Indian Statistical Institute
M. Math 2nd year
Academic year 2025-2026
Third semester
Endsem Examination
Course: Riemann surfaces
24 - 11 - 25
3 hours

- *Answer as many questions as you can.*
- *You may use results proved in class, but make sure to state them clearly.*
- *Maximum marks is 60.*
- *Unless otherwise mentioned, all Riemann surfaces are assumed to be connected.*

1. For an irreducible nonconstant polynomial $P \in \mathbb{C}[z, w]$ let $C_P = \{(z, w) \in \mathbb{C}^2 \mid P(z, w) = 0\} \subset \mathbb{C}^2, \overline{C_P} \subset \mathbb{CP}^2$ denote the corresponding affine and projective algebraic curves respectively, and let X_P denote the normalization of $\overline{C_P}$. For the polynomial $P(z, w) = w^n + z^n - 1$, where $n \geq 2$:

   a) Find the points at infinity and singular points of $\overline{C_P}$.

   b) Compute the genus of X_P.

   (5+10 = 15 marks)

2. Let $P(z, w) = w^2 - Q(z)$ where Q is a polynomial of degree 3 with distinct roots.

   a) Show that the projective algebraic curve $\overline{C_P}$ is nonsingular.

   b) Let $x, y \in \mathcal{M}(\overline{C_P})$ be the meromorphic functions on $\overline{C_P}$ induced by the coordinate functions on $\mathbb{C}^2$. Show that the meromorphic 1-form $\omega = dx/y$ is in fact a holomorphic 1-form which is nowhere zero.

   (4+8 = 12 marks)

3. Let X be a compact Riemann surface of genus $g \geq 0$.

a) Let f be a nonconstant meromorphic function on X. Show that

$$\sum_{p \in X} \mathrm{ord}_p(df) = 2g - 2.$$

b) Hence show that for any meromorphic 1-form ω on X we have

$$\sum_{p \in X} \mathrm{ord}_p(\omega) = 2g - 2.$$

(8+4 = 12 marks)

4. Let X, Y be Riemann surfaces and let $J_X : TX \to TX, J_Y : TY \to TY$ denote the canonical bundle endomorphisms defining the complex structures on the tangent spaces of X and Y respectively. Show that a C^1 map $f : X \to Y$ is holomorphic if and only if $df \circ J_X = J_Y \circ df$.

(5 marks)

5. Let X be a Riemann surface, and let ω be a C^∞ complex 1-form on X. Recall that ω is said to be a holomorphic 1-form if it is locally of the form $\omega = df$ for some locally defined holomorphic function f. Recall that ω is said to be a $(1,0)$ form if $\omega_p : T_pX \to \mathbb{C}$ is $\mathbb{C}$-linear for all $p \in X$. Show that the following are equivalent:

a) ω is a holomorphic 1-form.

b) For any local coordinate $z : U \subset X \to \mathbb{C}$, $\omega = g dz$ on U where g is a holomorphic function on U.

c) ω is a $(1,0)$-form and $d\omega = 0$.

(12 marks)

6. Let X be a compact Riemann surface of genus g and let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be simple closed curves on X forming a canonical homology basis of X (so the algebraic intersection numbers satisfy $a_i \cdot b_j = \delta_{ij}$).

a) Show that for any two C^∞, closed, complex 1-forms α, β on X we have

$$\int_X \alpha \wedge \beta = \sum_{j=1}^{g} \left(\int_{a_j} \alpha \int_{b_j} \beta - \int_{b_j} \alpha \int_{a_j} \beta \right)$$

b) Show that if a holomorphic 1-form ω on X satisfies $\int_{a_j} \omega = 0$ for all $j = 1, \ldots, g$, then $\omega \equiv 0$.

c) Show that there exist holomorphic 1-forms $\omega_1, \ldots, \omega_g$ on X such that $\int_{a_j} \omega_i = \delta_{ij}$ for all $1 \leq i, j \leq g$.

(8+3+3 = 14 marks)