishers, The Netherlands, 1994.

[89] Van Le T., 'A fuzzy temporal logic scheme for fuzzy dynamic systems', in Proc. ICCN '95, ICEC '95, ANZIIS '95, Perth, 1995, pp. 152-157.

[90] Vere S., 'Planning in time : windows and durations for activities and goals', *IEEE Pattern Analysis and Machine Intelligence*, 1983, vol.3, pp. 246-267.

[91] Virant J., N.Zimic, 'Attention to time in fuzzy logic', *Fuzzy Sets and Systems*, 1996, vol.82, no.1, pp. 39-49.

[92] Vitek M., 'Fuzzy information and fuzzy time', in Proc. IFAC Symp. Fuzzy Information, Knowledge representation and decision analysis, Marseille, France, 1983, pp. 159-162.

[93] Wang Z., G.J.Klir, 'Fuzzy measure theory', Plenum press, NY, 1992.

[94] Wang W.J., 'New similarity measures on fuzzy sets and on elements', *Fuzzy Sets and Systems*, 1997, vol.85, pp. 305-309.

[95] Yager R.R., 'An approach to inference in approximate reasoning', *Internat. J. Man-Machine Stud.*, 1980, vol.13, pp. 323-338.

[96] Yager R.R., 'Measuring tranquility and anxiety in decision making: An application of fuzzy sets', *Int. J. General Systems*, 1982, vol.8, pp. 139-146.

[97] Yager R.R., 'Recent advances in fuzzy set and possibility theory', Elmsford, New York, Pergamon press, 1982.

[98] Yager R.R., 'Approximate reasoning as basis for rule-based expert systems', *IEEE Trans. Systems, Man and Cybernet.*, 1984, vol.14, no.4, pp. 636-646.

[99] Yager R.R., 'A characterisation of the fuzzy extension principle', *Fuzzy sets and Systems*, 1986, vol.18, pp. 205-217.

[100] Yager R.R., 'Using approximate reasoning to represent default knowledge', *Artificial Intelligence*, 1987, vol.31, pp. 99-112.

[101] Yager R.R., 'Expert systems using fuzzy logic', in R.R.Yager and L.A.Zadeh eds., An introduction to fuzzy logic applications in intelligent systems, Boston, Kluwer Academic, 1992.

[102] Yager R.R., 'Fuzzy sets and approximate reasoning in decision and control', in Proc. IEEE Int. Conf. Fuzzy Syst., San Diego, 1992, pp. 415-428.

[103] Yager R.R., D.P.Filev, 'Essentials of fuzzy modeling and control', New York, John Wiley, 1994.

[104] Yeung D.S., E.C.C.Tsang, 'Improved fuzzy knowledge representation and rule evaluation using fuzzy petri nets and degree of subsethood', *Intelligent Systems*, 1994, vol.9, no.12, pp.1083-1100.

[105] Yeung D.S., E.C.C.Tsang, 'A comparative study on similarity based fuzzy reasoning methods', *IEEE Trans. on SMC - Part B: Cyber.*, 1997, vol.27, no.2, pp. 216-227.

[106] Zadeh L.A., 'Fuzzy sets', *Information and Control*, 1965, vol.8, pp. 338-353.

[107] Zadeh L.A., 'Towards a theory of fuzzy systems', in R.E.Kalman, N.DeClaris, eds., *Aspects of network and system theory*, Holt, Rinehart and Winston, 1971, pp. 209-245.

[108] Zadeh L.A., 'Quantitative fuzzy semantics', *Information Science*, 1971, vol.3, pp. 159-176.

[109] Zadeh L.A., 'Outline of a new approach to the analysis of complex systems and decision processes', *IEEE Trans. SMC*, 1973, vol.3, pp. 28-44.

[110] Zadeh L.A., 'Fuzzy logic and approximate reasoning', *Synthese*, 1975, vol.30, pp. 407-428.

[111] Zadeh L.A., 'The concept of a linguistic variable and its application to approximate reasoning I, II, III', *Information Science*, 1975, vol.8, pp. 199-249, pp. 301-357, vol.9, pp. 43-93.

[112] Zadeh L.A., 'Fuzzy sets as a basis for a theory of possibility', *Fuzzy Sets and Systems*, 1978, vol.1, pp. 3-28.

[113] Zadeh L.A., 'PRUF-a meaning representation language for natural languages', *Internat. J. Man-Machine Stud.*, 1978, vol.10, pp. 395-460.

[114] Zadeh L.A., 'A theory of approximate reasoning', in J.E.Hayes, D.Mitchie, L.I.Mikulich, eds., Machine Intelligence, Vol.9, 1979, pp. 149-194, Wiley, New York.

[115] Zimmermann H.J., 'Description and optimization of fuzzy systems', *Int. Jour. Gen. Syst.*, 1976, vol.2, pp. 209-215.

[116] Zimmermann H.J., 'Fuzzy sets in pattern recognition', in P.A. Devijer and J. Kittler, eds., Pattern recognition : theory and applications, Springer-Verlag, Berlin, 1987, pp. 383-391.

[117] Zimmermann H.J., 'Fuzzy set theory and its applications', 2nd revised edition, Boston, Kluwer Academic, 1991.

[118] Zimmermann H.J., 'First European Congress on fuzzy and intelligent techniques - EUFIT '93', vol.1, Aachen, September, 1993, Verlag der Augustinus Buchhandlung, 1993.

[119] Zimmermann H.J., 'First European Congress on fuzzy and intelligent techniques - EUFIT '93', vol.2, Aachen, September, 1993, Verlag der Augustinus Buchhandlung, 1993.

[120] Zimmermann H.J., 'First European Congress on fuzzy and intelligent techniques - EUFIT '93', vol.3, Aachen, September, 1993, Verlag der Augustinus Buchhandlung, 1993.

[121] Zwick R., E.Carlstein, D.V.Budescu, 'Measures of similarity among fuzzy concepts : A comparative analysis', *Inter. Jour. of App. Reasoning*, 1987, vol.1, pp. 221-242.