INDIAN STATISTICAL INSTITUTE

First Semester Examination: 2025–2026

Master of Science in Quantitative Economics. Year 1. Semester 1

Mathematical Methods

Date: November 19, 2025 | Maximum Marks: 60 | Duration: 3 hours

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$\boxed{1}$ The questions are divided into two groups. Answer both the groups. $\boxed{2}$ You should present all your arguments while answering a question. $\boxed{3}$ You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

$\boxed{\text{GROUP A: Answer ANY four questions.}}$

1. Show that the vectors $\mathbf{v}_1 := (0, 1, \alpha), \mathbf{v}_2 := (\alpha, 1, 0), \mathbf{v}_3 := (1, \alpha, 1)$ in $\mathbb{R}^3$ are linearly dependent (over $\mathbb{R}$) iff $\alpha^3 = 2\alpha$. [10]

2. Let S, T be subspaces of $\mathbb{R}^4$ (over $\mathbb{R}$) given by

$$S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 = 0,\ x_3 + x_4 = 0\},$$
$$T = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_3 = 0,\ x_2 + x_4 = 0,\ x_1 + x_4 = 0\}.$$

   (a) Show that $S \oplus T = \mathbb{R}^4$. [6]

   (b) Let $\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$. Find the projection of $\mathbf{u}$ into S along T. [4]

3. Let $\mathcal{P}_4$ denote the set of polynomials in $t \in \mathbb{R}$ with real coefficients and of degree at most 3. Consider the differentiation transformation f from $\mathcal{P}_4$ to $\mathcal{P}_4$ defined by

$$[f(p)](t) = a_1 + 2a_2 t + 3a_3 t^2,\ \ t \in \mathbb{R},$$

where $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. Let $\mathcal{X}$ be the basis $\{1, t, t^2, t^3\}$ of $\mathcal{P}_4$. Find the matrices of (1) f, (2) $f \circ f$, (3) $f \circ f \circ f \circ f$ with respect to $\mathcal{X}$ and $\mathcal{X}$. [5+3+2=10]

4. Let $\mathbf{A}$ be and $\mathbf{B}$ be $m \times n$ and $n \times p$ matrices, respectively. Let $\mathbf{A}$ be of full column rank. Show that $\rho(\mathbf{AB}) = \rho(\mathbf{B})$. [10]

5. Let $\mathbf{A}$ be and $\mathbf{B}$ be $m \times n$ and $m \times p$ matrices, respectively. Show that $\rho[\,\mathbf{A} : \mathbf{B}\,] = \rho(\mathbf{A})$ iff $\mathbf{B} = \mathbf{AC}$ for some matrix $\mathbf{C}$. [10]

6. Consider the 4×4 matrix $\mathbf{A} = ((a_{ij}))$, where $a_{ij} = i + 4(j-1)$.

   (a) Obtain a rank-factorization of $\mathbf{A}$. [5]

   (b) Show that $\{(1, -2, 1, 0), (2, -3, 0, 1)\}$ is a basis of $\mathcal{N}(\mathbf{A})$. [5]

[P.T.O.]

GROUP B: Answer <u>ANY</u> two questions.

7. Let (X, d) be a complete metric space and let Y be a subspace of X (i.e., $Y \subseteq X$). Show that Y is complete $\Leftrightarrow$ it is closed. [10]

8. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover. You have to justify all your assertions. [10]

9. Suppose f is a continuous one-to-one mapping of a compact metric space X onto a metric space Y. Then show that the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x$, $x \in X$, is a continuous mapping of Y onto X. [10]
