Indian Statistical Institute
Second Semestral Examination
2024-2025 Academic Year
M.Stat. First Year
Large Sample Statistical Methods

Date : 28.04.2025 Maximum Marks: 50 Duration :- 3 hours

Answer as many questions as you can. The maximum you can score is 50.

1. Suppose you have iid observations from a distribution having a unique median. Prove that the (smallest) sample median converges to the population median almost surely. [8]

2. Suppose $X_1, \ldots, X_n$ are iid with a common density $f(x, \theta)$, where $\theta \in \Theta$, Θ consisting of only finitely many real numbers. Assume also that the densities under different θ's have the same support and that the distributions under different θ's are different. If θ_0 is the true value of θ, prove that with probability tending to 1 (under θ_0) as $n \to \infty$, the likelihood function will be maximized at the value $\theta = \theta_0$. [7]

3. Assume the usual regularity conditions used in the proof of asymptotic normality of consistent roots of the likelihood equation in the setup of iid sampling from a distribution on the real line with a density. Assume that the parameter space Θ is an open interval. Suppose $\hat{\theta}_n$ is a consistent sequence of roots of the likelihood equation. Prove that with probability tending to 1 (under θ_0) as $n \to \infty$, $\hat{\theta}_n$ is a local maximum of the likelihood function. Here θ_0 is the true value of the parameter. [6]

4. Suppose you have a random sample of size n from a multinomial population with 6 cells where the cell probabilities are all equal to $\frac{1}{6}$. Derive the asymptotic distribition of $\sum_{i=1}^{6} \frac{(n_i - n/6)^2}{n_i}$, where n_i is the observed frequency of the ith class for $i = 1, \ldots, 6$. If a particular n_i is zero, you may redefine the corresponding term in the above sum as 0. [6]

5. (a) Let $\{X_i\}_{i \geq 1}$ be iid with a distribution F such that $\sup\{x \in \mathbb{R} : \mathrm{F(x)} < 1\} = +\infty$. Assume also that for all $x \geq 2$, $F(x)$ is differentiable with $f(x) = F'(x)$. Assume further that as $x \to \infty$, $\frac{xf(x)}{1-F(x)}$ tends to 3. Does $\frac{X_{(n)} - b_n}{a_n}$ have a non-degenerate asymptotic distribution for some appropriate sequences of constants $\{a_n\}$ and $\{b_n\}$? Prove your answer. [5]

   (b) Suppose $\{X_i\}_{i \geq 1}$ are iid $N(0, 1)$ random variables. Show that $\frac{X_{(n)} - b_n}{a_n}$ converges in distribution to a non-degenerate limit, where $\{b_n\}$ and $\{a_n > 0\}$ are real sequences to be determined by you. Here $X_{(n)}$ is the maximum of $\{X_1, \ldots, X_n\}$. Justify all your steps. [8]

6. Suppose $X_1, \ldots, X_{2n}$ are iid $N(0, 1)$. Find the asymptotic distribution of

$$\frac{n(X_1X_2 + X_3X_4 + \ldots + X_{2n-1}X_{2n})^2}{(\sum_{i=1}^{2n} X_i^2)^2}.$$

Prove your assertion. [7]

7. Suppose F_n for $n \geq 1$ and F are distribution functions on the real line such that $F_n(a_n x + b_n) \to F(x)$ as $n \to \infty$ for each continuity point x of F, where a_n and b_n are real constants for each $n \geq 1$. Prove that $a_n > 0$ when n is sufficiently large. [6]